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WEAK CONVERGENCE TO THE STUDENT AND LAPLACE DISTRIBUTIONS

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Abstract

One often observed empirical regularity is a power-law behavior of the tails of some distribution of interest. We propose a limit law for normalized random means that exhibits such heavy tails irrespective of the distribution of the underlying sampling units: the limit is a t-distribution if the random variables have finite variances. The generative scheme is then extended to encompass classic limit theorems for random sums. The resulting unifying framework has wide empirical applicability which we illustrate by considering two empirical regularities in two different fields. First, we turn to urban geography and explain why city-size growth rates are approximately t-distributed, using a model of random sector growth based on the central place theory. Second, turning to an issue in finance, we show that high-frequency stock index returns can be modeled as a generalized asymmetric Laplace process. These empirical illustrations elucidate the situations in which heavy tails can arise.

Keywords: Limit theorem; city-size growth; high-frequency return process

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Secondary 62E20; 91B30; 91B70

1. Introduction

Many empirical regularities are tied to limit laws, i.e. limit distributions which do not depend on the original sampling distribution. But well-known limit laws such as the central limit theorem cannot explain one often observed empirical regularity, which is the power-law behavior of the tails of some distribution of interest. We address this gap by proposing a limit law for normalized random means that exhibits such heavy tails irrespective of the distribution of the underlying sampling units: the limit is a t-distribution if the random variables have finite variances when sample sizes are drawn from a negative binomial distribution (the leading count model that accommodates overdispersion), independent of the summands, and tend to ∞. Unlike limit theorems in extreme value theory, we do not rely on a Fréchet domain-of-attraction assumption; here heavy tails of the limit distribution are obtained when the sampling distribution has finite variance. Moreover, our theory characterizes the entire distribution and not just the behavior of the tail. In our empirical application in the field of urban geography, we explain why city-size growth rates are approximately t-distributed, using a model of random sector growth based on the central place theory.

As we consider random sample sizes, we briefly relate our work to the classic literature on random sums, noting that the resulting limit laws do not exhibit heavy tails unless the

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sampling distribution is heavy-tailed. In particular, if the random summands are independent and identically distributed (i.i.d.), nonnegative, and with finite mean, the sample size is geometrically distributed, and independent of the summands, then the (nonrandomly) normalized sum converges weakly to an exponential variate [19]. If the summands are symmetric random variables with finite variance, the limit distribution is the Laplace distribution [13]. With no assumptions on the distribution of the summands, the limit is a geometric stable law [11], and in the case of the nonrandom sum, a stable law [10]. We show how these classic theorems and our limit theorem can be embedded in a unifying framework, which is then further generalized to stochastic processes.

The unifying framework shows how different normalization schemes of a random sum result in different limit distributions. Which scheme is of empirical relevance depends on the application, and we examine two important empirical regularities in two different fields, which pertain to core issues therein. First, we turn to the literature on urban geography, and explain why city-size growth rates are approximately t-distributed, using a model of random sector growth based on the central place theory. Second, we turn to the finance literature, and show that high-frequency stock index returns can be modeled in a natural way as a generalized asymmetric Laplace motion. We verify these model-based predictions in two empirical investigations, using data for city sizes in Germany, and high-frequency data from the German Deutscher Aktienindex stock index (DAX®). These empirical illustrations elucidate the situations in which heavy tails can arise.

This paper is organized as follows. In Section 2 we state the limit theorem. In Section 3 we present two applications.

2. Limit distributions of random means and sums

In this section we present the limit theorem for a normalized random mean, relate it to existing limit theorems for normalized random sums, and present a general unifying framework.

2.1. Weak convergence to the t-distribution

Let $X_1, X_2, \ldots,$ be a sequence of i.i.d. random variables with finite mean $\mathbb{E}(X_k) = \mu < \infty$, finite variance $\operatorname{var}(X_k) = \sigma^2 < \infty$, and $\mathbb{E}(X_k^2 \ln |X_k|) < \infty$ for $k \in \mathbb{N}$. Define the random sum S_p and the random mean $\bar{X}_p$:

$$S_p = \sum_{k=1}^{N_p} X_k,$$
$$\bar{X}_p = N_p^{-1} S_p, \tag{1}$$

where N_p is a positive random variable whose distribution depends on the index p. Let N_p have a negative binomial distribution $\mathrm{NB}_p(\gamma)$, independent of the X_k, with parameters $\gamma > 0$ and $0 < p < 1$. The probability function is

$$\mathbb{P}(N_p = n) = \frac{\Gamma(n+\gamma)}{n!\,\Gamma(\gamma)} p^\gamma (1-p)^n \quad \text{for } n \in \mathbb{N}_0.$$

If $N_p = 0$ then $\bar{X}_p$ in (1) is undefined. Since $\mathbb{P}(N_p = 0) \to 0$ as $p \to 0$, this is not a problem. The negative binomial distribution is one of the two leading cases for count models, it accommodates the overdispersion typically observed in count data (which the Poisson model cannot), and the special case of the geometric distribution ($\gamma = 1$) is the leading case in the literature on random sums. The expectation of N_p is $\mathbb{E}(N_p) = (1-p)\gamma/p$, so, for γ given, $\mathbb{E}(N_p) \to \infty$ as $p \to 0$.

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Theorem 1. *Under the stated assumptions, the normalized mean converges weakly to a Student's t-distribution with 2γ degrees of freedom,*

$$\sqrt{\frac{\gamma}{p}}\frac{\bar{X}_p - \mu}{\sigma} \to T \overset{\mathrm{D}}{\to} t_{2\gamma} \quad as\ p \to 0,$$

where $t_{2\gamma}$ is Student's t-distribution with 2γ degrees of freedom.

The special case of the geometric distribution follows immediately with $\gamma = 1$: the normalized mean then converges weakly to a t-distribution with 2 degrees of freedom.

Theorem 1 is shown to be valid in Section 2.2. The distributional assumption concerning the random sample size can be weakened. The result still holds as long as $2pN_p$ converges weakly to a $\chi^2_{2\gamma}$-distribution. In particular, the distribution of N_p can be shifted. Moreover, the assumptions about the distribution of $X_1, X_2, \ldots$ can also be weakened. Provided that the central limit theorem still applies for a deterministically increasing sample size n, Theorem 1 remains valid. For example, we could attach different weights to the sample elements such that the weighted sample elements are no longer identically distributed. We only need to assume that the weights $W_1, W_2, \ldots$ have finite variance and be independent of each other, the observations $X_1, X_2, \ldots$, and the sample size N_p.

Turning to estimation, our theorem implies that, for small but nonnegligible values of p, the mean $\bar{X}_p$ is approximately distributed as a variate that has distribution $t_{2\gamma,\mu,s}$, i.e. a $t_{2\gamma}$-distribution with location parameter μ and scale parameter $s = \sigma(p/\gamma)^{1/2}$. The shape parameter γ can be identified from the degrees of freedom of the limit distribution and the location parameter from its mean. The parameters σ and p are not individually identifiable from the scale parameter s unless the sample elements $X_1, X_2, \ldots$ were observed and their standard deviation σ estimable. In case the above weighting scheme is applied, the weighted mean is, of course, $\bar{X}_p = \sum_{i=1}^{N_p} W_i X_i / \sum_{i=1}^{N_p} W_i$. The scale parameter of the approximate $t_{2\gamma,\mu,s}$ limit distribution is then $s = \sigma(p/\gamma)^{1/2}(\mathrm{var}(W_1/\bar{W}) + \mathbb{E}(W_1/\bar{W}))^{1/2}$, where $\bar{W} = N_p^{-1}\sum_{i=1}^{N_p} W_i$. Hence, the inflation of the scale parameter depends on the distribution of the weights.

Regarding the tail behavior of the limit distribution, the next result follows immediately.

Corollary 1. *The limit distribution of the normalized random mean is heavy-tailed, and for $\gamma \leq 1$ has infinite variance.*

The limit law thus exhibits heavy tails but does not require the summands to be sampled from a heavy-tailed distribution. By contrast, existing limit laws for random sums do not exhibit heavy tails unless the summands themselves have infinite variance. Here, heavy-tailedness of the random mean is obtained through the normalization of the random sum by the random variable $p^{1/2}N_p$. Building on this insight, when applied to specific contexts such as those explored in the applications below, the new limit law might suggest the mechanism which generates the heavy-tailed limit (without presupposing it for the summands).

2.2. A unifying framework

In Theorem 1 we have considered the limit law for the suitably normalized random mean $\bar{X}_p$. For the special case of the geometric distribution, Rényi [19] showed that for nonnegative X_k, pS_p converges to an exponential distribution as $p \to 0$. Kotz *et al.* [13] extended this result by showing that for symmetric X_k with finite variance, $p^{1/2}S_p$ converges to a Laplace distribution. A further extension is given in Kozubowski *et al.* [15] who considered asymmetric X_k with possibly nonzero expectation and let the number of summands have a negative binomial

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distribution. The random sums converge to the generalized asymmetric Laplace distribution. All limiting schemes can be embedded in a unifying framework as follows.

Consider integer-valued random variables N_p, $p > 0$, that are increasing without bound as $p \to 0$. Consider also functions g_p with the property that the random variables $g_p(N_p)$ converge in distribution to some distribution G. Then, by the Skorokhod representation theorem, N_p can be chosen so that $g_p(N_p)$ converges almost surely to a random variable Γ whose distribution coincides with G. In the same way, assume that the random variables Y_n converge almost surely to Y_0, and that N_p is independent of Y_{N_p}. Let $F(g, y)$ be a continuous function in g and y. Then, almost surely,

$$\lim_{p\to 0} F(g_p(N_p), Y_{N_p}) = F(\Gamma, Y_0), \tag{2}$$

where Γ and Y_0 are independent.

In particular, let N_p be distributed according to a negative binomial distribution $\mathrm{NB}_p(\gamma)$, and consider the function $g_p(n) = pn$. As $p \to 0$, $g_p(N_p)$ converges to a standard gamma-distributed random variable Γ with shape parameter γ. Consequently, $2g_p(N_p)/(2\gamma)$ converges to the random variable whose distribution is $\chi^2_{2\gamma}/(2\gamma)$. Let $Y_{N_p} = \sigma^{-1}\sqrt{N_p}\bar{X}_p$, which, by the central limit theorem, converges to a standard normally distributed random variable Z. Since $g_p(N_p)$ is independent of Y_{N_p}, the hypotheses of the generic scheme (2) are verified.

For the specific scheme considered in Theorem 1, we set $F(g, y) = y/\sqrt{2g/(2\gamma)}$, and the limiting distribution is that of $Z/\sqrt{\chi^2_{2\gamma}/(2\gamma)}$, i.e. Student's $t_{2\gamma}$-distribution. By contrast, if we set $F(g, y) = \sqrt{g}y$, we have the setting examined in Kotz *et al.* (2001). The limiting distribution of this scheme is that of $\sqrt{\Gamma}Z$, i.e. we have convergence to the generalized symmetric Laplace distribution in the case of symmetric random variables X_k. If the distribution of X_k is asymmetric with $\mathbb{E}(X_k) = \sqrt{p}\mu$ and variance σ^2, the limit for $F(g, y) = \sqrt{g}y$ is the generalized asymmetric Laplace distribution with density function [13, p. 189]

$$f_{\mathrm{GAL}}(x) = \frac{\sqrt{2}\exp(\sqrt{2}(1/\kappa - \kappa)x/(2\sigma))}{\sqrt{\pi}\sigma^{\gamma+1/2}\Gamma(\gamma)}\left(\frac{\sqrt{2}|x|}{\kappa + 1/\kappa}\right)^{\gamma-1/2} K_{\gamma-1/2}\left(\frac{\sqrt{2}}{2\sigma}\left(\frac{1}{\kappa}+\kappa\right)|x|\right), \tag{3}$$

where $\kappa = (\sqrt{2\sigma^2 + \mu^2} - \mu)/(\sqrt{2}\sigma)$ is the asymmetry parameter.

For the scheme considered in Rényi [19], we set $F(g, y) = gy$, but now assume that N_p follows $\mathrm{NB}_p(1)$ and $Y_{N_p} = \bar{X}_{N_p}$ converges in probability to a positive value. The limit distribution is that of an exponential random variable.

Seen from the perspective of the generic limiting scheme (2), the specific schemes are equivalent and express the same general limiting behavior. The specific choice of F, however, depends on the application of interest. We proceed to develop in Section 3 two examples for which the schemes $F(g, y) = y/\sqrt{2g/(2\gamma)}$ and $F(g, y) = y\sqrt{g}$ are appropriate.

2.2.1. *Further extensions: stochastic processes.* The limiting scheme (2) can be further generalized by replacing the negatively binomially distributed random variable N_p and the asymptotically Gaussian random variable Y_n by appropriate stochastic processes. In particular, let $Y_n(\cdot)$ be a stochastic process which converges to a Gaussian process $Y(\cdot)$, and let $N_p(\cdot)$ now represent a negative binomial process $\{N_p(t),\ t \geq 0\}$, i.e. the continuous-time process starting at 0 with independent and strictly stationary increments that follow the $\mathrm{NB}_p(\gamma)$-distribution. As before, $g_p(n) = pn$. It follows immediately from the results above that the finite-dimensional distributions of $g_p(N_p(\cdot)) = pN_p(\cdot)$ converge to those of a standard gamma process $\Gamma(\cdot)$ as $p \to 0$. Since $N_p(\cdot)$ is a Lévy process, this yields weak convergence of $g_p(N_p(\cdot))$ to $\Gamma(\cdot)$ in the

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Skorokhod metric J_1 by standard theorems [18, p. 134]. Recently, Kozubowski and Podgórski [14] have also shown uniform convergence over a compact set. Further assume that $Y_n(\cdot)$ is independent of $N_p(\cdot)$. Then weak convergence of the scheme (2) is obtained by the continuous mapping theorem. In the previous section, we have derived the finite-dimensional distributions for the specific schemes considered.

3. Applications

We present two distinct applications, one taken from urban geography and economics, the other from finance, in order to demonstrate how our unifying framework of Section 2 can be applied to explain some striking empirical regularities.

3.1. Growth rates of cities

The size and growth rate distribution of cities are topics that are still hotly debated [21]. One popular view, Gibrat's law of proportional growth, is based on the assumption that random growth rates are i.i.d. and of finite variance. Here we show that the last assumption might fail as the unconditional growth rate distribution is very heavy-tailed.

We consider cities composed of economic sectors that grow randomly. The numbers of sectors themselves are random, leading to cities of different sizes. Usually, the individual sectors are not observable by the researcher. We link the number of economic sectors and the size of the city by Christaller's central place theory [4]. In particular, assume that the number of economic sectors N_i in city i depends on the city size W_i according to

$$N_i = C + \lambda \ln W_i \tag{4}$$

with $\lambda > 0$ and constant C. Empirical evidence in support of (4) is reported in, e.g. [17]. Let w_{ij} be the size of sector j in city i. Total city size is $W_i = \sum_j w_{ij}$. Each sector j in city i grows at the random rate r_{ij} with $\mathbb{E}(r_{ij}) = \mu$ and $\operatorname{var}(r_{ij}) = \sigma^2 < \infty$. Then the growth rate R_i of city i is the weighted sum of the N_i sectors' growth rates

$$R_i = \sum_{j=1}^{N_i} \frac{w_{ij}}{W_i} r_{ij}. \tag{5}$$

If all sectors are equally large, the city growth rate is just $N_i^{-1} \sum_{j=1}^{N_i} r_{ij}$.

The size distribution of cities is often found to be of the Pareto type, an empirical regularity known as Zipf's law; see, e.g. [7] and [9]. Accordingly, assume that the size W_i of city i, measured as the number of inhabitants, follows a Pareto distribution with scale parameter $w_0 > 0$ and shape parameter $\alpha > 0$,

$$\mathbb{P}(W_i > w) = \begin{cases} w_0^{\alpha} w^{-\alpha} & \text{for } w > w_0, \\ 1 & \text{for } w \le w_0. \end{cases} \tag{6}$$

We note that the exact Pareto assumption of (6) has been made mainly for expositional simplicity. This could be generalized to a domain-of-attraction assumption so that (6) holds for sufficiently large w.

Together, (4) and (6) imply that N_i has a shifted exponential distribution with cumulative density function

$$\mathbb{P}(N_i \le n) = \begin{cases} 1 - B \exp\left(-\dfrac{\alpha}{\lambda} n\right) & \text{for } n > C + \lambda \ln w_0, \\ 0 & \text{otherwise,} \end{cases}$$

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where $B = w_0^{\alpha} e^{\alpha C/\lambda}$. The discrete counterpart of the (shifted) exponential distribution is a (shifted) geometric distribution with parameter $p = \alpha/\lambda$.

What is the limit distribution of the growth rate of cities as the numbers of sectors become large? Theorem 1 can be invoked in this setting to explain the unconditional growth rate distribution of city sizes: the city-size growth rate distribution should follow approximately a shifted and scaled t_2-distribution.

Proposition 1. *Let R denote the city-size growth rate (5), and assume that (4) and (6) hold. Then by Theorem 1,*

$$R \xrightarrow{\mathrm{D}} t_{2,\mu,s} \quad \text{as } p \to 0. \tag{7}$$

The scale parameter s depends not only on p and σ^2 but also on the particular weighting scheme.

Since the unconditional growth rate distribution has infinite variance, an important corollary at least for the urban economics literature, is that one fundamental assumption underpinning Gibrat's law of proportional growth fails to hold.

Empirical investigation. We consider the distribution of city-size growth rates using data for German cities (or, more precisely, 'Gemeinden'). The data provided by the German Federal Statistical Office are very accurate, given the legal obligation of citizens to register with their respective town halls. In this illustration, we select two representative years, 1995 and 1996, in which the number of cities is $n = 14\,551$. Our earlier work [21] confirms that the upper tail of the city-size distribution is indeed heavy and thus of the Pareto type so that (6) applies. In particular, in an encompassing framework based on the generalized extreme value distribution, we have used the moment estimator of Dekkers *et al.* [8], and the maximum likelihood estimator of Smith [22], as well as the classic Hill estimator [2]. All estimators consider excesses over a threshold, and the threshold value was chosen using the mean-squared error criterion. The estimates of the shape parameter α (or, more precisely, of the index of regular variation, or the extreme value or tail index) range between 1.24 and 1.39 depending on the estimation method, and did not change much over the period under investigation, 1995–2006. Turning from the size data to the annual growth rate of cities, in Figure 1(a) we show the histogram of the growth rates.

If the hypotheses of Proposition 1 are met, growth rates are approximately distributed as a t_2-variate by (7). We depict the histogram of city-size growth rates along with the fitted t_2-density in Figure 1(a), having estimated the scale parameter s and location parameter μ in (7) by maximum likelihood, the estimates being $\hat{s} = 0.016\,26$ (with standard error (se) 0.000 190) and $\hat{\mu} = 0.005\,22$ (se $= 0.000\,175$). It is evident that the limit law fits the actual data closely. If the degrees of freedom parameter is estimated along the scale and location parameters, using an expectation–maximization algorithm [20], the maximum likelihood estimated degrees of freedom is $\widehat{df} = 2.005$ with an se of 0.043. Hence, the hypothesis that the number of degrees of freedom equals 2 cannot be rejected at usual significance levels.

Finally, we also estimate the tail index of the growth rate distribution. Of course, the t_2-distribution has a tail index of 2. In Figure 1(b) we depict the usual Hill plot of the Hill estimator on threshold level k. It is evident that the tail index is close to 2, and the estimate is fairly robust as to the number of extremes k included. Repeating the estimation procedure for other years we find that the t-distribution still fits very well. However, if the degrees of freedom are estimated as an additional parameter it is not always as close to 2 as in 1995–1996 but fluctuates over the interval $[2, 2.7]$. We conclude this illustration by observing that our theory characterizes the entire distribution and not just its tail behavior.

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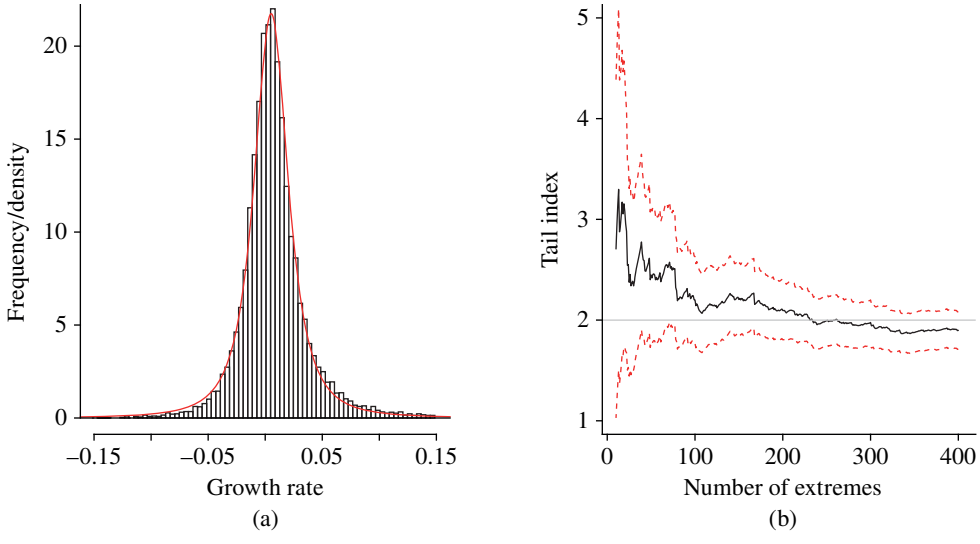


FIGURE 1: (a) Histogram of the city-size growth rates 1995–1996 and a fitted t-distribution with two degrees of freedom. (b) Hill plot of the estimate of the extremal index; the dashed lines are point-wise 0.95-confidence intervals.

3.2. High-frequency stock returns

Stock return distributions are well known to be nonnormally distributed, in particular if high-frequency returns are considered [6]. Barndorff-Nielsen and Shephard advocate Lévy processes to model financial time series [1]. We proceed to propose a transaction-based model which, in conjunction with the unifying framework, yields a generalized asymmetric Laplace motion.

Let $K(t)$ denote a stock price at time $t \geq 0$. Suppose that there is a random number $N_p(t)$ of market transactions in the interval $[0, t]$. Let each transaction i cause a return r_i, where r_i are i.i.d. with $\mathbb{E}(r_i) = \mu$ and $\operatorname{var}(r_i) = \sigma^2 < \infty$, $i = 1, \ldots, N_t$. For given $K(0)$, the stock price at time t is

$$K(t) = K(0) \exp\left(\sum_{i=1}^{N_p(t)} r_i \right). \tag{8}$$

Note that the number of transactions $N_p(t)$ has an impact on the return distribution. The more transactions that occur during $[0, t]$, the larger the variance. This may be thought of as a market activity effect [5].

The distribution of the return over the period $[0, t]$,

$$R(t) = \ln\left(\frac{K(t)}{K(0)} \right) = \sum_{i=1}^{N_p(t)} r_i,$$

is a random sum of a random number of returns. Following Cameron and Trivedi [3], assume that the number of transactions $N_p(t)$ is a negative binomial process with parameters p and γ, i.e. $N_p(t) \sim \mathrm{NB}_p(t\gamma)$.

Proposition 2. *For small p, the logarithm of the price process (8) can be approximated by a Lévy process with independent increments (i.e. returns) that follow the generalized asymmetric*

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Laplace distribution. Over the unit period, its density is given by (3) with location parameter $\mu' = \mu/p$*, scale parameter* $\sigma' = \sigma p^{-1/2}$*, and shape parameter* γ*.*

Due to the Lévy property the temporal aggregation of returns is straightforward. The approximate distribution of the return over an interval of arbitrary length Δ is a generalized asymmetric Laplace with shape parameter $\gamma\Delta$, while both σ' and μ' remain unchanged.

Empirical investigation. We consider high-frequency returns of the German stock index DAX. The data are provided by the EUREX® database, the sample includes 2 612 189 observations (at the 1-second sampling frequency) from 2 January 2006 until 28 April 2006, observed for 90 trading days over 18 weeks. The daily trading phase starts at 9 a.m. and ends at 5:45 p.m. In order to prevent noise due to the market's microstructure and to ensure a sufficiently large number of transactions, we compute returns over 5 minutes or more. Overnight returns are discarded.

In Figure 2 we depict the histograms of the returns over 5 minutes (left) and 30 minutes (right) together with densities of fitted generalized asymmetric Laplace distributions.

The generalized asymmetric Laplace distribution apparently fits both return distributions rather well. This finding is in line with other studies of financial data, e.g. [12], [13], and [16]. A closer look at the maximum likelihood estimates reported in Table 1 reveals some limitations of the Lévy model. The shape parameter estimate hardly changes when the return period is extended from 5 minutes to 30 minutes. This finding is not compatible with the temporal aggregation property of a Lévy process. Under the Lévy assumption the shape parameter has to be linear in the length of the return period. We conclude that the limit theorem does provide an explanation for the marginal distributions but the complex time series properties of financial market data, e.g. volatility clustering, cannot be entirely captured by the corresponding limiting Lévy-type stochastic process.

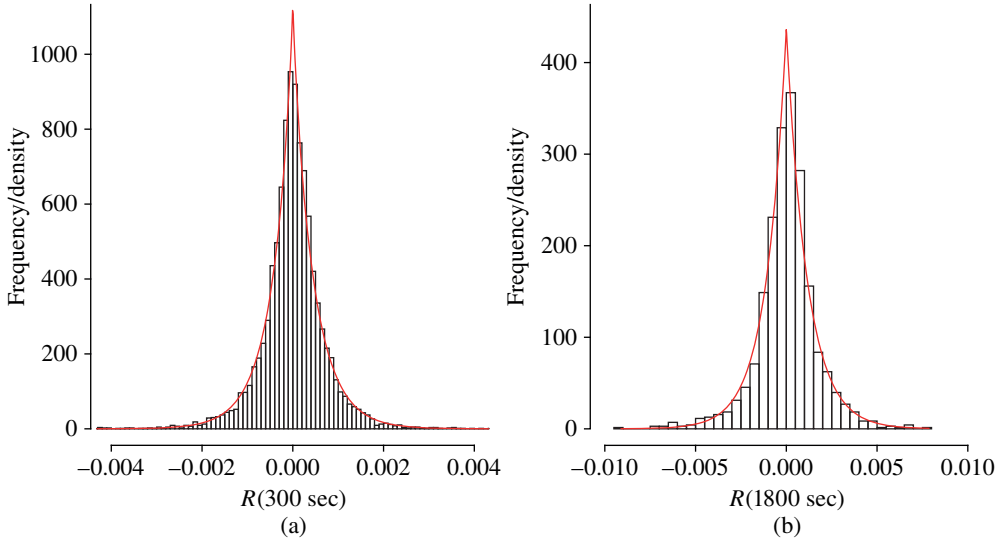


FIGURE 2: (a) Histograms of 5-minute and (b) 30-minute returns of the DAX index and density of fitted generalized asymmetric Laplace distributions.

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TABLE 1: Maximum likelihood estimates of generalized asymmetric Laplace distributions fitted to 5-minute and 30-minute returns.

Parameter	5-minute returns		30-minute returns	
	Estimate	Standard error	Estimate	Standard error
Shape ($\hat{\gamma}$)	0.9111	0.0303	0.9976	0.0848
Scale ($\hat{\sigma} \times 10^3$)	0.7093	0.0168	1.6218	0.0953
Asymmetry ($\hat{\kappa}$)	0.9899	0.0079	0.9799	0.0185

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