Math. Proc. Camb. Phil. Soc. (1979), **85**, 281
Printed in Great Britain

Remarks on generalized analytic independence

By GIUSEPPE VALLA
Istituto Matematico, University of Genoa, Italy

(*Received* 10 *March* 1978)

This paper is concerned with the notion of independence relating sets of elements in a ring A to a proper ideal $\mathfrak{a}$ of A. A set of elements $a_1, \ldots, a_n \in A$ is called $\mathfrak{a}$-independent if every form in $A[X_1, \ldots, X_n]$ vanishing at $a_1, \ldots, a_n$ has all its coefficients in $\mathfrak{a}$. This notion leads to many questions (cf. (2) and (12)), which are of some interest in their own right, several of which are considered here. On the other hand, this independence is related to the structure of the graded ring associated to the ideal generated by the set of elements, hence is often relevant to some problems concerning regular sequences and Koszul complex.

A brief summary of the results obtained in this paper will now be given.

Let $\mathfrak{a} = (a_1, \ldots, a_n)$ be an ideal of a commutative ring A; then it is well known that if A is noetherian, $a_1, \ldots, a_n$ are $\mathfrak{a}$-independent if and only if $H_1(a_1, \ldots, a_n; A) = 0$, where $H_1(a_1, \ldots, a_n; A)$ is the first homology group of the Koszul complex $K(a_1, \ldots, a_n; A)$. In Section 1 we extend this result to the non noetherian case, but we need some additional conditions that are considered in Proposition 1·9 and Theorem 1·11. Furthermore, in Theorem 1·4 it is shown that $H_1(a_1, \ldots, a_n; A) = 0$ if and only if $\mathfrak{a}/\mathfrak{a}^2$ is a free $A/\mathfrak{a}$-module of rank n and the symmetric and Rees algebras of the ideal $\mathfrak{a}$ are isomorphic. In particular we answer a question asked by C. P. L. Rhodes in (9), p. 5.

In Section 2 it is shown that if $\mathfrak{a} = (a_1, \ldots, a_n)$ is an ideal of a noetherian ring A and $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{a}$, then $h(\mathfrak{p}) = n$ if and only if $a_1, \ldots, a_n$ are $\mathfrak{p}$-independent (Theorem 2·6). This gives a generalisation, from local to semilocal rings, of the theorem of analytic independence of systems of parameters. The key tool in this section is Proposition 2·3, which extends the result of Eisenbud, Herrmann and Vogel in (4) concerning the so called lifting property for regular sequences.

In Section 3, if $\mathfrak{a}$ is an ideal of a neotherian ring A, $\sup(\mathfrak{a})$ is defined as the maximum number of $\mathfrak{a}$-independent elements in $\mathfrak{a}$. Then it is proved that S_1 (Serre condition) rings can be characterized by the equality of sup and height for all principal ideals of A (Proposition 3·1), while it is shown that, for a ring A of dimension $\geqslant 2$, the condition $\sup(\mathfrak{a}) = h(\mathfrak{a})$ for all ideals $\mathfrak{a}$ of A is not equivalent either to A being Cohen-Macaulay, or to A being S_1. Furthermore evidence is given for the conjecture that, if A is a local ring of dimension 2, then the equality $\sup(\mathfrak{a}) = h(\mathfrak{a})$ holds for all ideals $\mathfrak{a}$ of A, if and only if the ideal (0) is isobathic (cf. Definition 3·3).

1. Throughout this paper 'ring' will mean a commutative ring with unity and 'ideal' will mean a proper ideal. We will often be concerned with the notion of independence of a set of elements with respect to an ideal of A. Hence we begin with the following definition.

https://doi.org/10.1017/S0305004100055699 Published online by Cambridge University Press

DEFINITION 1·1. *If $a_1, \dots, a_n$ are elements of A and $\mathfrak{a}$ is an ideal of A, we say that $a_1, \dots, a_n$ are $\mathfrak{a}$-independent if any form $F(X_1, \dots, X_n) \in A[X_1, \dots, X_n]$ such that*

$$F(a_1, \dots, a_n) = 0,$$

must have all of its coefficients in $\mathfrak{a}$.

The notion of $\mathfrak{a}$-independence is related to the structure of the Rees algebra of the ideal generated by the set of elements. Recall that the Rees algebra of an ideal $\mathfrak{a}$ in a ring A, is the graded ring $R_A(\mathfrak{a}) = \bigoplus_{n=0}^{\infty} \mathfrak{a}^n$, where $\mathfrak{a}^0 = A$. If $a_1, \dots, a_n$ is a basis of $\mathfrak{a}$, then $R_A(\mathfrak{a}) = A[a_1 T, \dots, a_n T] \subset A[T]$.

PROPOSITION 1·2. (Cf. (12), teorema 2·1.) *Let $\mathfrak{a} = (a_1, \dots, a_n) \subseteq \mathfrak{b}$ be ideals of A; then $a_1, \dots, a_n$ are $\mathfrak{b}$-independent if and only if $R_A(\mathfrak{a}) \otimes_A (A/\mathfrak{b})$ is isomorphic to*

$$(A/\mathfrak{b})[X_1, \dots, X_n].$$

Let us consider the form ring of A relative to $\mathfrak{a}$; this is the graded $A/\mathfrak{a}$-algebra

$$G_A(\mathfrak{a}) = \bigoplus_{n=0}^{\infty} \mathfrak{a}^n/\mathfrak{a}^{n+1}.$$

It is clear that $G_A(\mathfrak{a}) = R_A(\mathfrak{a}) \otimes_A (A/\mathfrak{a})$; thus if $\mathfrak{a} = (a_1, \dots, a_n)$ then $a_1, \dots, a_n$ are $\mathfrak{a}$-independent if and only if $G_A(\mathfrak{a})$ is isomorphic to $(A/\mathfrak{a})[X_1, \dots, X_n]$.

One also has the notion of the symmetric algebra $S_A(\mathfrak{a})$ of an ideal $\mathfrak{a}$ of the ring A. The map $\mathfrak{a} \to R_A(\mathfrak{a})$ given by $a \to aT$ induces an A-algebra epimorphism $\phi: S_A(\mathfrak{a}) \to R_A(\mathfrak{a})$.

Furthermore, if $a_1, \dots, a_n$ are elements of A, let $K(a_1, \dots, a_n; A)$ be the Koszul complex $A\langle X_1, \dots, X_n\rangle: \partial X_i = a_i$, that is the exterior algebra generated by $\{X_1, \dots, X_n\}$ with boundary map $\partial X_i = a_i$. The homology groups of $K(a_1, \dots, a_n; A)$ are denoted by $H_i(a_1, \dots, a_r; A)$. Let $M = [a_1, \dots, a_n]$ and $N = [b_1, \dots, b_n]$, where $b_i \in A$, be $1 \times n$ matrices. Then $\sum_{i=1}^n b_i X_i$ is a one-cycle of $K(a_1, \dots, a_n; A)$ if and only if $NM^t = 0$, and $\sum_{i=1}^n b_i X_i$ is a one-boundary of $K(a_1, \dots, a_n; A)$ if and only if $N = MS$ for some $n \times n$ skew-symmetric matric S, where $S = [s_{ij}]$ is skew-symmetric if and only if $s_{ii} = 0$ and $s_{ij} = -s_{ji}$.

PROPOSITION 1·3 (*cf.* (7), *proposition* 1). *Let $\mathfrak{a} = (a_1, \dots, a_n)$ be an ideal of A. Then*:

(1) *If $H_1(a_1, \dots, a_n; A) = 0$, $\phi: S_A(\mathfrak{a}) \to R_A(\mathfrak{a})$ is bijective.*

(2) *If $\phi: S_A(\mathfrak{a}) \to R_A(\mathfrak{a})$ is bijective, any one-cycle $\sum b_i X_i$ of $K(a_1, \dots, a_n; A)$ such that $b_i \in \mathfrak{a}$ for each $i = 1, \dots, n$, is a one-boundary.*

We remark that, even for an ideal $\mathfrak{a} = (a_1, \dots, a_n)$ in a noetherian local ring A, the condition $S_A(\mathfrak{a})$ is isomorphic to $R_A(\mathfrak{a})$ does not imply $H_1(a_1, \dots, a_n; A) = 0$. In fact let $A = k[\![X, Y]\!]$, $\mathfrak{a} = (X^2, XY)$; then it is easy to check that $R_A(\mathfrak{a}) \simeq S_A(\mathfrak{a})$ but $YX_1 - XX_2$ is a nonbounding cycle.

The next theorem fills in the gap in Proposition 1·3.

THEOREM 1·4. *Let $\mathfrak{a} = (a_1, \dots, a_n)$ be an ideal of A. Then $H_1(a_1, \dots, a_n; A) = 0$ if and only if $S_A(\mathfrak{a})$ is isomorphic to $R_A(\mathfrak{a})$ and $\mathfrak{a}/\mathfrak{a}^2$ is a free $A/\mathfrak{a}$-module of rank n.*

https://doi.org/10.1017/S0305004100055699 Published online by Cambridge University Press

Proof. If $H_1(a_1, \ldots, a_n; A) = 0$, $S_A(\mathfrak{a}) \simeq R_A(\mathfrak{a})$ by Proposition 1·3. Denote by $\bar{a}$ the image in $\mathfrak{a}/\mathfrak{a}^2$ of $a \in \mathfrak{a}$; then $\bar{a}_1, \ldots, \bar{a}_n$ is a set of generators of $\mathfrak{a}/\mathfrak{a}^2$. Now if

$$\sum_{i=1}^{n} b_i a_i \in \mathfrak{a}^2 \quad \text{then} \quad \sum_{i=1}^{n} (b_i - c_i) a_i = 0,$$

with $c_i \in \mathfrak{a}$; but $H_1(a_1, \ldots, a_n; A) = 0$, so $b_i - c_i \in \mathfrak{a}$ and this proves that $\bar{a}_1, \ldots, \bar{a}_n$ is a basis of the $A/\mathfrak{a}$-module $\mathfrak{a}/\mathfrak{a}^2$.

Conversely, since $\mathfrak{a}/\mathfrak{a}^2$ is a free $A/\mathfrak{a}$-module of rank n and $\bar{a}_1, \ldots, \bar{a}_n$ is a set of n generators of $\mathfrak{a}/\mathfrak{a}^2$, $\bar{a}_1, \ldots, \bar{a}_n$ is a basis of $\mathfrak{a}/\mathfrak{a}^2$ (cf. (1), p. 45). Thus if $\Sigma_{i=1}^{n} b_i X_i$ is a one-cycle of $K(a_1, \ldots, a_n; A)$, we get

$$\sum_{i=1}^{n} b_i a_i = 0$$

hence $b_i \in \mathfrak{a}$ for each $i = 1, \ldots, n$. Therefore $\Sigma_{i=1}^{n} b_i X_i$ is a one-boundary by (2) of Proposition 1·3.

COROLLARY 1·5. *Let* $\mathfrak{a} = (a_1, \ldots, a_n)$ *be an ideal of* A *such that* $H_1(a_1, \ldots, a_n; A) = 0$. *Then for every basis* $b_1, \ldots, b_n$ *of* $\mathfrak{a}$ *consisting of* n *elements,* $H_1(b_1, \ldots, b_n; A) = 0$.

Remark 1·6. The preceding result answers a question asked by C. P. L. Rhodes in (9), p. 5.

Another consequence of Theorem 1·4 is the following result (cf. (6) and (9)).

COROLLARY 1·7. *Let* $\mathfrak{a} = (a_1, \ldots, a_n)$ *be an ideal of* A *such that* $H_1(a_1, \ldots, a_n; A) = 0$. *Then* $G_A(\mathfrak{a})$ *is isomorphic to* $(A/\mathfrak{a})[X_1, \ldots, X_n]$, *hence* $a_1, \ldots, a_n$ *are* $\mathfrak{a}$*-independent.*

Proof. We have $R_A(\mathfrak{a}) \simeq S_A(\mathfrak{a})$, hence $R_A(\mathfrak{a}) \otimes_A (A/\mathfrak{a}) \simeq S_A(\mathfrak{a}) \otimes_A (A/\mathfrak{a})$. Therefore $G_A(\mathfrak{a}) \simeq S_{A/\mathfrak{a}}(\mathfrak{a}/\mathfrak{a}^2) = (A/\mathfrak{a})[X_1, \ldots, X_n]$, since $\mathfrak{a}/\mathfrak{a}^2$ is a free $A/\mathfrak{a}$-module of rank n.

Remark 1·8. As is remarked in (9), theorem 2, the converse of Corollary 1·7 holds if A is noetherian. However, in (6), T. Kabele constructed a non noetherian local domain $(A, \mathfrak{m})$ for which $\mathfrak{m}$ is generated by two elements a_1, a_2 such that a_1, a_2 are $\mathfrak{m}$-independent but $H_1(a_1, a_2; A) \neq 0$.

This section will be closed by studying, in the non noetherian case, under which additional conditions the converse of Corollary 1·7 holds.

The case of a principal ideal is covered by the following result which is used also in the general situation.

PROPOSITION 1·9. *Let* $\mathfrak{a} = (a)$ *be an ideal of* A. *Then* $H_1(a; A) = 0$ *if and only if* a *is* $\mathfrak{a}$*-independent and the chain of ideals* $0 : a^n$ *is stationary.*

Proof. Since $H_1(a; A) = 0 : a$ the 'only if' part is obvious. Conversely we assume that a is $\mathfrak{a}$-independent and let $r \geqslant 1$ be an integer such that $0 : a^r = 0 : a^t$ for all $t \geqslant r$. Since a is $\mathfrak{a}$-independent, $0 : a^p \subseteq (a)$ for each $p \geqslant 1$; it follows that

$$0 : a^p = (a) \cap (0 : a^p) = a(0 : a^{p+1})$$

for each $p \geqslant 1$. Therefore $0 : a = a(0 : a^2) = \ldots = a^r(0 : a^{r+1}) = a^r(0 : a^r) = 0$.

https://doi.org/10.1017/S0305004100055699 Published online by Cambridge University Press

Remark 1·10. Given any element $a \in A$, we denote by a^* the initial form of a relative to $\mathfrak{a}$; we have $a^* = 0$ if $a \in \bigcap_{n=0}^{\infty} \mathfrak{a}^n$ and if $a \in \mathfrak{a}^t$, $a \notin \mathfrak{a}^{t+1}$, a^* is the residue class of a in $\mathfrak{a}^t/\mathfrak{a}^{t+1}$. Furthermore if $\mathfrak{b}$ is an ideal in A, we shall denote by $\mathfrak{b}^*$ the homogeneous ideal of $G_A(\mathfrak{a})$ generated by all the initial forms of the elements of $\mathfrak{b}$. It is easy to see that $\mathfrak{b}^*$ is the kernel of the natural epimorphism of graded rings $f: G_A(\mathfrak{a}) \to G_{A/\mathfrak{b}}(\mathfrak{a}+\mathfrak{b}/\mathfrak{b})$.

If $\mathfrak{b} = (b_1, \ldots, b_n)$ it is not necessarily true that $\mathfrak{b}^* = (b_1^*, \ldots, b_n^*)$; however, in (11), proposition 2·1, it is proved that if $b_1^*, \ldots, b_n^*$ is a regular sequence in $G_A(\mathfrak{a})$, then $\mathfrak{b}^* = (b_1^*, \ldots, b_n^*)$.

THEOREM 1·11. *Let* $\mathfrak{a} = (a_1, \ldots, a_n)$ *be an ideal of* A. *Then* $H_1(a_1, \ldots, a_n; A) = 0$ *if and only if the following conditions are satisfied*:

(1) $a_1, \ldots, a_n$ *are* $\mathfrak{a}$*-independent.*

(2) *The chain of ideals* $(a_1, \ldots, a_{n-1}) : a_n^t$ *is stationary.*

(3) $H_1(a_1, \ldots, a_{n-1}; A) = a_n H_1(a_1. \ldots, a_{n-1})$.

Proof. We have the long exact sequence (cf. (8), proposition 2, p. 365):

$$\ldots \longrightarrow H_1(a_1, \ldots, a_{n-1}; A) \xrightarrow{(-a_n)} H_1(a_1, \ldots, a_{n-1}; A) \longrightarrow H_1(a_1, \ldots, a_n; A)$$
$$\longrightarrow H_0(a_1, \ldots, a_{n-1}; A) = A/(a_1, \ldots, a_{n-1}) \xrightarrow{(a_n)} H_0(a_1, \ldots, a_{n-1}; A) \longrightarrow \ldots .$$

Hence, if $H_1(a_1, \ldots, a_n; A) = 0$, we have $H_1(a_1, \ldots, a_{n-1}; A) = a_n H_1(a_1, \ldots, a_{n-1}; A)$ and a_n is a non zero-divisor modulo $(a_1, \ldots, a_{n-1})$; thus (2) and (3) follow while (1) is proved in Corollary 1·7.

Conversely, since (3) holds, we need only prove that a_n is a non-zero-divisor modulo $(a_1, \ldots, a_{n-1})$. Now if $\bar{A} = A/(a_1, \ldots, a_{n-1})$ and $\bar{\mathfrak{a}} = \mathfrak{a}/(a_1, \ldots, a_{n-1})$, we have

$$G_{\bar{A}}(\bar{\mathfrak{a}}) = G_A(\mathfrak{a})/(a_1, \ldots, a_{n-1})^*.$$

Since $a_1, \ldots, a_n$ are $\mathfrak{a}$-independent, $G_A(\mathfrak{a}) = (A/\mathfrak{a})[X_1, \ldots, X_n]$ and in this isomorphism $X_1, \ldots, X_n$ correspond to the initial forms $a_1^*, \ldots, a_n^*$ of $a_1, \ldots, a_n$; therefore $a_1^*, \ldots, a_{n-1}^*$ is a regular sequence in $G_A(\mathfrak{a})$, hence, by the preceding remark,

$$(a_1, \ldots, a_{n-1})^* = (a_1^*, \ldots, a_{n-1}^*).$$

Thus $$G_{\bar{A}}(\bar{\mathfrak{a}}) = (A/\mathfrak{a})[X_1, \ldots, X_n]/(X_1, \ldots, X_{n-1}) = (A/\mathfrak{a})[X_n];$$

this proves that $\bar{a}_n$ is $(\bar{a}_n)$-independent and since by (2) the chain of ideals $0 : \bar{a}_n^t$ is stationary, it follows from Proposition 1·9 that $\bar{a}_n$ is a non zero-divisor in $\bar{A}$, as required.

2. In this section all rings are assumed to be noetherian. The main result in this section is a generalisation of the following well-known result proved essentially by Zariski in his fundamental paper (13): if $a_1, \ldots, a_n$ is a system of parameters of a noetherian local ring $(A, \mathfrak{m})$, $a_1, \ldots, a_n$ are $\mathfrak{m}$-independent (analytically independent).

We start with some remarks and examples.

Let $\mathfrak{a}$ be an ideal of A; following (12) sup $(\mathfrak{a})$ is defined as the maximum number of $\mathfrak{a}$-independent elements in $\mathfrak{a}$.

Remark. 2·1. Let A be a semilocal ring of dimension d and J be the Jacobson radical of A; if $a_1, \ldots, a_d$ is a system of parameters of A, it is not necessarily true that $a_1, \ldots, a_d$

https://doi.org/10.1017/S0305004100055699 Published online by Cambridge University Press

are J-independent. In fact in (12), teorema 5·4, we proved that for any radical ideal $\mathfrak{a}$ of a ring A, $\sup(\mathfrak{a}) = h(\mathfrak{a})$; hence if $h(J) < d$, $a_1, \ldots, a_d$ cannot be J-independent. However putting $h(J) = h$, one can construct elements $a_1, \ldots, a_h$ belonging to some system of parameters of R such that $a_1, \ldots, a_h$ are J-independent (cf. (12), proposition 5·3). The following example shows that this does not hold for every set of h elements belonging to some system of parameters of A.

Example 2·2. Let $R = k[X, Y, Z, T]/(XZ, YZ) = k[x, y, z, t]$, $\mathfrak{p}_1 = (x, y, t)$,

$$\mathfrak{p}_2 = (y, z, t); \quad \text{then} \quad h(\mathfrak{p}_1) = 1, \quad h(\mathfrak{p}_2) = 2.$$

Now let M be the complement of $\mathfrak{p}_1 \cup \mathfrak{p}_2$ in R and $A = R_M$; then R_M is a semilocal ring of dimension 2 whose Jacobson radical J has height 1. It is clear that $\dim A/(y) = 1$ but y is not J-independent since $yz = 0$ and $z \notin J$.

The next elementary result is often useful in dealing with the notion of $\mathfrak{a}$-independence. As a corollary we obtain very easily the so called 'lifting' property for regular sequences' (cf. (4) and (5)).

PROPOSITION 2·3. *Let $f: A \to B$ be a ring homomorphism and $\mathfrak{a} = (a_1, \ldots, a_n) \subseteq \mathfrak{b}$ ideals of A such that $\mathfrak{b}^{ec} = \mathfrak{b}$. If $f(a_1), \ldots, f(a_n)$ are $\mathfrak{b}^e$-independent, then $a_1, \ldots, a_n$ are $\mathfrak{b}$-independent.*

Proof. Denote by $\overline{F}(X_1, \ldots, X_n)$ the image in $B[X_1, \ldots, X_n]$ of the element

$$F(X_1, \ldots, X_n) \in A[X_1, \ldots, X_n].$$

If $\quad F(a_1, \ldots, a_n) = 0 \quad$ then $\quad 0 = f(F(a_1, \ldots, a_n)) = \overline{F}(f(a_1), \ldots, f(a_n))$;

hence $\overline{F}(X_1, \ldots, X_n)$ has its coefficients in $\mathfrak{b}^e$ and $F(X_1, \ldots, X_n)$ has its coefficients in $\mathfrak{b}^{ec} = \mathfrak{b}$, as required.

In the following if M is an A-module and $a_1, \ldots, a_n$ are elements of A, then we denote by $H(a_1, \ldots, a_n; M)$ the homology of the complex $K(a_1, \ldots, a_n; A) \otimes_A M$.

COROLLARY 2·4. *Let $f: A \to B$ be a ring homomorphism and $\mathfrak{a} = (a_1, \ldots, a_n)$ an ideal of A such that $\mathfrak{a}^{ec} = \mathfrak{a}$. If $H_1(a_1, \ldots, a_n; B) = 0$ then $H_1(a_1, \ldots, a_n; A) = 0$.*

Proof. We have $H_1(a_1, \ldots, a_n; B) = H_1(f(a_1), \ldots, f(a_n); B)$. By Corollary 1·7 if

$$H_1(f(a_1), \ldots, f(a_n); B) = 0,$$

$f(a_1), \ldots, f(a_n)$ are $\mathfrak{a}^e$-independent, hence $a_1, \ldots, a_n$ are $\mathfrak{a}$-independent. But since A is noetherian, this implies $H_1(a_1, \ldots, a_n; A) = 0$.

COROLLARY 2·5. *Let $f: A \to B$ be a ring homomorphism and let $\mathfrak{a} = (a_1, \ldots, a_n)$ be an ideal of A, such that $a_1, \ldots, a_n$ belong to the Jacobson radical of A and $f(a_1), \ldots, f(a_n)$ is a regular sequence in B. If $\mathfrak{a}^{ec} = \mathfrak{a}$, then $a_1, \ldots, a_n$ is a regular sequence in A.*

Proof. It is enough to remark that if $a_1, \ldots, a_n$ are elements of a noetherian ring A such that $a_1, \ldots, a_n$ is a regular sequence, then $H_1(a_1, \ldots, a_n; A) = 0$; furthermore, if $a_1, \ldots, a_n$ belong to the Jacobson radical of A, then the converse holds.

Another consequence of Proposition 2·3 is the following theorem, which is the main result of this section.

https://doi.org/10.1017/S0305004100055699 Published online by Cambridge University Press

THEOREM 2·6. *Let* $\mathfrak{a} = (a_1, \ldots, a_n)$ *be an ideal of* A *and let* $\mathfrak{p}$ *be a minimal prime ideal of* $\mathfrak{a}$. *Then* $h(\mathfrak{p}) = n$ *if and only if* $a_1, \ldots, a_n$ *are* $\mathfrak{p}$*-independent.*

Proof. If $a_1, \ldots, a_n$ are $\mathfrak{p}$-independent we have (Remark 2·1) $n \leqslant \sup(\mathfrak{p}) = h(\mathfrak{p}) \leqslant n$, hence $h(\mathfrak{p}) = n$. Conversely, assume that $h(\mathfrak{p}) = n$; then $A_{\mathfrak{p}}$ is an n-dimensional local ring and, since $\mathfrak{p}$ is minimal over $\mathfrak{a}$, $\dim A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}} = 0$. It follows that $a_1, \ldots, a_n$ is a system of parameters in $A_{\mathfrak{p}}$, hence $a_1, \ldots, a_n$ are $\mathfrak{p}A_{\mathfrak{p}}$-independent. Since $\mathfrak{p}A_{\mathfrak{p}} \cap A = \mathfrak{p}$, it follows by Proposition 2·3, that $a_1, \ldots, a_n$ are $\mathfrak{p}$-independent, completing the proof.

Remark 2·7. As a consequence of Theorem 2·6 and Proposition 1·2, we see that if $\mathfrak{a} = (a_1, \ldots, a_n)$ has some minimal component $\mathfrak{p}$ of height n, then $G_A(\mathfrak{a})/\mathfrak{p}G_A(\mathfrak{a})$ is isomorphic to a polynomial ring on n generators over $A/\mathfrak{p}$. This result was proved, when A is local, in (4), lemma 1.

Remark 2·8. In (3) Davis has shown that if $a_1, \ldots, a_n$ is a minimal basis of an ideal $\mathfrak{a}$ of the principal class, then $a_1, \ldots, a_n$ are $\sqrt{\mathfrak{a}}$-independent. We shall show, as an immediate corollary to Theorem 2·6, that the converse of this property is also valid.

COROLLARY 2·9. *Let* $\mathfrak{a} = (a_1, \ldots, a_n)$ *be an ideal of* A. *Then* $h(\mathfrak{a}) = n$ *if and only if* $a_1, \ldots, a_n$ *are* $\sqrt{\mathfrak{a}}$*-independent.*

Proof. If $h(\mathfrak{a}) = n$, then $h(\mathfrak{p}) = n$ for all minimal primes of $\mathfrak{a}$. Hence, by Theorem 2·6, $a_1, \ldots, a_n$ are $\sqrt{\mathfrak{a}}$-independent. Conversely, if $a_1, \ldots, a_n$ are $\sqrt{\mathfrak{a}}$-independent and $\mathfrak{p}$ is a minimal prime of $\mathfrak{a}$, $a_1, \ldots, a_n$ are $\mathfrak{p}$-independent; hence $h(\mathfrak{p}) = n$ and so $h(\mathfrak{a}) = n$.

As an easy application of the preceding corollary, we answer the question discussed in Remark 2·1.

COROLLARY 2·10. *Let* A *be a semilocal ring of dimension* d *and let* $a_1, \ldots, a_d$ *be a system of parameters of* A. *Then* $a_1, \ldots, a_d$ *are* J*-independent if and only if the Jacobson radical* J *of* A *is unmixed.*

Proof. Immediate from Corollary 2·9 and the fact that J is unmixed if and only if $h(J) = d$.

3. In this section we consider a noetherian ring A and a proper ideal $\mathfrak{a}$ in A. We proved in (12) that $\sup(\mathfrak{a})$ is bounded above and below by the height of $\mathfrak{a}$ and the grade of $\mathfrak{a}$, respectively; furthermore if $\mathfrak{a} = \sqrt{\mathfrak{a}}$, then $\sup(\mathfrak{a}) = h(\mathfrak{a})$. It follows that Cohen–Macaulay rings are characterized by the equality of sup and grade for all ideals, or just for all prime ideals. On the other hand, it is clear that, if A is Cohen–Macaulay, then $h(\mathfrak{a}) = \sup(\mathfrak{a})$ for all ideals $\mathfrak{a}$ of A and it was first thought that, perhaps, the converse holds. In this section we prove that this is true if A has dimension $\leqslant 1$, but we give an example of a 2-dimensional local ring A which is not Cohen–Macaulay and with the property $h(\mathfrak{a}) = \sup(\mathfrak{a})$ for all ideals $\mathfrak{a}$ of A. Some other related questions are discussed.

We recall that a ring A is said to be S_1 (Serre condition) if the zero ideal of A is unmixed. This notion is strictly related to independence, as the following proposition shows.

PROPOSITION 3·1. *The ring* A *is* S_1 *if and only if* $\sup(a) = h(a)$ *for every principal ideal* (a) *of* A.

https://doi.org/10.1017/S0305004100055699 Published online by Cambridge University Press

Proof. Assume that A is S_1; if a is a non zero-divisor in A then a is (a)-independent, hence $1 \leqslant \sup(a) \leqslant h(a) \leqslant 1$. If a is a zero-divisor in A, then a is not (a)-independent by Proposition 1·9, hence there exist $b \notin (a)$ and $n \geqslant 1$ such that $ba^n = 0$. Then, for every element $ac \in (a)$, $b(ac)^n = 0$, thus $\sup(a) = 0$; but, since (0) is unmixed, $h(a) = 0$. Conversely, if A is not S_1, let $\mathfrak{p}$ be an embedded prime of (0) and $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ the minimal primes in A. Then

$$\mathfrak{p} \not\subset \bigcup_{i=1}^{r} \mathfrak{p}_i,$$

hence there exists $a \in \mathfrak{p}$, $a \notin \mathfrak{p}_i$ for all $i = 1, \ldots, r$; it follows that $h(a) = 1$ but, since a is zero-divisor in A, $\sup(a) = 0$, as before.

The following result is an obvious consequence of the previous proposition.

PROPOSITION 3·2. *Let A be a ring of dimension $\leqslant 1$; the following conditions are equivalent*:

(1) $\sup(\mathfrak{a}) = h(\mathfrak{a})$ *for all ideals* $\mathfrak{a}$ *of* A.

(2) $\sup(a) = h(a)$ *for every principal ideal* (a) *of* A.

(3) A *is* S_1.

(4) A *is Cohen–Macaulay.*

Proof. We use Proposition 3·1 and the fact that a ring of dimension $\leqslant 1$ is Cohen–Macaulay if is S_1.

Now we examine what can be said for rings of dimension $\geqslant 2$. We need the following definition.

Definition 3·3. *We say that an ideal* $\mathfrak{a}$ *of a ring* A *is isobathic if for every prime ideal* $\mathfrak{p}$ *belonging to* $\mathfrak{a}$, $\dim A/\mathfrak{p} = \dim A/\mathfrak{a}$.

It is clear that if A is local Cohen–Macaulay then (0) is isobathic and if (0) is isobathic then A is S_1.

On the other hand, if $\sup(\mathfrak{a}) = h(\mathfrak{a})$ for all ideals $\mathfrak{a}$ of a 2 dimensional local ring A, then (0) is isobathic. This can be seen as a consequence of the following more general result.

THEOREM 3·4. *Let $(A, \mathfrak{m})$ be a local ring such that there exists a prime ideal $\mathfrak{p}$, belonging to (0), with $\dim A/\mathfrak{p} \leqslant 1$. Then, for all ideals $\mathfrak{a}$ of A, $\sup(\mathfrak{a}^n) < 2$ for some positive integer n.*

Proof. We have $\mathfrak{p} = 0 : x$ for some $x \neq 0$; hence $x \notin \bigcap_{n=0}^{\infty} \mathfrak{a}^n$ and there exists a positive integer n such that $x \in \mathfrak{a}^{n-1}$, $x \notin \mathfrak{a}^n$. Since $\dim A/\mathfrak{p} \leqslant 1$, $A/\mathfrak{p}$ is a local Cohen–Macaulay ring; hence, by a theorem of Rees (cf. (10), proposition 2·1), there exists an integer r such that every ideal of $A/\mathfrak{p}$ can be generated by r elements. Now suppose $a, b \in A$ and denote by $\bar{c}$ the image in $A/\mathfrak{p}$ of $c \in A$; then the ideal $(\bar{a}, \bar{b})^r$ in $A/\mathfrak{p}$ can be generated by r elements, hence for some $i = 0, \ldots, r$,

$$a^i b^{r-i} + \sum_{j \neq i} c_j a^j b^{r-j} \in \mathfrak{p} = 0 : x.$$

It follows that

$$x a^i b^{r-i} + \sum_{j \neq i} c_j x a^j b^{r-j} = 0,$$

and since $x \notin \mathfrak{a}^n$, this shows that a, b are not $\mathfrak{a}^n$-independent and $\sup(\mathfrak{a}^n) < 2$, as required.

https://doi.org/10.1017/S0305004100055699 Published online by Cambridge University Press

COROLLARY 3·5. *Let* $(A, \mathfrak{m})$ *be a local ring of dimension* 2. *If* $\sup(\mathfrak{a}) = h(\mathfrak{a})$ *for all ideals* $\mathfrak{a}$ *of* A, *then* (0) *is isobathic.*

Example 3·6. Let $A = k[\![X, Y, Z]\!]/(XY, XZ) = k[\![x, y, z]\!]$, $\mathfrak{m} = (x, y, z)$. Then $\dim A = 2$, A is S_1 and (0) is not isobathic. If $\mathfrak{p} = (y, z)$, we have $\mathfrak{p} = 0 : x$ and $x \in \mathfrak{m}, x \notin \mathfrak{m}^2$. It follows, from the proof of Theorem 3·4, that $\sup(\mathfrak{m}^2) < 2$, while $h(\mathfrak{m}^2) = 2$.

Example 3·7. Let $A = k[\![X, Y, Z, T]\!]/(X, Y) \cap (Z, T) = k[\![x, y, z, t]\!]$, $\mathfrak{m} = (x, y, z, t)$; then A is S_1, $\dim A = 2$ and A is not Cohen–Macaulay. We prove that $\sup(\mathfrak{a}) = h(\mathfrak{a})$ for all ideals $\mathfrak{a}$ of A. If $h(\mathfrak{a}) = 1$, since A is S_1, there exists an element $a \in \mathfrak{a}$ which is a non zero-divisor in A, hence $\sup(\mathfrak{a}) = 1$. If $h(\mathfrak{a}) = 2$, then $\mathfrak{m}^n \subseteq \mathfrak{a}$ for some positive integer n; thus we need to prove that $\sup(\mathfrak{m}^n) = 2$ for all positive integers n. Now suppose that

$$\sum_{i=0}^{p} f_i(x, y, z, t)(x^n + z^n)^i (y^n + t^n)^{p-i} = 0;$$

then

$$\sum_{i=0}^{p} f_i(X, Y, Z, T) X^{ni} Y^{n(p-i)} + \sum_{i=0}^{p} f_i(X, Y, Z, T) Z^{ni} T^{n(p-i)}$$

is an element of $(X, Y) \cap (Z, T)$. From this it follows that

$$\sum_{i=0}^{p} f_i(X, Y, Z, T) X^{ni} Y^{n(p-i)} \in (Z, T);$$

since X^n, Y^n is a regular sequence modulo (Z, T), we get $f_i \in (X^n, Y^n, Z, T)$ and in the same way we can prove $f_i \in (X, Y, Z^n, T^n)$. Hence

$$f_i \in (X^n, Y^n, Z, T) \cap (X, Y, Z^n, T^n) = (X^n, Y^n, Z^n, T^n) + (X, Y) \cap (Z, T);$$

thus $f_i(x, y, z, t) \in \mathfrak{m}^n$, for all i. This proves that $x^n + z^n$, $y^n + t^n$ are $\mathfrak{m}^n$-independent for all positive integers n, and the required conclusion follows.

Remark 3·8. In the set F of 2-dimensional local rings, we consider:

M = subset of elements of F which are Cohen–Macaulay,

S = subset of elements of F such that $\sup(\mathfrak{a}) = h(\mathfrak{a})$ for all ideals $\mathfrak{a}$,

I = subset of elements of F such that (0) is isobathic,

S_1 = subset of elements of F which satisfy property S_1.

Then we have $M \subset S \subseteq I \subset S_1$. In fact, by Corollary 3·5 we see that $S \subseteq I$, while Example 3·7 and 3·6 show that $M \subset S$ and $I \subset S_1$.

Question 3·9. Is it true that $S = I$?

This work was supported by the Consiglio Nazionale delle Ricerche.

REFERENCES

(1) ATIYAH, M. and MACDONALD, G. *Introduction to commutative algebra* (Reading, Mass; Addison-Wesley, 1969).

(2) BARSHAY, J. Generalized analytic independence. *Proc. Amer. Math. Soc.* **58** (1976), 32–36.

(3) DAVIS, E. D. Ideals of the principal class, R-sequences and a certain monoidal transformation. *Pacific J. Math.* **20** (1967), 197–205.

https://doi.org/10.1017/S0305004100055699 Published online by Cambridge University Press

(4) Eisenbud, D., Herrmann, M. and Vogel, W. Remarks on regular sequences. *Nagoya Math. J.* **67** (1977), 177–180.
(5) Herrmann, M. and Schmidt, R. Regular sequences and lifting property. (Preprint.)
(6) Kabele, T. Regularity conditions in non-noetherian rings. *Trans. Amer. Math. Soc.* **155** (1971), 363–374.
(7) Micali, A., Salmon, P. and Samuel, P. Integrité et factorialité des algèbres symètriques. *Atas do IV Coloquio Brasileiro de Matematica*, São Paulo, 1965.
(8) Northcott, D. G. *Lessons on rings, modules and multiplicities* (Cambridge University Press, 1968).
(9) Rhodes, C. P. L. A multiplicative property of R-sequences and H_1-sets. *Math. Proc. Cambridge Philos. Soc.* **78** (1975), 1–6.
(10) Sally, J. and Vasconcelos, W. Stable rings and a problem of Bass. *Bull. Amer. Math. Soc.* **79** (1973), 574–576.
(11) Valabrega, P. and Valla, G. Form rings and regular sequences. *Nagoya Math. J.* **72** (1978). (In the press.)
(12) Valla, G. Elementi independenti rispetto ad un ideale. *Rend. Sem. Mat. Univ. Padova* **44** (1970), 339–354.
(13) Zariski, O. Foundations of a general theory of birational correspondences. *Trans. Amer. Math. Soc.* **53** (1943), 497–542.

https://doi.org/10.1017/S0305004100055699 Published online by Cambridge University Press