THE HILBERT FUNCTION OF TWO IDEALS

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Introduction. It is well known that Hilbert's function $\chi[r, \bar{\mathfrak{a}}]$ of a homogeneous ideal $\bar{\mathfrak{a}}$ in the ring of polynomials $K[x_0, \ldots, x_m]$, where K is a field and $x_0, \ldots, x_m$ are independent indeterminates over K, is, for large values of r, a polynomial in r of degree equal to the projective dimension of $\bar{\mathfrak{a}}$ (1). Samuel (4) and Northcott (2) have both shown that if the field K is replaced by an Artin ring A, $\chi[r, \bar{\mathfrak{a}}]$ is still a polynomial in r for large values of r. Applying this generalization Samuel (4) has shown that in a local ring Q the length of an ideal $\mathfrak{q}^\rho$, where $\mathfrak{q}$ is a primary ideal belonging to the maximal ideal $\mathfrak{m}$ of Q, is, for sufficiently large values of ρ, a polynomial in ρ whose degree is equal to the dimension of Q.

In this paper we have followed the methods of van der Waerden (5) and Northcott (2) to prove that Hilbert's function $\chi[r, s; \bar{\mathfrak{a}}]$ of a bihomogeneous ideal $\bar{\mathfrak{a}}$ in the ring $A[x_0, \ldots, x_m; y_0, \ldots, y_n]$, where $x_0, \ldots, x_m; y_0, \ldots, y_n$ are $m+n+2$ indeterminates and A is a primary Artin ring, is for sufficiently large values of r, s a polynomial in r and s. We have made use of this result to show that in a local ring Q, the length $\chi_{\mathfrak{q}_1, \mathfrak{q}_2}(\rho, \sigma)$ of the ideal $\mathfrak{q}_1^\rho \mathfrak{q}_2^\sigma$, where $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are two primary ideals belonging to the maximal ideal $\mathfrak{m}$ of Q, is, for sufficiently large values of ρ and σ, a polynomial in ρ and σ of total degree d which is the dimension of Q. In proving these results we have followed a technique which has been developed by Rees (3).

1. Let Q be a local ring and $\mathfrak{m}$ its maximal ideal. Let $\mathfrak{q}$ be a $\mathfrak{m}$-primary ideal of Q. Then $A = Q/\mathfrak{q}$ is a primary Artin ring having $\mathfrak{p} = \mathfrak{m}/\mathfrak{q}$ as its unique prime ideal. Consider the ring $A[x_0, \ldots, x_m; y_0, \ldots, y_n]$ of polynomials over A with $m+n+2$ indeterminates $x_0, \ldots, x_m; y_0, \ldots, y_n$. Let $\bar{\mathfrak{p}}$ denote the extension of $\mathfrak{p}$ in $A[x_0, \ldots, x_m; y_0, \ldots, y_n]$. Then $\bar{\mathfrak{p}}$ is a prime ideal of $A[x_0, \ldots, x_m; y_0, \ldots, y_n]$ and it is the only prime ideal belonging to the zero ideal of $A[x_0, \ldots, x_m; y_0, \ldots, y_n]$ (2). Hence every prime ideal in $A[x_0, \ldots, x_m; y_0, \ldots, y_n]$ contains $\bar{\mathfrak{p}}$ and, further, if a form $\bar{\phi}(x_0, \ldots, x_m; y_0, \ldots, y_n)$ do not belong to $\bar{\mathfrak{p}}$, then $\bar{\phi}(x_0, \ldots, x_m; y_0, \ldots, y_n)$ is not a zero divisor in $A[x_0, \ldots, x_m; y_0, \ldots, y_n]$.

Henceforth $A[x, y]$ will denote the ring $A[x_0, \ldots, x_m; y_0, \ldots, y_n]$; $\bar{\mathfrak{a}}$, $\bar{\mathfrak{b}}$, etc., will denote bihomogeneous ideals of $A[x, y]$.

2. Let $\mathfrak{f}(r, s)$ denote the A-module of all forms of $A[x, y]$ of degree (r, s). Since the A-module $\mathfrak{f}(r, s)$ has a finite basis, $\dim_A \mathfrak{f}(r, s)$, the length of a composition series of A-submodules of $\mathfrak{f}(r, s)$, is defined and finite. If $\bar{\mathfrak{a}}$ be a bihomogeneous ideal of $A[x, y]$, then $\mathfrak{f}(r, s; \bar{\mathfrak{a}})$ will denote the A-submodule of all forms of degree (r, s) contained in $\bar{\mathfrak{a}}$. If $\bar{\mathfrak{b}}$ be another bihomogeneous ideal of $A[x, y]$, then it is evident that

$$\mathfrak{f}(r, s; \bar{\mathfrak{a}} + \bar{\mathfrak{b}}) = \mathfrak{f}(r, s; \bar{\mathfrak{a}}) + \mathfrak{f}(r, s; \bar{\mathfrak{b}}) \tag{2·1}$$

and

$$\mathfrak{f}(r, s; \bar{\mathfrak{a}} \cap \bar{\mathfrak{b}}) = \mathfrak{f}(r, s; \bar{\mathfrak{a}}) \cap \mathfrak{f}(r, s; \bar{\mathfrak{b}}). \tag{2·2}$$

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Consequently

$$\begin{aligned}\dim_A \mathfrak{f}(r,s;\bar{\mathfrak{a}}+\bar{\mathfrak{b}})-\dim_A \mathfrak{f}(r,s;\bar{\mathfrak{a}}) &= \dim_A[\mathfrak{f}(r,s;\bar{\mathfrak{a}})+\mathfrak{f}(r,s;\bar{\mathfrak{b}})]/\mathfrak{f}(r,s;\bar{\mathfrak{a}})\\ &= \dim_A \mathfrak{f}(r,\bar{\mathfrak{b}})/\mathfrak{f}(r,s;\bar{\mathfrak{a}}\cap\bar{\mathfrak{b}}),\end{aligned}$$

that is to say,

$$\dim_A \mathfrak{f}(r,s;\bar{\mathfrak{a}}+\bar{\mathfrak{b}})+\dim_A \mathfrak{f}(r,s;\bar{\mathfrak{a}}\cap\bar{\mathfrak{b}}) = \dim_A \mathfrak{f}(r,s;\bar{\mathfrak{a}})+\dim_A \mathfrak{f}(r,s;\bar{\mathfrak{b}}). \tag{2·3}$$

The Hilbert function $\chi[r,s;\bar{\mathfrak{a}}]$ of the bihomogeneous ideal $\bar{\mathfrak{a}}$ is defined as follows:

$$\chi[r,s;\bar{\mathfrak{a}}] = \dim_A \mathfrak{f}(r,s)-\dim_A \mathfrak{f}(r,s;\bar{\mathfrak{a}}). \tag{2·4}$$

From (2·3) and (2·4) it follows that

$$\chi[r,s;\bar{\mathfrak{a}}+\bar{\mathfrak{b}}] = \chi[r,s;\bar{\mathfrak{a}}]+\chi[r,s;\bar{\mathfrak{b}}]-\chi[r,s;\bar{\mathfrak{a}}\cap\bar{\mathfrak{b}}]. \tag{2·5}$$

THEOREM 1. *Let $\bar{\mathfrak{a}}$ be a proper bihomogeneous ideal of $A[x,y]$ and let $\bar{\phi}(x,y)$ be a form of degree (a,b) and such that $\bar{\phi}(x,y)$ is relatively prime to $\bar{\mathfrak{a}}$. Then for $r \geqslant a$, $s \geqslant b$,*

$$\chi[r,s;\bar{\mathfrak{a}}+(\bar{\phi})] = \chi[r,s;\bar{\mathfrak{a}}]-\chi[r-a,s-b;\bar{\mathfrak{a}}]. \tag{2·6}$$

Proof. Since $\bar{\phi}$ is relatively prime to $\bar{\mathfrak{a}}$, therefore $\bar{\phi} \notin \bar{\mathfrak{p}}$. This shows that $\bar{\phi}$ is not a zero divisor in $A[x,y]$. Now

$$\mathfrak{f}(r,s;\bar{\mathfrak{a}}\cap(\bar{\phi})) = \bar{\phi}\mathfrak{f}(r-a,s-b;\bar{\mathfrak{a}})$$

and

$$\mathfrak{f}(r,s;(\bar{\phi})) = \bar{\phi}\mathfrak{f}(r-a,s-b).$$

Since $\bar{\phi}$ is not a zero divisor, we have the isomorphisms between A-modules

$$\mathfrak{f}(r,s;\bar{\mathfrak{a}}\cap(\bar{\phi})) \simeq \mathfrak{f}(r-a,s-b;\bar{\mathfrak{a}})$$

and

$$\mathfrak{f}(r,s;(\bar{\phi})) \simeq \mathfrak{f}(r-a,s-b).$$

Hence

$$\begin{aligned}\chi[r,s;\bar{\mathfrak{a}}+(\bar{\phi})] &= \dim_A \mathfrak{f}(r,s)-\dim_A \mathfrak{f}(r,s;\bar{\mathfrak{a}}+(\bar{\phi}))\\ &= \dim_A \mathfrak{f}(r,s)-\dim_A \mathfrak{f}(r,s;\bar{\mathfrak{a}})-\dim_A \mathfrak{f}(r,s;(\bar{\phi}))+\dim_A \mathfrak{f}(r,s;\bar{\mathfrak{a}}\cap(\bar{\phi}))\\ &= \chi[r,s;\bar{\mathfrak{a}}]-\chi[r-a,s-b;\bar{\mathfrak{a}}].\end{aligned}$$

This completes the proof of the theorem.

Let $\mathfrak{X} = (\bar{\mathfrak{p}}; x_0,\ldots,x_m; y_0,\ldots,y_n)$. Then $\mathfrak{X}$ is a bihomogeneous ideal; further, it is a maximal ideal of $A[x,y]$ and contains every proper bihomogeneous ideal of $A[x,y]$. Since $\bar{\mathfrak{p}}$ is the unique prime ideal belonging to the zero ideal of $A[x,y]$, therefore, for a suitable integer a, $\bar{\mathfrak{p}}^a = (0)$. Therefore $\mathfrak{X}^a \subseteq (x_0,\ldots,x_m; y_0,\ldots,y_n)$. Hence if a proper bihomogeneous ideal $\bar{\mathfrak{a}}$ contains a power of $(x_0,\ldots,x_m; y_0,\ldots,y_n)$, then $\bar{\mathfrak{a}}$ is $\mathfrak{X}$-primary.

DEFINITION. *A proper bihomogeneous ideal $\bar{\mathfrak{a}}$ which contains the ideal $(x_0,\ldots,x_m)^a(y_0,\ldots,y_n)^b$ for some non-negative integers a and b, is called a projectively irrelevant ideal of $A[x_0,\ldots,x_m; y_0,\ldots,y_n]$.*

It follows that if a bihomogeneous ideal $\bar{\mathfrak{a}}$ is $\mathfrak{X}$-primary, then $\bar{\mathfrak{a}}$ is necessarily projectively irrelevant. It also follows that if $\bar{\mathfrak{a}}$ is a projectively irrelevant bihomogeneous ideal of $A[x,y]$, then for sufficiently large values of r and s, $\chi[r,s;\bar{\mathfrak{a}}] = 0$.

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We now come to our key theorem:

THEOREM 2. *If $\bar{\mathfrak{a}}$ is a bihomogeneous ideal of $A[x_0, \ldots, x_m; y_0, \ldots, y_n]$ then $\chi[r, s; \bar{\mathfrak{a}}]$ is represented, for large values of r and s, by an expression of the form*

$$\chi[r, s; \bar{\mathfrak{a}}] = \sum_{i+j \leqslant d-2} a_{ij} \binom{r}{i} \binom{s}{j}, \tag{2·7}$$

where a_{ij} are integers and d denotes the dimension of $\bar{\mathfrak{a}}$.*

Proof. If $d = 0$ or $d = 1$ then $\bar{\mathfrak{a}}$ is projectively irrelevant and, therefore, for large values of r and s, $\chi[r, s; \bar{\mathfrak{a}}] = 0$. If, therefore, we consider 0 as a polynomial of total degree not greater than -1 in r and s, then the theorem holds for $d = 0$ and $d = 1$.

We suppose that $d > 1$, and we assume that the theorem has been proved for bihomogeneous ideals of smaller dimensions. Suppose first that $\bar{\mathfrak{a}}$ is primary, and let it be $\bar{\mathfrak{p}}_1$-primary. If $\bar{\mathfrak{a}}$ is projectively irrelevant, then we have already shown that $\chi[r, s, \bar{\mathfrak{a}}] = 0$ for sufficiently large values of r and s, and therefore the theorem holds. If $\bar{\mathfrak{a}}$ is not projectively irrelevant, then there exists x_i and y_j such that $x_i \notin \bar{\mathfrak{p}}_1$, $y_j \notin \bar{\mathfrak{p}}_1$. Then the ideals $\bar{\mathfrak{a}} + (x_i)$, $\bar{\mathfrak{a}} + (y_j)$ have the dimensions $d-1$ at most. Hence by Theorem 1,

$$\chi[r, s; \bar{\mathfrak{a}} + (x_i)] = \chi[r, s; \bar{\mathfrak{a}}] - \chi[r-1, s; \bar{\mathfrak{a}}], \tag{2·8}$$

$$\chi[r, s; \bar{\mathfrak{a}} + (y_j)] = \chi[r, s; \bar{\mathfrak{a}}] - \chi[r, s-1; \bar{\mathfrak{a}}]. \tag{2·9}$$

By hypothesis, there exist integers r_0 and s_0, such that for $r \geqslant r_0$, $s \geqslant s_0$,

$$\chi[r, s; \bar{\mathfrak{a}} + (x_i)] = \sum_{i+j \leqslant d-3} b_{ij} \binom{r}{i} \binom{s}{j} \tag{2·10}$$

and

$$\chi[r, s; \bar{\mathfrak{a}} + (y_j)] = \sum_{i+j \leqslant d-3} c_{ij} \binom{r}{i} \binom{s}{j}. \tag{2·11}$$

From (2·8) and (2·10) we get by summation

$$\chi[r, s; \bar{\mathfrak{a}}] - \chi[r_0, s; \bar{\mathfrak{a}}] = \sum_{i+j \leqslant d-3} b_{ij} \left\{ \binom{r+1}{i+1} - \binom{r_0+1}{i+1} \right\} \binom{s}{j}.$$

Similarly from (2·9) and (2·11) we get

$$\chi[r_0, s; \bar{\mathfrak{a}}] - \chi[r_0, s_0; \bar{\mathfrak{a}}] = \sum_{i+j \leqslant d-3} c_{ij} \binom{r_0}{i} \left\{ \binom{s+1}{j+1} - \binom{s_0+1}{j+1} \right\}.$$

Adding the last two equations we get

$$\begin{aligned} \chi[r, s; \bar{\mathfrak{a}}] = \chi[r_0, s_0; \bar{\mathfrak{a}}] &+ \sum_{i+j \leqslant d-3} b_{ij} \binom{r+1}{i+1} \binom{s}{j} - \sum_{i+j \leqslant d-3} b_{ij} \binom{r_0+1}{i+1} \binom{s}{j} \\ &+ \sum_{i+j \leqslant d-3} c_{ij} \binom{r_0}{i} \binom{s+1}{j+1} - \sum_{i+j \leqslant d-3} c_{ij} \binom{r_0}{i} \binom{s_0+1}{j+1}, \end{aligned}$$

which has the form (2·7).

Let us now remove the restriction that $\bar{\mathfrak{a}}$ is primary. Let

$$\bar{\mathfrak{a}} = \bar{\mathfrak{q}}_1 \cap \bar{\mathfrak{q}}_2 \cap \ldots \cap \bar{\mathfrak{q}}_l$$

be the normal decomposition of $\bar{\mathfrak{a}}$ into bihomogeneous primary ideals. Since $\dim \bar{\mathfrak{a}} = d$ we may suppose that $\dim \bar{\mathfrak{q}}_l = d$. Let $\bar{\mathfrak{b}} = \bar{\mathfrak{q}}_1 \cap \bar{\mathfrak{q}}_2 \cap \ldots \cap \bar{\mathfrak{q}}_{l-1}$. Then $\bar{\mathfrak{a}} = \bar{\mathfrak{b}} \cap \bar{\mathfrak{q}}_l$, where $\bar{\mathfrak{b}}$ has

* By the dimension of $\bar{\mathfrak{a}}$ we mean the ordinary and not the projective dimension of $\bar{\mathfrak{a}}$.

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less than l primary components and $\bar{\mathfrak{q}}_l$ is primary of dimension d. If $l = 1$ the theorem has already been proved. We assume, therefore, that the theorem holds good for d-dimensional bihomogeneous ideals with fewer than l primary components. By (2·5), we may write

$$\chi[r, s; \bar{\mathfrak{a}}] = \chi[r, s; \bar{\mathfrak{b}}] + \chi[r, s; \bar{\mathfrak{q}}_l] - \chi[r, s; \bar{\mathfrak{q}}_l + \bar{\mathfrak{b}}].$$

The first two terms on the right-hand side refer to ideals with fewer than l primary components and the third term to an ideal of dimensions less than d. Hence all the functions on the right-hand side have the form (2·7), and therefore $\chi[r, s; \bar{\mathfrak{a}}]$ has the form (2·7) for sufficiently large values of r and s.

3. *Degrees of a bihomogeneous ideal.* Let us call the coefficients a_{ij} of the terms of the Hilbert function $\chi[r, s; \bar{\mathfrak{a}}]$ such that $i+j = d-2$, where d is the dimension of the bihomogeneous ideal $\bar{\mathfrak{a}}$, the degrees of the bihomogeneous ideal $\bar{\mathfrak{a}}$ and let us denote them by $e_{ij}(\bar{\mathfrak{a}})$. Then the following theorems follow easily.

THEOREM 3. *Any degree $e_{ij}(\bar{\mathfrak{a}})$ of a bihomogeneous ideal $\bar{\mathfrak{a}}$ is the sum of the corresponding degrees of the primary components of $\bar{\mathfrak{a}}$ with the same dimension.*

THEOREM 4. *If $\bar{\mathfrak{q}}$ be $\bar{\mathfrak{p}}_1$-primary ideal and $\bar{l}_1$ be a linear form in the χ_i such that $\bar{l}_1 \notin \bar{\mathfrak{p}}_1$, then $e_{ij}(\bar{\mathfrak{q}}) = e_{i-1,\,j}(\bar{\mathfrak{q}} + (\bar{l}_1))$. Similarly if $\bar{l}_2$ be a linear form in the y_i and $\bar{l}_2 \notin \bar{\mathfrak{p}}_1$, then $e_{ij}(\bar{\mathfrak{q}}) = e_{i,\,j-1}(\bar{\mathfrak{q}} + (\bar{l}_2))$.*

A generalization of Theorem 4 is

THEOREM 5. *If $\bar{\phi}$ be $\bar{\mathfrak{p}}_1$-primary and $\bar{\phi}$ be a form of degree (γ, δ) such that $\bar{\phi} \notin \bar{\mathfrak{p}}_1$, then $e_{ij}(\bar{\mathfrak{q}} + (\bar{\phi})) = \gamma \cdot e_{i+1,\,j}(\bar{\mathfrak{q}}) + \delta \cdot e_{i,\,j+1}(\bar{\mathfrak{q}})$.*

We now prove the following theorem about the degrees of a bihomogeneous ideal:

THEOREM 6. *If $\bar{\mathfrak{a}}$ is a bihomogeneous ideal of $A[x_0, \ldots, x_m; y_0, \ldots, y_n]$ which is not projectively irrelevant, then $e_{ij}(\bar{\mathfrak{a}}) > 0$.*

Proof. Let $\dim \bar{\mathfrak{a}} = d$. By Theorem 3, we can confine ourselves to the case where $\bar{\mathfrak{a}}$ is primary. Let $\bar{\mathfrak{a}}$ be $\bar{\mathfrak{p}}_1$-primary. If $d = 2$, the Hilbert function $\chi[r, s; \bar{\mathfrak{a}}]$ is a constant which is equal to $e_{0,0}(\bar{\mathfrak{a}})$. Since $\bar{\mathfrak{a}}$ is not projectively irrelevant, therefore $\bar{\mathfrak{a}} \not\supseteq (x_0, \ldots, x_m)^a (y_0, \ldots, y_n)^b$, whatever a and b may be and hence $e_{0,0}(\bar{\mathfrak{a}}) = \chi[r, s; \bar{\mathfrak{a}}] > 0$. Suppose next that $d > 2$. Let us assume that the theorem is valid for bihomogeneous ideals $\bar{\mathfrak{a}}$ of dimension less than d. Since $\bar{\mathfrak{a}}$ is not projectively irrelevant and $\bar{\mathfrak{a}}$ is $\bar{\mathfrak{p}}_1$-primary, therefore, there exists x_k, y_l such that $x_k \notin \bar{\mathfrak{p}}_1$, $y_l \notin \bar{\mathfrak{p}}_1$. Hence

$$\begin{aligned}\chi[r, s; \bar{\mathfrak{a}} + (x_k)] &= \chi[r, s; \bar{\mathfrak{a}}] - \chi[r-1, s; \bar{\mathfrak{a}}] \\ &= \sum_{i+j \leqslant d-2} a_{ij} \left\{ \binom{r}{i}\binom{s}{j} - \binom{r-1}{i}\binom{s}{j} \right\} \\ &= \sum_{i+j \leqslant d-2} a_{ij} \binom{r-1}{i-1}\binom{s}{j}.\end{aligned}$$

Now $\dim(\bar{\mathfrak{a}} + (x_k)) = d-1$, since $\dim \bar{\mathfrak{a}} = d$ and $x_k \notin \bar{\mathfrak{p}}_1$. Therefore

$$e_{ij}(\bar{\mathfrak{a}} + (x_i)) = e_{i+1,\,j}(\bar{\mathfrak{a}}) \quad (i+j = d-3);$$

similarly

$$e_{ij}(\bar{\mathfrak{a}} + (y_l)) = e_{i,\,j+1}(\bar{\mathfrak{a}}) \quad (i+j = d-3).$$

By hypothesis, $e_{ij}(\bar{\mathfrak{a}} + (x_k))$ and $e_{ij}(\bar{\mathfrak{a}} + (y_i))$ are both greater than zero. Hence $e_{ij}(\bar{\mathfrak{a}}) > 0$.

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4. Let $\mathfrak{q}_1$, $\mathfrak{q}_2$ be two proper ideals of Q, one of which, say $\mathfrak{q}_2$, is $\mathfrak{m}$-primary. Let $(u_0, \ldots, u_m)$ be a basis of $\mathfrak{q}_1$ and let $(v_0, \ldots, v_n)$ be a basis of $\mathfrak{q}_2$. Consider the ring

$$R = Q[t_1 u_0, \ldots, t_1 u_m; t_2 v_0, \ldots, t_2 v_n, t_2^{-1}],$$

where t_1, t_2 are indeterminates over Q. The elements of R are polynomials

$$\sum_{i=0}^{s} \sum_{j=-p}^{q} c_{ij} t_1^i t_2^j,$$

where $c_{ij} \in \mathfrak{q}_1^i \mathfrak{q}_2^j$ and $\mathfrak{q}_1^i = \mathfrak{q}_2^j = Q$ if i, j are zero or negative. Consider the ideal $t_2^{-1}R$. This consists of polynomials

$$\sum_{i=0}^{s} \sum_{j=-p}^{q} c_{ij} t_1^i t_2^j, \quad c_{ij} \in \mathfrak{q}_1^i \mathfrak{q}_2^{j+1}.$$

Hence $R/t_2^{-1}R$ consists of all polynomials

$$\sum_{i=0}^{s} \sum_{j=0}^{q} \bar{c}_{ij} t_1^i t_2^j, \quad \text{where} \quad \bar{c}_{ij} \in \mathfrak{q}_1^i \mathfrak{q}_2^j / \mathfrak{q}_1^i \mathfrak{q}_2^{j+1}.$$

But $$R/t_2^{-1}R = Q/\mathfrak{q}_2[\overline{t_1 u_0}, \ldots, \overline{t_1 u_m}; \overline{t_2 v_0}, \ldots, \overline{t_2 v_n}]$$

since $Q \cap t_2^{-1}R = \mathfrak{q}_2$, where $\overline{t_1 u_i}$, $\overline{t_2 v_j}$ are respectively the images of $t_1 u_i$ and $t_2 v_j$ in $R/t_2^{-1}R$.

Now consider the homomorphism

$$Q/\mathfrak{q}_2[x_0, \ldots, x_m; y_0, \ldots, y_n] \quad \text{onto} \quad Q/\mathfrak{q}_2[\overline{t_1 u_0}, \ldots, \overline{t_1 u_m}; \overline{t_2 v_0}, \ldots, \overline{t_2 v_n}],$$

where $x_0, \ldots, x_m; y_0, \ldots, y_n$ are independent indeterminates over $Q/\mathfrak{q}_2$. If an element α of $Q/\mathfrak{q}_2[x_0, \ldots, x_m; y_0, \ldots, y_n]$ maps onto the zero element of $Q/\mathfrak{q}_2[\overline{t_1 u_0}, \ldots, \overline{t_1 u_m}; \overline{t_2 v_0}, \ldots, \overline{t_2 v_n}]$, then each form of α maps onto the zero element of $Q/\mathfrak{q}_2[\overline{t_1 u_0}, \ldots, \overline{t_1 u_m}; \overline{t_2 v_0}, \ldots, \overline{t_2 v_n}]$, since the latter is a graduated ring, and hence each form of α belongs to the kernel of the homomorphism. Hence the kernel of the above homomorphism is a bihomogeneous ideal $\bar{\mathfrak{a}}$, say, of $Q/\mathfrak{q}_2[x_0, \ldots, x_m; y_0, \ldots, y_n]$. Hence

$$Q/\mathfrak{q}_2[\overline{t_1 u_0}, \ldots, \overline{t_1 u_m}; \overline{t_2 v_0}, \ldots, \overline{t_2 v_n}] \simeq Q/\mathfrak{q}_2[x_0, \ldots, x_m; y_0, \ldots, y_n]/\bar{\mathfrak{a}}. \tag{4·1}$$

Let us put $A = Q/\mathfrak{q}_2$. Then A is a primary Artin ring and $\mathfrak{p} = \mathfrak{m}/\mathfrak{q}_2$ is its single prime ideal. By virtue of the isomorphism (4·1) we have

$$\dim_A(\mathfrak{q}_1^r \mathfrak{q}_2^s / \mathfrak{q}_1^r \mathfrak{q}_2^{s+1}) = \dim_A[\mathfrak{f}(r, s) + \bar{\mathfrak{a}}]/\bar{\mathfrak{a}},$$

where $\mathfrak{f}(r, s)$ is the A-module of all forms of $A[x_0, \ldots, x_m; y_0, \ldots, y_n]$ of degree (r, s); thus

$$\dim_A(\mathfrak{q}_1^r \mathfrak{q}_2^s / \mathfrak{q}_1^r \mathfrak{q}_2^{s+1}) = \dim_A \mathfrak{f}(r, s) - \dim_A \mathfrak{f}(r, s; \bar{\mathfrak{a}})$$

or $$\dim_A(\mathfrak{q}_1^r \mathfrak{q}_2^s / \mathfrak{q}_1^r \mathfrak{q}_2^{s+1}) = \chi[r, s; \bar{\mathfrak{a}}], \tag{4·2}$$

where $\chi[r, s; \bar{\mathfrak{a}}]$ is the Hilbert function of the bihomogeneous ideal $\bar{\mathfrak{a}}$ in the ring of polynomials $A[x_0, \ldots, x_m; y_0, \ldots y_n]$.

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Since we have already proved that for sufficiently large values of r, s, $\chi[r, s; \bar{\mathfrak{a}}]$ is a polynomial in r, s of degree $d-2$, where d is the dimension of $\bar{\mathfrak{a}}$, we have the following result:

THEOREM 7. *If* $\mathfrak{q}$ *is any* $\mathfrak{m}$*-primary ideal and* $\bar{\mathfrak{a}}$ *is an arbitrary, proper, ideal in the local ring* Q *whose maximal ideal is* $\mathfrak{m}$*, then for sufficiently large values of* ρ *and* σ, $\dim_{Q/\mathfrak{q}}(\mathfrak{a}^\rho \mathfrak{q}^\sigma/\mathfrak{a}^\rho \mathfrak{q}^{\sigma+1})$ *is a polynomial in* ρ *and* σ.

It is to be noted that if $\bar{\mathfrak{a}} \supset (x_0, \ldots, x_m)^a (y_0, \ldots, y_n)^b$ for some non-negative integers a and b, then $\mathfrak{q}_1^r \mathfrak{q}_2^s \subseteq \mathfrak{q}_1^r \mathfrak{q}_2^{s+1}$ for all $r \geqslant a$, $s \geqslant b$. If $\mathfrak{q}$, is $\mathfrak{m}$-primary, this implies that Q is a primary Artin ring and hence for sufficiently large values of ρ and σ, $\chi_{\mathfrak{q}_1, \mathfrak{q}_2}(\rho, \sigma)$, the length of the ideal $\mathfrak{q}_1^\rho \mathfrak{q}_2^\sigma$, is a constant.

THEOREM 8. *Let* $\mathfrak{q}_1$, $\mathfrak{q}_2$ *be two* $\mathfrak{m}$*-primary ideals of* Q*. Then* $\chi_{\mathfrak{q}_1, \mathfrak{q}_2}(\rho, \sigma)$*, the length of the ideal* $\mathfrak{q}_1^\rho \mathfrak{q}_2^\sigma$*, is, for sufficiently large values of* ρ *and* σ*, a polynomial in* ρ *and* σ*, whose total degree is equal to the dimension of* Q.

Proof. We have already remarked that if Q is a primary Artin ring then the above theorem is trivially true. Let us confine ourselves to the case where Q is not a primary ring. Put $A = Q/\mathfrak{q}_2$, $B = Q/\mathfrak{q}_1$. Then A and B are both primary rings. We have already shown that

$$\dim_A(\mathfrak{q}_1^r \mathfrak{q}_2^s/\mathfrak{q}_1^r \mathfrak{q}_2^{s+1}) = \chi[r, s; \bar{\mathfrak{a}}].$$

where $\bar{\mathfrak{a}}$ is a bihomogeneous ideal of $A[x_0, \ldots, x_m; y_0, \ldots, y_n]$ which is not projectively irrelevant. In the same way we can show that

$$\dim_B(\mathfrak{q}_1^r \mathfrak{q}_2^s/\mathfrak{q}_1^{r+1} \mathfrak{q}_2^s) = \chi[r, s; \bar{\mathfrak{a}}'],$$

where $\bar{\mathfrak{a}}'$ is a bihomogeneous ideal of $B[x_0, \ldots, x_m; y_0, \ldots, y_n]$ which is not projectively irrelevant. Hence there exist integers ρ_0 and σ_0 such that for all values of $r \geqslant \rho_0, s \geqslant \sigma_0$

$$\chi[r, s; \bar{\mathfrak{a}}] = \sum_{i+j \leqslant d-2} a_{ij} \binom{r}{i} \binom{s}{j} \tag{4·3}$$

and

$$\chi[r, s; \bar{\mathfrak{a}}'] = \sum_{i+j \leqslant d'-2} a'_{ij} \binom{r}{i} \binom{s}{j}, \tag{4·4}$$

where a_{ij} and a'_{ij} are integers and d and d' are respectively the dimensions of $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{a}}'$.

Now for $\rho > \rho_0$, $\sigma > \sigma_0$,

$$\begin{aligned}
\chi_{\mathfrak{q}_1, \mathfrak{q}_2}(\rho, \sigma) - \chi_{\mathfrak{q}_1, \mathfrak{q}_2}(\rho_0, \sigma_0) &= \sum_{r=\rho_0}^{\rho-1} \dim_B(\mathfrak{q}_1^r \mathfrak{q}_2^{\sigma_0}/\mathfrak{q}_1^{r+1} \mathfrak{q}_2^{\sigma_0}) + \sum_{s=\sigma_0}^{\sigma-1} \dim_A(\mathfrak{q}_1^\rho \mathfrak{q}_2^s/\mathfrak{q}_1^\rho \mathfrak{q}_2^{s+1}) \\
&= \sum_{r=\rho_0}^{\rho-1} \left[\sum_{i+j \leqslant d'-2} a'_{ij} \binom{r}{i} \binom{\sigma_0}{j} \right] + \sum_{s=\sigma_0}^{\sigma-1} \left[\sum_{i+j \leqslant d-2} a_{ij} \binom{\rho}{i} \binom{s}{j} \right] \\
&= \sum_{i+j \leqslant d'+2} a'_{ij} \left\{ \binom{\rho}{i+1} - \binom{\rho_0}{i+1} \right\} \binom{\sigma_0}{j} \\
&\qquad + \sum_{i+j \leqslant d-2} a_{ij} \binom{\rho}{i} \left\{ \binom{\sigma}{j+1} - \binom{\sigma_0}{j+1} \right\},
\end{aligned}$$

or

$$\begin{aligned}
&\chi_{\mathfrak{q}_1, \mathfrak{q}_2}(\rho, \sigma) - \chi_{\mathfrak{q}_1, \mathfrak{q}_2}(\rho_0, \sigma_0) \\
&= \sum_{i+j \leqslant d-2} a_{ij} \binom{\rho}{i} \binom{\sigma}{j+1} - \sum_{i+j \leqslant d-2} a_{ij} \binom{\rho}{i} \binom{\sigma_0}{j+i} + \sum_{i+j \leqslant d'-2} a'_{ij} \binom{\rho}{i+1} \binom{\sigma_0}{j} \\
&\qquad + \text{a constant integer.}
\end{aligned} \tag{4·5}$$

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We have also

$$\chi_{\mathfrak{q}_1,\mathfrak{q}_2}(\rho,\sigma)-\chi_{\mathfrak{q}_1,\mathfrak{q}_2}(\rho_0,\sigma_0)=\sum_{s=\sigma_0}^{\sigma-1}\dim_A(\mathfrak{q}_1^{\rho_0}\mathfrak{q}_2^{s}/\mathfrak{q}_1^{\rho_0}\mathfrak{q}_2^{s+1})+\sum_{r=\rho_0}^{\rho-1}\dim_B(\mathfrak{q}_r^1\mathfrak{q}_2^{\sigma}/\mathfrak{q}_1^{r+1}\mathfrak{q}_2^{\sigma})$$

$$=\sum_{s=\sigma_0}^{\sigma-1}\Big[\sum_{i+j\leqslant d-2}a_{ij}\binom{\rho_0}{i}\binom{s}{j}\Big]+\sum_{r=\rho_0}^{\rho-1}\Big[\sum_{i+j\leqslant d'-2}a'_{ij}\binom{r}{i}\binom{\sigma}{j}\Big]$$

$$=\sum_{i+j\leqslant d-2}a_{ij}\binom{\rho_0}{i}\Big\{\binom{\sigma}{j+1}-\binom{\sigma_0}{j+1}\Big\}$$

$$+\sum_{i+j\leqslant d'-2}a'_{ij}\Big\{\binom{\rho}{i+1}-\binom{\rho_0}{i+1}\Big\}\binom{\sigma}{j},$$

or $\chi_{\mathfrak{q}_1,\mathfrak{q}_2}(\rho,\sigma)-\chi_{\mathfrak{q}_1,\mathfrak{q}_2}(\rho_0,\sigma_0)$

$$=\sum_{i+j\leqslant d'-2}a'_{ij}\binom{\rho}{i+1}\binom{\sigma}{j}-\sum_{i+j\leqslant d'-2}a'_{ij}\binom{\rho_0}{i+1}\binom{\sigma}{j}+\sum_{i+j\leqslant d-2}a_{ij}\binom{\rho_0}{i}\binom{\sigma_0}{j+1}$$

$$+\text{a constant integer.}\qquad(4\cdot6)$$

Since $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{a}}'$ are not projectively irrelevant ideals in

$$A[x_0,\ldots,x_m;y_0,\ldots,y_n]\quad\text{and}\quad B[x_0,\ldots,x_m;y_0,\ldots,y_n]$$

respectively, therefore all the degrees of both $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{a}}'$ are greater than zero. Hence comparing (4·5) and (4·6) we get $d=d'$, that is,

$$\dim\bar{\mathfrak{a}}\quad\text{in}\quad A[x_0,\ldots,x_m;y_0,\ldots,y_n]=\dim\bar{\mathfrak{a}}'\quad\text{in}\quad B[x_0,\ldots,x_m;y_0,\ldots,y_n].$$

We can now rewrite (4·5) as follows:

$$\chi_{\mathfrak{q}_1,\mathfrak{q}_2}(\rho,\sigma)=\sum_{i+j\leqslant d-2}a_{ij}\binom{\rho}{i}\binom{\sigma}{j+1}-\sum_{i+j\leqslant d-2}a_{ij}\binom{\rho}{i}\binom{\sigma_0}{j+1}$$

$$+\sum_{i+j\leqslant d-2}a'_{ij}\binom{\rho}{i+1}\binom{\sigma_0}{j}+\text{a constant integer},$$

or
$$\chi_{\mathfrak{q}_1,\mathfrak{q}_2}(\rho,\sigma)=\sum_{i+j\leqslant d-1}c_{ij}\binom{\rho}{i}\binom{\sigma}{j},\qquad(4\cdot7)$$

where c_{ij} are integers and $c_{ij}>0$, $i+j=d-1$.

If $\mathfrak{q}'_1$, $\mathfrak{q}'_2$ are any two other m-primary ideals of Q,

then
$$\mathfrak{q}_1^{\rho}\mathfrak{q}_2^{\sigma}\supset\mathfrak{q}_1'^{m\rho}\mathfrak{q}_2'^{m\sigma}$$

and
$$\mathfrak{q}_1'^{\rho}\mathfrak{q}_2'^{\sigma}\supset\mathfrak{q}_1^{n\rho}\mathfrak{q}_2^{n\sigma}$$

for suitable integers m and n. Therefore

$$\chi_{\mathfrak{q}_1,\mathfrak{q}_2}(\rho,\sigma)\leqslant\chi_{\mathfrak{q}'_1,\mathfrak{q}'_2}(m\rho,m\sigma)$$

and
$$\chi_{\mathfrak{q}'_1,\mathfrak{q}'_2}(\rho,\sigma)\leqslant\chi_{\mathfrak{q}_1,\mathfrak{q}_2}(n\rho,n\sigma).$$

These two equations show that the degree of $\chi_{\mathfrak{q}_1,\mathfrak{q}_2}(\rho,\sigma)$ is the same as that of $\chi_{\mathfrak{q}'_1,\mathfrak{q}'_2}(\rho,\sigma)$.

In particular, if $\mathfrak{q}_1=\mathfrak{q}_2$, then $\chi_{\mathfrak{q}_1,\mathfrak{q}_2}(\rho+\sigma)=\chi_{\mathfrak{q}_1}(\rho+\sigma)$. But the degree of $\chi_{\mathfrak{q}_1}(\rho+\sigma)$, which is a polynomial in $\rho+\sigma$, is equal to the dimension of the local ring Q (4). Hence we have proved that the degree of the polynomial $\chi_{\mathfrak{q}_1,\mathfrak{q}_2}(\rho,\sigma)$ is equal to the dimension of Q.

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