A METHOD FOR CONSTRUCTING AND INTERPRETING SOME WEIGHTED PREMIUM PRINCIPLES

BY

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ABSTRACT

We present a method for constructing and interpreting weighted premium principles. The method is based on modifying the underlying risk distribution in such a way that the risk-adjusted expected value (or premium) is greater than the expected value of some conveniently chosen function of claims, which defines the insurer's perception of the risk. Under some assumptions on the function of claims, the method produces distortion premium principles. We provide several examples under different assumptions on the claim arrival process and different functions of claims, including record claims and kth record claims.

KEYWORDS

Weighted premium principle, distortion premium principle, equilibrium distribution, order statistics, record value.

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1. INTRODUCTION

The actuarial literature provides a wide variety of approaches for pricing insurance risks (see Goovaerts and Laeven, 2008, for a review). Some theories are based on the idea that the price of a risk should be derived from a risk-adjusted distribution, which is obtained by modifying the underlying risk distribution to include risk aversion or the insurer's perception of the risk. Relevant examples of risk-adjusted distributions include distorted distributions (see Yaari, 1987, Denneberg, 1990, and Wang, 1996) and weighted distributions (see Furman and Zitikis, 2008a,b, 2009 and Zhu *et al*., 2016).

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Given a nonnegative random variable (or risk) X with distribution function F_X, under the distorted expectation theory, the insurer uses a distortion function h (i.e., a nondecreasing function $h:[0,1]\to[0,1]$ such that $h(0)=0$ and $h(1)=1$) and values the risk X as its distorted expectation, defined as

$$P_h(X)=\int_0^\infty h\big(\overline{F}_X(x)\big)dx, \tag{1.1}$$

where $\overline{F}_X=1-F_X$ is the tail function of X and h is assumed to be independent of F_X. The functional (1.1) is known as the distortion premium principle. When h is concave, the premium is loaded (i.e., $P_h(X)\geq E[X]$) and satisfies many desirable actuarial properties (see Section 2.6 in Denuit *et al.*, 2005, for a review). If h is continuous, $\overline{F}_h(x)=h\left(\overline{F}_X(x)\right)$ can be interpreted as the (distorted) tail function of certain risk-adjusted random variable X_h called distorted random variable (if h is not continuous, $\overline{F}_h(x)$ may not be right continuous).

Under the weighted distributions approach, the insurer uses a function $\omega:[0,\infty)\longrightarrow[0,\infty)$ such that the expectation $E[\omega(X)]$ is strictly positive and finite. The risk X is valued as the expectation of a risk-adjusted random variable X_ω called weighted random variable (see Rao, 1965), with distribution function

$$F_\omega(x)=\frac{E\left[I\{X\leq x\}\,\omega(X)\right]}{E\left[\omega(X)\right]},\quad x>0, \tag{1.2}$$

where, given a statement S, the indicator $I\{S\}$ is equal to 1 if S is true and 0 otherwise. When X is absolutely continuous with density function $f_X(x)$, the weighted density function of X_ω is given by

$$f_\omega(x)=\frac{\omega(x)}{E\left[\omega(X)\right]}f_X(x),\quad x>0.$$

The corresponding weighted premium is given by

$$P(X_\omega)=\int_0^\infty \overline{F}_\omega(x)\,dx=\frac{E\left[X\omega(X)\right]}{E\left[\omega(X)\right]}. \tag{1.3}$$

When ω is nondecreasing, Equation (1.3) satisfies the loading and other convenient actuarial properties (see Furman and Zitikis, 2008a).

A number of well-known premiums are special cases of (1.3) for appropriate choices of ω. In particular, a distortion premium principle with differentiable distortion function h can be interpreted as a weighted premium principle of the form (1.3), where the weight function ω is defined by $\omega(x)=h'(\overline{F}_X(x))$, for $x\geq 0$. It will be useful in the sequel to note that, under this interpretation, if the distortion function h is independent of the distribution function F_X (as is generally assumed), the corresponding ω depends on F_X.

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In this paper, we provide an appealing interpretation of a class of premium principles (which turn to be weighted premium principles) when the purpose is to evaluate the risk of a portfolio of claims. A premium principle of this class is interpreted as the expectation of a risk-adjusted distribution in which the risk perception of the insurer is captured by a real function Ψ with domain in the space of real sequences. For given such a function Ψ, let P_Ψ be a functional that maps a risk X (which, throughout this paper, is assumed to be continuous with strictly increasing distribution function F_X) to a nonnegative real number $P_\Psi(X)$, representing the premium to be charged to make the risk acceptable. A risk X can be priced through this approach whenever, given a sequence $X_1, X_2, \ldots$ of successive independent claims with the same distribution as X, the support of the random variable,

$$Z = \Psi\left[\{X_i\}_{i\geq 1}\right], \tag{1.4}$$

is contained in the support of X. For different choices of Ψ, the random variable Z may be, for example, the largest claim in a set of n independent claims or the first claim whose value exceeds that of all previous claims. Observe that, given X, the concepts Ψ and Z can be used interchangeably via (1.4) to determine the premium. A key difference between our approach and those initially based on distortions or weights is how we reflect the attitude of the insurer toward risk. The key element in this approach is that, given X, the insurer requires a premium principle to charge, at least, the expected payout of Z. To formalize this idea, we interpret the premium $P_\Psi(X)$ as being $P_\Psi(X) = E[Y]$, where Y is a risk-adjusted random variable such that

$$E[Y \mid Z = x] \geq x, \text{ for all } x \in Supp(Z). \tag{1.5}$$

The meaning of this equation is that for a sequence of independent claims $x_1, x_2, \ldots$ such that $\Psi\left[\{x_i\}_{i\geq 1}\right] = x$, the premium is expected to be greater or equal than x. Then, if F_Z is the distribution function of Z, we have

$$P_\Psi(X) = \int_0^\infty E[Y \mid Z = x] dF_Z(x) \geq \int_0^\infty x dF_Z(x) = E[Z],$$

that is, the premium is at least as large as the expected value of Z, as claimed.

The next step in this approach is to find a reasonable risk-adjusted random variable Y satisfying (1.5). A first candidate for such a risk-adjusted random variable is $Y = Z$ itself. However, the premium principle $P_\Psi(X) = E[Z]$ derived from this choice may have undesirable properties. For example, it may hold that $P_\Psi(X) < E[X]$, which violates the risk loading property (this is the case, e.g., when Z is the smallest claim in a set of n independent claims with the same distribution as X). Instead, we propose (among other possible candidates) a risk-adjusted random variable Y such that

$$[Y \mid Z = x] \overset{d}{=} [X \mid X > x], \text{ for all } x \in Supp(Z), \tag{1.6}$$

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(where $\stackrel{d}{=}$ denotes "equal in distribution"). This equation uniquely determines a random variable Y satisfying (1.5) that offers the following benefits.

(a) Since $E[X \mid X > x] \geq \max\{x, E[X]\}$, then

$$P_\Psi(X) = \int_0^\infty E[Y \mid Z = x] dF_Z(x) = \int_0^\infty E[X \mid X > x] dF_Z(x) \qquad (1.7)$$
$$\geq \max\{E[X], E[Z]\},$$

that is, the premium satisfies the risk loading property and $P_\Psi(X) \geq E[Z]$, as claimed. Moreover, given a sequence of independent claims $x_1, x_2, \ldots$ such that $\Psi[\{x_i\}_{i\geq 1}] = x$, the premium $P_\Psi(X)$ depends on the distribution of all the claims greater than x, which makes the premium principle sensitive to the tail risk.

(b) The premium principle P_Ψ derived from (1.6) is a weighted premium principle, as it will be shown in Section 2, that can be expressed as a mixture of tail value at risks. Recall that the tail value at risk at level p of X is defined by

$$\mathrm{TVaR}_p(X) = \frac{1}{1-p} \int_p^1 F_X^{-1}(t)\, dt, \; p \in (0, 1),$$

where $F_X^{-1}(t) = \inf\{x : F_X(x) \geq t\}$, $t \in [0, 1]$, is the quantile function of X. Making the change of variable $F_X(x) = p$ in (1.7) and using the continuity of $\overline{F}_X$, we see that

$$P_\Psi(X) = \int_0^1 \mathrm{TVaR}_p(X) dF_Z(F_X^{-1}(p)).$$

Observe that the mixing distribution depends on the composition of F_Z with the quantile function of X. However, when the mixing distribution $F_Z(F_X^{-1}(p))$ does not depend on the distribution function F_X (which happens when $\overline{F}_Z$ is obtained from $\overline{F}_X$ via distortion), this representation guarantees that P_Ψ is a distortion premium principle with concave distortion function, as shown in Lemma 4.63 of Föllmer and Schied (2004) or in Proposition 2.64 of Pflug and Römisch (2007). Moreover, when this is the case (i.e., when there exists a distortion function H such that $\overline{F}_Z(x) = H(\overline{F}_X(x))$ for all $x \geq 0$), we can obtain an upper bound for the risk loading of the premium that only depends on H (which is determined by the function Ψ) and the standard deviation of the risk X (this is shown in Section 2). Some examples are studied in detail in Section 3.

(c) Under Equation (1.6), the stochastic behavior of Z determines the attitude of the insurer toward risk, as claimed. To see that, let us consider $Z_1 = \Psi_1[\{X_i\}_{i\geq 1}]$ and $Z_2 = \Psi_2[\{X_i\}_{i\geq 1}]$. Assume that $Z_1 \leq_{st} Z_2$, that is, that $E[\phi(Z_1)] \leq E[\phi(Z_2)]$ for all nondecreasing real function ϕ. Since

$$P_\Psi(X) = \int_0^\infty \phi_X(x) dF_Z(x) = E[\phi_X(Z)], \qquad (1.8)$$

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where the function $\phi_X(x) = E[X \mid X > x]$ is nondecreasing, $Z_1 \leq_{st} Z_2$ clearly implies $P_{\Psi_1}(X) \leq P_{\Psi_2}(X)$.

The risk-adjusted distribution Y studied in this paper is a natural extension of the original one proposed by Sordo *et al.* (2016), which was restricted to the case where Z is the ith-order statistic of a sample of n identical copies of X. Specifically, the risk-adjusted distribution $Y_{X,i,n}$ considered by Sordo *et al.* (2016) verifies

$$\left[Y_{X,i,n} \mid X_{i:n} = x\right] \stackrel{d}{=} [X \mid X > x] \text{ for all } x \geq 0,$$

where $X_{i:n}$ is the ith-order statistic of n independent copies of X. As we will show throughout the paper, the extension (1.4) offers more flexibility to fix the premium than the original $Z = X_{i:n}$ and incorporates, in a natural way, record and kth record random variables within a risk-adjusted measurement framework.

The remainder of the paper is structured as follows. In Section 2, we study the main properties of the premium principle $P_\Psi(X)$. In Section 2.1, we focus on the risk loading of the premium, providing general bounds that can be useful to know the highest possible premium to be charged. In Section 2.2, we study the case where $P_\Psi(X)$ is a distortion premium principle and show that, in this case, the bound on the risk loading only depends on σ (the standard deviation of X) and a distortion function H that is exogenous to X. Finally, in Section 3, we particularize the premium principle $P_\Psi(X)$ to different contexts and choices of Ψ (or Z). In Section 3.1, we suppose that the claims arrive according to a classical continuous time risk model and Z is defined as the random variable whose distribution function is the equilibrium distribution function of F_X. In Sections 3.2, 3.3, and 3.4, we consider, respectively, the cases where Z is the ith-order statistic of a sample of n identical copies of X (Section 3.2), the jth record claim of a sequence of independent identical copies of X (Section 3.3), and the jth kth record claim of a sequence of independent identical copies of X (Section 3.4).

2. THE PREMIUM PRINCIPLE

Given a real function Ψ with domain in the space of real sequences and a risk X, in the remainder of this section, we assume that Z is a random variable with tail function $\overline{F}_Z$ associated to X and Ψ via (1.4) and that Y is a random variable defined by

$$[Y \mid Z = x] \stackrel{d}{=} [X \mid X > x], \text{ for all } x \in Supp(Z).$$

Our aim is to study the premium principle P_Ψ defined as

$$P_\Psi(X) = E[Y]. \tag{2.1}$$

We first show that the risk-adjusted random variable Y is a weighted random variable.

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Lemma 1. *The risk-adjusted random variable Y is a weighted random variable X_ω with weight function given by*

$$\omega(x) = \int_0^x \frac{dF_Z(t)}{\overline{F}_X(t)}, \quad x \geq 0. \tag{2.2}$$

Proof. We obtain the tail function $\overline{F}_Y$ of Y. For $y \geq 0$,

$$\begin{aligned}
\overline{F}_Y(y) &= \int_0^\infty P(Y > y \mid Z = x)\, dF_Z(x) \\
&= \int_0^\infty P(X > y | X > x) dF_Z(x) \\
&= \int_0^\infty \frac{P(X > \max\{x,\ y\})}{P(X > x)} dF_Z(x) \\
&= \overline{F}_X(y) \int_0^y \frac{dF_Z(x)}{\overline{F}_X(x)} + \overline{F}_Z(y) \qquad (2.3) \\
&= \overline{F}_X(y)\omega(y) + \overline{F}_Z(y).
\end{aligned}$$

It is immediate to check that $\overline{F}_Y(y) = \overline{F}_\omega(y)$, where $\overline{F}_\omega$ is the tail function of a weighted random variable X_ω with ω given by (2.2) and with $F_\omega(y)$ defined by (1.2). ■

Remark 2. *When X and Z are absolutely continuous with density functions f_X and f_Z, respectively, then Y is absolutely continuous with density function*

$$f_Y(y) = f_X(y)\omega(y), \quad y \geq 0,$$

where $\omega(y) = \int_0^y \frac{f_Z(x)}{\overline{F}_X(x)}\, dx$.

Remark 3. *It is easy to see that the function $\omega(x)$ defined by (2.2) satisfies the following properties: (a) $\omega(0) = 0$, (b) $\omega(x)$ is increasing and right continuous (and if F_Z is continuous, it is also continuous) and (c) $E[\omega(X)] = 1$.*

The following result is a sort of reciprocal of Lemma 1. It states that any function $\omega : [0, \infty) \longrightarrow [0, \infty)$ satisfying the properties (a), (b) and (c) given in Remark 3 can be expressed in the form (2.2) for some distribution function G.

Theorem 4. *Let X be a risk with continuous distribution function F_X and let $\omega : [0, \infty) \longrightarrow [0, \infty)$ a function such that $\omega(0) = 0$, $\omega(x)$ is increasing and right continuous and $E[\omega(X)] = 1$. Then, there exists a distribution function G, such that*

$$\omega(x) = \int_0^x \frac{dG(y)}{\overline{F}_X(y)}, \quad x \geq 0.$$

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Proof. It suffices to show that the function G defined by

$$G(x) = \int_0^x \overline{F}_X(t) d\omega(t), \quad x \geq 0, \tag{2.4}$$

is a distribution function. Clearly, $G(0) = 0$. Since $\overline{F}_X$ is continuous, G inherits the increasing monotonicity and the right continuity from ω. Since $\omega(0) = 0$, integration by parts of (2.4) gives

$$G(x) = \overline{F}_X(x)\omega(x) + \int_0^x \omega(t) dF(t), \quad x \geq 0.$$

Finally, since ω is increasing, it follows by standard arguments that $\lim_{x\to\infty} \overline{F}_X(x)\omega(x) = 0$, which, together with $E[\omega(X)] = 1$, implies that $\lim_{x\to\infty} G(x) = 1$. This proves that G is a distribution function. ■

2.1. The risk loading: formula and bounds

Now we focus on the risk loading of the premium principle (2.1). Integration by parts of (2.3) gives

$$\overline{F}_Y(x) = \overline{F}_X(x) \left(1 + \int_0^x \frac{\overline{F}_Z(t)}{(\overline{F}_X(t))^2} dF_X(t) \right). \tag{2.5}$$

Therefore,

$$P_\Psi(X) = \int_0^\infty \overline{F}_Y(x) dx = E[X] + \epsilon_\Psi(X),$$

where the risk loading is given by

$$\epsilon_\Psi(X) = \int_0^\infty \overline{F}_X(x) \left(\int_0^x \frac{\overline{F}_Z(t)}{(\overline{F}_X(t))^2} dF_X(t) \right) dx.$$

The following result provides an upper bound on the risk loading $\epsilon_\Psi(X)$.

Theorem 5. *For a risk X with mean μ and finite variance σ^2, it holds*

$$\epsilon_\Psi(X) \leq \sigma \, SD[\omega(X)], \tag{2.6}$$

where $SD[\omega(X)]$ denotes the standard deviation of the weight function $\omega(X)$ defined by (2.2). Moreover, the bound is reached when

$$F_Z(x) = \frac{1}{\mu} \int_0^x \overline{F}_X(t) dt, \quad x \geq 0. \tag{2.7}$$

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Proof. Since $\epsilon_\Psi(X) = E[Y - \mu]$, using (2.3), we can write

$$\epsilon_\Psi(X) = \int_0^\infty (y-\mu) dF_Y(y)$$
$$= \int_0^\infty (y-\mu)\omega(y) dF_X(y).$$

Given $\alpha \in \mathbb{R}$, this can also be written as

$$\epsilon_\Psi(X) = \int_0^\infty (y-\mu)\,(\omega(y)-\alpha)\, dF_X(y)$$
$$= E[(X-\mu)(\omega(X)-\alpha)].$$

Using the Cauchy–Schwarz inequality, it follows that

$$(\epsilon_\Psi(X))^2 \leq \sigma^2 \cdot E[(\omega(X)-\alpha)^2],$$

and taking the infimum over $\alpha \in \mathbb{R}$, we get

$$(\epsilon_\Psi(X))^2 \leq \sigma^2 Var[\omega(X)].$$

Finally, observe that the Cauchy–Schwarz inequality becomes equality if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\omega(x) - \alpha = \lambda(x-\mu),$$

with $\alpha = 1$. It is easy to see from (2.2) that this happens when (2.7) holds. ■

Remark 6. *A random variable Z with distribution function given by (2.7) is known in the literature as the equilibrium (or integrated tail) distribution of X (see Section 3.4.5 in Klugman et al., 2008) and has important applications in the context of the ruin model. We come back to this random variable in Section 3.1.*

Example 7. *For the case where Z is an independent copy of X, Sordo et al. (2016) showed that the risk loading $\epsilon_\Psi(X)$ coincides with the cumulative residual entropy of X, denoted by $\epsilon(X)$ (introduced by Rao et al. (2004)). In this case, the function $\omega(X) = -log\overline{F}_X(X)$ follows a standard exponential distribution and from Theorem 5 we have*

$$\epsilon(X) \leq \sigma.$$

This upper bound for the cumulative residual entropy was also obtained by Psarrakos and Sordo (2019) using a different method of proof.

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2.2. A distortion premium principle

In general, Equation (2.1) is not a distortion premium principle. However, when $\overline{F}_Z$ is a distortion of $\overline{F}_X$, then $P_\Psi(X)$ is a distortion premium principle with concave distortion function, as it is shown in the following result.

Theorem 8. *Suppose that there exists a distortion function H such that*

$$\overline{F}_Z(x) = H(\overline{F}_X(x)), \quad \text{for all } x \geq 0. \tag{2.8}$$

Then, $P_\Psi(X)$ is a distortion premium principle with concave distortion function

$$h(u) = u \int_u^1 \frac{dH(v)}{v} + H(u), \quad u \in [0, 1], \tag{2.9}$$

where $h(0)$ is defined by continuity.

Proof. Combining (2.3) and (2.8), we have

$$\begin{aligned}
\overline{F}_Y(y) &= -\overline{F}_X(y) \int_0^y \frac{d\overline{F}_Z(x)}{\overline{F}_X(x)} + \overline{F}_Z(y) \\
&= -\overline{F}_X(y) \int_0^y \frac{dH(\overline{F}_X(x))}{\overline{F}_X(x)} + H(\overline{F}_X(y)) \\
&= \overline{F}_X(y) \int_{\overline{F}_X(y)}^1 \frac{dH(v)}{v} + H(\overline{F}_X(y)) \\
&= h(\overline{F}_X(y)),
\end{aligned}$$

where h is given by (2.9). Now, we show that h is a concave distortion function. Since H is nondecreasing and $h'(u) = \int_u^1 \frac{dH(v)}{v}$ for all $u \in [0, 1]$, h is increasing and concave. Clearly, $h(1) = 1$. To see that $h(0) = 0$, from (2.9), we can write

$$\begin{aligned}
h(u) &= u \int_u^1 \frac{dH(v)}{v} + 1 - \int_u^1 dH(v) \\
&= 1 - \int_u^1 \left(1 - \frac{u}{v}\right) dH(v) \\
&= 1 - \int_u^1 \left(\int_u^v \frac{1}{v} dt\right) dH(v).
\end{aligned} \tag{2.10}$$

Observe that (2.10) is well defined for all $u \in [0, 1]$. In particular, we see that $h(0) = 1 - \int_0^1 dH(v) = 0$, which ends the proof. ∎

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When (2.1) is a distortion premium principle, the upper bound (2.6) is particularly useful because only depends on σ and the function $H(u)$. For simplicity in the proof, we assume that Z is absolutely continuous.

Corollary 9. *Let X and Z be absolutely continuous random variables with respective tail functions $\overline{F}_X$ and $\overline{F}_Z$ verifying (2.8) and density functions f_X and f_Z, respectively. Then,*

$$\epsilon_\Psi(X) \leq \sigma \sqrt{2 \int_0^1 \frac{H(u)}{u} dH(u) - 1}, \tag{2.11}$$

where σ is the standard deviation of X and $H(u)$ is the distortion function defined by (2.8). The bound is sharp.

Proof. Since Z is absolutely continuous, it follows from Remark 2 that Y is absolutely continuous with density function $f_Y(x) = f_X(x)\omega(x)$, where $\omega(x)$ is given by (2.2). Moreover,

$$\begin{aligned} E[\omega^2(X)] &= \int_0^\infty \omega^2(x) f_X(x) dx \\ &= \int_0^\infty \omega(x) f_Y(x) dx \\ &= E[\omega(Y)]. \end{aligned}$$

Since $E[\omega(X)] = 1$ (see Remark 3), we have

$$SD\,[\omega(X)] = \sqrt{E[\omega(Y)] - 1}. \tag{2.12}$$

Now, from (2.2),

$$\begin{aligned} E[\omega(Y)] &= \int_0^\infty \omega(y) f_Y(y) dy \\ &= \int_0^\infty \left(\int_0^y \frac{f_Z(x)}{\overline{F}_X(x)} dx \right) f_Y(y) dy \\ &= \int_0^\infty \frac{f_Z(x)}{\overline{F}_X(x)} \left(\int_x^\infty f_Y(y) dy \right) dx \\ &= \int_0^\infty \frac{f_Z(x)}{\overline{F}_X(x)} \overline{F}_Y(x) dx \\ &\overset{(*)}{=} - \int_0^\infty \frac{h(\overline{F}_X(x))}{\overline{F}_X(x)} dH(\overline{F}_X(x)) \\ &\overset{(**)}{=} \int_0^1 \frac{h(u)}{u} dH(u), \end{aligned}$$

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where we have used that $\overline{F}_Y(x) = h(\overline{F}_X(x))$ and $\overline{F}_Z(x) = H(\overline{F}_X(x))$ in (*) (see Theorem 8) and the change of variables $u = \overline{F}_X(x)$ in (**). Replacing $h(u)$ according to (2.9), we get

$$\begin{aligned} E[\omega(Y)] &= \int_0^1 \left(\int_u^1 \frac{dH(v)}{v} \right) dH(u) + \int_0^1 \frac{H(u)}{u} dH(u) \\ &= \int_0^1 \left(\int_0^v dH(u) \right) \frac{dH(v)}{v} + \int_0^1 \frac{H(u)}{u} dH(u) \\ &= 2 \int_0^1 \frac{H(u)}{u} dH(u). \end{aligned} \tag{2.13}$$

The inequality (2.11) follows by combining (2.6), (2.12), and (2.13).

To show that the upper bound in (2.11) can be attained, note that the Cauchy–Schwarz inequality used in the proof of Theorem 2.6 becomes equality when there exists $\lambda \in \mathbb{R}$ such that

$$F_X^{-1}(u) - \mu = \lambda \left(\int_{1-u}^1 \frac{dH(v)}{v} - 1 \right), \quad u \in [0, 1],$$

where λ is determined by the condition $Var[X] = \sigma^2$. Therefore, the bound is attained for a random variable X with mean μ, variance σ^2, and quantile function given by

$$F_X^{-1}(u) = \mu + \sigma \frac{\displaystyle\int_{1-u}^1 \frac{dH(v)}{v} - 1}{\sqrt{\displaystyle 2\int_0^1 \frac{H(v)}{v} dH(v) - 1}}, \quad 0 \le u \le 1.$$

■

3. PARTICULAR CASES

Given a risk X, the insurer wishes to charge the expected value of $Z = \Psi[\{X_i\}_{i\ge1}]$, where the set $\{X_i\}_{i\ge1}$ can be interpreted as a sequence of independent claims with the same distribution as X. In this section, we consider various scenarios and different choices of the function Ψ giving rise to different families of weighted premium principles. The first one is not a distortion premium principle, while the others are distortion premium principles.

3.1. Equilibrium distributions

Let X be a risk with continuous distribution function F_X and finite moment of order $n+1$ denoted by $\alpha_{n+1} = E[X^{n+1}]$, with $n \ge 1$ and $\alpha_1 = \mu$. The equilibrium distribution function of F_X, denoted by $F_1(x)$, is defined by

$$F_1(x) = \frac{1}{\mu} \int_0^x \overline{F}_X(t) dt, \quad x \ge 0.$$

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The equilibrium distribution plays an important role in the classical continuous time risk model (see Lin and Willmot, 2000). In this context, the number N_t of claims of an insurance portfolio follows a Poisson process with mean λ. The individual claims $X_1, X_2, ...$ are independent, identically distributed with common distribution function F_X, and are also independent of N_t. The insurer's surplus process is $\{U_t : t \geq 0\}$, with $\{U_t = u + ct - \sum_{i=1}^{N_t} X_i\}$, where u is the initial surplus and c is the premium rate per unit time. The equilibrium distribution is interpreted as the distribution function of the random variable Z_1 representing the amount of the drop in the surplus level, given that there is a drop (see Section 13.5 in the book by Bowers *et al.*, 1997). Clearly, Z_1 can be considered as some function Ψ_1 of the claims $X_1, X_2, ...$

In this context, suppose that the insurer requires a premium principle to charge, at least, the amount $E[Z_1]$. In our approach, the insurer takes the premium principle $P_{\Psi_1}(X)$ which, according to Lemma 1, is a weighted premium principle with a weight function given by

$$\omega_1(x) = \int_0^x \frac{dF_1(t)}{\overline{F}_X(t)} = \frac{x}{\mu}.$$

This premium principle is the modified variance premium (see Furman and Zitikis, 2008a). It is easy to see that $P_{\Psi_1}(X) = \mu + \sigma^2/\mu = \alpha_2/\mu$, where $\sigma^2 = \alpha_2 - \alpha_1^2$.

In order to introduce a family of premium principles, we consider the notion of higher order equilibrium distribution function introduced by Lin and Willmot (2000). Let F_2 be the equilibrium distribution function of F_1, and let Z_2 be the random variable with distribution function $F_2(x)$. Then, F_2 is called the second-order equilibrium distribution function of F_X. Similarly, the ith-order equilibrium distribution function of F_X is defined by

$$F_i(x) = \frac{\int_0^x \overline{F}_{i-1}(t)dt}{\int_0^\infty \overline{F}_{i-1}(t)dt} \quad x \geq 0, \tag{3.1}$$

for $1 \leq i \leq n$, with $\overline{F}_0(t) = \overline{F}_X(t)$. Let Z_i be the random variable with distribution function $F_i(x)$, for $1 \leq i \leq n$. By a recursive argument, it follows that Z_i is some function Ψ_i of the claims $X_1, X_2, ...$ It is immediate to prove that

$$\overline{F}_i(x) = \frac{1}{\alpha_i} \int_x^\infty (t-x)^i dF_X(t), \quad x \geq 0$$

(see Lin and Willmot, 2000). The sequence $\{P_{\Psi_i}(X), i = 1, 2, ...\}$ defines a sequence of weighted premium principles with associated weight functions (see Lemma 1) given by

$$\omega_i(x) = \int_0^x \frac{dF_i(y)}{\overline{F}_X(y)}, \quad i = 1, 2, .., n.$$

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In order to give an alternative representation for $\omega_i(x)$, observe that

$$\begin{aligned}\overline{F}_i(x) &= \frac{1}{\alpha_i} E\left[(X-x)^i | X > x\right] \overline{F}_X(x) \\ &= \frac{1}{\alpha_i} m_i(x) \overline{F}_X(x), \end{aligned} \tag{3.2}$$

where $m_i(x)$ denotes the ith moment of the residual life of X, for $1 \le i \le n$, $m_1(x) = m(x)$ is the mean residual life of X, and $m_0(x) = 1$. Using that $E[Z_i] = \frac{\alpha_{i+1}}{(i+1)\alpha_i}$, it follows from (3.1) and (3.2) that

$$\begin{aligned}\frac{dF_i(y)}{\overline{F}_X(y)} &= \frac{\overline{F}_{i-1}(y)}{E[Z_{i-1}]\overline{F}_X(y)} dy \\ &= \frac{m_{i-1}(y)}{\alpha_{i-1} E[Z_{i-1}]} dy. \end{aligned}$$

Using that $\alpha_{i-1} E[Z_{i-1}] = \alpha_i / i$, we have

$$\omega_i(x) = \frac{i}{\alpha_i} \int_0^x m_{i-1}(y) dy, \quad 1 \le i \le n.$$

In particular, for $i = 1$, we obtain the weight function $\omega_1(x) = x/\mu$ corresponding to the equilibrium distribution function, and for $i = 2$, we obtain $\omega_2(x) = \frac{2}{\alpha_2} \int_0^x m(y) dy$. In general, $P_{\Psi_i}(X)$ can be represented in terms of the moments of the residual life of X as follows.

Theorem 10. *Given* $1 \le i \le n$,

$$P_{\Psi_i}(X) = \mu + \frac{1}{\alpha_i} \int_0^\infty m(t) m_i(t) dF_X(t).$$

Proof. By definition,

$$P_{\Psi_i}(X) = \int_0^\infty \overline{F}_{Y_i}(x) dx,$$

where, according to (2.5),

$$\overline{F}_{Y_i}(x) = \overline{F}_X(x) \left(1 + \int_0^x \frac{\overline{F}_i(t)}{(\overline{F}_X(t))^2} dF_X(t)\right).$$

Therefore,

$$P_{\Psi_i}(X) = \mu + \int_0^\infty \overline{F}_X(x) \left(\int_0^x \frac{\overline{F}_i(t)}{(\overline{F}_X(t))^2} dF_X(t)\right) dx,$$

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and changing the order of integration, we get

$$P_{\Psi_i}(X) = \mu + \int_0^\infty \left(\int_t^\infty \frac{\overline{F}_X(x)}{\overline{F}_X(t)} dx \right) \frac{\overline{F}_i(t)}{\overline{F}_X(t)} dF_X(t)$$

$$= \mu + \int_0^\infty m(t) \frac{\overline{F}_i(t)}{\overline{F}_X(t)} dF_X(t)$$

$$= \mu + \frac{1}{\alpha_i} \int_0^\infty m(t) m_i(t) dF_X(t),$$

where the last equality follows from (3.2). ■

From Theorem 10, we see that the risk loading $\epsilon_{\Psi_i}(X)$ of $P_{\Psi_i}(X)$ is given by

$$\epsilon_{\Psi_i}(X) = \frac{1}{\alpha_i} \int_0^\infty m(t) m_i(t) dF_X(t), \quad i = 1, ..., n.$$

In particular, for $i = 1$ (which produces the modified variance premium principle), we have $\epsilon_{\Psi_1}(X) = \int_0^\infty m^2(t) dF_X(t) = \sigma^2$. We can use Theorem 5 to obtain an upper bound on the risk loading $\epsilon_{\Psi_i}(X), i = 1, 2, ...n$. For the case $i = 1$, we recall from (2.7) that the risk loading reaches the bound. In general, for $i = 2, ..., n$, the upper bound is not useful because it depends on m_i and α_i. This is due to the fact that $P_{\Psi_i}(X)$ is not a distortion premium principle.

The following result shows that when the risk X is DFR (decreasing failure rate), then the sequence $\{P_{\Psi_i}(X), i = 1, 2, ..., n\}$ defines an ordered sequence of weighted premium principles, which allows to incorporate the attitude toward risk of the insurer when choosing Z_i. Recall that an absolutely continuous random variable X is DFR if its failure rate, given by

$$r(x) = \frac{f_X(x)}{\overline{F}_X(x)}, \quad x \geq 0,$$

is decreasing. Examples of DFR distributions relevant to actuarial science include Gamma and Weibull distributions with shape parameters less than one and Pareto distributions.

Theorem 11. *If X is a DFR random variable, then $P_{\Psi_i}(X) \leq P_{\Psi_j}(X)$ for $1 \leq i \leq j \leq n$.*

Proof. On the one hand, since X is DFR, then $X \leq_{st} Z_1$ (see Theorem 1.C.55 of Shaked and Shanthikumar, 2007). On the other hand, note that the mean residual life $m(t)$ of X is increasing (see Proposition 1.7.10 in Denuit *et al.*, 2005). Using (3.2) for $i = 1$, we see that the failure rate of Z_1, denoted by $r_1(t)$, satisfies

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$$r_1(t) = \frac{\overline{F}_X(t)}{\mu \overline{F}_1(t)} = 1/m(t), \quad t \geq 0,$$

which implies that Z_1 is DFR. A recursive reasoning shows that the random variables Z_i, for $1 \leq i \leq n$, are DFR and $X \leq_{st} Z_1 \leq_{st} \cdots \leq_{st} Z_n$, which proves the result. ■

Consequently, when X is DFR, the choice of the index i reflects the degree of risk aversion of the insurer: the higher i, the more risk averse the insurer is.

3.2. Order statistics, $Z = X_{i:n}$

The choice $Z = X_{i:n}$, $1 \leq i \leq n$, $n \geq 1$, considered by Sordo *et al.* (2016) and Castaño-Martínez *et al.* (2019), leads to the distortion premium principle

$$P_\Psi(X) = P_{i,n}(X) = \int_0^1 \mathrm{TVaR}_p(X) d\beta_{i,n-i+1}(p), \tag{3.3}$$

where

$$\beta_{i,j}(p) = \int_0^p \frac{(i+j-1)!}{(i-1)!\,(j-1)!} t^{i-1} (1-t)^{j-1} dt, \ 0 \leq p \leq 1,$$

is the Pearson's incomplete beta function (see Pearson, 1934) with parameters (i, j). For $1 \leq i \leq n-1$ and $n \geq 2$, the premium (3.3) coincides with the expected average of the $n-i$ largest claims, that is,

$$P_{i,n}(X) = \frac{1}{n-i} \sum_{j=i+1}^{n} E[X_{j:n}], \quad 0 \leq i \leq n-1, \quad n \geq 2.$$

Clearly, fixed n, the choice of i reflects the degree of risk aversion of the insurer: the higher the value of i (with $1 \leq i \leq n-1$) the more risk averse the insurer is.

Since the explicit form of the risk loading $\epsilon_{i,n}(X)$ corresponding to $P_{i,n}(X)$ is rather complicated (see Sordo *et al.*, 2016), it is useful to know the highest possible premium, which corresponds to the upper bound of $\epsilon_{i,n}(X)$.

Theorem 12. *Let X be an absolutely continuous random variable with distribution function F_X, density function f_X, and finite variance σ^2. Let $\hat{\epsilon}_{i,n}(X)$ denote the upper bound of $\epsilon_{i,n}(X)$. Then,*

$$\hat{\epsilon}_{i,n}(X) = \begin{cases} \sigma \sqrt{\dfrac{2n}{n-i} \dfrac{\sum_{k=0}^{i-1} \binom{i-1+k}{k} \binom{2n-i-k-1}{n-k}}{\binom{2n-1}{n}} - 1}, & 1 \leq i < n, \\ \sigma \sqrt{2n \displaystyle\sum_{j=0}^{n-1} \frac{1}{n+j} - 1}, & i = n, n \geq 1. \end{cases} \tag{3.4}$$

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Proof. Let $n \geq 2$ and consider first the case $1 \leq i < n$. Denote by $\overline{F}_{i:n}$ the survival function of $Z = X_{i:n}$, given by (see Arnold *et al.*, 2008)

$$\overline{F}_{i:n}(x) = \sum_{k=0}^{i-1} \binom{n}{k} [F(x)]^k [\overline{F}(x)]^{n-k}.$$

Clearly, $\overline{F}_{i:n}$ is obtained from $\overline{F}_X$ by the distortion function

$$H_{i,n}(u) = \sum_{k=0}^{i-1} \binom{n}{k} u^{n-k}(1-u)^k, \quad u \in [0,1]. \tag{3.5}$$

Using (3.5) and taking into account that

$$\frac{dH_{i,n}(u)}{u} = \frac{n!}{(i-1)!(n-i)!} u^{n-i-1}(1-u)^{i-1} du,$$

we have

$$\begin{aligned}
\int_0^1 \frac{H_{i,n}(u)}{u} dH_{i,n}(u) &= \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{i-1} \binom{n}{k} \int_0^1 u^{2n-i-k-1}(1-u)^{i+k-1} du \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{i-1} \binom{n}{k} \frac{(2n-i-k-1)!(i+k-1)!}{(2n-1)!} \\
&= \frac{n}{n-i} \frac{\displaystyle\sum_{k=0}^{i-1} \binom{i-1+k}{k} \binom{2n-i-k-1}{n-k}}{\displaystyle\binom{2n-1}{n}},
\end{aligned}$$

where the last equality is obtained reordering the factorial numbers.

The first line of (3.4) follows from this expression and Corollary 9.

Now consider the case $i = n$. Since $H_{n,n}(u) = 1 - (1-u)^n$, it follows that

$$\begin{aligned}
\int_0^1 \frac{H_{n,n}(u)}{u} dH_{n,n}(u) &= n \int_0^1 \frac{1-(1-u)^n}{u} (1-u)^{n-1} du \\
&= n \int_0^1 \frac{1-v^n}{1-v} v^{n-1} dv \\
&= n \int_0^1 \sum_{j=0}^{n-1} v^{j+n-1} dv \\
&= n \sum_{j=0}^{n-1} \frac{1}{n+j},
\end{aligned}$$

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TABLE 1

$\epsilon_{i,n}(X)$, $X \sim Pa(4, 3\sqrt{2})$.

	$n=1$	$n=2$	$n=5$	$n=10$	$n=20$	$n=50$	$n=100$
$i=1$	1.8856	0.8081	0.2977	0.1450	0.0716	0.0284	0.0141
$i=2$		2.9631	0.6947	0.3108	0.1479	0.0575	0.0285
$i=5$			4.9015	1.0092	0.4124	0.1497	0.0727
$i=10$				6.7852	1.0392	0.3217	0.1503
$i=20$					9.0711	0.7625	0.3231
$i=50$						12.8109	1.0640
$i=100$							16.2846

TABLE 2

$\hat{\epsilon}_{i,n}(X)$, $X \sim Pa(4, 3\sqrt{2})$.

	$n=1$	$n=2$	$n=5$	$n=10$	$n=20$	$n=50$	$n=100$
$i=1$	2	1.1547	0.6666	0.4588	0.3202	0.2010	0.1417
$i=2$		3.0550	1.1818	0.7629	0.5185	0.3207	0.2251
$i=5$			5.0818	1.6202	0.97736	0.5737	0.3967
$i=10$				7.3144	1.7347	0.8961	0.6015
$i=20$					10.4369	1.4952	0.9272
$i=50$						16.5912	1.8842
$i=100$							23.5058

where we have used that $\frac{1-v^n}{1-v} = \sum_{j=0}^{n-1} v^j$. The second line of (3.4) follows from this expression and (2.11). ■

Example 13. *Given an absolutely continuous random variable X, the risk loading $\epsilon_{1,n}(X)$, with $n \geq 1$, coincides with the cumulative Tsallis entropy of order $\alpha = n$ (see Rajesh and Sunoj, 2019). From (3.4), we see that*

$$\hat{\epsilon}_{1,n}(X) = \sigma \sqrt{\frac{1}{2n-1}}.$$

Example 14. *Let X be a Pareto distribution with distribution function given by*

$$F(x) = 1 - \left(\frac{\beta}{x+\beta}\right)^{\alpha}, \quad x > 0. \tag{3.6}$$

Tables 1 and 2 show $\epsilon_{i,n}(X)$ and $\hat{\epsilon}_{i,n}(X)$, respectively, for a Pareto distribution, $X \sim Pa(4, 3\sqrt{2})$, for which $\sigma = 2$, and some choices of i and n.

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3.3. Record claims

The formal study of record values, which are successive extremes occurring in a sequence of random variables, started with Chandler (1952). Basic references are Ahsanullah (1995), Arnold *et al.* (1998), and Nevzorov (2001). They arise in a wide variety of practical situations, which include industrial stress testing, meteorological analysis, hydrology, seismology, sporting events, and mining investigations. In the context of time series of insurance claims, record values describe sequences of successive largest claims. Therefore, we refer them as record claims. Other recent studies focussing on stochastic properties of functions of extreme claims include Barmalzan *et al.* (2016, 2017) and Zhang *et al.* (2019).

Let $\{X_i\}_{i\geq 1}$ be a sequence of independent claims having the same distribution as X and denote by F_X and $\overline{F}_X(x)$, respectively, the common distribution function and tail function. We say that X_j is a (upper) record claim if it is larger than all the previous claims of the sequence. The indices at which the record claims arise are given by the record times $\{U_n\}_{n\geq 1}$, where $U_1 = 1$ and

$$U_n = \min\{j : j > U_{n-1},\ X_j > X_{U_{n-1}}\}, \quad n \geq 2.$$

We denote $R_j = X_{U_j}$ as the jth record value of the sequence. Note that $R_1 = X_1$ is considered always a record, so the first nontrivial record value is R_2. Since the record times of the sequence $\{X_i\}_{i\geq 1}$ are the same as those for the sequence $\{F_X(X_i)\}_{i\geq 1}$ and $F_X(X)$ has a uniform distribution, it is clear that the distribution of U_n, $n \geq 1$ does not depend on F_X.

The distribution function of R_j, for $j \geq 1$, is

$$F_{R_j}(x) = Ga(-\log(\overline{F}_X(x)); j),$$

where

$$Ga(x; j) = \int_0^x \frac{t^{j-1}}{(j-1)!} e^{-t} dt.$$

Using the relation

$$\int_z^\infty \frac{t^{j-1}}{(j-1)!} e^{-t} dt = e^{-z} \sum_{i=0}^{j-1} \frac{z^i}{i!}, \tag{3.7}$$

we have

$$\overline{F}_{R_j}(x) = \overline{F}_X(x) \sum_{i=0}^{j-1} \frac{(-\log \overline{F}_X(x))^i}{i!},$$

see Arnold *et al.* (1998).

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Now consider the function $Z = R_j$ and a risk-adjusted distribution Y satisfying

$$[Y \mid R_j = x] \stackrel{d}{=} [X \mid X > x].$$

Denote by $P_j(X) = E[Y]$, the premium principle associated to the random variable $Z = R_j$. The intuition under this premium principle is the following. Consider, for example, the case $j = 2$ (recall that R_2 is the first nontrivial record). Given a sequence of claims recorded over a period of time, if the insurer charges the amount $P_2(X)$, this will be sufficient to cover the payout of the first claim that is larger than all the preceding ones. In general, the amount $P_j(X)$, with $j > 2$, will be sufficient to cover the payout of the $(j-1)$th claim that is larger than all the preceding ones. Since $R_j \leq_{st} R_{j+1}$ (see Kochar, 1990), then, according to (1.8), $P_j(X) \leq P_{j+1}(X)$. Therefore, the choice of j reflects the degree of risk aversion of the insurer (the larger the risk aversion, the greater the parameter j).

The next theorem gives explicit expressions for the premium principle $P_j(X)$ and its associated distortion function.

Theorem 15. *Given* $j \geq 1$,

$$P_j(X) = E[R_{j+1}],$$

is a distortion premium principle with concave distortion function[1] *given by*

$$h_j(u) = 1 - Ga(-\log u; j+1), \ 0 \leq u \leq 1.$$

Proof. Since $H_j(u) = 1 - Ga(-\log u; j)$ is a distortion function such that $\overline{F}_{R_j}(x) = H_j(\overline{F}_X(x))$, the assumptions of Theorem 8 are satisfied. Since

$$\frac{dH_j(v)}{v} = \frac{(-\log v)^{j-1}}{(j-1)!v}dv,$$

after a change of the order of integration in (2.10), we obtain

$$\begin{aligned} h_j(u) &= 1 - \int_u^1 \left(\int_t^1 \frac{dH_j(v)}{v} \right) dt \\ &= 1 - \int_u^1 \left(\int_t^1 \frac{(-\log v)^{j-1}}{(j-1)!v} dv \right) dt \\ &= 1 - \int_u^1 \frac{(-\log t)^j}{j!} dt \\ &= 1 - Ga(-\log u; j+1). \end{aligned}$$

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Therefore,

$$\begin{aligned} P_j(X) &= \int_0^\infty h_j(\overline{F}_X(x))dx \\ &= \int_0^\infty \overline{F}_{R_{j+1}}(x)dx \\ &= E[R_{j+1}], \end{aligned}$$

as stated. ■

Thus, the premium principle $P_j(X)$ is the expected $(j+1)$th record claim. In particular, $P_1(X)$ is the first nontrivial expected record claim, and the corresponding risk loading is given by

$$\epsilon_{R_1}(X) = E\left[R_2 - X\right].$$

Observe that $h_1(u) = u(1 - \log u)$ for $0 < u \leq 1$ (with $h_1(0) = 0$ by continuity). This is the same as the distortion function (13) in Sordo *et al.* (2016), which implies that

$$P_1(X) = E[R_2] = \int_0^1 TVaR_p(X)dp,$$

and

$$\epsilon_{R_1}(X) = -\int_0^\infty \overline{F}(x)\log\overline{F}(x)dx.$$

The measure $\epsilon_{R_1}(X)$ is the cumulative residual entropy introduced by Rao *et al.* (2004) and recently used in an actuarial context by Psarrakos and Sordo (2019). More generally, for $j \geq 1$, we have

$$P_j(X) - P_{j-1}(X) = \epsilon_j(X),$$

where

$$\epsilon_j(X) = \int_0^\infty \overline{F}_X(x)\frac{\left(-\log\overline{F}_X(x)\right)^j}{j!}dx,$$

is the generalized cumulative residual entropy studied by Psarrakos and Navarro *et al.* (2013). The risk loading of the premium principle $P_j(X)$ is given by

$$\epsilon_{R_j}(X) = \sum_{i=1}^{j}\epsilon_i(X).$$

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The next theorem provides an upper bound for $\epsilon_{R_j}(X)$. This bound is known in the literature as the universal bound for the expected value of a record value, see Arnold *et al*. (1998).

Theorem 16. *Given an absolutely continuous random variable X with survival function $\overline{F}_X$ and finite variance, σ^2, the risk loading $\epsilon_{R_j}(X)$ is bounded by*

$$\hat{\epsilon}_{R_j}(X) = \sigma\sqrt{\binom{2j}{j} - 1}, \quad j \geq 1.$$

Proof. The proof consists of applying Theorem 5 to $Z = R_j$. Let $j \geq 1$ and denote by $\omega_{R_j}(x)$ the weight function $\omega(x)$ defined in (2.2). Using the density function of R_j, given by

$$f_{R_j}(x) = \frac{(-\log \overline{F}_X(x))^{j-1}}{(j-1)!} f_X(x),$$

(2.2) gives

$$\begin{aligned}
\omega_{R_j}(x) &= \frac{1}{(j-1)!}\int_0^x \frac{\left[-\log\left(\overline{F}_X(t)\right)\right]^{j-1}}{\overline{F}_X(t)} f_X(t)dt \\
&= \frac{1}{(j-1)!}\int_{\overline{F}_X(x)}^1 \frac{(-\log u)^{j-1}}{u} du \\
&= \frac{\left[-\log\left(\overline{F}_X(x)\right)\right]^j}{j!},
\end{aligned}$$

where we have used the change of variables $\overline{F}_X(t) = u$ in the second equality. Then,

$$\begin{aligned}
E[\omega_{R_j}^2(X)] &= \int_0^\infty \left(\frac{\left[-\log\left(\overline{F}_X(x)\right)\right]^j}{j!}\right)^2 f_X(x)dx \\
&= \frac{1}{(j!)^2}\int_0^1 (-\log u)^{2j}\, du \\
&= \frac{(2j)!}{(j!)^2}.
\end{aligned}$$

The desired result follows from this expression and (2.6) using that $E[\omega_{R_j}(X)] = 1$. ■

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TABLE 3

$\epsilon_{R_j}(X)$ AND $\hat{\epsilon}_{R_j}(X)$, $X \sim Pa(4, 3\sqrt{2})$.

	$j=1$	$j=2$	$j=3$	$j=4$
$\epsilon_{R_j}(X)$	1.88562	4.39978	7.75199	12.2216
$\hat{\epsilon}_{R_j}(X)$	2	4.47214	8.7178	16.6132

Example 17. *Table 3 shows $\epsilon_{R_j}(X)$ and $\hat{\epsilon}_{R_j}(X)$ for a Pareto random variable, $X \sim Pa(4, 3\sqrt{2})$, with distribution function (3.6), for which $\sigma = 2$ and different values of j.*

3.4. *k*th record claims

Since the occurrences of record claims are very rare in practice, they are sometimes viewed as outliers. To overcome this drawback, in this section, we focus on the second or third (or, in general, kth) largest claims to fix the premium. The concept of kth record values was introduced by Dziubdziela and Kopociński (1976). Suppose that $\{X_i\}_{i\geq 1}$ is a sequence of independent and identically distributed claims. If for a positive integer k, we collect those claims in the sequence which occupy the kth largest position but exceeds in value for the first time the just previously recorded kth largest claim, then the resulting sequence is known as the sequence of kth upper records or simply kth records. We denote the times at which kth record claims occur as $U_{n(k)}$ for $n = 1, 2, \ldots$ and are defined by $U_{1(k)} = k$, and for $n > 1$,

$$U_{n(k)} = \min\{j : j > U_{n-1(k)},\ X_{j-k+1:j} > X_{U_{n-1(k)}-k+1:U_{n-1(k)}}\},$$

where $X_{i:m}$ is the ith-order statistic in a random sample of size m. Then, we define the sequence of kth record claims by $R_n^{(k)} = X_{U_{n(k)}-k+1:U_{n(k)}}$. Setting $k = 1$, we obtain ordinary record claims.

The distribution function $F_{j(k)}(x)$ of the jth value of kth record claim $R_j^{(k)}$ of a sequence of independent and identically distributed claims $\{X_i\}_{i\geq 1}$ is given by

$$F_{j(k)}(x) = \mathrm{Ga}(-k \log(\overline{F}_X(x)); j),\ x \geq 0,\ j \geq 1, \tag{3.8}$$

where

$$Ga(x; j) = \int_0^x \frac{t^{j-1}}{(j-1)!} e^{-t}\, dt,\ x \geq 0,$$

see Arnold *et al.* (1998).

Now we consider the function $Z = R_j^{(k)}$ and the risk-adjusted distribution Y satisfying

$$[Y \mid R_j^{(k)} = x] \stackrel{d}{=} [X \mid X > x].$$

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Denote by $P_{j(k)}(X) = E[Y]$ the premium principle associated to the random variable $Z = R_j^{(k)}$. This premium principle is interpreted in a similar manner as the premium principle $P_j(X)$ defined in Section 3.3, except that the record claim R_j is replaced by the jth value of kth record claim $R_j^{(k)}$. Observe that this replacement adds flexibility to model risk aversion, since we focus on the kth largest claims (with $k \geq 1$) rather than on the largest one. Moreover, $j \leq j'$ and $k \geq k'$ imply $R_j^{(k)} \leq_{st} R_{j'}^{(k')}$ (see Theorem 5.1 in Kamps, 1994), which, according to (1.8), implies $P_{j(k)}(X) \leq P_{j'(k')}(X)$. In particular, fixed k, the larger the risk aversion, the greater the parameter j and fixed j, the larger the risk aversion, the smaller the parameter k.

In the next result, we provide the explicit expressions of the premium principle $P_{j(k)}(X)$ and its associated distortion function.

Theorem 18. *Given* $k > 1$,

$$P_{j(k)}(X) = \left(\frac{k}{k-1}\right)^j E[X] - \frac{1}{k}\sum_{i=1}^{j} E[R_i^{(k)}]\left(\frac{k}{k-1}\right)^{j-i+1}$$

is a distortion premium principle with concave distortion function[2]

$$h_{j(k)}(u) = 1 - Ga(-k\log u; j) + \left(\frac{k}{k-1}\right)^j uGa(-(k-1)\log u; j), \qquad (3.9)$$

for $0 \leq u \leq 1$.

Proof. From (3.8), it is clear that $\overline{F}_{j(k)}$ can be obtained from $\overline{F}_X$ via the distortion function

$$H_{j(k)}(u) = 1 - \mathrm{Ga}(-k\log u; j),\ 0 \leq u \leq 1. \qquad (3.10)$$

It follows from Theorem 8 that

$$P_{j(k)}(X) = \int_0^\infty h_{j(k)}(\overline{F}_X(x))dx, \qquad (3.11)$$

where

$$h_{j(k)}(u) = u\int_u^1 \frac{dH_{j(k)}(v)}{v} + H_{j(k)}(u), \qquad (3.12)$$

is a concave distortion function. Deriving (3.10), we have

$$\begin{aligned}\frac{dH_{j(k)}(v)}{v} &= \frac{k^j(-\log v)^{j-1}v^{k-2}}{(j-1)!}dv \\ &= \left(\frac{k}{k-1}\right)^j \frac{(-(k-1)\log v)^{j-1}v^{k-2}(k-1)}{(j-1)!}dv,\end{aligned}$$

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and therefore

$$\int_u^1 \frac{dH_{j(k)}(v)}{v} = \left(\frac{k}{k-1}\right)^j Ga(-(k-1)\log u; j). \tag{3.13}$$

By combining (3.13) with (3.10) and (3.12), we obtain (3.9).

To obtain explicitly $P_{j(k)}(X)$, we develop (3.11) by using a more convenient representation of $h_{j(k)}$. First, note from (3.7) that

$$H_{j(k)}(u) = u^k \sum_{i=0}^{j-1} \frac{(-k\log u)^i}{i!}, \tag{3.14}$$

from which

$$u^k \frac{(-k\log u)^i}{i!} = H_{i+1(k)}(u) - H_{i(k)}(u), \tag{3.15}$$

with $H_{0(k)}(u) = 0$.

From (3.10) and (3.14), we can write

$$Ga(-(k-1)\log u; j) = 1 - u^{k-1} \sum_{i=0}^{j-1} \frac{(-(k-1)\log u)^i}{i!},$$

this expression, together with (3.12), (3.13), and (3.15) yields

$$\begin{aligned} h_{j(k)}(u) &= H_{j(k)}(u) + u\left(\frac{k}{k-1}\right)^j \left(1 - u^{k-1}\sum_{i=0}^{j-1}\frac{(-(k-1)\log u)^i}{i!}\right) \\ &= u\left(\frac{k}{k-1}\right)^j + H_{j(k)}(u) - u^k \sum_{i=0}^{j-1}\frac{(-k\log u)^i}{i!}\left(\frac{k}{k-1}\right)^{j-i} \\ &= u\left(\frac{k}{k-1}\right)^j + H_{j(k)}(u) \\ &\quad - \sum_{i=0}^{j-1}\left[H_{i+1(k)}(u) - H_{i(k)}(u)\right]\left(\frac{k}{k-1}\right)^{j-i}. \end{aligned}$$

After some algebra, we get

$$h_{j(k)}(u) = u\left(\frac{k}{k-1}\right)^j - \frac{1}{k}\sum_{i=1}^{j} H_{i(k)}(u)\left(\frac{k}{k-1}\right)^{j-i+1}. \tag{3.16}$$

From (3.11), using (3.16) and taking account that

$$E[R_i^{(k)}] = \int_0^\infty H_{i(k)}(\overline{F}_X(x))dx,$$

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we have

$$P_{j(k)}(X) = \left(\frac{k}{k-1}\right)^j E[X] - \frac{1}{k}\sum_{i=1}^{j} E[R_i^{(k)}]\left(\frac{k}{k-1}\right)^{j-i+1},$$

and the result is proved. ■

It is easy to check that the family of premium principles $P_{j(k)}(X)$ verifies the next recurrence relation for $k \geq 1$ and $j \geq 0$,

$$kP_{j(k)}(X) - (k-1)P_{j+1(k)}(X) = E[R_{j+1}^{(k)}],$$

with $P_{0(k)}(X) = E[X]$.

The risk loading associated to $P_{j(k)}(X)$ is

$$\epsilon_{j(k)}(X) = \frac{1}{k}\sum_{i=1}^{j}\left(E[X] - E[R_i^{(k)}]\right)\left(\frac{k}{k-1}\right)^{j-i+1}, \quad k > 1,\ j \geq 1.$$

The next theorem provides an upper bound for $\epsilon_{j(k)}$.

Theorem 19. *Given an absolutely continuous random variable X with survival function $\overline{F}_X$ and finite variance σ^2, the risk loading $\epsilon_{j(k)}(X)$, with $j \geq 1$ and $k \geq 1$, is bounded by*

$$\hat{\epsilon}_{j(k)}(X) = \sigma\sqrt{2\sum_{i=0}^{j-1}\binom{j-1+i}{j-1}\left(\frac{k}{2k-1}\right)^{i+j} - 1}.$$

Proof. The proof follows from Corollary 9 with $Z = R_j^{(k)}$. Using (3.14), we see that

$$\begin{aligned}\int_0^1 \frac{H_{j(k)}(u)}{u} dH_{j(k)}(u) &= \frac{k^j}{(j-1)!}\sum_{i=0}^{j-1}\frac{k^i}{i!}\int_0^1 u^{2k-2}(-\log u)^{i+j-1}du \\ &= \sum_{i=0}^{j-1}\frac{(j-1+i)!}{i!(j-1)!}\left(\frac{k}{2k-1}\right)^{i+j},\end{aligned}$$

and the proof follows by direct application of (2.11). ■

Example 20. *Tables 4 and 5 show $\epsilon_{j(k)}(X)$ and $\hat{\epsilon}_{j(k)}(X)$ for a Pareto random variable, $X \sim Pa(4, 3\sqrt{2})$, with distribution function (3.6), for which $\sigma = 2$ and for different values of j and k.*

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TABLE 4

$\epsilon_{j(k)}(X)$, $X \sim Pa(4, 3\sqrt{2})$.

	$k=2$	$k=3$	$k=4$
$j=1$	0.808122	0.514259	0.377124
$j=2$	1.73169	1.07527	0.779389
$j=3$		1.68728	1.20847

TABLE 5

$\hat{\epsilon}_{j(k)}(X)$, $X \sim Pa(4, 3\sqrt{2})$.

	$k=2$	$k=3$	$k=4$
$j=1$	1.1547	0.894427	0.755929
$j=2$	2.07275	1.5284	1.26399
$j=3$		2.13796	1.72515

4. CONCLUSIONS

In this paper, we have studied a special class of weighted premium principles. The premiums of this class capture the insurer's perception of the risk in terms of functions of claims and, under some assumptions on these functions, are distortion premium principles. We have provided several sequences of premium principles under different assumptions on the claim arrival process and different functions of claims, including record claims and kth record claims, which offer flexibility and gradually incorporate risk aversion in their parameters. We have studied the main properties of the derived premium principles and provided bounds for the corresponding risk loadings.

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NOTES

1. For $u=0$, $h_j(u)$ is defined by continuity.
2. For $u=0$, $h_{j(k)}(u)$ is defined by continuity.

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