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SIMPLE EXTENSIONS OF COMBINATORIAL STRUCTURES

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Abstract. An interval in a combinatorial structure R is a set I of points that are related to every point in $R\setminus I$ in the same way. A structure is simple if it has no proper intervals. Every combinatorial structure can be expressed as an inflation of a simple structure by structures of smaller sizes—this is called the substitution (or modular) decomposition. In this paper we prove several results of the following type: an arbitrary structure S of size n belonging to a class $\mathcal{C}$ can be embedded into a simple structure from $\mathcal{C}$ by adding at most $f(n)$ elements. We prove such results when $\mathcal{C}$ is the class of all tournaments, graphs, permutations, posets, digraphs, oriented graphs and general relational structures containing a relation of arity greater than two. The functions $f(n)$ in these cases are 2, $\lceil \log_2(n+1)\rceil$, $\lceil (n+1)/2\rceil$, $\lceil (n+1)/2\rceil$, $\lceil \log_4(n+1)\rceil$, $\lceil \log_3(n+1)\rceil$ and 1, respectively. In each case these bounds are the best possible.

§1. *Introduction.* Relational structures—objects governed by a given set of relations over some ground set—provide a general setting in which to study a wide variety of combinatorial structures including graphs, tournaments, posets and permutations. For each of these types of structure, the *substitution decomposition* (also called the *modular decomposition*) can be used to describe larger objects in terms of smaller ones. The elemental building blocks arising in this decomposition are the *simple*† structures.

In this paper, rather than decompose a combinatorial structure to describe it in terms of smaller simple ones, we shall be embedding structures into larger simple ones. In particular, we consider the following question: given a class $\mathcal{C}$ of relational structures, what is the minimal function $f(n)$ so that any structure in $\mathcal{C}$ of size n can be embedded in a simple structure from $\mathcal{C}$ containing at most $f(n)$ additional elements?

This question has its origins in the early 1970s with Erdős *et al* [**8**], who showed in the case of tournaments that $f(n) = 2$ for all n. In a subsequent paper [**9**] most of the same authors also showed that the only tournaments that could not be embedded in a simple tournament with one additional vertex were transitive tournaments with an odd number of vertices. Around the same time, Sumner showed in his thesis [**23**] that $f(n) = \lceil \log_2(n+1)\rceil$ for the subcase of complete graphs (when allowing extensions to arbitrary graphs). In this paper

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† The terms indecomposable, prime and primitive have been used in place of simple in other contexts.

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we answer the question completely for many standard combinatorial structures, in addition to more general relational structures containing relations of higher arity. These results rest on a unifying framework that uses the substitution decomposition, but each class of structures presents different problems and this is reflected in the variety of answers attained.

In the rest of this section we shall introduce important definitions and present the standard framework for our approach to the problem. In §2 we show how the case of tournaments fits into the framework, and then in §§3–6 we demonstrate the cases of graphs, permutations, posets, digraphs and oriented graphs. Section 7 discusses more general results for higher-arity relational structures, and some concluding remarks are given in §8.

A *k-ary relation* R on a set A is a subset of A^k. Informally, a *relational structure* is an ordered sequence of relations over some set A. More specifically, we may define a *relational language*, $\mathcal{L}$, to be a set of *relational symbols*, and for each symbol R there is a positive integer n_R denoting the arity of R. A relational structure $\mathcal{A}$ whose relational symbols are those of $\mathcal{L}$ is then defined by its *ground set* $A = \mathrm{dom}(\mathcal{A})$ and a set of subsets $R^{\mathcal{A}} \subseteq A^{n_R}$ for each $R \in \mathcal{L}$. Such a structure may also be called an $\mathcal{L}$*-structure*. When it does not cause any confusion, we shall omit the superscript $\mathcal{A}$ from relations.

Most standard combinatorial structures can be viewed as relational structures; they differ in the number of relations involved, their arities, and extra properties they are required to possess. This viewpoint offers a unifying framework for investigating different combinatorial structures "in parallel".

Example 1.1. A graph G can be viewed as a relational structure with a single binary operation that is required to be symmetric and irreflexive. Thus, in this case the language is $\mathcal{L} = \{E, n_E = 2\}$, $\mathrm{dom}(\mathcal{A}) = V(G)$, and $(x, y) \in E$ if and only if $x \sim y$ in G.

Example 1.2. Some further combinatorial structures viewed as relational structures are:

(i) oriented graphs: $\mathcal{L} = \{\rightarrow, n_{\rightarrow} = 2\}$ where $\rightarrow$ is asymmetric, i.e., $(x \rightarrow y) \Rightarrow (y \not\rightarrow x)$;
(ii) tournaments: $\mathcal{L} = \{\rightarrow, n_{\rightarrow} = 2\}$, but where $\rightarrow$ is trichotomous, i.e., for any x, y, precisely one of $x = y$, $x \rightarrow y$ or $y \rightarrow x$ is true;
(iii) digraphs: $\mathcal{L} = \{\rightarrow, n_{\rightarrow} = 2\}$, a "generic" structure with a single binary relation;
(iv) posets: $\mathcal{L} = \{<, n_{<} = 2\}$, where $<$ is asymmetric and transitive;
(v) linear orders: $\mathcal{L} = \{<, n_{<} = 2\}$, where $<$ is asymmetric, transitive and trichotomous;
(vi) permutations: $\mathcal{L} = \{<, \prec, n_{<} = 2, n_{\prec} = 2\}$, where each of $<$ and $\prec$ are linear orders. For a permutation π of $[n]$ these two orders are defined as follows: $<$ is the normal ordering on $[n]$, while $i \prec j$ if and only if $\pi(i) < \pi(j)$.

A set $I \subseteq A = \mathrm{dom}(\mathcal{A})$ is an *interval* if for every $R \in \mathcal{L}$ and all $x, y \in A^{n_R} \setminus I^{n_R}$ where, for all i, either $x_i = y_i$ or $x_i, y_i \in I$, we have $x \in R^{\mathcal{A}} \Leftrightarrow y \in R^{\mathcal{A}}$. Intuitively, this means that in relationships involving elements outside

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an interval I, any two elements within I are interchangeable. Every singleton set $\{x\} \subseteq A$ is an interval, as is all of A and the empty set. All other intervals are said to be *proper*, and a structure is *simple* if it has no proper intervals.

Let $\mathcal{A}$ and $\mathcal{B}$ be two structures over the same language $\mathcal{L}$. We say that $\mathcal{A}$ is a *substructure* of $\mathcal{B}$, or, equivalently, that $\mathcal{B}$ is an *extension* of $\mathcal{A}$, if $A \subseteq B$ and $R^{\mathcal{A}} = R^{\mathcal{B}}\restriction_A$ for every $R \in \mathcal{L}$. If, in addition, $\mathcal{B}$ is simple, we say that $\mathcal{B}$ is a *simple extension* of $\mathcal{A}$. Our aim is therefore to show that in a class of $\mathcal{L}$-structures $\mathcal{C}$, for every $\mathcal{A} \in \mathcal{C}$ there exists a simple extension $\mathcal{B} \in \mathcal{C}$ of $\mathcal{A}$, and, more importantly, to bound $|B \setminus A|$ as a function of $|A|$.

To establish embeddings into simple $\mathcal{L}$-structures we typically need to look at the intervals of $\mathcal{A}$. This is easier to establish by first defining the reverse process. Given an $\mathcal{L}$-structure $\mathcal{S}$, we say that $\mathcal{A}$ is an *inflation* of $\mathcal{S}$ by the $\mathcal{L}$-structures $\{\mathcal{A}_s : s \in S\}$, denoted $\mathcal{A} = \mathcal{S}[\mathcal{A}_s : s \in S]$, if $\mathcal{A}$ is obtained by replacing each element $s \in S$ with a set of elements $A_s = \mathrm{dom}(\mathcal{A}_s)$ (assuming beforehand, of course, that all of these domains are disjoint) that form an interval in $\mathcal{A}$. More precisely, $A = \mathrm{dom}\,\mathcal{A} = \bigcup_{s \in S} A_s$, and for every $R \in \mathcal{L}$, and every n_R-tuple $(a_1, \ldots, a_{n_R}) \in A$ where $a_i \in A_{s_i}$, we have

$$R^{\mathcal{A}}(a_1, \ldots, a_{n_R}) \Leftrightarrow \begin{cases} R^{\mathcal{A}_s}(a_1, \ldots, a_{n_R}) & \text{if } s_1 = \cdots = s_{n_R} = s \text{ or} \\ R^{\mathcal{S}}(s_1, \ldots, s_{n_R}) & \text{otherwise.} \end{cases}$$

If $\mathcal{A}$ is an inflation of $\mathcal{S}$, we also say that $\mathcal{S}$ is a *quotient* of $\mathcal{A}$. We are particularly interested in the case where $\mathcal{S}$ is simple—Theorem 1.4 below gives the uniqueness of such an $\mathcal{S}$, which will be called the *simple quotient* of $\mathcal{A}$. This decomposition is the *substitution* (or *modular*) *decomposition*.

The notion of modular decomposition dates back at least to a 1953 talk of Fraïssé. However, only the abstract of this talk [**11**] survives. The first article using modular decompositions seems to be by Gallai [**12**] (for an English translation, see [**13**]), who applied them to the study of transitive orientations of graphs. Since then it has reappeared in settings ranging from game theory to combinatorial optimization—see Möhring [**18**] or Möhring and Radermacher [**20**] for extensive references. Before stating and proving the general uniqueness theorem, we quote two basic facts about intervals, which follow from elementary considerations. The first of these facts is proved explicitly by Földes [**10**].

PROPOSITION 1.3. *For any two intervals I and J of the $\mathcal{L}$-structure $\mathcal{A}$:*

(a) *if $I \cap J \neq \emptyset$ then $I \cap J$ and $I \cup J$ are intervals of $\mathcal{A}$; and*
(b) *$I \setminus J$ is an interval of $\mathcal{A}$.*

THEOREM 1.4. *Let $\mathcal{A}$ be an $\mathcal{L}$-structure for some language $\mathcal{L}$. There exists a unique simple $\mathcal{L}$-structure $\mathcal{S}$ such that $\mathcal{A} = \mathcal{S}[\mathcal{A}_s : s \in S]$. Moreover, when $|S| > 2$, the structures $\mathcal{A}_s$ are also uniquely determined.*

Proof. Let M denote the set of all intervals, except $A = \mathrm{dom}(\mathcal{A})$, which are not contained in another proper interval.

If two intervals $I, J \in M$ intersect, then Proposition 1.3(a) shows that $I \cup J$ is also an interval, which, unless $I \cup J = A$, contradicts the definition of M.

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If $I \cup J = A$, then Proposition 1.3(b) shows that $J \backslash I$ is an interval, so $\mathcal{A}$ can be written as the inflation $\mathcal{S}[\mathcal{A}_1, \mathcal{A}_2]$ of a two-element structure $\mathcal{S}$ (with $S = \{1, 2\}$). Note that every two-element structure is obviously simple.

For uniqueness in this case, suppose that $\mathcal{A} = \mathcal{T}[\mathcal{A}'_t : t \in T]$ is another modular decomposition of $\mathcal{A}$. Let $t_1, t_2 \in T$ be such that $A'_{t_1} \cap A_1 \neq \emptyset$, $A'_{t_2} \cap A_2 \neq \emptyset$. Suppose first that $|T| > 2$, and let $t \in T \backslash \{t_1, t_2\}$. Clearly, at least one of the sets $A'_t \cap A_1$ or $A'_t \cap A_2$ is non-empty; without loss of generality we assume that $A'_t \cap A_1 \neq \emptyset$. Let $U = \{u \in T : A'_u \cap A_1 \neq \emptyset\} \backslash \{t_2\}$. Then it is easy to see that U is an interval in $\mathcal{T}$. But $t, t_1 \in U$, contradicting the simplicity of $\mathcal{T}$. Thus $T = \{t_1, t_2\}$, and it is easy to check that $\mathcal{T} \cong \mathcal{S}$ under the isomorphism $1 \mapsto t_1$, $2 \mapsto t_2$.

Now consider the case where no two intervals in M intersect. Clearly, now the sets in M partition A. For each $I \in M$ choose a representative $x_I \in I$, and define the $\mathcal{L}$-structure $\mathcal{S}$ to be the restriction of the structure $\mathcal{A}$ to the set $S = \mathrm{dom}(\mathcal{S}) = \{x_I : I \in M\}$. Clearly $\mathcal{A}$ is the inflation of $\mathcal{S}$ by the corresponding intervals of $\mathcal{A}$. The simplicity of $\mathcal{S}$ follows from the observation that if $\mathcal{S}$ contained a proper interval K, then $\bigcup\{I : x_I \in K\}$ would be a proper interval of $\mathcal{A}$ contradicting the definition of M.

To show uniqueness of $\mathcal{S}$ and all $\mathcal{A}_s$ in this case, suppose that we have another modular decomposition $\mathcal{A} = \mathcal{T}[\mathcal{A}'_t : t \in T]$. Since the intervals in M are maximal and disjoint, each A'_t is contained in some member of M. Suppose $I \in M$ contains several such intervals, $I = A'_{t_1} \cup \cdots \cup A'_{t_j}$. It is then easy to verify that $\{t_1, \ldots, t_j\}$ is an interval in $\mathcal{T}$, which is a contradiction. □

The two cases in the above proof will be referred to as the *degenerate* and *non-degenerate* substitution decompositions. The degenerate case can give rise to non-uniqueness of this decomposition, and this may be dealt with in several ways, either by specifying the properties that one of the two substructures must possess (for example, the substructure must not itself be decomposable in the same way), or by decomposing the structure into as many pieces as possible (for example, dividing a disconnected graph into its connected components).

We use the former treatment here, but note that the latter is often better suited for certain problems, particularly in relation to the *substitution decomposition tree* of a structure $\mathcal{A}$: each node of this tree corresponds to a substructure of $\mathcal{A}$ whose ground set is an interval, with the root of the tree being $\mathcal{A}$ and the leaves being the singleton ground sets. For a given node with corresponding non-singleton structure $\mathcal{A}'$, the children of $\mathcal{A}'$ are the substructures $\mathcal{A}'_s$ in the decomposition $\mathcal{A}' = \mathcal{S}'[\mathcal{A}'_s : s \in \mathrm{dom}(\mathcal{S})]$. The complexity of computing these trees for particular relational structures has received considerable attention in recent years, particularly in the case of graphs because of its applications in optimization. See, for example, [**1**, **4**, **6**, **15**, **16**].

The proofs of our main theorems below all follow the same basic pattern: we take an arbitrary structure $\mathcal{A}$ of the type under consideration, identify its substitution decomposition, and then employ an inductive argument to show how adding extra points can be used to break the intervals of $\mathcal{A}$. The degenerate case usually needs to be considered separately, and it is precisely this case that normally gives our upper bounds.

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§2. *Tournaments.* Recall that a *tournament* is an oriented complete graph. An interval in a tournament (often called a *convex set* in the literature) is a set I of vertices such that for every pair $x, y \in I$ and any $z \in V(T)\backslash I$, either $x \to z$ and $y \to z$ or $z \to x$ and $z \to y$.

The name "tournament" derives from its use to describe a competition where every two players x, y must meet each other in a match, the outcome being either that x wins, $y \to x$, or x loses, $x \to y$. This viewpoint then defines an algebra with two idempotent binary operations $A_T = \langle T, \vee, \wedge \rangle$, so that if $x \to y$, then $x \vee y = y \vee x = x$ and $x \wedge y = y \wedge x = y$. An extensive survey of tournaments from this algebraic viewpoint has been written by Crvenković *et al* [5]. Of interest for us here is that a tournament is simple if and only if its corresponding abstract algebra is also simple, meaning that it has no proper homomorphic images. Simple extensions of tournaments were studied in a string of papers in the early 1970s, and in particular it is known that at most two additional vertices are needed to extend any tournament into a simple tournament.

THEOREM 2.1 (Erdős *et al* [8]). *Every tournament on n vertices has a simple extension with at most two additional vertices.*

A *chain* in this context is a transitive tournament.

PROPOSITION 2.2 (Erdős *et al* [9]). *A tournament T has a one-point simple extension if and only if T is not a finite odd chain with five or more vertices.*

Note that these results hold for tournaments of arbitrary cardinality, though they were originally proved for finite tournaments by Moon [21]. We give here another proof of the finite case using the substitution decomposition, in order to introduce the methods used in the following sections, and to be able to compare those later results with the tournament case.

Proof. We proceed by induction on $|T| = n \geq 2$, by constructing a one-point extension $T^x = T \cup \{x\}$ of T. We shall show that T^x is simple unless $n = 3$ or T is a finite odd chain, in which case we shall prove that all proper intervals of T^x must contain x. From this construction, in the exceptional cases it will be possible to build a two-point simple extension $T^{x\bar{x}}$ by adding another vertex $\bar{x}$ satisfying $x \to \bar{x}$, and $\bar{x} \to v$ if and only if $v \to x$ for all $v \in V(T)$.

The base case is $n = 2$, for which we may assume $T = \{u, v\}$ with $u \to v$. We define x by $v \to x \to u$ and observe that T^x is simple. For the exceptional case $n = 3$, note that there are no simple tournaments on four vertices, so a simple extension requires at least two additional points. For the induction, however, we must create T^x so that every proper interval contains x, and this is done in Figure 1. Note also that the two-vertex extension $T^{x\bar{x}}$ is simple in both cases.

Suppose now that $n \geq 4$. We take the substitution decomposition $T = S[A_s : s \in S]$, where the simple quotient S of T has $m \geq 2$ vertices. First, we shall consider the case where $m > 2$, in which case the tournaments A_s are defined uniquely. If every A_s contains a single vertex then $T = S$ is simple. Since we have assumed $n \geq 4$, the addition of x will preserve simplicity provided it does not have the same neighbourhoods as any existing vertex of T and it does not satisfy $x \to T$ or $T \to x$. (Note that there are $2^n - n - 2$ ways of doing this.)

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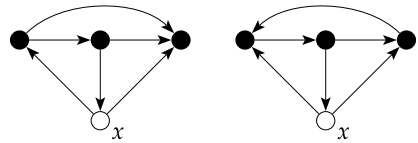


Figure 1: One-vertex (non-simple) extensions of tournaments on three vertices.

Thus we may assume that at least one interval A_s contains more than one vertex. In this case, we add a vertex x to T so that for each non-singleton interval A_s, the subtournament $A_s \cup \{x\}$ is by induction equal to A_s^x. Now fix any non-singleton interval A_{s^*}, and for each singleton interval A_s form A_s^x to satisfy $x \to A_s$ if and only if $A_s \to A_{s^*}$. We claim in this case that T^x is simple. Note first that by the inductive construction we cannot have $x \to T$ or $T \to x$ in T^x, as otherwise the extension A_s^x of any non-singleton block A_s would contradict the inductive hypothesis. Now suppose I is a non-singleton interval of T^x, and let $u, v \in I$ be two distinct vertices. The possible positions of u and v in T^x give rise to three cases.

Case 1: $u \in A_s$, $v \in A_{s'}$ *for some* $s, s' \in S$ *with* $s \neq s'$. Then by the simplicity of the quotient S we have $V(T) \subseteq I$, and x must be included since we cannot have $x \to T$ or $T \to x$.

Case 2: $u \in A_s$, $v = x$, *for some* $s \in S$. If there exists some other block $A_{s'}$ of T with $|A_{s'}| \geq 2$, then we have either $u \to A_{s'}$ or $A_{s'} \to u$, but not $v \to A_{s'}$ or $A_{s'} \to v$, reducing to Case 1. If every other $A_{s'}$ is a singleton, then A_s is the only non-singleton block, so $s = s^*$, and, by the construction, u and v will disagree on every singleton block, again reducing to Case 1.

Case 3: $u, v \in A_s$ *for some* $s \in S$. By induction, the only possible proper intervals of A_s^x are those that involve x, reducing immediately to Case 2.

This leaves the case where $T = S[A_1, A_2]$ with $|S| = 2$, $A_1 \to A_2$, and A_1 and A_2 not necessarily uniquely defined. Suppose first that T is not a finite odd chain, and note in this case that we can always choose A_1 and A_2 so that neither is an odd chain of length greater than one. We shall also try to choose A_1 and A_2 so that they are not singletons, and when this is possible we construct T^x so that $A_1 \cup \{x\} = A_1^x$ and $A_2 \cup \{x\} = A_2^x$. By induction, A_1^x and A_2^x are simple unless one or both contains only three vertices. In every case, however, it is easy to check that T^x is simple. However, if we are forced to take $|A_2| = 1$, then there is no $a \in A_1$ for which $A_1 \backslash \{a\} \to a$, and so we create T^x by setting $A_1 \cup \{x\} = A_1^x$, and $A_2 \to x$. Note that A_1^x is simple unless $|A_1| = 3$, and in either case it is now easy to check that T^x is simple. A similar argument applies when $|A_1| = 1$.

Finally, if T is a finite odd chain then we pick A_1 to be a finite even chain and A_2 a finite odd chain of length greater than one. Given a one-point extension T^x of T, we define another one-point extension $T^{\bar{x}} = T \cup \{\bar{x}\}$ by $\bar{x} \to v$ if and only if $v \to x$ in T^x. (Note that if every proper interval of T^x contains x, then every proper interval of $T^{\bar{x}}$ must contain $\bar{x}$.) Denote by a the vertex of A_1 for which $A_1 \backslash \{a\} \to a$, and by b the vertex of A_2 for which $b \to A_2 \backslash \{b\}$. In order

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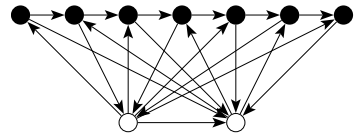


Figure 2: A two-vertex simple extension of a transitive tournament on seven vertices.

to form the one-point extension T^x required by the induction, we adjoin a vertex x to T so that $A_1 \cup \{x\} = A_1^x$ is a simple one-point extension. To adjoin x to A_2, we force $x \to b$ by taking either the edges given by A_2^x or the edges between $\bar{x}$ and A_2 from $A_2^{\bar{x}}$. The resulting extension T^x is not simple, but we note that any proper interval must contain the vertex x. The only other candidate we need to check is $\{a, b\}$, which, since we fixed $x \to b$, cannot be an interval unless $x \to a$, which would contradict the simplicity of A_1^x. Since $T^{\bar{x}}$ must satisfy the same property, it follows that the two-point extension $T^{x\bar{x}}$ is simple. □

It is initially surprising that the case where T is a finite odd chain with five or more vertices requires two additional vertices rather than one. Figure 2 shows a two-point simple extension of a chain on seven vertices.

§3. *Graphs.* In a graph G, an interval I is a set of vertices for which $N(u)\backslash I = N(v)\backslash I$ for every $u, v \in I$. A graph is simple if it contains no proper intervals, but it is worth noting that the word "simple" is commonly used to describe graphs with no multiple edges or loops, and our notion is more often referred to as *prime* or *indecomposable*. These graphs have been the subject of considerable study; see for example work by Ehrenfeucht *et al* [7], Ille [14], and Sabidussi [22]. A survey of indecomposability and the substitution decomposition in graphs can be found in Brandstädt *et al*'s text [2]. Simple extensions of graphs have received some attention in the past. In particular, we have the following lemma.

LEMMA 3.1 (Sumner [23, Theorem 2.45]). *The complete graph K_n has a simple extension with $\lceil \log_2(n+1) \rceil$ additional vertices.*

The bound $m = \lceil \log_2(n+1) \rceil$ is also the smallest possible, for were we to add a set B of m vertices with $n > 2^m - 1$, then either two vertices in G would have the same neighbourhood in $G \cup B$, or one vertex of G would be connected to every other vertex in $G \cup B$, both of which give an interval.

THEOREM 3.2. *Every graph G has a simple extension with at most $m = \lceil \log_2(|V(G)| + 1) \rceil$ additional vertices.*

Proof. First note that when $|G| = 1$, we may trivially extend by one vertex to form K_2. We shall prove, by induction on $|G| = n \geq 2$, the following statement: G has a simple extension with a set B of m independent vertices†, where $2 \leq m \leq \lceil \log_2(n+1) \rceil$.

† The assumption that B is independent may be replaced by the requirement that B be any graph specified in advance, but this adds considerably to the complexity and length of the proof.

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Figure 3: Two-point simple extensions in the cases when $|G| = 2$. In each case the filled nodes correspond to G, and the empty nodes to B.

The base case $n = 2$ may be seen in Figure 3, so now suppose $n \geq 3$. Write $G = H[J_v : v \in V(H)]$ where H is the simple quotient of G, and suppose first that $|H| > 2$. If $H = G$ then the graph is already simple, but for the induction we need to add two vertices. As in the tournament case, we may preserve simplicity by adding vertices whose neighbourhoods in G are distinct and neither the same as any vertex in G, empty, nor all of $V(G)$. As in the case of tournaments, there are $2^n - n - 2$ possible neighbourhoods, and for $n \geq 4$ we may pick any two of these for the new vertices.

Thus we may now suppose that $G \neq H$. By induction, each J_v of size $|J_v| \geq 2$ can be extended by some independent set B_v of size at most $\lceil \log_2(|V(J_v)| + 1) \rceil$ so that $J_v \cup B_v$ is simple. Picking $x \in V(H)$ for which $B_x = B$ is largest, for every $v \neq x$ in H with $|J_v| \geq 2$ we identify B_v with any subset of B of the correct size. For any $v \in H$ for which $|J_v| = 1$, we connect J_v to any vertex in B that is not adjacent to all of J_x, and label this singleton set B_v.

CLAIM. *The extension $G \cup B$ is simple.*

Proof. Consider an interval I with at least two vertices a and b. There are four cases.

Case 1: $a \in J_u$, $b \in J_v$ *for* $u \neq v$ *in* H. The simplicity of H implies that $G \subseteq I$. In particular this gives $J_x \subseteq I$, which implies $B \subseteq I$ since $J_x \cup B$ is simple.

Case 2: $a, b \in J_v \cup B_v$, $v \neq x$. By the simplicity of $J_v \cup B_v$ we have $J_v \cup B_v \subseteq I$. If $|J_v| \geq 2$ then by induction $|B_v| \geq 2$, implying that $J_x \subseteq I$, and reducing to Case 1. If, say, $J_v = \{a\}$ then a is adjacent either to everything in J_x or to nothing in J_x, while by the construction $b \in B_v$ is adjacent to some but not all of J_x, reducing to Case 1.

Case 3: $a, b \in J_x \cup B$. We immediately have $J_x \cup B \subseteq I$, and so if there is some $v \neq x$ for which $|J_v| \geq 2$ then $J_v \subseteq I$ gives Case 1, so we now assume $|J_v| = 1$ for all $v \neq x$ in H. Picking any such $J_v = \{c\}$ we know there exist $b_1, b_2 \in B$ with $b_1 \sim c$ but $b_2 \not\sim c$, implying $c \in I$, which again reduces to Case 1.

Case 4: $a \in J_v$, $b \in B \backslash B_v$. Since $J_v \cup B_v$ is simple, we know that a is adjacent to some $c \in J_v \cup B_v$, while b is not, reducing to Case 2. □

For the degenerate cases, let us first assume that $H = \overline{K_2}$, i.e., that G is disconnected, so $G = H[J_1, J_2]$ where J_1 and J_2 may possibly be picked in a number of ways. We arrange it so that J_1 is a largest connected component, and suppose first that $|J_1| \geq 2$. By induction, each of J_1 and J_2 may be extended by adding a set of vertices B_1 and B_2, respectively, unless $|J_2| = 1$, where

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we pick any vertex of B to act as the singleton set B_2. Fix $x \in \{1, 2\}$ so that $B = B_x$ is the larger of B_1 and B_2, and associate the other set $B_{x'}$, $x' \in \{1, 2\}$, $x' \neq x$, with any subset of B. If $|J_1| = 1$ then G is in fact $\overline{K_n}$, and but for the induction we could appeal to Lemma 3.1. By induction, there exists a set B_2 of $\lceil \log_2 n \rceil$ vertices for which $J_2 \cup B_2$ is a simple extension of J_2. Unless $\lceil \log_2(n+1) \rceil = \lceil \log_2 n \rceil + 1$, we may assign a non-empty neighbourhood of B_2 to the single vertex of J_1 that is different from the neighbourhood of every vertex in J_2. In the exceptional case where $\lceil \log_2(n+1) \rceil = \lceil \log_2 n \rceil + 1$, we assign any non-empty neighbourhood of B_2 to J_1, observing that there exists a vertex $v \in J_2$ with the same neighbourhood. However, we are now permitted to add another new vertex b, attached to J_1 and nothing else, so that v no longer has the same neighbourhood as the single vertex in J_1.

In the case that $H = K_2$, only a few modifications need to be made to the above. We first pick J_1 so that $\overline{J_1}$ is the largest possible connected component. When J_1 is not a singleton but J_2 is, we select $B_2 = \{b\}$ by choosing any $b \in B$ for which b is not connected to all of J_1 (as in the non-degenerate case). When J_1 is a singleton (so G is a complete graph), it can have any non-complete neighbourhood of $B = B_2$ (including the empty neighbourhood), provided no vertex of J_2 already has that neighbourhood. If no such neighbourhood exists, then we may add a new vertex to B and connect it to J_1. The analysis of these degenerate cases is similar to and easier than the analysis in the non-degenerate case above. □

Although this bound is tight for complete and independent graphs, the inductive construction used in the proof actually does somewhat better in many cases. Unless the graph is complete or empty, at each stage of the induction we add only as many vertices as are required by any single block J_v of the substitution decomposition. This may be thought of iteratively in terms of the substitution decomposition tree: recalling that the degenerate cases are handled in this tree by breaking up the graph into as many parts as possible, we see that the largest such degenerate node all of whose children are leaves will dictate how many additional vertices are needed. If no such degenerate node exists, then the inductive construction above requires only two additional vertices.

§4. *Permutations.* In a permutation π of $[n]$, an interval is a collection of entries that are contiguous by both position and value. If a permutation is viewed as a set with two linear orders $<$ and $\prec$, an interval is a set of points that is contiguous under both $<$ and $\prec$. Intervals are easily identified in the plot of a permutation as sets of points enclosed in an axis-parallel rectangle, with no points lying in the regions above, below, to the left or to the right (see Figure 4). To embed a given π in a simple permutation, therefore, we must ensure that every axis-parallel rectangle containing at least two points of π is "broken" by at least one of the extension points.

THEOREM 4.1. *Every permutation π on n symbols has a simple extension with at most $\lceil (n+1)/2 \rceil$ additional points.*

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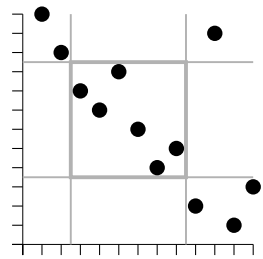


Figure 4: An interval in the plot of a permutation.

Proof. We proceed by induction on $n \geq 2$, claiming that for each permutation π of length n we may construct two extensions $\pi^{\uparrow}$ and $\pi^{\downarrow}$ both containing at most $\lceil (n+1)/2 \rceil$ new points, satisfying the following.

- Viewed as an extension, each π^a ($a \in \{\uparrow, \downarrow\}$) has a new leftmost point that is neither a new maximum nor a new minimum. This is called the *entry point* and denoted $\mathrm{In}(\pi^a)$.
- Each π^a ($a \in \{\uparrow, \downarrow\}$) has a new *exit point* $\mathrm{Out}(\pi^a)$. For $\pi^{\uparrow}$ this is a new maximum, while for $\pi^{\downarrow}$ it is a new minimum, and in both cases it is neither a leftmost nor a rightmost point.
- Any proper interval of π^a contains the exit point $\mathrm{Out}(\pi^a)$, but does not contain the rightmost point of π.
- At least one of $\pi^{\uparrow}$ and $\pi^{\downarrow}$ is simple.

Let us call the permutation obtained from π^a by removing the entry and exit points the *core* of π^a and denote it by $\mathrm{Core}(\pi^a)$. Note that $\mathrm{Core}(\pi^a)$ still contains the original copy of π.

In the base case $n = 2$, either $\pi = 12$ or $\pi = 21$. When $\pi = 12$, $\pi^{\uparrow} = 2413$ is simple, and the only non-singleton intervals of $\pi^{\downarrow} = 3124$ are 12 and 312, both of which contain the exit point. The case $\pi = 21$ is dealt with by symmetry.

Now suppose $n \geq 3$. Write $\pi = \sigma[\pi_1, \pi_2, \ldots, \pi_m]$ where the simple quotient σ of π is of length $m \geq 2$, and $\pi_1, \pi_2, \ldots, \pi_m$ are permutations. First suppose $m > 2$ so that the substitution decomposition is unique. If $\pi_i = 1$ for all i, then $\pi = \sigma$ is already simple. We construct $\pi^{\uparrow}$ and $\pi^{\downarrow}$ by adding precisely two points. The first is a new leftmost point, which may be positioned vertically anywhere, except as a new maximum or minimum, or adjacent to the leftmost entry of π. The new maximum or minimum is inserted similarly, and it is easily checked that the resulting permutation is simple.

Now suppose that at least one π_i contains at least two points. The permutations $\pi^{\uparrow}$ and $\pi^{\downarrow}$ will be constructed by extending each non-trivial block π_i of π to one of $\pi_i^{\uparrow}$ or $\pi_i^{\downarrow}$ and then "repositioning" the entry and exit points so that they "link" successive blocks. The choice between $\pi_i^{\uparrow}$ and $\pi_i^{\downarrow}$ is made so as to enable this linking. More formally, to define π^a ($a \in \{\uparrow, \downarrow\}$), first write $\sigma = s_1 \cdots s_m$, and suppose that the non-trivial blocks of π are, from left to

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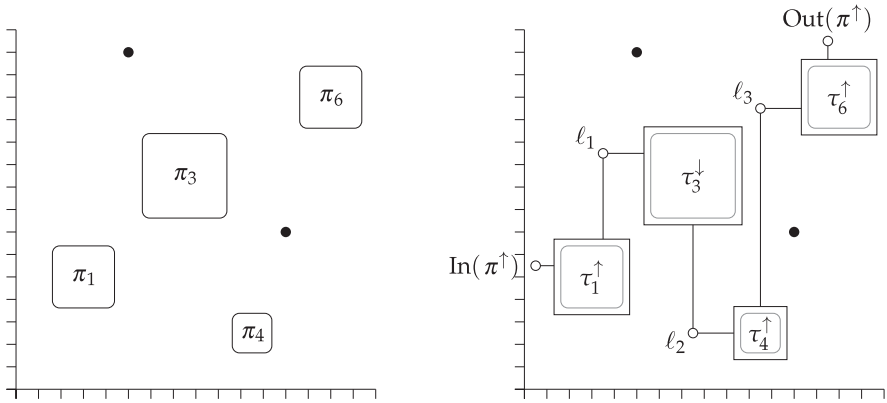


Figure 5: On the left, the substitution decomposition of $\pi = 264135[\pi_1, 1, \pi_3, \pi_4, 1, \pi_6]$. On the right, the extension $\pi^{\uparrow}$ formed from the blocks τ_i^a connected by linking points ℓ_1, ℓ_2 and ℓ_3.

right, $\pi_{i_1}, \ldots, \pi_{i_t}$. Next, for each $i = 1, \ldots, m$ define

$$\tau_i = \begin{cases} \mathrm{Core}(\pi_i^{\uparrow}) & \text{if } i = i_r,\ 1 \le r \le t-1 \text{ and } s_{i_r} < s_{i_{r+1}}, \\ \mathrm{Core}(\pi_i^{\downarrow}) & \text{if } i = i_r,\ 1 \le r \le t-1 \text{ and } s_{i_r} > s_{i_{r+1}}, \\ \mathrm{Core}(\pi_i^{a}) & \text{if } i = i_t, \\ 1 & \text{otherwise,} \end{cases}$$

and form the permutation $\tau = \sigma[\tau_1, \ldots, \tau_m]$. For each $r = 1, \ldots, t-1$ insert a linking point ℓ_r, the position of which is uniquely determined by the requirement that ℓ_r is the exit point for τ_{i_r} and the entry point for $\tau_{i_{r+1}}$. Finally, complete the construction of π^a by defining $\ell_0 = \mathrm{In}(\pi^a)$ to be the leftmost point, which at the same time serves as the entry point to τ_{i_1}, while $\ell_t = \mathrm{Out}(\pi^a)$ is a new minimum or a new maximum (depending on a) that is at the same time the exit point for τ_{i_t}. This construction is illustrated in Figure 5.

CLAIM. *If $\pi^{\uparrow}$ is not simple then the rightmost non-singleton block π_{i_t} is the maximal block by value, and every non-trivial proper interval of $\pi^{\uparrow}$ contains* $\mathrm{Out}(\pi^{\uparrow})$.

Proof. Let I be a non-singleton interval of $\pi^{\uparrow}$, and let $u, v \in I$ be two distinct points. We consider several cases, depending on the position of u and v in $\pi^{\uparrow}$.

Case 1: $u \in \tau_j$, $v \in \tau_k$, $j \neq k$. The simplicity of σ implies that I contains all τ_i $(1 \le i \le m)$. Each of the remaining points $\ell_0, \ell_1, \ldots, \ell_t$ overlaps at least one τ_i, and so these points must belong to I too. We conclude that $I = \pi^{\uparrow}$.

Case 2: $u, v \in \tau_{i_r}$ *for some* $r \in \{1, \ldots, t-1\}$. The set $I \cap (\tau_{i_r} \cup \ell_{r-1} \cup \ell_r)$ is an interval of the permutation $\pi_{i_r}^b = \tau_{i_r} \cup \ell_{r-1} \cup \ell_r$ $(b \in \{\uparrow, \downarrow\})$, and so by induction it contains the exit point ℓ_r. But ℓ_r overlaps horizontally with $\tau_{i_{r+1}}$, and so I must contain a point from $\tau_{i_{r+1}}$. Thus we have reduced this case to Case 1.

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Case 3: $u, v \in \tau_{i_t}$. As in Case 2 we see that I must contain the corresponding exit point, which this time is $\mathrm{Out}(\pi^{\uparrow})$. Now, if π_{i_t} is the maximal block by value there is nothing further to prove, while if π_{i_t} is not maximal we see that I must contain a point from some other block, and that reduces this case again to Case 1.

Case 4: $u \in \{\ell_0, \ell_1, \ldots, \ell_t\}$, $v \in \tau_j$. From the construction it follows that u and v are separated either horizontally or vertically by at least one other point belonging to some τ_k. Thus this case reduces to one of the first three cases.

Case 5: $u, v \in \{\ell_0, \ell_1, \ldots, \ell_t\}$. Again, we can see that I must contain at least one element from some τ_j, reducing this to Case 4. □

Returning to the proof of the theorem, we note that the analogous assertion holds for $\pi^{\downarrow}$: if $\pi^{\downarrow}$ is not simple then the rightmost non-singleton block π_{i_t} is the minimal block by value, and every non-trivial proper interval of $\pi^{\downarrow}$ contains $\mathrm{Out}(\pi^{\downarrow})$. Since σ is simple and of length at least four, neither the maximal nor the minimal block by value can be the rightmost block of π, and so the rightmost point of π cannot appear in any proper interval of π^a. Moreover, as π_{i_t} cannot be at the same time maximal and minimal, we conclude that at least one of $\pi^{\uparrow}$ or $\pi^{\downarrow}$ is simple.

To complete the proof in the non-degenerate case, let us estimate the number of points we have added to π. To each non-trivial block π_{i_r} $(1 \le r \le t)$ we have added at most $\lceil (|\pi_{i_r}| + 1)/2 \rceil$ points. Moreover, we merged $t - 1$ pairs of exit/entry points. Thus the total number of additional points does not exceed

$$\sum_{r=1}^{t} \left\lceil \frac{|\pi_{i_r}| + 1}{2} \right\rceil - (t-1) \le \sum_{r=1}^{t} \frac{|\pi_{i_r}| + 2}{2} - (t-1)$$
$$= 1 + \sum_{r=1}^{t} \frac{|\pi_{i_r}|}{2} \le 1 + \frac{n}{2}.$$

Since the number of points is a whole number, we conclude that it does not exceed $\lceil (n+1)/2 \rceil$, as required.

In the case where $m = 2$ we may assume without loss that $\sigma = 12$, and write $\pi = 12[\pi_1, \pi_2]$, where π_1 and π_2 may be chosen in a number of different ways. Fix π_1 so that if possible it does not end with its largest element. If this is possible, and if π_2 is not a singleton, we may construct π^a $(a \in \{\uparrow, \downarrow\})$ by extending π_1 to $\pi_1^{\uparrow}$ and π_2 to π_2^a, and identifying the exit point of the first to the entry point of the second. If π_2 is a singleton, then the exit point of $\pi_1^{\uparrow}$ may be placed above π_2 to form $\pi^{\uparrow}$ without producing a proper interval between the top of π_1 and π_2: the intersection of any such interval with $\pi_1^{\uparrow}$ would have to form a proper interval in $\pi_1^{\uparrow}$ containing the rightmost point of π_1, which is impossible by the inductive hypothesis. We form $\pi^{\downarrow}$ simply by taking $12[\pi_1^{\downarrow}, \pi_2]$, with the exit point of π_1 acting as the exit point of π. Note that $\pi^{\downarrow}$ is not simple since $\pi_1^{\downarrow}$ is an interval, but it contains $\mathrm{Out}(\pi^{\downarrow})$ and not the single point of π_2, and so satisfies the inductive hypothesis.

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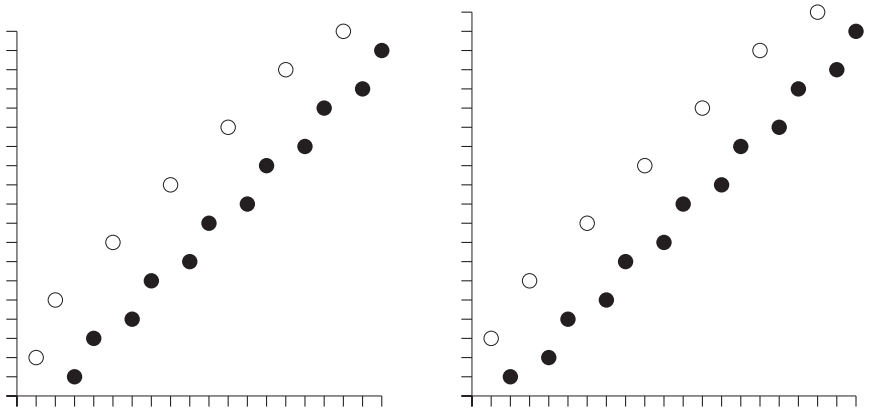


Figure 6: Simple extensions of $12\cdots n$, for $n = 12$ (left) and $n = 13$ (right).

If it is not possible to pick π_1 so that it does not end with its largest element, then we first attempt to pick π_2 so that it does not start with its smallest element. If this is possible, and if π_1 is not a singleton, an analogous argument to that given above applies. On the other hand, if π_1 is a singleton, place all additional points of π_2^a that lie to the left of every point in π_2 to the left of the singleton π_1. (This happens, for example, if $\pi = 132$, and we form $\pi^{\uparrow} = 35142$ and $\pi^{\downarrow} = 41253$.) If it is not possible to pick π_2 so that it does not start with its smallest element, then $\pi = 12\cdots n$. If $n > 3$, we set $\pi_1 = 12\cdots n-2$ so that π_2 forms the pattern 12, whence there is a two-point extension π_2^a, $a \in \{\uparrow, \downarrow\}$, in which both the entry and exit points occur to the left of $\mathrm{Core}(\pi_2)$. This permits us to connect the exit point of the extension $\pi_1^{\uparrow}$ to the entry point of π_2^a, and the exit point of π_2^a both acts as the exit point of π^a and ensures that there are no intervals between π_1 and π_2. Finally, when $\pi = 123$, we set $\pi^{\uparrow} = 31524$ and $\pi^{\downarrow} = 42135$. In each case, checking that the so-constructed extensions $\pi^{\uparrow}$ and $\pi^{\downarrow}$ satisfy the required conditions is similar to (and easier than) the non-degenerate case. □

See Figure 6 for examples of extensions of monotone permutations. Note that in this case the bound is tight: every adjacent pair $i, i+1$ must be "separated" either horizontally or vertically by one of the additional points, and the points $\pi(1) = 1$ and $\pi(n) = n$ of π must not lie in the "corners" of the simple extension—this gives a total of $n+1$ gaps to be filled, and each additional point can be used to fill at most two of these gaps (one horizontally and one vertically).

§5. *Posets.* Although posets are very naturally described as relational structures, we have delayed considering their simple extensions until now, since with posets we encounter a "mixture" of the results related to the degenerate decompositions corresponding to permutations and graphs. For permutations recall that the degenerate cases correspond to the increasing and decreasing permutations, which (viewing them as relational structures) occur when the two linear orders are equal or are the opposites of each other. For graphs, the

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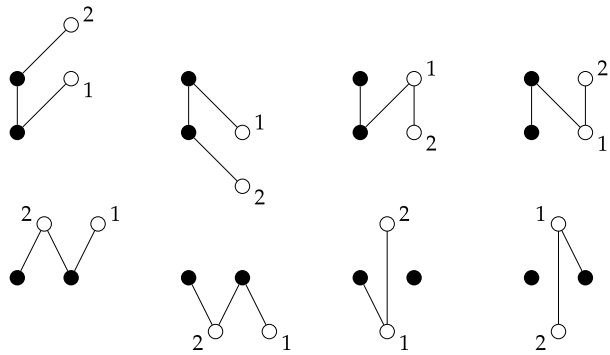


Figure 7: The eight two-element extensions when $|P| = 2$. The points labelled 1 and 2 correspond to the distinguished elements $\mathrm{Ext}_1(P^a)$ and $\mathrm{Ext}_2(P^a)$, respectively.

non-uniqueness comes in the form of complete and independent graphs. Posets can be decomposed non-uniquely either through total comparability (linear orders) or total incomparability (antichains), and the simple extension in each case is significantly different.

We begin with a result that holds for all posets; afterwards we shall discuss special cases where fewer additional elements are required. Our approach takes much the same form as the permutation case, inductively identifying entry and exit points from the simple extensions of the intervals in the substitution decomposition, and this gives rise to the same bound. Because of this similarity, we shall omit some of the case-by-case details in the proof.

THEOREM 5.1. *A poset $(P, <)$ on n elements has a simple extension with at most $\lceil (n+1)/2 \rceil$ additional elements.*

Proof. Extending a poset of size one is trivial, so we proceed by induction on n assuming $n \geq 2$. Our claim is that we may form four extensions $P^{\uparrow}$, $P^{\downarrow}$, $P^{\uparrow\downarrow}$ and $P^{\downarrow\uparrow}$ of a poset $(P, <)$, satisfying the following.

- Each extension P^a $(a \in \{\uparrow, \downarrow, \uparrow\downarrow, \downarrow\uparrow\})$ has at most $\lceil (n+1)/2 \rceil$ new elements.
- Among these new points lie two *distinguished elements*, $\mathrm{Ext}_1(P^a)$ and $\mathrm{Ext}_2(P^a)$. For $P^{\downarrow}$ these are both new minima, for $P^{\uparrow}$ they are new maxima, for $P^{\uparrow\downarrow}$ the point $\mathrm{Ext}_1(P^{\uparrow\downarrow})$ is a maximum and $\mathrm{Ext}_2(P^{\downarrow\uparrow})$ is a minimum, while for $P^{\downarrow\uparrow}$ the point $\mathrm{Ext}_1(P^{\downarrow\uparrow})$ is a minimum and $\mathrm{Ext}_2(P^{\downarrow\uparrow})$ is a maximum. Note that $P^{\uparrow\downarrow}$ and $P^{\downarrow\uparrow}$ need not be different extensions—it is only the labelling of the distinguished elements that matters. The remaining points of P^a will be called the *core*, denoted $\mathrm{Core}(P^a)$.
- Every proper interval of P^a $(a \in \{\uparrow, \downarrow, \uparrow\downarrow, \downarrow\uparrow\})$ contains $\mathrm{Ext}_2(P^a)$. Additionally, in $P^{\uparrow}$ and $P^{\uparrow\downarrow}$ there is a minimal element that is not contained in any proper interval, and in $P^{\downarrow}$ and $P^{\downarrow\uparrow}$ there is a maximal element not contained in any proper interval.
- At least one P^a is simple.

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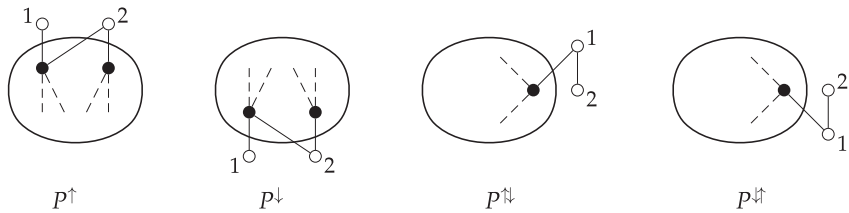


Figure 8: The four two-element simple extensions of an arbitrary simple poset. The points labelled 1 and 2 correspond to the distinguished elements $\mathrm{Ext}_1(P^a)$ and $\mathrm{Ext}_2(P^a)$, respectively.

The base case is $n = 2$, when the poset is either linear or an antichain. The extensions P^a may be seen in Figure 7, and it is straightforward to check that these satisfy the inductive hypothesis. Now suppose that $n > 2$ and decompose the poset as $P = S[A_s : s \in S]$. Assuming first that $|S| > 2$, we proceed in essentially the same way as in the permutation case.

If every A_s is a singleton, then P is already simple, but for the purposes of the induction we must add two elements to form the four extensions $P^{\downarrow}$, $P^{\uparrow}$, $P^{\uparrow\downarrow}$ and $P^{\downarrow\uparrow}$. First observe that P can contain neither a unique maximum nor a unique minimum element. To form $P^{\uparrow}$ we find any two maxima of P, place $\mathrm{Ext}_1(P^{\uparrow})$ above one of them, and place $\mathrm{Ext}_2(P^{\uparrow})$ above both. We form $P^{\downarrow}$ analogously, and $P^{\uparrow\downarrow}$ and $P^{\downarrow\uparrow}$ are formed by adjoining two elements to any chosen element of P; see Figure 8. Note that all four extensions are necessarily simple.

When at least one A_s has more than one element, induction may be used on each such interval to form the extensions $A_s^{a_s}$ ($a_s \in \{\uparrow, \downarrow, \uparrow\downarrow, \downarrow\uparrow\}$), which we utilize to construct P^a. The basic idea is to consider the non-trivial blocks in turn, and for each of them determine which extension $A_s^{a_s}$ to use so that $A_s^{a_s}$ can share one distinguished point with the previous block and one with the subsequent block. The first and the last blocks will contribute one distinguished point each to become the distinguished points of P^a. To be more precise, assume (without loss of generality) that the groundset of S is $\{1, \ldots, |S|\}$, and that the first t of these elements correspond to the non-trivial blocks of P. Define elements $a_s \in \{\uparrow, \downarrow, \uparrow\downarrow, \downarrow\uparrow\}$, $1 \le s \le t$, as follows:

$$a_s = \begin{cases} \begin{cases} \uparrow & \text{if } a \in \{\uparrow, \uparrow\downarrow\} \text{ and } 1 \not>_S 2, \\ \downarrow & \text{if } a = \downarrow \text{ and } 1 >_S 2, \\ \uparrow\downarrow & \text{if } a \in \{\uparrow, \uparrow\downarrow\} \text{ and } 1 >_S 2, \\ \downarrow\uparrow & \text{otherwise} \end{cases} & \text{if } s = 1, \\ \begin{cases} \uparrow & \text{if } s-1 \not<_S s \text{ and } s \not>_S s+1, \\ \downarrow & \text{if } s-1 <_S s \text{ and } s >_S s+1, \\ \uparrow\downarrow & \text{if } s-1 \not<_S s \text{ and } s >_S s-1, \\ \downarrow\uparrow & \text{otherwise} \end{cases} & \text{if } 2 \le s \le t-1, \text{ or} \\ \begin{cases} \uparrow & \text{if } t-1 \not<_S t \text{ and } a = \uparrow, \\ \downarrow & \text{if } t-1 <_S t \text{ and } a \in \{\downarrow, \uparrow\downarrow\}, \\ \uparrow\downarrow & \text{if } t-1 \not<_S t \text{ and } a \in \{\downarrow, \uparrow\downarrow\}, \\ \downarrow\uparrow & \text{otherwise} \end{cases} & \text{if } s = t. \end{cases}$$

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For $1 \leq s \leq t$, we define new, extended blocks $B_s = \mathrm{Core}(A_s^{a_s})$. For the remaining elements of S, $t+1 \leq s \leq |S|$, A_s is the trivial poset on one element, and we set $B_s = A_s$.

Now we form the extension $S[B_s : s \in S]$ by adding $t+1$ linking points ℓ_s $(s = 0, \ldots, t)$, as follows: ℓ_0 is related to the elements of B_1 in exactly the same way as $\mathrm{Ext}_1(A_1^{a_1})$ is related to the elements of $B_1 = \mathrm{Core}(A_1^{a_1})$; ℓ_s $(1 \leq s \leq t-1)$ is related to the elements of B_s in the same way as $\mathrm{Ext}_2(A_s^{a_s})$ is related to the elements of $B_s = \mathrm{Core}(A_s^{a_s})$, and ℓ_s is related to the elements of B_{s+1} in the same way as $\mathrm{Ext}_1(A_{s+1}^{a_{s+1}})$ is related to the elements of $B_{s+1} = \mathrm{Core}(A_{s+1}^{a_{s+1}})$; ℓ_t is related to the elements of B_t in exactly the same way as $\mathrm{Ext}_2(A_t^{a_t})$ is related to the elements of $B_t = \mathrm{Core}(A_t^{a_t})$; these comparisons and their consequences via transitivity are the only comparisons that the elements ℓ_s satisfy. Finally, we stipulate $\mathrm{Ext}_1(P^a) = \ell_0$, $\mathrm{Ext}_2(P^a) = \ell_t$.

To see that one such P^a $(a \in \{\uparrow, \downarrow, \uparrow\downarrow, \downarrow\uparrow\})$ is simple, a case analysis needs to be carried out, but as this is similar to that for permutations, we omit the details. The only intervals that can arise in any such P^a must contain ℓ_t and be contained in $B_t \cup \{\ell_t\}$, and these intervals are permitted by the inductive hypothesis for all but one of the extensions. To see that there is indeed one extension with no such interval, however, note that by the simplicity of S, B_t must be related to some other block B_s of P, and by symmetry suppose $B_s < B_t$. Taking ℓ_t to be a new minimum (i.e., the extension P^a that we are considering satisfies $a \in \{\downarrow, \uparrow\downarrow\}$), we note that any interval I in $B_t \cup \{\ell_t\}$ cannot be an interval in P^a, since there is some $x \in I \cap B_t$ with $x > B_s$, while $\ell_t \not> B_s$.

In the degenerate case we have $P = S[A_1, A_2]$, where S is the two-element chain (so $A_1 < A_2$) or the two-element antichain (so that A_1 and A_2 are incomparable). Supposing first that S is a chain, if possible pick A_1 so that it has no unique maximum. Assuming this is possible and that the resulting A_2 is not a singleton, identify the distinguished points of the extensions $A_1^{a_1}$ and $A_2^{a_2}$ in the same way as in the non-degenerate case, with $\mathrm{Ext}_1(P^a) = \mathrm{Ext}_1(A_1^{a_1})$ and $\mathrm{Ext}_2(P^a) = \mathrm{Ext}_2(A_2^{a_2})$. On the other hand, if the resulting A_2 is a singleton, we form $S[\mathrm{Core}(A_1^a), A_2]$ and then add the distinguished points to A_1^a to serve as the distinguished points of P^a. (Note that by the requirement that $A_1^{\downarrow\uparrow}$ contained a maximal element that is not contained in any proper interval and that $\mathrm{Ext}_2(P^{\downarrow\uparrow})$ was contained in every proper interval, there are no proper intervals in $P^{\downarrow\uparrow}$.)

If we cannot pick A_1 to have no unique maximum but we can instead pick A_2 to have no unique minimum, then we may argue by symmetry. This leaves only the case where P is a chain, which is handled by appealing directly to the permutation case; the chain P corresponds to an increasing permutation $\pi = 12 \cdots n$ under the following mapping: $i < j$ in P if and only if both $i < j$ and $\pi(i) < \pi(j)$. Under this mapping the simple extensions of increasing permutations map to simple extensions of linear orders, and by appealing to the inductive construction of Theorem 4.1 this allows us to form $P^{\downarrow}$ and $P^{\downarrow\uparrow}$, one of which is simple. We form $P^{\uparrow}$ and $P^{\uparrow\downarrow}$ by using an analogous mapping that connects chains to decreasing permutations.

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This leaves the case where S is a two-element antichain. By analogy with the case where S is a two-element chain, first attempt to pick A_1 so that it is not itself decomposable by an antichain and, if possible, so that A_1 is not a singleton. If this can be done leaving A_2 with $|A_2| \geq 2$, then we proceed by using the same construction as in the non-degenerate case, while if A_2 is a singleton then we form $P^{\downarrow\uparrow}$ and $P^{\uparrow\downarrow}$ simply by taking the corresponding extensions of A_1 and leaving the single element of A_2 unrelated to any other elements. We ensure that $P^{\uparrow}$ and $P^{\downarrow}$ are simple extensions of P by adding the relation $\mathrm{Ext}_2(A_1) > A_2$ in $P^{\uparrow}$ and $\mathrm{Ext}_2(A_1) < A_2$ in $P^{\downarrow}$. This leaves only the case where P is an antichain, which is covered by appealing to the connection between posets and permutations. □

As we may expect by its similarity to the permutation case, precisely $\lceil (n+1)/2 \rceil$ additional elements are required in the case when $(P, <)$ is a linear order. Note further that this bound is tight, because of its connection with permutations.

On the other hand, when $(P, <)$ is an antichain, a simple extension does not require this many points, by the following connection with graphs: the *comparability graph* $G(P, <)$ of a poset $(P, <)$ is a graph with vertex set P, and edges $p \sim q$ if and only if either $p < q$ or $q < p$†. It is then easy to appeal to Theorem 3.2 to create simple extensions of antichains that require just $\lceil \log_2(n+1) \rceil$ additional points, and again this bound is tight. In fact, if the simple extension of a graph admits a transitive orientation then this extension could be interpreted as a simple extension of a poset.

§6. *Digraphs and oriented graphs.* An arbitrary digraph corresponds to the most general type of binary relational structure: i.e., the structure $\mathcal{A}$ contains just one binary relation R for which there are no restrictions. Elements of the relation may be viewed as directed edges on a graph, so that two vertices u and v of a digraph may be related in one of four different ways: $u \to v$, $u \leftarrow v$, $u \leftrightarrow v$ or u and v have no edge between them.

We obtain a general bound on the number of additional vertices required by analogy with the graph case: where we had two possible "types" of connection (i.e., an edge or no edge) between an old and a new vertex for graphs, we now have four. The result of this is that a new vertex in a digraph can do the work of two new vertices in a graph, so half as many vertices are required.

THEOREM 6.1. *A digraph D has a simple extension with at most $m = \lceil \log_4(|V(D)| + 1) \rceil$ additional vertices.*

Proof. The *underlying graph* G_D of a digraph D is the graph formed by replacing each directed edge with an undirected one: i.e., $uv \in E(G)$ if and only if $u \to v$ or $v \to u$. Moreover, it is easy to see that a set of vertices that forms an interval in D will also form an interval in G_D. By Theorem 3.2, we can

† The converse operation—forming a poset from a graph—is called a *transitive orientation* of (the edges of) G. This connection between posets and graphs arises in a number of combinatorial problems—see Möhring [**19**] for a survey.

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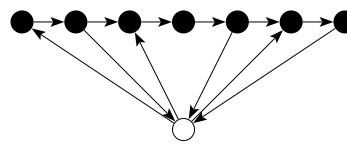


Figure 9: A one-vertex simple extension of a linear digraph on seven vertices.

construct a simple extension G_D^* of G_D by adding a set $B = \{b_1, b_2, \ldots, b_m\}$ of $m \leq \lceil \log_2(|V(G_D)| + 1) \rceil$ vertices. Now, noting that $2\lceil \log_4(|V(D)| + 1) \rceil \geq \lceil \log_2(|V(G_D)|) + 1 \rceil$, we add a set $C = \{c_1, \ldots, c_\ell\}$ of $\ell = \lceil m/2 \rceil$ additional vertices to G_D, satisfying, for every $v \in V(D)$ and $i = 1, 2, \ldots, \ell$,

$$v \to c_i \text{ if and only if } vb_{2i-1} \in E(G_D^*),$$

and

$$c_i \to v \text{ if and only if } vb_{2i} \in E(G_D^*) \text{ and } 2i \leq \ell.$$

Thus every vertex c_i of C has outneighbourhood matching the neighbourhood of b_{2i-1}, and inneighbourhood matching b_{2i}. It is now straightforward to check that this is a simple extension. □

This bound can be seen to be tight for a complete digraph—a digraph for which $u \leftrightarrow v$ for every pair of distinct vertices u and v—and an empty digraph, again by analogy with the graph case.

However, if our digraph forms a linear order (and is hence transitive), our structure forms a transitive tournament. Appealing to Theorem 2.1, we see that only one additional point is required to form a simple tournament unless the tournament has an odd number of vertices, when two are required. Here, however, we do not require the resulting simple extension to be a tournament, and this added freedom allows us to form a simple one-point extension in every case. See Figure 9 for an example.

Extending the connection with tournaments, any digraph that is also a tournament must consequently have a simple one-vertex extension. A natural question then arises: which digraphs have simple extensions requiring only one additional vertex? Clearly structures that are "close" to tournaments will have this property, and again the answer must lie in the substitution decomposition tree, and in particular how we decompose a structure with a quotient of size two.

Oriented graphs. An *oriented graph* G is a graph in which each edge has a specified direction. It can be described as a relational structure $\mathcal{A}_G$ on the language $\mathcal{L} = \{\to, n_\to = 2\}$ where $\to$ is an asymmetric binary relation; i.e., for each $u, v \in \mathrm{dom}(\mathcal{A}_G) = V(G)$, $u \to v$ implies $v \not\to u$. The difference between these and digraphs is that oriented graphs permit only three types of connection between two vertices, and so in particular there is no oriented graph corresponding to a complete graph. There is, however, an oriented graph corresponding to the empty graph, and it is this that gives rise to the upper bound.

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THEOREM 6.2. *An oriented graph G has a simple extension with at most $m = \lceil \log_3(|V(G)| + 1) \rceil$ additional vertices.*

When the oriented graph forms a tournament, it is easy to check that only one additional vertex is required—note in particular that the simple extension in Figure 9 is also an oriented graph.

§7. *Relations with higher arity.* When a relational structure is defined on a language containing a k-ary relation for some $k \geq 3$, simple extensions are often trivial to find and depend only on the type of relation. We begin with a straightforward observation.

LEMMA 7.1. *A relational structure $\mathcal{A}$ on the language*

$$\mathcal{L} = \{R_1, R_2, \ldots, n_{R_1}, n_{R_2}, \ldots\}$$

is simple if there is some k for which the structure $\mathcal{A}_{R_k}$, defined on the ground set $\mathrm{dom}(\mathcal{A}_{R_k}) = \mathrm{dom}(\mathcal{A})$ and language $\mathcal{L}_{R_k} = \{R_k, n_{R_k}\}$, is simple.

Proof. Suppose, for a contradiction, that I is a non-trivial interval of $\mathcal{A}$, and in particular we may choose two distinct elements $x, y \in I$. Since $\mathcal{A}_{R_k}$ is simple, there exists $z \in \mathrm{dom}(\mathcal{A}_{R_k}) \backslash I = \mathrm{dom}(\mathcal{A}) \backslash I$ for which x and y are not interchangeable in any relationship of R_k containing z and at least one of x and y. This, however, is also true in $\mathcal{A}$. □

Consequently, if a relational structure $\mathcal{A}$ contains an arbitrary k-ary relation R with $k \geq 3$, only one additional element is required, regardless of what other relations the structure may hold.

THEOREM 7.2. *A relational structure $\mathcal{A}$ containing an arbitrary n_R-ary relation R with $n_R \geq 3$ has a one-element simple extension.*

Proof. By Lemma 7.1 we need only prove that we may form a one-element simple extension for a structure $\mathcal{A}$ defined on the language $\mathcal{L} = \{R, n_R\}$ consisting of only one relation. Furthermore, we assume that $n_R = 3$, noting that the result is easily extended if our chosen arbitrary relation has higher arity.

Choose any simple binary relational structure $\mathcal{B}$ on the set $\mathrm{dom}(\mathcal{A})$ with language $\{B, n_B = 2\}$—such a structure exists for every size of set and can be found by taking, for example, a path in the graph-theoretic sense, unless $|\mathcal{A}| \leq 3$, in which case we may choose a simple tournament. We now add an additional vertex x to $\mathcal{A}$ and define relations between x and $\mathrm{dom}(\mathcal{A})$ to form $\mathcal{A}^*$ as follows: for every $(u, v) \in B^{\mathcal{B}}$, put $(x, u, v) \in R^{\mathcal{A}^*}$. (Note that if $n_R > 3$ we set $(x, \ldots, x, u, v) \in R^{\mathcal{A}^*}$.) We claim that $\mathcal{A}^*$ is simple. By the simplicity of $\mathcal{B}$, any interval containing more than one element of $\mathrm{dom}(\mathcal{A})$ must contain all of $\mathrm{dom}(\mathcal{A})$, so the only intervals we need to rule out are $I = \{w, x\}$ for some $w \in \mathrm{dom}(\mathcal{A})$, and $I = \mathrm{dom}(\mathcal{A})$.

For the first case, by the construction there exists at least one relationship of the form (x, w, v) or (x, v, w) in $R^{\mathcal{A}^*}$ for some $v \in \mathrm{dom}(\mathcal{A})$ with $v \neq w$, and without loss we may assume the former. Since I is an interval, this implies

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that $(x, x, v) \in R^{\mathcal{A}^*}$, a contradiction as there are no relationships of the form $(x, x, \cdot)$ in $R^{\mathcal{A}^*}$. In the second case, $I = \mathrm{dom}(\mathcal{A})$ implies that $(x, u, v) \in R^{\mathcal{A}^*}$ for every $u, v \in \mathrm{dom}\,\mathcal{A}$, which implies that $\mathcal{B}$ can be thought of as a complete graph, a contradiction since $\mathcal{B}$ was chosen explicitly to be simple. □

Structures that have more restrictive relations do also sometimes have one-point extensions. For example, a suitably large relational structure containing a k-ary relation for some $k \geq 3$ that is irreflexive in its entries is easily found to have such an extension.

THEOREM 7.3. *For $k \geq 3$, a relational structure with $n \geq k$ elements on a k-ary irreflexive relation has a one-vertex simple extension.*

Proof. Fix $k \geq 3$, and let R be a k-ary irreflexive relation. By Lemma 7.1 it suffices to form a one-element simple extension $\mathcal{A}^*$ of a relational structure $\mathcal{A}$ on the language $\mathcal{L} = \{R, k\}$ whose ground set consists of $n \geq k$ vertices. Set $\mathrm{dom}\,\mathcal{A}^* = \mathrm{dom}\,\mathcal{A} \cup \{x\}$, and define the set of relations $R^{\mathcal{A}^*}$ as follows: $(x_1, \ldots, x_k) \in R^{\mathcal{A}^*}$ if and only if either $(x_1, \ldots, x_k) \in R^{\mathcal{A}}$ or $x = x_i$ for exactly one i and $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ is a $(k-1)$-tuple of distinct elements from $\mathrm{dom}(\mathcal{A})$.

It follows routinely that $\mathcal{A}^*$ is simple: suppose that I is a proper interval of $\mathcal{A}^*$ containing $u, v \in \mathrm{dom}(\mathcal{A}^*)$. By the construction there is at least one relation in $R^{\mathcal{A}^*}$ of the form $(u, v, x_3, \ldots, x_k)$ where $x_3 \notin I$, but this then means that $(u, u, x_3, \ldots, x_k) \in R^{\mathcal{A}^*}$, a contradiction since R is irreflexive. □

Note in particular that the above proof immediately answers the problem for hypergraphs since the construction ensures that if $\mathcal{A}$ is a hypergraph then so is $\mathcal{A}^*$ (i.e., $R^{\mathcal{A}^*}$ is symmetric in its entries).

§8. *Concluding remarks.*

Tight bounds. Although the bounds in this paper are tight in the sense that there are certain structures that require as many points as the bound permits, on a typical example our constructions do quite a bit better. As noted in §3, more than two vertices are required in the graph case only if there is a large degenerate node in the substitution decomposition tree, all of whose children are leaves. Similar analysis of the substitution decomposition tree may be used in the cases of permutations, oriented graphs and digraphs to determine exactly how many points will be required in any particular case. More complicated is the poset case: we cannot simply appeal to the permutation case because degenerate decompositions whose quotients are antichains can be treated as if they are empty graphs, which require only $O(\log_2 n)$ additional vertices.

Average number of additional points required. In the case of graphs and other relational structures defined by a single asymmetric relation (e.g. tournaments, posets and oriented graphs), almost all structures are already simple (see Möhring [**17**]). In other words, in these contexts, the proportion of objects that are already simple tends to one, while the results in this paper bound the number of points needed in the other cases, from which it can be shown that

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the expected number of points needed to make such an structure simple tends to zero. The permutation case is more interesting, because here one expects to find two intervals of size two (see, for example [3]).

Simple extensions of infinite structures. Proposition 2.2 as proved by Erdős, Hajnal and Milner applies to tournaments of arbitrary cardinality, and it is natural to wonder what (if anything) can be said for other structures. Note that for many such structures it is easy to construct examples that require infinitely many additional points (e.g., the countably infinite complete graph). On the other hand, Theorem 7.2 holds so long as the structure contains an arbitrary n_R-ary relation with $n_R \geq 3$.

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