(E) when $2 \leqslant \mu \leqslant \sqrt{5}$,

$$f_1' = \tfrac{1}{4} \left| \{\alpha - \epsilon(\alpha^2 - 2\mu)\}^2 x^2 + \{\alpha + \epsilon(\alpha^2 - 2\mu)\}^2 y^2 \right|,$$

$$f_2' = \left| xy \right|,$$

where $\sqrt{(2\mu)} \leqslant \alpha \leqslant \tfrac{1}{2}(2+\mu)$;

(F) when $\mu \geqslant 5$,

$$f_1' = \left| x^2 - \epsilon xy - y^2 \right|,$$

$$f_2' = \left| (\alpha^2 - \alpha^{-2}) x^2 - \{\alpha^2(\sqrt{5}+\epsilon) + \alpha^{-2}(\sqrt{5}-\epsilon)\} xy + \tfrac{1}{4}\{(\sqrt{5}+\epsilon)^2 - \alpha^{-2}(\sqrt{5}-\epsilon)\} y^2 \right|,$$

where $-\mu + \sqrt{(\mu^2+1)} \leqslant \alpha^2 \leqslant \mu + \sqrt{(\mu^2+1)}$.

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TOTALLY HETEROGENEOUS CONTINUA

BY A. S. BESICOVITCH

A possible degree of heterogeneity in the structure of continua is investigated in this paper.

DEFINITION. *If there are no two points of a continuum such that a neighbourhood of one is homeomorphic to a part or to the whole of a neighbourhood of the other, then we call the continuum a totally heterogeneous continuum.*

The aim of this paper is to establish the existence of totally heterogeneous continua. The solution is arrived at by considering a special class of continua, that of dendrites; i.e. of locally connected continua of which no Jordan curve is a subset. The existence of totally heterogeneous dendrites gives an answer in the negative to an old

Problem of Zarankiewicz. Given any dendrite D, does there always exist a subdendrite D_1 homeomorphic to D? *

But we proceed in the opposite direction. In solving the problem of Zarankiewicz we develop the first stage of the method of constructing heterogeneous continua and then, by elaborating the method, we arrive at a final solution.

* Problem of Zarankiewicz, *Fund. Math.* 7 (1925), 381.

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If a dendrite is such that a neighbourhood of none of its multiple points can be mapped on a part or the whole of a neighbourhood of another point, then we call it a basically heterogeneous dendrite. We shall give a negative answer to the problem by constructing a certain dendrite of this nature. For a special class of dendrites an answer to the problem has been given in the affirmative by N. Aronszajn*. We give two theorems in which an answer, in the affirmative, is given for two other classes of dendrites.

1. DEFINITION 1. *If a point a of a dendrite separates it into $n+1$ (or ∞) dendrites, then we say that a is of multiplicity n (or ∞); for $n = 1$ we say that a is a simple point.*

The problem only arises when the set E of all the multiple points is infinite. For, if a dendrite D has a finite number of multiple points, let a be one of the end-points of D. Then any point b of the dendrite sufficiently near a separates D into two parts, of which the one containing a is an arc. Then, taking an interior point a' of ab, denote by D' the dendrite $D-a'a+a'$. D' is obviously homeomorphic to D, since ba' is to ba.

THEOREM 1. *If the derived set E' of the set E of all the multiple points of a dendrite D has an isolated point, then the dendrite D is homeomorphic to a true subdendrite D'.*

Proof. Let c be an isolated point of E' and $\{c_n\}$ a sequence of points of E different from c and convergent to c. Then there exists a system of different subdendrites $\{D_n\}$, D_n starting from c_n, converging to the point c. As c is an isolated point of E', there exists an n_0 such that, for all $n > n_0$, D_n does not contain points of E' and consequently contains, at most, a finite number of points of E. Consequently, there exists a subdendrite D'_n of D_n starting from c_n and homeomorphic to D_n, from which the theorem follows.

THEOREM 2. *If all the points of the set E' of Theorem 1 are end-points of D, then D is homeomorphic to its true subdendrite D'.*

Proof. When we consider subdendrites that can be separated from D by a single point, we denote them by $\Delta, \Delta', \ldots$, and we call them *subdendrites of the type* Δ. If one of such subdendrites Δ has a finite number of multiple points, then it is homeomorphic to its part obtained by cutting off a small arc terminating at one of the ends of Δ. This reduces the dendrite D to the dendrite D', which is obviously homeomorphic to D, and the theorem follows.

Assume now that every subdendrite has an infinite number of multiple points. Let k be the highest order of multiple points of D; then k will be called the norm of D. If the order of multiplicity is not bounded we say that the norm of D is ∞. We shall speak also of norms of subdendrites Δ. Observe that if D has a point of infinite multiplicity the theorem is true. For let c_0 be such a point. As c_0 does not belong to E', an infinite number of subdendrites which start at c_0 have no multiple points and thus are simple arcs, from which the theorem follows immediately.

It is sufficient to consider only the case when the norm of every subdendrite Δ is equal to that of D. For if not, let Δ_1 be a subdendrite whose norm is less than that of D (and consequently finite), and let c_1 be the point separating it from D. Suppose that Δ_1

* Ein Urdbildproblem, *Fund. Math.* 17 (1931), 92–121.

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has a subdendrite Δ_2, which does not contain the point c_1 and whose norm is less than that of Δ_1. Let c_2 be the point separating it from D. After a finite number of steps we shall arrive at a subdendrite Δ_k, separated from D by a point c_k, and such that every subdendrite Δ of Δ_k which does not contain c_k has the same norm as Δ_k. We can always assume c_k to be a simple point of the dendrite D. *If we show that Δ_k can be mapped* on one of its true subsets, so that c_k remains fixed, then the theorem will be proved.*

If D itself is such that its norm is equal to that of any of its subdendrites Δ, take a simple point c of D and denote by Δ one of the two subdendrites into which D is separated by c. *Thus in every case our problem is reduced to mapping Δ on one of its true subsets, so that c is mapped on itself.*

We shall now show how such a mapping can be carried out. Let d be the nearest multiple point of Δ to c in the sense that the arc cd contains no other multiple point of Δ, and let d' be another multiple point of Δ of multiplicity greater than or equal to that of d.

We map the arc cd on the arc cd'.

Let $\Delta_1, \Delta_2, \ldots, \Delta_m$ be all the subdendrites of Δ, apart from cd, starting from d. There are at least as many subdendrites starting from d'. Let m of them be $\Delta'_1, \Delta'_2, \ldots, \Delta'_m$. Let d_i, for $i = 1, 2, \ldots, m$, be the nearest multiple point of Δ_i to d, and d'_i a point of Δ'_i of multiplicity greater than or equal to that of d_i.

We map the arcs dd_i on the arcs $d'd'_i$.

Let $\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{imi}$ be all the subdendrites of Δ_i, apart from dd_i, starting from d_i. There are at least as many subdendrites of Δ'_i, starting from d'_i. Let m_i of them be $\Delta'_{i1}, \Delta'_{i2}, \ldots, \Delta'_{imi}$. Let d_{ij}, for $j = 1, 2, \ldots, m_i$, be the nearest multiple point of Δ_{ij} to d_i and d'_{ij} a point of Δ'_{ij} of multiplicity greater than or equal to that of d_{ij}.

We map the arcs $d_i d_{ij}$ on the arcs $d'_i d'_{ij}$, and so on.

In this way a mapping of Δ on one of its true subsets is performed and the theorem is proved.

2. The theorem which we have proved in particular cases is not true in general. This will be shown by an example.

Take a horizontal segment and a set A_i, $i = 1, 2, \ldots$, of interior points of the segment, whose only limit points are the end-points of the segment. We shall call the points A_i 'points of the first order' and the set 'a set of the first order'. Through every point A_{i_1}, draw a perpendicular upwards, of length tending to 0 as $i_1 \to \infty$. For every i_1 take a set ('set of the second order') of interior points $A_{i_1 i_2}$, $i_2 = 1, 2, \ldots$ ('points of the second order') on the perpendicular from A_{i_1}, so that the two ends of the perpendicular are limit points of the set. We assume the set to consist of isolated points only. Apart from the ends of the perpendicular, there may be also other limit points. From every point $A_{i_1 i_2}$ draw a perpendicular to $A_{i_1} A_{i_1 i_2}$ to the right, of length tending to 0 as $i_2 \to \infty$, and always less than half the distance from A_{i_1} to any other point A_i. For every i_1, i_2 take a 'set of the third order' of interior points $A_{i_1 i_2 i_3}$, $i_3 = 1, 2, \ldots$, on the perpendicular from $A_{i_1 i_2}$, so that the two ends of the perpendicular are limit points of the set. We

* When we speak of mapping we always mean a topological mapping.

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assume the set to consist of isolated points only. From every point $A_{i_1i_2i_3}$ draw upwards a perpendicular to $A_{i_1i_2}A_{i_1i_2i_3}$, of length tending to 0, as $i_3 \to \infty$, and always less than half the distance from $A_{i_1i_2i_3}$ to any other point $A_{i_1i_2i}$; and so on. In this way a dendrite is defined, as the closure of the set of all these segments. Additional conditions will be imposed later on the sets of points employed in the construction of the dendrite. Considering the set of the points $A_{i_1i_2\dots i_ki}$ for fixed values of $i_1, i_2, \dots, i_k$ and for $i = 1, 2, \dots$, we shall say that $A_{i_1i_2\dots i_k}$ is an s_0-point if it belongs to the s_0th derived set but does not belong to the (s_0+1)th one. We shall also say that the s-value of the set is s_0. *We assume that for no set is the s-value greater than* 4.

If the point $A_{i_1i_2\dots i_k}$ is an s_1-point we say that the index of the first order of $A_{i_1i_2\dots i_k}$ is s_1 and we write $I_1(A_{i_1i_2\dots i_k}) = (s_1)$. If $I_1(A_{i_1i_2\dots i_k}) = (s_1)$ and if, for all points $A_{i_1i_2\dots i_ki}$ in a neighbourhood of $A_{i_1i_2\dots i_k}$, $I_1(A_{i_1i_2\dots i_ki}) = (s_2)$, we say that the index of the second order of $A_{i_1i_2\dots i_k}$ exists and is equal to (s_1, s_2) and we write $I_2(A_{i_1i_2\dots i_k}) = (s_1, s_2)$. Similarly, if $I_1(A_{i_1i_2\dots i_k}) = (s_1)$, and if for all points $A_{i_1i_2\dots i_ki}$ in a neighbourhood of $A_{i_1i_2\dots i_k}$, $I_2(A_{i_1i_2\dots i_ki})$ exists and is equal to (s_2, s_3), we write $I_3(A_{i_1i_2\dots i_k}) = (s_1, s_2, s_3)$, and so on. Also we write $I_\infty(A_{i_1i_2\dots i_k}) = (s_1, s_2, s_3, \dots)$ if, for any n, $I_n(A_{i_1i_2\dots i_k}) = (s_1, s_2, \dots, s_n)$. If $(s_1, s_2, s_3, \dots)$, $(s'_1, s'_2, s'_3, \dots)$ are either two infinite sequences of integers or two finite sequences of the same number of terms, we write

$$(s_1, s_2, s_3, \dots) \leqslant (s'_1, s'_2, s'_3, \dots)$$

if $s_i \leqslant s'_i$ for any i.

If $I_l(A) = (s_1, s_2, s_3, \dots)$, $I_l(A') = (s'_1, s'_2, s'_3, \dots)$, where l is either finite or infinite, and if

$$(s_1, s_2, s_3, \dots) \leqslant (s'_1, s'_2, s'_3, \dots),$$

then we write

$$I_l(A) \leqslant I_l(A').$$

If the dendrite is mapped on itself or on a part of itself, so that the point A is mapped on the point A′, and if $I_l(A)$, $I_l(A')$ *exist, then*

$$I_l(A) \leqslant I_l(A').$$

Every double point of the dendrite is an isolated interior double point on one of the segments forming the dendrite and is a limit point of double points on the perpendicular from this point. Therefore if A' is mapped on A a part of the perpendicular from A' is mapped on a part of the perpendicular from A, and from this the statement follows easily. It is this fact that leads us to the solution of our problem.

Observe that if

$$I_\infty(A_{i_1i_2\dots i_k}) = (s_1, s_2, \dots),$$

then s_1 is the s-value of the set of order $k+1$ of points $A_{i_1i_2\dots i_ki}$, $i = 1, 2, \dots$, s_2 is the s-value of some of the sets of order $k+2$ of points $A_{i_1i_2\dots i_{k+1}i}$, $i = 1, 2, \dots$, and generally s_j is the s-value of some of the sets of order $k+j$ of points $A_{i_1i_2\dots i_{k+j-1}i}$, $i = 1, 2, \dots$. Under the conditions so far imposed on sets of multiple points, an index of order higher than 1 need not exist for every point; for instance, if $I_1(A_{1i})$ is not constant in a neighbourhood of A_1, then $I_2(A_1)$ does not exist.

We shall arrange the s-values of all the sets of various orders by stages, so that an index of infinite order exists for any multiple point of the dendrite, and that for no pair of multiple points A, B of the dendrite is there a relation of the form

$$I_\infty(A) \leqslant I_\infty(B).$$

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If this condition is satisfied, then no neighbourhood of any multiple point of the dendrite can be mapped on a part or the whole of a neighbourhood of another multiple point, and thus the dendrite cannot be mapped on one of its true subsets, since all the end-points of the dendrite are limit points of the set of multiple points.

Take an increasing sequence of positive integers $k_1 < k_2 < \ldots$, and denote by C_p the set of the points $A_{i_1 i_2 \ldots i_k}$ for $k = 1, 2, \ldots, k_p$ and $i_1, i_2, \ldots \leqslant k_p$, so that C_p contains $n_p = k_p + k_p^2 + \ldots + k_p^{k_p}$ points. For any p define the integer l_p so that $2^{l_p - 1} < n_p \leqslant 2^{l_p}$.

So far sets of various orders of double points have been subject only to the condition of consisting exclusively of isolated points and of having the end-points of the segments on which they are constructed as their limit points. We shall start now assigning indices of order $2l_1$ to the points of C_1 by specifying the nature of some of the above sets by their s-values.

Let a_i, for any $i = 1, 2, \ldots$, denote one of the two pairs of numbers $(1, 4)$ and $(2, 3)$; then, the sequence $a_1, a_2, \ldots, a_{l_1}$ is a sequence of $2l_1$ integers $s_1, s_2, s_3, s_4, \ldots, s_{2l_1 - 1}, s_{2l_1}$ and thus 2^{l_1} distinct sequences of this kind can be defined. As neither of the relations $(1, 4) \leqslant (2, 3)$, $(2, 3) \leqslant (1, 4)$ is true for the two pairs $(1, 4)$, $(2, 3)$, it follows that for no pair of the above sequences is the relation

$$(s_1, s_2, \ldots, s_{2l_1}) \leqslant (s'_1, s'_2, \ldots, s'_{2l_1})$$

true. Take n_1 of these sequences and assign one sequence to each point of C_1. Let $A^{(1)}, A^{(2)}, \ldots, A^{(n_1)}$ be the points of C_1 and $I^{(1)} = (s_1^{(1)}, s_2^{(1)}, \ldots, s_{2l_1}^{(1)})$, $I^{(2)}, \ldots, I^{(n_1)}$ the corresponding sequences. Let $D^{(1)}$ be the subdendrite springing from the point $A^{(1)} = A_{i_1 i_2 \ldots i_k}$, i.e. the subdendrite of the type Δ built on the perpendicular at $A^{(1)}$ to the segment of which $A^{(1)}$ is an interior point. If there are some other points of C_1 belonging to $D^{(1)}$, then remove from $D^{(1)}$ the subdendrites, springing from those points, and denote by $D_1^{(1)}$ the remainder. Otherwise $D^{(1)}$ coincides with $D_1^{(1)}$. Let the set of order $k+1$ of the points $A_{i_1 i_2 \ldots i_k i}$, $i = 1, 2, \ldots$, have the s-value $s_1^{(1)}$, the set of order $k+2$ of the points $A_{i_1 i_2 \ldots i_{k+1} i}$, $i = 1, 2, \ldots$, belonging to $D_1^{(1)}$ have the s-value $s_2^{(1)}$, the set of order $k+3$ of the points $A_{i_1 i_2 \ldots i_{k+2} i}$, $i = 1, 2, \ldots$, belonging to $D_1^{(1)}$ have the s-value $s_3^{(1)}$ and so on, up to $s_{2l_1}^{(1)}$. We shall have then

$$I_{2l_1}(A^{(1)}) = (s_1^{(1)}, s_2^{(1)}, \ldots, s_{2l_1}^{(1)}) = I^{(1)}.$$

Similarly, by specifying the nature of some other sets by their s-values we arrange for any other point $A^{(i)}$ of C_1 to have an index of order $2l_1$ equal to $I^{(i)}$, i.e. $I_{2l_1}(A^{(i)}) = I^{(i)}$. In this way some sets of the 2nd, the 3rd, ..., the $(k_1 + 2l_1)$th order have been assigned an s-value.

For no pair $A^{(i)}, A^{(j)}$ of points of C_1 is

$$I_{2l_1}(A^{(i)}) \leqslant I_{2l_1}(A^{(j)}),$$

and thus *a neighbourhood of no point of C_1 can be mapped on a part or the whole of a neighbourhood of another point of C_1.*

We shall now pass to C_2. It has n_2 points, where $2^{l_2 - 1} < n_2 \leqslant 2^{l_2}$. As before, consider sequences $a_{l_1+1}, a_{l_1+2}, \ldots, a_{l_1+l_2}$ of l_2 pairs of numbers $(1, 4)$, $(2, 3)$. There are 2^{l_2} different ones and they define as many different sequences of $2l_2$ numbers $s_{2l_1+1}, s_{2l_1+2}, \ldots, s_{2l_1+2l_2}$.

Take n_2 of them and assign one sequence to each of the points $A^{(1)}, A^{(2)}, \ldots, A^{(n_2)}$ of C_2, so that $I^{(j)} = (s_{2l_1+1}^{(j)}, \ldots, s_{2l_1+2l_2}^{(j)})$ is assigned to $A^{(j)}$.

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In the course of the preceding operations indices of order $2l_1$ were assigned to the points $A^{(1)}, A^{(2)}, \ldots, A^{(n_1)}$, and some of the remaining points of C_2 may have been assigned indices of lower order. For any $1 \leqslant j \leqslant n_2$ take, as before, the subdendrite $D^{(j)}$, springing from $A^{(j)}$, remove from it the subdendrites starting from the points of C_2 belonging to $D^{(j)}$ (if any), and denote the rest by $D_1^{(j)}$. By assigning s-values to the sets of various orders belonging to $D_1^{(j)}$, we arrange for $A_2^{(j)}$ to have an index of order $2l_1 + 2l_2$ defined by

$$I_{2l_1+2l_2}(A^{(j)}) = (s_1, s_2, \ldots, s_{2l_1+2l_2}),$$

where some of the numbers $s_1, s_2, \ldots, s_{2l_1}$, or all of them, may have been defined in the course of operations with respect to C_1, say the first i_1 ones, $0 \leqslant i_1 \leqslant 2l_1$; then $s_{i_1+1}, \ldots, s_{2l_1}$ are chosen arbitrarily and $s_{2l_1+1}, s_{2l_1+2}, \ldots, s_{2l_1+2l_2}$ are given the values $s_{2l_1+1}^{(j)}, s_{2l_1+2}^{(j)}, \ldots, s_{2l_1+2l_2}^{(j)}$. For no pair of points $A^{(i)}, A^{(j)}$ of C_2 are the sequences of the last $2l_2$ terms of their indices the same and, therefore, the relation

$$I_{2l_1+2l_2}(A^{(i)}) \leqslant I_{2l_1+2l_2}(A^{(j)})$$

is never satisfied, and thus: *a neighbourhood of no point of C_2 can be mapped on a part or the whole of a neighbourhood of another point of C_2.*

Observe that our specifications of various sets are not conflicting.

We proceed similarly with $C_3, C_4, \ldots$, so that for any p *a neighbourhood of no point of C_p can be mapped on a part or the whole of a neighbourhood of another point of C_p.*

As any pair of multiple points of the dendrite belongs to C_p for sufficiently large p, we conclude that *a neighbourhood of no multiple point of the dendrite can be mapped on a part or the whole of a neighbourhood of another multiple point.*

Thus, any subdendrite of the type Δ can be mapped only on itself; i.e.:

No subdendrite is homeomorphic to the whole of the dendrite.

The above property of neighbourhoods of multiple points is possessed also by limit points of the set of the multiple points. But the dendrite contains segments consisting entirely of simple points and, of course, neighbourhoods of all interior points of such a segment are homeomorphic.

3. We shall now proceed to the construction of a totally heterogeneous continuum. We start with the dendrite we have just constructed. Denote it by I_0. We have to make two remarks with regard to its construction. It is obviously irrelevant that every segment of the dendrite should be perpendicular to the segment from which it starts. This was done only for convenience in the description of the dendrite, and in the coming construction this condition will not be kept.

Further, all the double points of any order have been considered as distinct. We shall now assume that all the double points of any odd order are distinct, but the pairs of points of the second order $A_{i,2k-1}$, $A_{i,2k}$, $k = 1, 2, \ldots$, are coincident (any pair thus forming a single treble point) and so are the pairs of any even order, for instance $A_{i_1,i_2,i_3,2k-1}$, $A_{i_1,i_2,i_3,2k}$. From every pair of coincident points start two different subdendrites (that is why the perpendicularity condition has to be abandoned in this construction). This alteration need not affect the dendrite from our main point of view. The indices of infinite order are all the same defined for all the multiple points and the dendrite is basically heterogeneous.

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Take now the largest segment, or one of the largest segments, of I_0 that is free from multiple points, and from its mid-point construct a dendrite I_1 of diameter less than $\frac{1}{2}$, so that it does not meet I_0 at any other point. The dendrite I_1 is assumed to be similar to I_0, with the difference that the points of the orders 1, 2, 5, 6, 9, 10, 13, 14, ... are distinct and the points of all the other orders are coincident in pairs.

Take now again the largest segment, or one of the largest segments, of the dendrite I_0+I_1 that is free from multiple points, and from its mid-point construct a basically heterogeneous dendrite I_2, of diameter less than 2^{-2}, so that it does not meet I_0+I_1 at any other point. The multiple points of the first four orders of I_2 are assumed distinct, then those of the next four orders coincident in pairs, then those of the following four orders again distinct, and so on. Similarly, we construct basically heterogeneous dendrites I_3, I_4, ..., so that I_n is of diameter less than 2^{-n}, and that the multiple points of the first 2^n orders of I_n are distinct, those of the following 2^n orders are coincident in pairs, and so on.

Denote by H the closure of $I_0+I_1+I_2+\ldots$. H is a dendrite and its multiple points are everywhere dense on it. Let B and C be two multiple points of H. B is an interior point of one segment of H and the starting point of one or two other segments of H, according as B is one double point or a pair of two coincident double points. Denote by H_B a subdendrite of H which starts from B and contains a segment of H starting from B. If B is one double point of H, then H_B is defined uniquely, and if B is a pair of two coincident double points then there are two subdendrites, each of which can be taken for H_B. Similarly, let H_C be a subdendrite starting at C. *We shall show that a neighbourhood of C in H_C cannot be mapped into* (*i.e. on a part or on the whole of*) *a neighbourhood of B in H_B so that C is mapped on B.*

Suppose this statement is not true. Let B be the starting point or a multiple point of the dendrite I_k and C of the dendrite I_l. Let B' be the other end of the straight segment of I_k, starting at B and belonging to H_B, and let CC' be a similar segment of I_l. The multiple points of BB' consist of multiple points of I_k and of the starting points of some of I_i for $i>k$. The suffix i is large in the neighbourhood of B. A similar position exists on CC'. When we map a neighbourhood of C in H_C into a neighbourhood of B in H_B, transforming C into B, a certain part of CC', say CC'', must be mapped on a certain part BB'' of BB'. We may assume B'' and C'' as near respectively to B and C as we like. Any multiple point C_{i_1} ('a point of the first order with respect to H_C') of CC'' will be mapped on a multiple point B_{i_1} of BB'', of the same or of larger multiplicity. Then, as before, a certain part $C_{i_1}C''_{i_1}$ of a segment $C_{i_1}C'_{i_1}$ of H_C whose origin is C_{i_1} will be mapped on a certain part $B_{i_1}B''_{i_1}$ of a similar segment $B_{i_1}B'_{i_1}$ of H_B. Any multiple point $C_{i_1i_2}$ ('a point of the second order with respect to H_C') of $C_{i_1}C''_{i_1}$ will be mapped on a multiple point $B_{i_1i_2}$ of $B_{i_1}B''_{i_1}$, and so on. Denote by H'_C the part of H_C consisting of CC'', of $C_{i_1}C''_{i_1}$ for all C_{i_1} on CC'', of $C_{i_1i_2}C''_{i_1i_2}$ for all $C_{i_1i_2}$ on $C_{i_1}C''_{i_1}$ and so on, and by H'_B the image of H'_C on H_B. *If C'' is sufficiently near C, then $I_lH'_C$ is mapped on $I_kH'_B$.* For if a point C^0 of H'_C is mapped on a point B^0 of H'_B, then the order of B^0 with respect to H_B is equal to the order of C^0 with respect to H_C. Also any pair of coincident double points of H'_C is mapped on a pair of coincident double points of H'_B. If C'' is near enough to C, the double points of H'_B and H'_C are either double points of I_k, I_l or of I_i for

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large i. The double points of the first l_0 orders of $I_l H'_C$, $1 \leqslant l_0 \leqslant 2^l$, are all of the same kind, say all distinct, the double points of the next 2^l orders are all coincident in pairs, the double points of the following 2^l orders are all distinct, and so on. The intervals of successive orders of distinct double points of I_i and of intervals of successive orders of coincident double points are longer than for I_k, I_l, and therefore no subdendrite of I_k or I_l can be mapped on I_i.

Thus $I_l H'_C$ is mapped on $I_k H'_B$. By a similar argument we conclude that k must be equal to l. Now the part of I_l belonging to H'_C, strictly speaking, cannot be called a neighbourhood of C, since there may be parts of I_l as near C as we please that do not belong to H'_C. But, still, the set $I_l H'_C$ is sufficient for defining the index of infinite order of I_l at the point C. But I_l, being a basically heterogeneous dendrite $I_l H'_C$, cannot be mapped on $I_l H'_B$ unless H_C coincides with H_B.

From that it follows easily that H is a totally heterogeneous continuum. For, if not, suppose that a neighbourhood of a double point V in H can be mapped into a neighbourhood of a double point U in H. Let U, V lie on the segments MM', NN' of H respectively and let UU', VV' and possibly UU'_1, VV'_1, be the segments of H starting from U, V. By the above argument the subdendrite of H starting from V and containing VV' cannot be mapped into the subdendrite starting from U and containing UU' (or UU'_1) and thus must be mapped into a subdendrite starting from U and containing either UM or UM', say UM'.

Then a certain part VV'' of VV' is mapped on a part UM'' of UM'. If C is a double point on VV'' and CC' a segment of H starting from C, then the subdendrite starting from C and containing CC' must be mapped into a subdendrite starting from a double point B of UM'' and containing a segment BB' starting from B, which is impossible.

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A GENERAL FORM OF THE COVERING PRINCIPLE AND RELATIVE DIFFERENTIATION OF ADDITIVE FUNCTIONS

BY A. S. BESICOVITCH

The Vitali covering principle is a powerful method in a wide class of problems of the theory of functions of a real variable and of the theory of sets.

Various problems demand the covering principle under various forms of conditions. In the present paper I consider in a certain sense the most general form of condition. The covering principle in Vitali's own case is concerned with n-dimensional Lebesgue measure in n-dimensional space: I have extended it to m-dimensional Hausdorff measure for any $m \leqslant n$.* I give here an extension of the covering principle to any non-negative additive function.

* A. S. Besicovitch, On the fundamental geometrical properties of linearly measurable plane sets of points (II). *Math. Ann.* 115 (1938), 295–329. Lemma 2 establishes the principle for $m = 1$ and $n = 2$, but the same argument is applicable for any n and $m < n$.

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