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N-BERKELEY CARDINALS AND WEAK EXTENDER MODELS

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Abstract. For a given inner model N of ZFC, one can consider the relativized version of Berkeley cardinals in the context of ZFC, and ask if there can exist an "N-Berkeley cardinal." In this article we provide a positive answer to this question. Indeed, under the assumption of a supercompact cardinal δ, we show that there exists a ZFC inner model N such that there is a cardinal which is N-Berkeley, even in a strong sense. Further, the involved model N is a weak extender model of δ is supercompact. Finally, we prove that the strong version of N-Berkeley cardinals turns out to be inconsistent whenever N satisfies closure under ω-sequences.

§1. Introduction. Set theory is presently at a fundamental crossroads, reflecting the two sides of a crucial dichotomy theorem due to Woodin—the HOD Dichotomy Theorem—which informally states that, under standard large cardinal assumptions (namely, ZFC plus the existence of large cardinals), either the ZFC inner model HOD is "close" to the set-theoretic universe V, or is "far" from it. There is a program aimed at establishing the first side of the dichotomy: on the one hand, there is the program of inner model theory; in particular, if this program succeeded, then this would mean that there exists an ultimate inner model of ZFC—Woodin's Ultimate-L—which recognizes all of the traditional large cardinals existing in V, which satisfies the CH and the axiom $V = \text{HOD}$, and which would provide us with a fairly complete analysis of the set-theoretic universe itself. On the other hand, there is an opposite program, one aimed at establishing the second side of the dichotomy: the program of large cardinals beyond Choice; in particular, if this program succeeded, then this would mean that the choiceless large cardinals, from Reinhardt cardinals upward, are consistent with ZF, and there can be no ultimate inner model giving us a definitive structural insight into V.

Among the choiceless large cardinals, Berkeley cardinals play a decisive role in the scenario described above; in fact:

If there exists a HOD*-Berkeley cardinal then the second side of the* HOD *Dichotomy Theorem holds, where* HOD *is far from* V.[1]

The definition of HOD-Berkeley cardinal is simply the relativization to HOD of the notion of being a Berkeley cardinal. A HOD-Berkeley cardinal would constitute for HOD an analogue of what $0^{\#}$ constitutes for L, that is, a large cardinal axiom transcending HOD, thus making $V = \text{HOD}$ a limiting principle.

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[1] See [1].

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So, should it turn out there is a HOD-Berkeley, we could actually decide which of the two sides of the HOD dichotomy holds. In such a case, we would know that there is no Ultimate-L. In other words, inner model theory would fail at the level of a supercompact cardinal; it is in fact at this level that one gets the surprising universal properties making the model an "ultimate" inner model. Woodin isolated the key features of what an inner model for a supercompact cardinal should be, stating the definition and calling it a "weak extender model for a supercompact."[2] His Universality Theorem lies at the heart of several remarkable results showing that a weak extender model is capable to recognize all ZFC large cardinals, far beyond the level of supercompactness.

In this article we investigate the relativized version of Berkeley cardinals to inner models of ZFC; in particular, we will be interested in the case of weak extender models. We stress that while Berkeley cardinals, being inconsistent with the Axiom of Choice, are defined just in ZF, in considering the relativized version one can instead keep AC, and eventually show that V satisfies there is an N-Berkeley cardinal for some inner model N of ZFC (and obviously, the more structurally rich the model the more interesting the result). Indeed, we will provide the first example of a ZFC inner model N for which there exists an N-Berkeley cardinal; remarkably, and rather surprisingly, this N is a weak extender model.

For more on the HOD Dichotomy Theorem the reader can refer to [5] (where the theorem was originally proved) or [7]. For an account of the choiceless large cardinals see [1].

§2. N-Berkeley cardinals and universality. We start by stating the definition of the relativized version of Berkeley cardinals, first proposed in [1]. The aim of this section is to illustrate the limitation imposed by Woodin's Universality Theorem to the existence of cardinals satisfying such definition in the case of a weak extender model.

The results below, which first appeared in [5], are revisited and enhanced in [6].

DEFINITION 2.1. For any transitive set M, let $\mathcal{E}(M)$ be the collection of all nontrivial elementary embeddings $j : M \to M$.

DEFINITION 2.2. Let N be an inner model of ZFC. A cardinal δ is a N-Berkeley cardinal if for all transitive sets $M \in N$ such that $\delta \in M$, and for every ordinal $\eta < \delta$, there exists $j \in \mathcal{E}(M)$ with $\eta < crit(j) < \delta$.

DEFINITION 2.3. Suppose that δ is a supercompact cardinal and that N is an inner model of ZFC. Then N is a *weak extender model* of δ is supercompact if for all $\lambda > \delta$ there exists a δ-complete, normal, fine, ultrafilter $\mathcal{U}$ on $\mathcal{P}_\delta(\lambda) = \{X \subseteq \lambda : |X| < \delta\}$ such that

1. $\mathcal{P}_\delta(\lambda) \cap N \in \mathcal{U}$ (i.e., $\mathcal{U}$ concentrates on N) and
2. $\mathcal{U} \cap N \in N$ (i.e., $\mathcal{U}$ is amenable to N).

REMARK 2.4. If N is a weak extender model of δ is supercompact, then δ is supercompact in N.

[2] Throughout the rest of the article we will say just "weak extender model".

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THEOREM 2.5. *Suppose that N is a weak extender model of δ is supercompact. Then the following hold.*

1. *N has the δ-cover property, i.e., for all $\sigma \subset N$ with $|\sigma| < \delta$, there exists $\tau \in N$ such that $\sigma \subset \tau$ and $|\tau| < \delta$.*
2. *N has the δ-approximation property, i.e., for all $X \subset N$ the following are equivalent.*
   (a) *$X \in N$.*
   (b) *$X \cap \tau \in N$ for all $\tau \in N$ with $|\tau| < \delta$.*[3]

THEOREM 2.6. *Suppose that N is a weak extender model of δ is supercompact and that $\gamma > \delta$ is a regular cardinal in N. Then $(cof(\gamma))^V = |\gamma|^V$.*

THEOREM 2.7. *Suppose that N is a weak extender model of δ is supercompact and that $\gamma > \delta$ is a singular cardinal. Then γ is a singular cardinal in N and $(\gamma^+)^N = \gamma^+$.*

THEOREM 2.8 (Universality Theorem). *Suppose that N is a weak extender model of δ is supercompact. Suppose that $\gamma > \delta$ is an ordinal, and that $j : N \cap V_{\gamma+1} \to N \cap V_{j(\gamma)+1}$ is an elementary embedding with $crit(j) \geq \delta$. Then, $j \in N$.*

The results above display how a weak extender model realizes remarkable conditions of "closeness" to the set-theoretic universe V. In fact, as a consequence of the Universality Theorem, it can be shown that a weak extender model satisfies a sort of analogue of Kunen's Theorem, and that it actually inherits all of the large cardinal structure of V; we mention here the central result witnessing that.

THEOREM 2.9. *Suppose that N is a weak extender model of δ is supercompact. Then there is no nontrivial elementary embedding $j : N \to N$ such that $crit(j) \geq \delta$.*

DEFINITION 2.10. Suppose $n < \omega$. Then κ is *n-huge* if there exists an elementary embedding $j : V \to M$ such that $crit(j) = \kappa$ and such that $M^{\kappa_n} \subset M$ where $\langle \kappa_i : i < \omega \rangle$ is the sequence where $\kappa_0 = \kappa$ and for all $i < \omega$, $\kappa_{i+1} = j(\kappa_i)$.

THEOREM 2.11. *Suppose that N is a weak extender model of δ is supercompact. Suppose that for each $n < \omega$, there is a proper class of n-huge cardinals. Then in N, for each $n < \omega$, there is a proper class of n-huge cardinals.*

Let us finally note the following corollary of the Universality Theorem affecting the existence of N-Berkeley cardinals when N is a weak extender model.

COROLLARY 2.12. *Suppose that N is a weak extender model of δ is supercompact. Then there are no N-Berkeley cardinals above δ.*

PROOF. Assume for contradiction that $\gamma > \delta$ is N-Berkeley. Then, there exists a nontrivial elementary embedding $j : (V_{\gamma+2})^N \to (V_{\gamma+2})^N$ with $\delta < crit(j) < \gamma$. It follows from the Universality Theorem that $j \in N$, contradicting the local version of Kunen's Theorem inside N.[4] ⊣

[3] δ-cover and δ-approximation are two general properties that one can naturally impose on a ZFC inner model N; both properties were abstracted out by Hamkins in [2] motivated by the relationship between V and its forcing extensions.

[4] Recall that the local version of Kunen's Theorem states that for any ordinal λ, there is no nontrivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$.

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§3. On the existence of strong N-Berkeley. In this section we prove a somehow unexpected result concerning the existence of N-Berkeley cardinals when N has the remarkable feature of being a weak extender model. This is however the first established result concerning the existence of an N-Berkeley cardinal for a ZFC inner model N. Moreover, we isolate a strong version of the notion of being N-Berkeley. We then identify a condition which is sufficient to rule it out.

3.1. A strong N-Berkeley cardinal for N a weak extender model. We pointed out that in the case of a weak extender model N of δ is supercompact, as a straightforward consequence of the Universality Theorem there cannot be an N-Berkeley cardinal above δ. It remains open the possibility that δ itself can be N-Berkeley, or that there can exist an N-Berkeley cardinal below δ. We now show that the latter turns out to be the case.

THEOREM 3.1. *Suppose that δ is a supercompact cardinal. Then there exists an inner model N of ZFC such that N is a weak extender model of δ is supercompact and there exists an N-Berkeley cardinal.*

PROOF. Let $\kappa_0 < \delta$ be a measurable cardinal, and let μ_0 be a normal measure on κ_0. Let $\mathrm{Ult}(V, \mu_0)$ denote the ultrapower of V with respect to μ_0, and let $j_0 : V \to \mathrm{Ult}(V, \mu_0)$ be the ultrapower embedding. Thus, $\kappa_0 = crit(j_0)$; $\mathrm{Ult}(V, \mu_0)$ is well-founded and so we identify it with its transitive collapse, call it N_0. Let $V \to N_0 \to N_1 \to N_2 \to \cdots \to N_\omega$ be the internal iteration of V by μ_0 of length ω. So for $i \in \omega$, $i > 0$ we inductively define $\mu_i = j_{i-1}(\mu_{i-1})$, $\kappa_i = j_{i-1}(\kappa_{i-1}) = crit(j_i)$ where $j_i : N_{i-1} \to N_i$ is the map induced by taking the ultrapower of N_{i-1} by μ_i.[5]

$$\underset{\kappa_0}{V} \overset{\mu_0}{\underset{j_0}{\longrightarrow}} \mathrm{Ult}(V, \mu_0) = \underset{\kappa_1}{N_0} \overset{\mu_1}{\underset{j_1}{\longrightarrow}} \underset{\kappa_2}{N_1} \overset{\mu_2}{\underset{j_2}{\longrightarrow}} N_2 \longrightarrow \cdots \longrightarrow N_\omega,$$

where N_ω is the ω-iterate, i.e., the direct limit of the system $\langle N_i; j_i : i < \omega \rangle$. Let $j_{0,\omega} : V \to N_\omega$ and $j_{i,\omega} : N_{i-1} \to N_\omega$, for $i > 0$, be the induced embeddings. Note that $j_i(N_\omega) = N_\omega$ for each $i \in \omega$, and so $j_i \restriction N_\omega : N_\omega \to N_\omega$ for each i. In fact, one starts with $\langle \mu_0, N_0 = \mathrm{Ult}(V, \mu_0) \rangle$ and defines $\langle \mu_i, N_i \rangle$ for $i > 0$ by induction as $\mu_i = j_0(\mu_{i-1})$, $N_i = j_0(N_{i-1})$. But $j_0(j_0(x)) = j_0(j_0)(j_0(x))$, so the sequence $\langle \mu_i, N_i \rangle$ for $i > 0$ is just the sequence as defined in N_0 from $j_0(\mu_0)$, etc. Thus, N_ω is the same as N_ω defined in N_0 from $j_0(\mu_0)$, etc.

Since N_ω is well-founded,[6] every ordinal is eventually fixed, that is, for every ordinal α there exists $i \in \omega$ such that $j_m(\alpha) = \alpha$ for all $m > i$. In fact, if not, then we can construct a decreasing sequence of ordinals in N_ω as follows. Since $j_i(\alpha) \geq \alpha$ for all i, we have $j_{0,\omega}(\alpha) = j_{1,\omega}(j_0(\alpha)) \geq j_{1,\omega}(\alpha) = j_{2,\omega}(j_1(\alpha)) \geq j_{2,\omega}(\alpha)$; and in general, $j_{i,\omega}(\alpha) = j_{i+1,\omega}(j_i(\alpha)) \geq j_{i+1,\omega}(\alpha)$. Now, suppose that for all $i \in \omega$ there exists $m > i$ such that $j_m(\alpha) > \alpha$. Then, $j_{m+1,\omega}(\alpha) < j_{m+1,\omega}(j_m(\alpha)) = j_{m,\omega}(\alpha) \leq j_{i,\omega}(\alpha)$. Therefore, for all $i \in \omega$ there exists $m > i$ such that $j_{m,\omega}(\alpha) < j_{i,\omega}(\alpha)$, and so there exists a decreasing subsequence of $\langle j_{i,\omega}(\alpha) : i < \omega \rangle$, which is a decreasing sequence of ordinals in N_ω, contradicting the well-foundedness of N_ω.

Let us prove that every transitive set (in fact every set) in N_ω is eventually fixed; it will easily follow that $\kappa_\omega = \sup\{\kappa_i : i \in \omega\}$ is a N_ω-Berkeley cardinal.

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[5] For the basics of iterated ultrapowers see [3] or [4].

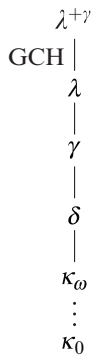
[6] In fact, iterations with a κ-complete measure are always well-founded (Gaifman. See, e.g., [4]).

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CLAIM 1. *For every transitive set $M \in N_\omega$ there exists $i \in \omega$ such that $j_m(M) = M$ for all $m > i$.*

PROOF. Fix a transitive set $M \in N_\omega$. Choose $\gamma > \delta$ such that $M \in V_\gamma$, $|V_\gamma| = \gamma$ and $\mathrm{cof}(\gamma) = \omega$. We want to code V_γ. Let $\langle x_\alpha : \alpha < \gamma \rangle$ enumerate V_γ. Let $B = \{\langle \alpha, \beta \rangle : x_\alpha \in x_\beta\} \subseteq \gamma \times \gamma$. Let $A = \pi^{-1}\text{“}B \subseteq \gamma$ where $\pi : \gamma \leftrightarrow \gamma \times \gamma$ is a bijection such that $\pi \in L$.

Now, fix a regular cardinal $\lambda > \gamma > \delta$. We can force GCH above λ without adding λ-sequences, and get a forcing extension $V[G]$ of V such that in $V[G]$, $V^\lambda \subseteq V$ and GCH holds in the interval $[\lambda, \lambda^{+\gamma}]$.



Now, in $V[G]$, define the partial order

$$\mathbb{P} = \prod_{\alpha \in A} \mathrm{Add}\left(\lambda^{+\omega(\alpha+1)+1}, \lambda^{+\omega(\alpha+1)+3}\right).^7$$

Let $H \subseteq \mathbb{P}$ be $V[G]$-generic. This forcing is $< \lambda^+$-closed and so, it adds no λ-sequences. Therefore in $V[G][H]$, $V^\lambda \subseteq V$ and

$$A = \left\{\alpha < \gamma : 2^{\left(\lambda^{+\omega(\alpha+1)+1}\right)} = 2^{\left(\lambda^{+\omega(\alpha+1)+3}\right)}\right\}.$$

So in $V[G][H]$, $A \in \mathrm{HOD}$. Then, since $B = \pi\text{“}A$ and $\pi \in L \subseteq \mathrm{HOD}$, we have that $B \in \mathrm{HOD}$, which implies that $V_\gamma \in \mathrm{HOD}$. Hence in $V[G][H]$, $M \in \mathrm{HOD}$.

Note that since the forcing is $< \lambda^+$-closed and adds no λ-sequences and $\lambda > \gamma$, $(V[G][H])_\gamma = V_\gamma$ and all the embeddings lift from V to $V[G][H]$ (in fact, by closure of V under λ-sequences, j_0 lifts trivially; but then $j_0(j_0)$ must lift by the elementarity of j_0, etc.).

$$\begin{array}{ccccccccccc}
V & \xrightarrow[j_0]{} & N_0 & \xrightarrow[j_1]{} & N_1 & \xrightarrow[j_2]{} & N_2 & \longrightarrow \cdots \longrightarrow & N_\omega \\
\cap & & \cap & & \cap & & \cap & & \cap \\
V[G][H] & \xrightarrow[\hat{j}_0]{} & \underset{N_0[j_0(G)][j_0(H)]}{\hat{N}_0} & \xrightarrow[\hat{j}_1]{} & \underset{N_1[\ldots]}{\hat{N}_1} & \xrightarrow[\hat{j}_2]{} & \hat{N}_2 & \longrightarrow \cdots \longrightarrow & \hat{N}_\omega
\end{array}$$

[7]Recall that if κ is a regular cardinal and ν is a cardinal, then $\mathrm{Add}(\kappa, \nu)$ is the partial order of all partial functions $f : \mathrm{dom}(f) \to \{0, 1\}$ such that $\mathrm{dom}(f) \subseteq \kappa \times \nu$ and $|\mathrm{dom}(f)| < \kappa$.

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Since $V_\gamma = (V[G][H])_\gamma \in \mathrm{HOD}^{V[G][H]}$, for some ordinal η there exists a bijection $e : V_\gamma \leftrightarrow \eta$ such that $e \in \mathrm{HOD}^{V[G][H]}$. Now, since $\hat{\jmath}_{0,\omega} : V[G][H] \to \hat{N}_\omega$ fixes γ (in fact, γ has cofinality ω and since $|V_\gamma| = \gamma > \kappa_0$, γ is closed under all the embeddings), we have that $\hat{\jmath}_{0,\omega}(e) = e_\omega : \hat{N}_\omega \cap V_\gamma \leftrightarrow \eta_\omega$ is a bijection in $\mathrm{HOD}^{V[G][H]} \cap \hat{N}_\omega$. It follows that $\hat{N}_\omega \cap V_\gamma \in \mathrm{HOD}^{V[G][H]}$ and there exists a definable well-ordering of V_γ in $\hat{N}_\omega$, which is a well-ordering definable from γ. Since all the embeddings $\hat{\jmath}_i$ fix both $\hat{N}_\omega$ and γ, they all fix this well-ordering, and since every ordinal is eventually fixed, they eventually fix every element of $\hat{N}_\omega \cap V_\gamma$: in fact, if $\sigma : \alpha \leftrightarrow V_\gamma$ is a bijection in $\hat{N}_\omega$ such that $\hat{\jmath}_i(\sigma) = \sigma$ for all $i \in \omega$, then for any $x \in \hat{N}_\omega \cap V_\gamma$, letting $\beta \in \alpha$ be such that $x = \sigma(\beta)$ and $i \in \omega$ be such that $\hat{\jmath}_m(\beta) = \beta$ for all $m > i$, we have that $\hat{\jmath}_m(x) = \hat{\jmath}_m(\sigma(\beta)) = \hat{\jmath}_m(\sigma)(\hat{\jmath}_m(\beta)) = \sigma(\beta) = x$ for all $m > i$. But the embeddings $\hat{\jmath}_i$ agree with the embeddings j_i of V on V_γ; therefore in V, every element of $N_\omega \cap V_\gamma$ is eventually fixed. Since $M \in N_\omega \cap V_\gamma$, the proof is complete. ⊣

Claim 2. *$\kappa_\omega = sup\{\kappa_i : i \in \omega\}$ is a N_ω-Berkeley cardinal.*

Proof. Let $M \in N_\omega$ be a transitive set such that $\kappa_\omega \in M$. Then there exists $i \in \omega$ such that $j_m(M) = M$ for all $m > i$. So for all such m, $j_m \restriction M : M \to M$; since the critical points of these embeddings are cofinal in κ_ω, we are done. ⊣

Finally, we sketch the proof that N_ω is a weak extender model of δ is supercompact (this is proved in details in [7]). Note that $j_{0,\omega} : V \to N_\omega$ fixes δ and for unboundedly many ordinals λ, $j_{0,\omega}(\lambda) = \lambda$. Fixing such a λ greater than δ, and letting $\mathcal{U}$ be a δ-complete, normal and fine ultrafilter on $\mathcal{P}_\delta(\lambda)$, it turns out that $\mathcal{U} \cap N_\omega = j_{0,\omega}(\mathcal{U})$, and so $\mathcal{U} \cap N_\omega \in N_\omega$; also, $\mathcal{P}_\delta(\lambda) \cap N_\omega \in \mathcal{U}$, in that $\mathcal{P}_\delta(\lambda) \cap N_\omega \in j_{0,\omega}(\mathcal{U}) \subseteq \mathcal{U}$. ⊣

Remark 3.2. Note that the proof of Theorem 3.1 actually shows more than what the theorem states: it shows that for every transitive set $M \in N_\omega$ such that $\kappa_\omega \in M$, there exists a sequence of elementary embeddings in $\mathcal{E}(M)$ with critical points less than κ_ω and well-founded limit (in fact, $\langle j_m \restriction M : m > i\rangle$ is such a sequence, where $i \in \omega$ is such that $j_m(M) = M$ for all $m > i$).

The above observation motivates the definition of the following strong version of an N-Berkeley cardinal, and allows us to state the final result.

Definition 3.3. Let N be an inner model of ZFC. A cardinal δ is an *strong N-Berkeley cardinal* if for every transitive set $M \in N$ such that $\delta \in M$ there exists a sequence of elementary embeddings $\langle j_i : j_i \in \mathcal{E}(M) \wedge crit(j_i) < \delta \wedge i < \omega\rangle$ such that the direct limit, $\lim_{i<\omega} j_i \circ \cdots \circ j_0(M)$, is well-founded.

Theorem 3.4. *If there exists a supercompact cardinal δ then there exists an inner model N of ZFC such that the following hold.*

1. *N is a weak extender model of δ is supercompact.*
2. *There exists a strong N-Berkeley cardinal.*

Remark 3.5. Note that every Berkeley cardinal δ satisfies the "strong Berkeley property," i.e., the property that for every transitive set M in V such that $\delta \in M$ there exists a sequence of elementary embeddings $\langle j_i : j_i \in \mathcal{E}(M) \wedge crit(j_i) < \delta \wedge i < \omega\rangle$ with well-founded direct limit. Let's see why. Let M be a transitive set such

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that $\delta \in M$. Pick γ such that $M \in V_\gamma$. Then there exists a nontrivial elementary embedding $j : V_{\gamma+1} \to V_{\gamma+1}$ with $crit(j) < \delta$ such that $j(M) = M$ and $j(\delta) = \delta$. So $j \restriction M : M \to M$. Now, consider $j(j) = j(j \restriction V_\gamma), j(j(j))$, and so forth; the restrictions of these embeddings to M give the required sequence.

Of course, that is no more true when δ is just an N-Berkeley cardinal for an arbitrary ZFC inner model N. The point is that in V, one can make any set definable in some transitive set and so (in the context of Berkeley cardinals) have nontrivial elementary embeddings fixing any desired set. In fact, as shown in [1], for any set x there exists a transitive set H such that $x \in H$ and x is definable (without parameters) in H; now, the way one gets j as above is by taking a transitive set M' such that $\langle V_{\gamma+1}, M, \delta\rangle$ is definable in M' and a $j' \in \mathcal{E}(M')$ with $crit(j') < \delta$, and finally putting $j = j' \restriction V_{\gamma+1}$.

3.2. A sharp inconsistency. In the proof of Theorem 3.1, the limit model N_ω is a weak extender model of δ is supercompact, and we have shown that there exists a strong N_ω-Berkeley cardinal. Note that N_ω is not closed under ω-sequences: in fact, $\langle \kappa_i : i < \omega\rangle \notin N_\omega$—where $\langle \kappa_i : i < \omega\rangle$ is the critical sequence of $j_0 : V \to \mathrm{Ult}(V, \mu_0)$—as $\kappa_\omega = \sup\{\kappa_i : i \in \omega\}$ is a measurable cardinal in N_ω. However, the model $N_\omega[\langle \kappa_i : i < \omega\rangle]$—call it N^*—is still a weak extender model of δ is supercompact (see [5], Section 5.1) and is closed under ω-sequences (in fact κ_0-sequences, because each N_i is and it turns out that $N^* = \bigcap_{i\in\omega} N_i$; see [4]); it also has nontrivial elementary embeddings $j : N^* \to N^*$ (in fact take $j = j_i$), and the question arises as to whether there can be a strong N^*-Berkeley cardinal.

With the next theorem we definitely provide a negative answer: indeed, we answer negatively to the more general question of whether there can exist a strong N-Berkeley cardinal when N is any ZFC inner model closed under ω-sequences.

THEOREM 3.6. *Suppose N is an inner model of ZFC such that $N^\omega \subseteq N$. Then there are no strong N-Berkeley cardinals.*

PROOF. Assume for contradiction that δ is a strong N-Berkeley cardinal. Fix $\gamma > \delta$ such that $|V_\gamma| = \gamma$ and $V_\gamma^\omega \subseteq V_\gamma$. Then, $|\gamma^\omega \cap N|^N = \gamma$. So fix a well-ordering $\prec \in N$ of $\gamma^\omega \cap N$ of length γ, and let $M = (V_\gamma \cap N) \cup \{\prec\}$. Then, M is a transitive set and so there exists a sequence of elementary embeddings $j_i : M \to M$, $i \in \omega$ with well-founded limit and $crit(j_i) < \delta$ for all i.

CLAIM. *The set $\{crit(j_i) : i \in \omega\}$ is infinite.*

PROOF. For contradiction, suppose not. Then there exists κ such that $\{i : crit(j_i) = \kappa\}$ is infinite. So, let κ be least. Note that since the limit is well-founded, every ordinal is eventually fixed, and so we can let i_κ be such that $j_i(\kappa) = \kappa$ for all $i > i_\kappa$. Then for all $i > i_\kappa$, $crit(j_i) \neq \kappa$, and so since $\{crit(j_i) : i > i_\kappa\}$ is finite, there must exist at least $\kappa' > \kappa$ such that $crit(j_i) = \kappa'$ for infinitely many $i > i_\kappa$. But κ' is eventually fixed, and so it follows that there exists $\kappa'' > \kappa'$ such that $\{i : crit(j_i) = \kappa''\}$ is infinite; continuing this way, we get that there are infinitely many κ for which $\{i : crit(j_i) = \kappa\}$ is infinite, a contradiction. ⊣

Now, let $s = \langle crit(j_i) : i < \omega\rangle$. Then, being N closed under ω-sequences, $s \in \delta^\omega \cap N$. Since $\prec$ is definable in M, we have that $j_i(\prec) = \prec$ for all $i \in \omega$, which implies that every element of $\gamma^\omega \cap N$ is eventually fixed; in fact: there exists a bijection $\sigma : \alpha \leftrightarrow \gamma^\omega \cap N$

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in N such that $j_i(\sigma) = \sigma$ for all $i \in \omega$, and so for any $x \in \gamma^\omega \cap N$, letting $\beta \in \alpha$ be such that $x = \sigma(\beta)$ and $i \in \omega$ be such that $j_m(\beta) = \beta$ for all $m > i$, we have that $j_m(x) = j_m(\sigma(\beta)) = j_m(\sigma)(j_m(\beta)) = \sigma(\beta) = x$ for all $m > i$.

Thus, $j_i(s) = s$ for all sufficiently large i. So, fix i_0 such that $j_i(s) = s$ for all $i > i_0$. Then for $i > i_0$, $j_i(crit(j_i)) = j_i(s(i)) = j_i(s)(j_i(i)) = s(i) = crit(j_i)$, which is a contradiction. ⊣

OPEN QUESTION. *If N is an inner model of ZFC such that $N^\omega \subseteq N$, can there exist an N-Berkeley cardinal*?

3.3. Conclusions. To conclude, let us point out some remarkable consequences that Theorem 3.1 entails:

1. Despite the fact that it captures most of the large cardinal structure of the universe V, a weak extender model N needs not be "fully overlapping" V, in the sense of having all V large cardinals (although inheriting all of the traditional V large cardinals): in fact in V, there can exist a strong N-Berkeley cardinal.
2. For a weak extender model N of δ is supercompact, there can be a nontrivial elementary embedding $j : N \to N$ with $crit(j) < \delta$ (like in the N_ω case).
3. For a weak extender model N of δ is supercompact, in the Universality Theorem the critical point of the embedding—$j : N \cap V_{\gamma+1} \to N \cap V_{j(\gamma)+1}$ with $\gamma > \delta$—has to be bigger than or equal to δ: otherwise N could recognize that there is an N-Berkeley cardinal, contradicting that N models Choice.

Finally, maybe, we have just opened the way to a higher analogue of $0^\#$, which would lead us to the second alternative of the HOD dichotomy.

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