105.49 Equality in mathematics

The basic properties of equality were known to Euclid around 300 B.C. and probably before that time. These properties are:

1. P = P, for all P. (Reflexive property)
2. If P = Q, then Q = P. (Symmetric property)
3. If P = Q and Q = R, then P = R. (Transitive property)

Thus equality is the prototype equivalence relation, and all equivalence relations stem from equality, because two elements are equivalent if, and only if, their equivalence classes are equal.

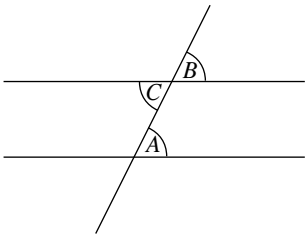
Property 3 is often rendered colloquially as "things that are equal to the same thing are equal to each other", i.e. if P = R and Q = R, then P = Q, and this version can easily be deduced from the above as follows:

We have Q = R, which implies R = Q, and P = R, R = Q together imply that P = Q, as claimed.

The purpose of this Note is to explore the following notion – if it is claimed that two things in mathematics are equal, and one cannot immediately see the reason why, it is often because both are equal to a third independent thing. This often unnoticed principle frequently occurs in geometry, but it also arises in algebra and throughout mathematics. We present several examples from basic mathematics.

Our examples have deliberately been presented to exemplify the equality principle; they do not *have* to be presented in this way. The emphasis throughout is on the equality step, and the familiar geometrical justifications for some of the arguments have been omitted for brevity.

Example 1: Parallel lines



$\angle A = \angle C$ and $\angle B = \angle C$ imply that $\angle A = \angle B$.

FIGURE 1

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Example 2: If $ABCD$ is a quadrilateral and X, Y, Z and W are the midpoints of the sides, then $XYZW$ is a parallelogram.

This is because $|WX| = \frac{1}{2}|DB|$ and $|ZY| = \frac{1}{2}|DB|$, so $|WX| = |ZY|$. Similarly, $|XY| = \frac{1}{2}|AC|$ and $|WZ| = \frac{1}{2}|AC|$, so $|XY| = |WZ|$.

Thus $XYZW$ is a parallelogram.

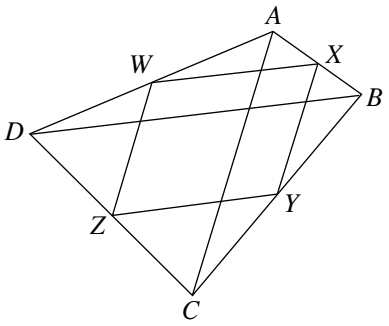


FIGURE 2

Example 3: The sum of the angles of a triangle is 180°.

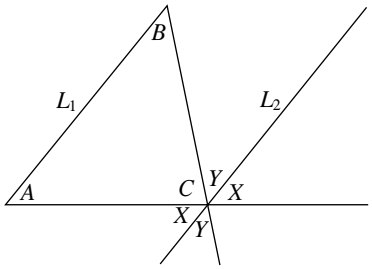


FIGURE 3

Line L_2 is parallel to L_1,

$$\angle B = \angle Y; \angle A = \angle X.$$

$$\angle A + \angle B + \angle C = \angle C + \angle Y + \angle X$$

$$180° = \angle C + \angle Y + \angle X$$

So $\angle A + \angle B + \angle C = 180°$.

Example 4: The theorem of Pythagoras

As areas, A = X, B = Y;

$$A + B = X + Y$$

$$C = X + Y.$$

So A + B = C.

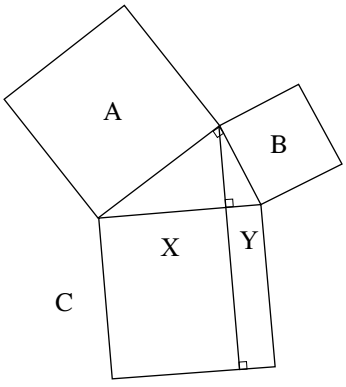


FIGURE 4

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Example 5: Angles on the same chord of a circle are equal.

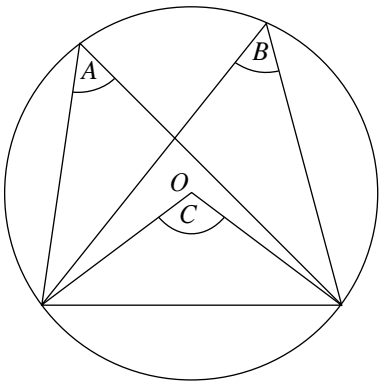


FIGURE 5

Let O be the centre of the circle.

$$\angle A = \tfrac{1}{2}\angle C;\ \angle B = \tfrac{1}{2}\angle C$$

So $\angle A = \angle B$.

Example 6: In a cyclic quadrilateral, the angle A is equal to the angle B.

$$\angle A = 180^\circ - \angle C,$$

$$\angle B = 180^\circ - \angle C,$$

so $\angle A = \angle B$.

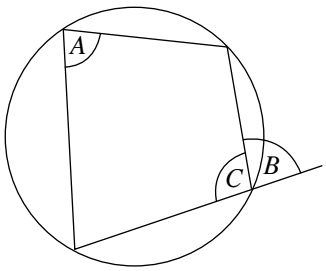


FIGURE 6

Example 7: If two chords of a circle intersect internally, the products of the segment lengths are equal, i.e. $ab = cx$.

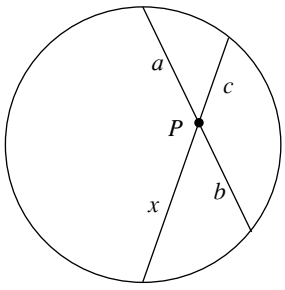


FIGURE 7a

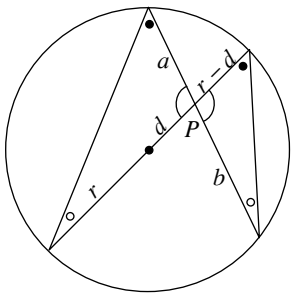


FIGURE 7b

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Consider the case where one of the chords is a diameter of the circle. Then by similar triangles

$$\frac{a}{r+d} = \frac{r-d}{b}, \text{ so } ab = r^2 - d^2.$$

This expression is called the 'power' of P, depending only on the distance d of P from the centre of the circle and the length of the radius of the circle. This power is the same for any other chord passing through P. So $ab = r^2 - d^2$, $cx = r^2 - d^2$, giving us $ab = cx$.

Example 8: If two chords of a circle AB and CD intersect externally at P, then $|PB| \times |PA| = |PC| \times |PD|$.

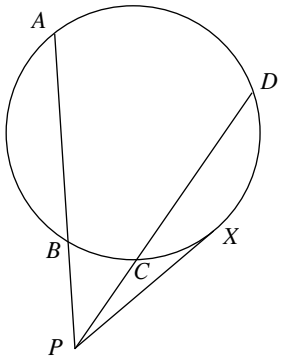


FIGURE 8a

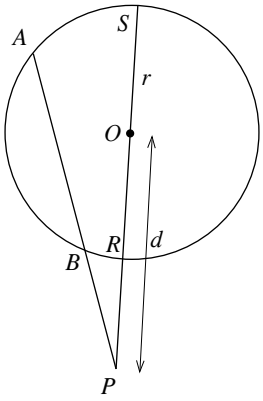


FIGURE 8b

Consider the case where PS contains the centre of the circle O. Then, by using similar triangles as in Example 7, we get

$$|PB| \times |PA| = (d + r)(d - r) = d^2 - r^2.$$

As this is true for any chord of the circle passing through P, we get

$$|PC| \times |PD| = d^2 - r^2,$$

so $|PB| \times |PA| = |PC| \times |PD|$, as claimed.

We note that if PX is a tangent to the circle, $|PB| \times |PA| = |PX|^2$ and $|PC| \times |PD| = |PX|^2$, so again $|PB| \times |PA| = |PC| \times |PD|$.

For completeness, we remark that if P lies on the circumference of the circle, then its power is $r^2 - d^2 = r^2 - r^2 = 0$, and the result still holds, the segment product being zero.

Our final two examples are from algebra.

Example 9: Prove by induction that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$.

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The result is clearly true for $n = 1$, since $1 = (1 \times 2 \times 3)/6$. Assume therefore that the result is true for some $n = k$, so that

$$\sum_{r=1}^{k} r^2 = \frac{k(k+1)(2k+1)}{6}.$$

We need to show that

$$A = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = B.$$

Impressionable students come to believe that this first expression A must be 'beaten into shape' to arrive precisely at expression B; in more complicated cases, this can require the expenditure of a great deal of time and unnecessary skill – it is very easy to get lost when one's target has to be B. But straightforward calculations show that

$$A = \frac{2k^3 + 9k^2 + 13k + 6}{6} = C$$

$$\text{and } B = \frac{2k^3 + 9k^2 + 13k + 6}{6} = C$$

so that $A = B$, and the result is established by induction. [Of course, the same conclusion can be arrived at by showing that $A - B = 0$.]

Example 10: Show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

We show that these two expressions are equal by showing that each is equal to the total number of subsets of a set S, with exactly n elements.

If $|S| = n$, then a subset of S has 0, 1, 2, . . . , or n elements; the number of subsets with k elements is $\binom{n}{k}$, so the total number of subsets is $\sum_{k=0}^{n} \binom{n}{k} = A$.

But we can also count the total number of subsets as follows: each element of S may be included or excluded from a given subset, giving two choices; thus for the n elements of S we have $2 \times 2 \times 2 \times \ldots \times 2$ choices, giving $2^n = B$ subsets. [Strictly speaking, we need to use induction here].

Thus $A = B$, as claimed.

We note that this can also be seen by putting $x = 1$ in the binomial expansion of $(1 + x)^n$.

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