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Computation of isomorphism invariants for stationary dimension groups

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Abstract. In this paper, which is a sequel to an earlier paper, we consider special examples, methods, and invariants for the isomorphism problem for ordered dimension groups built on square primitive integer matrices A. We use the algorithm of our first paper to derive techniques and computable invariants for deciding isomorphism for subclasses of the matrices A.

0. *Introduction*

In a previous paper [**BJKR99a**] we considered the isomorphism problem for the class of stationary ordered dimension groups $G(A)$ built on primitive integer matrices with non-negative matrix entries. The stationary case refers to the incidence matrix A being fixed as the group is built as an inductive limit, and we showed that the isomorphism problem is decidable. The non-stationary case refers to the inductive limit

$$\mathbb{Z}^{n_1} \xrightarrow{A_1} \mathbb{Z}^{n_2} \xrightarrow{A_2} \mathbb{Z}^{n_3} \xrightarrow{A_3} \cdots$$

with different incidence matrices (possibly non-square) as we pass from one step to the next in the limit construction. It is known that this more general isomorphism problem is not decidable; see [**MuPa98**]. In [**BJKR98**] we studied isomorphism of the stable AF-algebras associated with constant square primitive non-singular incidence matrices. This isomorphism is called C^*-equivalence of the matrices in [**BJKR98**] and weak equivalence of the (transposed) matrices in [**SwVo00**].

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In this paper, we use the same notation as in [**BJKR99a**]. The dimension group built on a primitive integer matrix A is denoted as $G(A)$. The Perron–Frobenius eigenvalue on A is denoted as $\lambda = \lambda_{(A)}$, and the respective left/right Perron–Frobenius eigenvectors for A are denoted as $v(A)$ and $w(A)$, respectively, i.e. positive vectors satisfying

$$v(A)A = \lambda_{(A)}v(A), \quad Aw(A) = \lambda_{(A)}w(A),$$

normalized so that the components are contained in the field $\mathbb{Q}[\lambda_{(A)}]$.

In [**BJKR99a**, §6] we gave an explicit but complicated algorithm for deciding isomorphism of the stable stationary AF-algebras. The next question is a presentation of the answer in terms of numerical invariants. We take this up in §§1–4, which are a continuation of [**BJO99**]. Here the answers are not yet complete in the generality of the previous paper, so we present in §1 (Proposition 1.1 and Corollary 1.2) a subclass of incidence matrices for which the C^*-equivalence question is decided by the value of a numerical invariant. The matrices A in the subclass allow a direct-sum decomposition, $A = A_0 \oplus (\lambda)$, such that A_0 is unimodular up to sign, and (λ) is multiplication by the Perron–Frobenius eigenvalue λ on the one-dimensional subspace spanned by the right Perron–Frobenius eigenvector. This property is equivalent to $|\det A| = \lambda$.

§§3 and 6 address symmetry properties, pointing out that there are non-symmetric primitive incidence matrices A which are C^*-equivalent to A^{tr}, the transposed matrix. However, even in the (2×2) case, there are also examples where A and A^{tr} are not C^*-equivalent. The related symmetry question for shift equivalence comes from the issue of reversibility for topological Markov chains, which was studied in [**PaTu82**] and [**CuKr80**].

While the dimension group $G(A)$ associated with an incidence matrix A is torsion-free, it has a certain torsion group quotient $G(A)/L$ by a lattice L in $G(A)$. We show in Proposition 4.2 that this quotient is natural in the sense that it is an invariant. It is well known that abelian torsion groups have explicitly computable and complete numerical invariants, and these invariants are thus also invariants for the dimension group (but not complete because they do not reflect order and some of the group structure). In this case they take an especially simple form, and they can be read off from the characteristic polynomial. This is proved in §4. §6 presents a formulation of C^*-equivalence for matrices A, B in terms of a certain explicit matrix factorization $B = CAD$, where the two factors C, D are specified in the statement of the result, Theorem 6.2.

While we have an algorithm in [**BJKR99a**] for deciding order isomorphism of a pair of ordered dimension groups $G(A)$ and $G(B)$ built on two given primitive integer matrices A and B, the question of numerical invariants is more difficult, and we therefore begin the study of the special case of matrices A such that the Perron–Frobenius eigenvalue λ_{PF} agrees with $|\det A|$.

1. *The case* $\lambda_{\mathrm{PF}} = |\det A|$

In [**BJKR99a**, Theorem 5.9 and §6] we gave a finite algorithm (but in general a long one) to decide whether two square, non-singular, integer primitive matrices A, B are C^*-equivalent or not. In special cases, like those considered in [**BJO99**], this algorithm can be substantially simplified. One nice feature of the algorithm is that it uses only 'elementary' algebraic results, and avoids using the deep results on decidability from

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[**GrSe80a, GrSe80b**]. Nevertheless, the implementation of the algorithm for general pairs A, B may of course be complicated. Let us pick up and generalize one special case from [**BJO99**]. In Theorem 17.18 and Corollary 17.21 there, it was proved that if A, B had a special form, and $\lambda_{(A)} = |\det(A)|$ and $\lambda_{(B)} = |\det(B)|$, then the ideal generated by $\langle v(A) \mid w(A)\rangle$ in $\mathbb{Z}[1/\det(A)]$ is a complete invariant, if the left and right Perron–Frobenius eigenvectors are taken to have integer components, and $\gcd(v(A)) = 1$, $\gcd(w(A)) = 1$, where gcd denotes the greatest common divisor of the components. We now prove that this is also true for more general matrices A, B.

In stating this more general result, there is a technical complication. In picking extension fields F and an associated ring R of algebraic integers, it is not automatically true that the ideals in R are principal. But by a result in [**Wei98**] or [**Ser79**], there is always a finite extension E of F in which the associated ideals are automatically principal. We refer to this in the statement of the proposition. To further simplify the terminology in the statement of the proposition, we denote the above-mentioned respective Perron–Frobenius column vectors as w, w', i.e. $Aw = \lambda w$ and $Bw' = \lambda' w'$, and similarly v, v' for the two respective Perron–Frobenius row vectors.

PROPOSITION 1.1. *Choose a finite extension E of the algebraic number field F of the eigenvalues of primitive non-singular integer matrices A, B in which all ideals of F become principal and consider primes in it.*

(i) *An isomorphism J on ordered dimension groups from the dimension group of A to that of B sends the row Perron–Frobenius eigenvector v' (normalized so all entries are algebraic integers with* gcd 1*) of B to a multiple c times the row Perron–Frobenius eigenvector v of A.*

(ii) *The two Perron–Frobenius eigenvalues generate the same algebraic number field and involve the same primes of that field.*

(iii) *Assume that for each non-Perron–Frobenius eigenvalue μ of A or B, the Perron–Frobenius eigenvalue λ is divisible by some algebraic prime not dividing μ. Then the Perron–Frobenius column eigenvector w is mapped to a multiple ξ times the other Perron–Frobenius eigenvector w'.*

(iv) *If λ satisfies the hypothesis in (iii), then the latter coefficient ξ factorizes into the primes dividing λ.*

(v) *If λ satisfies the hypothesis in (iii), then the former coefficient c factorizes into primes dividing λ. The inner products of left and right Perron–Frobenius eigenvectors are equal modulo normalization: $v'Jw' = cvw' = (c/\xi)vJw$. Therefore, the inner product of the normalized Perron–Frobenius eigenvectors is an invariant up to units in the algebraic number ring generated by* $1/\lambda$.

Proof. The first assertion is by [**BJKR98**, Theorem 6], and the second is by [**BJKR98**, Theorem 10].

The third assertion follows because the space of vectors in the dimension group $G \otimes E$ such that some fixed multiple is arbitrarily divisible by a given algebraic prime is sent to the corresponding subspace of the other dimension group, and this set is the sum of the generalized eigenspaces for all eigenvalues divisible by the prime. If these spaces

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are intersected over all primes dividing the Perron–Frobenius eigenvalue, we get, by our hypothesis, only the Perron–Frobenius eigenspace.

The fourth assertion follows because the eigenspace of w will consist precisely of those vectors in the dimension group which are divisible by arbitrary powers of primes occurring only in λ, so it must be preserved by any isomorphism of dimension groups. In addition, vectors in this one-dimensional space which are not divisible by primes other than those in λ will be unique up to multiplication by units and primes dividing λ, so they will be preserved by any isomorphism, up to such multiplication.

Next note that the fourth statement and the first part of the fifth statement are equivalent whenever we have an isomorphism from the dimension group to itself induced by an integer matrix. The reason is that since row and column Perron–Frobenius eigenvectors are preserved, this integer matrix in a basis corresponding to generalized eigenvectors becomes block diagonal, and the block for the Perron–Frobenius eigenvectors must be the same element for the row eigenvectors as for the column eigenvectors, and by (iv) it involves only primes dividing λ.

Now consider integer matrices R and S inducing mappings each way between two different column dimension groups, with Perron–Frobenius eigenvectors v and v' normalized over the algebraic number ring. We have $v \to c_1 v'$ and $v' \to c_2 v$, where c_1 and c_2 are algebraic integers since R and S are integer matrices. However $c_1 c_2 = c$ arises from a map of the dimension group to itself, so it divides a power of λ, hence so do c_1 and c_2. □

The following is a partial converse to Proposition 1.1.

COROLLARY 1.2. *Suppose A, B are non-singular primitive integer matrices such that their Perron–Frobenius eigenvalues are integers and that the inner products as above are equal, i.e. after normalization, that $\langle v(A) \mid w(A)\rangle = \langle v(B) \mid w(B)\rangle$, that the primes dividing the Perron–Frobenius eigenvalues are equal, and that the dimensions of the matrices are at least three. Suppose that the Perron–Frobenius eigenvalues are the determinants of A, B up to sign. Then there exists an isomorphism between the ordered dimension groups of A and B.*

Proof. By [**BJO99**, Lemma 17.19], there is a unimodular matrix J sending the Perron–Frobenius row eigenvector $v(A)$ of A to the Perron–Frobenius row eigenvector $v(B)$ of B and the Perron–Frobenius column eigenvector $w(A)$ of A to the Perron–Frobenius column eigenvector $w(B)$ of B (and we can choose signs for positivity). By [**BJKR98**, Theorem 6] this gives a positive mapping on dimension groups. If $V(A) = v(A)^{\perp}$, then $JV(A) = V(B)$, and also $v(B)J \subseteq \mathbb{Q}v(A)$ as noted in [**BJKR99a**, §1]. Write any vector v as a direct sum according to $\mathbb{C}w(A) \oplus V(A)$, $v = x + y$. This splitting can introduce certain fixed primes p in the denominator.

Note that the matrix A is unimodular and integer restricted to the integer vectors in $V(A)$ (and similarly for the matrix B), because each determinant is the product of its determinant on this space and its determinant on the Perron–Frobenius eigenspace, and because it is an integer matrix preserving this subspace. Multiplication by A is multiplication by λ on x (see Figure 1), and the same is true for B.

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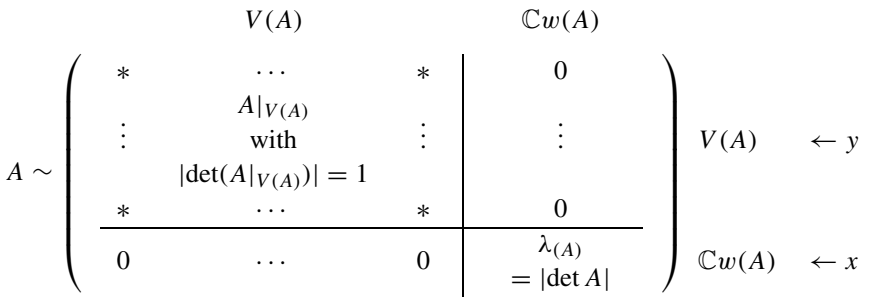


FIGURE 1. The case $|\det A| = \lambda_{(A)}$: decomposition relative to $\mathbb{C}^N = V(A) \oplus \mathbb{C}w(A)$ (see [**BJKR99a**, equation (1.20)]) and unimodular restriction.

For v to be in the dimension group means that, for all sufficiently large n, $A^n v$ has integer entries. Any prime p which does not divide λ will not occur in the denominator of the expression $B^m J A^{-n}(x + y)$.

Consider those primes p which divide λ. We claim that they cannot occur in denominators of y. Restricted to vectors y, the matrix A is unimodular, so modulo any powers of those primes it lies in a finite group, $\mathrm{GL}(N, \mathbb{Z}_{p^s})$. Thus we can choose arbitrarily large n so that A^n is congruent to the identity. But then in $A^n(x + y)$, the denominators in x have vanished, being multiplied by λ^n and those in y remain. So $x + y$ is not in the dimension group, a contradiction.

Therefore, in

$$\begin{aligned} B^m J A^{-n}(x + y) &= B^m J A^{-n} x + B^m J A^{-n} y \\ &= J\lambda'^m \lambda^{-n} x + B^m J A^{-n} y \end{aligned} \tag{1.1}$$

both terms are integers for sufficiently large m (with a symmetrical argument the other way) which verifies the conditions in [**BJKR99a**, §1] for isomorphism of ordered dimension groups. □

2. *The case of no infinitesimal elements and the case of rational eigenvalues*

In this section we will consider the C^*-equivalence problem in two extreme cases. To describe these two cases, let us recall some facts from [**Eff81, BJO99**]. We define a functional τ_A on $G(A)$ by the formula

$$\tau_A(g) = \langle v(A) \mid g \rangle, \quad g \in G(A), \tag{2.1}$$

where $v(A)$ is a left Perron–Frobenius eigenvector for A. This functional τ_A is called 'the' trace since it defines a trace on the corresponding C^*-algebra. It follows from the eigenvalue equation that $v(A)$ can be taken to have components in the field $\mathbb{Q}[\lambda] = \mathbb{Q}[1/\lambda]$, where λ is the Perron–Frobenius eigenvalue. However, multiplying $v(A)$ by a positive integer, we may assume that the components of $v(A)$ are contained in the ring $\mathbb{Z}[1/\lambda]$. It then follows from $v(A)A^{-n} = \lambda^{-n} v(A)$ that

$$\tau_A(G(A)) \subset \mathbb{Z}[1/\lambda]. \tag{2.2}$$

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Furthermore, $\tau_A(G(A))$ is invariant under multiplication by elements of $\mathbb{Z}$ and by $1/\lambda$, so it is a $\mathbb{Z}[1/\lambda]$-module. In particular, $\tau_A(G(A))$ is an ideal in the ring $\mathbb{Z}[1/\lambda]$. We need only verify that $\lambda^{-1}\tau_A(g)$ is in $\mathrm{ran}(\tau_A)$ for all $g \in G(A)$, where $\mathrm{ran}(\tau_A)$ denotes the range of the trace functional τ_A, i.e. the subgroup $\tau_A(G(A))$ from (2.2). Pick $g \in G(A)$, and set $g = A^{-n}m$, $n \in \mathbb{Z}_+$, $m \in \mathbb{Z}^N$. Then $\lambda^{-1}\tau_A(g) = \langle v(A)A^{-1} \mid g\rangle = \langle v(A) \mid A^{-1}g\rangle = \tau_A(A^{-(n+1)}m) \in \mathrm{ran}(\tau_A)$ as claimed. This is a very special feature of the constant-incidence-matrix situation which is not shared by the range of a trace on a general dimension group of general AF-algebras. This range is not even closed under multiplication in the general case when the incidence matrix is not assumed constant. We have the natural short exact sequence of groups

$$0 \longrightarrow \ker(\tau_A) \hookrightarrow G(A) \xrightarrow{\tau_A} \tau_A(G(A)) \longrightarrow 0 \tag{2.3}$$

and the order isomorphism

$$G(A)/\ker(\tau_A) \underset{\approx}{\xrightarrow{\tau_A}} \mathrm{ran}(\tau_A) \subset \mathbb{Z}[1/\lambda_{(A)}], \tag{2.4}$$

where $\mathrm{ran}(\tau_A)$ inherits the natural order from $\mathbb{Z}[1/\lambda]$. Note that for the particular matrices we considered in [**BJO99**], we had

$$\mathrm{ran}(\tau_A) = \mathbb{Z}[1/\lambda] \tag{2.5}$$

(see [**BJO99**, equations (5.21)–(5.22)]), but be warned that this is not a general feature. This will be discussed further in Remarks 3.5 and 3.7. Chapter 5 in [**BMT87**] also has a nice treatment of $\mathrm{ran}(\tau_A)$ in the general case. Let us already at this point state and prove the remarkable fact that *any* subset I of $\mathbb{Q}[\lambda]$ which is an ideal over $\mathbb{Z}[1/\lambda]$ occurs as the image of the trace for a suitable primitive non-singular matrix A (this is a version of [**BMT87**, Corollary 5.15] which is a consequence of results of Handelman, see [**Han81**] and [**Han87**]).

PROPOSITION 2.1. *Let λ be a real algebraic integer larger than the absolute value of any of its conjugates, and let $I \subset \mathbb{Q}[\lambda]$ be an ideal over $\mathbb{Z}[1/\lambda]$. Then I can occur as the image of the trace for some matrix whose Perron–Frobenius eigenvalue is a power of λ (the size of the matrix will be the degree $\mathbb{Q}[\lambda]/\mathbb{Q}$).*

Proof. Let $I_1 = I \cap \mathbb{Z}[\lambda]$; it will be a $\mathbb{Z}[\lambda]$-ideal which spans I over $\mathbb{Z}[1/\lambda]$.

Now define an integer matrix M which expresses the action of λ on I_2, that is, form an additive basis w_i for I_1, let $\lambda w_i = \sum_j m_{ij}w_j$, $m_{ij} \in \mathbb{Z}$. This matrix will have an eigenvalue λ, and we claim that at the corresponding eigenspace, the image of the trace is isomorphic to I. This is because the action of M on $\mathbb{Z}^N$ has been forced to be that of λ on I_1, and because the trace reflects this module structure, by means of the short (nearly exact) sequence.

Finally, we claim that we can conjugate M over $\mathrm{GL}(N,\mathbb{Z})$ to a matrix whose powers are eventually positive; then those powers will be non-negative matrices whose image of trace is the same. To get eventual positivity, given that λ is the largest eigenvalue (the largest of its Galois conjugates), it is necessary and sufficient that its row and column eigenvectors for this eigenvalue be positive, by a limit argument somewhat like that in

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[**BJKR99a**, Proposition 1.1]. Let v, w be row and column eigenvectors at the eigenvalue λ, with signs chosen so that their inner product is positive. Multiply each by a large integer, and then take relatively prime integers approximating its components. Such a pair of vectors can be mapped over $\mathrm{GL}(N, \mathbb{Z})$ to any vectors whose entries are relatively prime integers having the same inner product (by [**BJO99**, Lemma 17.19]), in particular, to ones which are positive, if $N > 2$. If $N = 2$ we use the same result and get a congruence condition, but that is compatible with positivity. □

Remark 2.2. The quotient of the ring $\mathbb{Z}[1/\lambda]$ by any of these ideals will be finite. The ideal can be lifted to an ideal inside the rank-N additive group $\mathbb{Z}[\lambda]$, and the quotient of two rank-N free abelian groups is finite—its order is given by the determinant of the map expressing the inclusion.

Let us return to the two special cases of C^*-equivalence we shall discuss in this section. These are the following.

(i) The kernel $\ker(\tau_A)$ is 0, i.e. $G(A)$ has no infinitesimal elements, i.e. the characteristic polynomial of A is irreducible over $\mathbb{Z}$ (or equivalently over $\mathbb{Q}$).

(ii) All the eigenvalues of A are rational (thus integer), each of them is relatively prime to the rest, and none is equal to ± 1.

In §§3 and 6 we will apply this to many examples. See, for example, Example 3.9 for an application in situation (ii) above.

THEOREM 2.3. *Two primitive $(N \times N)$ matrices A, B over $\mathbb{Z}_+$ with irreducible characteristic polynomials are C^*-equivalent if and only if the following three conditions all hold:*

(i) *the roots of their characteristic polynomials generate the same field;*

(ii) *their Perron–Frobenius eigenvalues are divisible by the same algebraic primes; and*

(iii) *their dimension groups, as modules over $\mathbb{Z}[1/\lambda]$ (or a full-rank subring), are isomorphic. These modules are isomorphic to the fractional ideals given by the image of the trace τ.*

Moreover, these three conditions are equivalent to the one condition:

(iv) *the two ordered additive subgroups in $\mathbb{Z}[1/\lambda]$ defined by the ranges of the respective traces are isomorphic.*

If in addition the characteristic polynomials of A, B are equal, then C^*-equivalence (isomorphism of ordered dimension groups) is the same as shift equivalence.

Note that taking powers of the matrix will preserve the $\mathbb{Z}[1/\lambda]$-module mentioned in (iii)*, i.e. the ideal in $\mathbb{Z}[1/\lambda]$, and not replace it by its powers.*

Remark 2.4. To say that the dimension groups $G(A)$ and $G(B)$ as modules over $\mathbb{Z}[1/\lambda]$ are isomorphic means that there is an isomorphism $\varphi\colon G(A) \to G(B)$ of abelian groups such that

$$\varphi(\omega g) = \omega\varphi(g) \tag{2.6}$$

for all $g \in G(A)$, $\omega \in \mathbb{Z}[1/\lambda]$. This is not the same as saying that $G(A)$ is isomorphic to $G(B)$ as ideals in $\mathbb{Z}[1/\lambda]$. The latter concept means that there is an automorphism φ of the ring $\mathbb{Z}[1/\lambda]$ such that $\varphi(G(A)) = G(B)$. When we talk about equivalence of ideals

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it is the *first* concept we are thinking about, i.e. there is an element of the quotient field $\mathbb{Q}[1/\lambda] = \mathbb{Q}[\lambda]$ mapping the one ideal into the other by multiplication.

Proof of Theorem 2.3. The first three statements are a reformulation of [**BJKR98**, Proposition 10], except for the relationship with the trace, which we next show. The definition (2.1), the properties $G_n(A) = A^{-n}(\mathbb{Z}^N)$ and $G(A) = \bigcup_{n=0}^{\infty} G_n(A)$, and

$$t_A \circ A^{-1} = \lambda^{-1}\tau_A$$

imply that the image of the trace is a module over $\mathbb{Z}[1/\lambda]$ and a subset of $\mathbb{Z}[1/\lambda]$. Using the standard basis for $\mathbb{Z}^N$, it is generated by $\langle v \mid e_i = v_i \rangle$ as a module over $\mathbb{Z}[1/\lambda]$ since $A^{-n}e_i$, $n \in \mathbb{Z}_+$, generate the dimension group. The trace mapping is an epimorphism if we pass to rational coefficients (that is, tensor dimension groups with $\mathbb{Q}$), just because its image is non-zero (consider v as a Perron–Frobenius column eigenvector) and closed under field operations in $\mathbb{Q}[1/\lambda]$. Its kernel is zero since the dimension group with rational coefficients is also a 1-dimensional vector space over $\mathbb{Q}[1/\lambda]$ (for instance, by [**BMT87**, Ch. 5]). Thus the trace mapping is an isomorphism to its image as asserted in the second part of (iii).

A theorem of Handelman (stated as Theorem 5.2 in [**BMT87**]) in fact allows us to replace the condition in Theorem 2.3 above that A and B be non-negative with the condition that instead they are *integral eventual positive* (IEP), i.e. that they are in $M_N(\mathbb{Z})$ and have respective powers with strictly positive entries.

Next we show that dimension group isomorphism in our sense implies shift equivalence if the irreducible characteristic polynomials of A, B are equal. The only difference with the isomorphisms used in [**BMT87**, Theorem 2.8] is that there the actions of A, B are the same, that is, the matrices themselves represent the field elements acting on this module. But the field element A, B represent are roots of the same irreducible characteristic polynomials, and are the unique Perron–Frobenius roots of these polynomials so they must be the same field element.

Equivalence to (iv): (iv) is, properly understood, a rephrasing of (iii), given the isomorphism in the first paragraph of the proof. We will clarify the kind of module structure which is involved. Assuming (iv), the images of the traces generate the fields $\mathbb{Q}[\lambda_{(A)}] = \mathbb{Q}[\lambda_{(B)}]$, but the rings $\mathbb{Z}[\lambda_{(A)}]$ and $\mathbb{Z}[\lambda_{(B)}]$ may be different, in which case we work with the full-rank subring $\mathbb{Z}[\lambda_{(A)}] \cap \mathbb{Z}[\lambda_{(B)}]$. Condition (i) is immediate, and condition (ii) follows since algebraic primes dividing λ are those primes which can divide elements of the dimension groups to arbitrary powers.

Conversely, suppose we are given (i), (ii) and (iii). The equality of fields asserted in [**BJKR98**, Proposition 10] is taken in the sense of 'equality of $\mathbb{Q}[\lambda_{(A)}]$ and $\mathbb{Q}[\lambda_{(B)}]$ as subfields of the real numbers', which gives embeddings of $\mathbb{Z}[\lambda_{(A)}]$ and $\mathbb{Z}[\lambda_{(B)}]$ into the real numbers. Thus, it also embeds the modules which can be considered as subsets of $\mathbb{Q}[\lambda_{(B)}]$. The isomorphism of modules as additive groups acted on multiplicatively (i.e. the action $y \mapsto xy$) by subrings of $\mathbb{Q}[\lambda_{(A)}]$ having full rank (in this case N) means there is some element of the quotient field mapping one to the other: if the isomorphism of (iii) maps some element y to z (considered as images in the real numbers) then the ratio y/z is independent of the choice of y by definition of the isomorphism in (iii), and we multiply by this ratio to get the isomorphism in (iv). □

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Note that this applies in particular to Example 3.8 below.

THEOREM 2.5. *Let A and B be matrices over $\mathbb{Z}_+$, all of whose eigenvalues are rational, each of which is divisible by some prime not dividing the other eigenvalues. Assume further that A and B have the same characteristic polynomial. Let E_a and E_b be their matrices of column eigenvectors normalized to be integer vectors having greatest common divisor one. Let D be a diagonal matrix whose entries involve only powers of primes in the respective eigenvalues, let D_s be a diagonal matrix consisting of precisely the diagonal eigenvalues. Then the following are equivalent:*

(i) *A and B are C^*-equivalent;*

(ii) *A and B are shift equivalent, as follows: for some choice of signs in E_a, E_b, and some choice of D, and for all sufficiently large n, $E_a D D_s^n E_b^{-1}$ and $E_b D^{-1} D_s^n E_a^{-1}$ are integer matrices.*

Proof. Consider an isomorphism of dimension groups. The eigenvectors generate the one-dimensional spaces of vectors such that some multiples of those vectors are in the dimension group and are divisible by arbitrary powers of the respective eigenvalues. Hence any dimension group isomorphism must preserve those subspaces. Moreover, we claim that a dimension group isomorphism must send normalized eigenvectors to normalized eigenvectors of the image. The rational multiples of a normalized eigenvector v with rational eigenvalue η which lie in the dimension group are the elements of $M_v = \{(n/m)v \mid n \in \mathbb{Z},\ \exists k \in \mathbb{Z}_+, m|\eta^k\}$: a vector $w \in M_v$ lies in the dimension group $G(A)$, since w is a multiple of an integer vector by negative powers of A, B. And if $w = (n/m)v \in G(A)$ then there is a $k \in \mathbb{Z}$ such that $A^k v = \eta^k v \in \mathbb{Z}^N$, so that $m|\eta^k$ in lowest terms, $w \in M_v$. It follows that the dimension group isomorphism implies the existence of an isomorphism of $\mathbb{Q}^N$ which sends each eigenvector to a multiple of the other eigenvector by a number which divides a power of η. Such a mapping must preserve the action of multiplication by A, given that the characteristic polynomials are equal, because this multiplies each eigenvector by its eigenvalue, and the eigenvalues are the same. So the mapping will be a shift equivalence. Let D be the diagonal matrix whose main diagonal entries are the multiples just mentioned. Then the isomorphism J of dimension groups will be, specifically, $E_b(E_a D)^{-1}$ if it exists. For if we multiply J and its inverse on the left by a large enough power of A or B, respectively, we see that the resulting matrix products must be integer matrices. Moreover, these multiples are the matrices stated in the theorem. □

3. *The transpose map and C^*-symmetry*

In this section we will study the behavior of the dimension group $(G(A), G(A)_+)$ under the transpose map $A \to A^{\mathrm{tr}}$. In particular, we say that A is C^*-symmetric if A is C^*-equivalent to A^{tr}, i.e. $G(A)$ and $G(A^{\mathrm{tr}})$ are isomorphic as ordered groups. We give several examples showing that A may be C^*-symmetric, (3.3), Remark 3.5; or not, Example 3.6 ((2×2) matrices with rational eigenvalues), Example 3.8 ((2×2) matrices with irrational eigenvalues) and Example 3.9. An interesting feature with these particular examples is that when A is a (2×2) matrix, then C^*-symmetry is equivalent to shift-symmetry (i.e. A and A^{tr} are shift equivalent). For (2×2) matrices, symmetry seems to be more common than non-symmetry. Our first example, while very simple, illustrates both C^*-symmetry

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and a non-trivial Ext-element. It has $\lambda = \lambda_{(A)} = 2$. The Ext-group represents another contrast between the two cases, λ rational (and hence integral), and the characteristic polynomial irreducible. In the first case, we generally have $\ker(\tau_A) \neq 0$, and as we note in Remark 3.5, $\mathrm{ran}(\tau_A) = \mathbb{Z}[1/\lambda]$. Hence this extra extension structure for $G(A)$ arises only in the reducible case: the corresponding short exact sequence

$$0 \longrightarrow \ker(\tau_A) \longrightarrow G(A) \xrightarrow{\tau_A} \mathbb{Z}[1/\lambda] \longrightarrow 0 \tag{3.1}$$

may be non-split, which means that $G(A)$ is then not the direct sum of the two groups $\ker(\tau_A)$ and $\mathbb{Z}[1/\lambda]$.

Recall that, for groups R and S, $\mathrm{Ext}(R, S)$ is again a group. Elements in $\mathrm{Ext}(R, S)$ are equivalence classes of short exact sequences

$$0 \longrightarrow S \longrightarrow E \xrightarrow{\psi} R \longrightarrow 0, \tag{3.2}$$

see [**CaEi56**], [**BJO99**, p. 62], and there are natural operations

$$E \longmapsto -E$$

and

$$E, E' \longmapsto E + E'$$

on $\mathrm{Ext}(R, S)$ which turn it into a group. We say that (3.2) splits if there is some $\varphi \in \mathrm{Hom}(R, E)$ such that $\psi \circ \varphi = \mathrm{id}_R$. Then (3.2) splits if and only if it represents the zero element in $\mathrm{Ext}(R, S)$. For example,

$$\mathrm{Ext}(\mathbb{Z}_2, \mathbb{Z}) \cong \mathbb{Z}_2,$$

where the non-split Ext-element is represented by

$$0 \longrightarrow 2\mathbb{Z} \longrightarrow \mathbb{Z} \xrightarrow{\mathrm{proj}} \mathbb{Z}_2 \longrightarrow 0$$

and the zero element is represented by

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{0 \oplus \mathrm{id}} \mathbb{Z}_2 \longrightarrow 0.$$

The next example illustrates how non-trivial Ext-elements are part of the invariant structure for the dimension groups.

Example 3.1. The dimension group defined by A may be order isomorphic to that defined by its transpose $B = A^{\mathrm{tr}}$. Hence an AF-C^*-algebra built on such a matrix A (i.e. from the corresponding stationary Bratteli diagram) has a non-trivial period-two symmetry corresponding to $A \mapsto A^{\mathrm{tr}}$. An example here is

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \quad A^{\mathrm{tr}} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}. \tag{3.3}$$

In this case A and A^{tr} have eigenvalues 2 and -1, and both of the dimension groups G and G^{tr} are in $\mathrm{Ext}(\mathbb{Z}[1/2], \mathbb{Z})$. It can be checked (by use of [**BJO99**, Corollary 11.28]) that this

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Ext-element is not zero. Here $\ker(\tau) = \mathbb{Z}$, $\mathrm{ran}(\tau) = \mathbb{Z}[1/2]$, and the corresponding short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow G(A) \xrightarrow{\tau} \mathbb{Z}[1/2] \longrightarrow 0 \tag{3.4}$$

does not split, i.e. it is not the zero element in Ext. Equivalently, $G(A)$ is not $\mathbb{Z} \oplus \mathbb{Z}[1/2]$ as a group. If it were, we would get $\tau(w)^{-1} \in \mathbb{Z}[1/2]$ by [**BJO99**, Corollary 11.28]. But we computed $\tau(w) = 3$, and $1/3$ is not in $\mathbb{Z}[1/2]$. Since $\lambda_{(A)} = 2 = |\det A|$, it is tempting to apply Corollary 1.2. In fact the inner-product invariants are

$$\langle v \mid w \rangle = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3, \quad \text{and} \quad \langle v' \mid w' \rangle = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3.$$

But since the dimension is two (less than three), Corollary 1.2 does not apply directly, and instead we will verify directly that A and A^{tr} are C^*-equivalent. Define matrices J, K by

$$J = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}. \tag{3.5}$$

One verifies that

$$A = KJ, \quad A^{\mathrm{tr}} = JK. \tag{3.6}$$

Thus A and A^{tr} are elementary shift equivalent, and it follows that they are shift equivalent and C^*-equivalent (see the discussion in [**BJKR98**]).

However, we will see in Examples 3.9 and 6.4 that this is not a general feature of the transpose map.

We may analyze the C^*-symmetry question by dimension-group analysis: if we show that the ordered group $G(A)$ is order isomorphic to $G(A^{\mathrm{tr}})$, then A is C^*-equivalent to A^{tr}, i.e. A is C^*-symmetric. Clearly then the two groups $G(A)$ and $\mathrm{ran}(\tau_A)$ are order isomorphic whenever $\ker(\tau_A) = 0$, and we have the result.

PROPOSITION 3.2. *Let $A \in M_N(\mathbb{Z})$ be non-singular and primitive, and suppose its characteristic polynomial $p_{(A)}(x)$ is irreducible, and* $\mathrm{ran}(\tau_A) = \mathrm{ran}(\tau_{A^{\mathrm{tr}}})$*: then A is C^*-equivalent to A^{tr}. Note, in particular, that this holds if:*

(i) $N = 2$*;*

(ii) *the Perron–Frobenius eigenvalue $\lambda_{(A)}$ is irrational; and*

(iii) $\mathrm{ran}(\tau_A) = \mathrm{ran}(\tau_{A^{\mathrm{tr}}})$.

Proof. This follows directly from Theorem 2.3. □

Remark 3.3. We saw that by scaling out denominators in the entries v_i of the left (row) Perron–Frobenius eigenvector $v(A) = (v_1, \ldots, v_N)$ we can arrange that $v_i \in \mathbb{Z}[1/\lambda]$ for all i. But then by a further scaling with a power of λ we can get each v_i in the subring $\mathbb{Z}[\lambda] \subset \mathbb{Z}[1/\lambda]$. Suppose that the characteristic polynomial of A is irreducible. Note that, as a group, $\mathbb{Z}[\lambda]$ is then a copy of the lattice $\mathbb{Z}^N$ so the entries v_i may therefore be viewed as vectors in $\mathbb{Z}^N$. Then pick $v(A)$ such that $\gcd(v_i) = 1$ for each i. In this case the matrix V with the v_i's as rows is in $M_N(\mathbb{Z})$ and is non-singular. If we could define greatest common divisors in the ring $\mathbb{Z}[\lambda]$ then we could divide v by this greatest common divisor and obtain some new v defined over $\mathbb{Z}[\lambda]$ which has gcd 1. Then the image of its trace would contain

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the span of its coordinates v_i over $\mathbb{Z}[\lambda]$, that is, the entire ring $\mathbb{Z}[\lambda]$. Moreover the image of the trace will be contained in this ring, so they are equal. In general, however, this ring will not be a principal ideal domain, so that the class of the ideal generated by the trace becomes an obstruction. In fact, the subgroup in $\mathbb{Z}[\lambda]$ which is generated by the v_i's is also an ideal in $\mathbb{Z}[\lambda]$. Indeed, for $m \in \mathbb{Z}^N$, $\sum_i m_i v_i = \tau(m) = \langle v \mid m \rangle$, so $\lambda \sum_i m_i v_i = \langle vA \mid m \rangle = \langle v \mid A^{\mathrm{tr}} m \rangle$, and $A^{\mathrm{tr}} m \in \mathbb{Z}^N$. As a consequence, we get that the special incidence matrices A which we considered in [**BJO99**] satisfy the condition $\mathrm{ran}(\tau_A) = \mathbb{Z}[1/\lambda_{(A)}]$. However, this fails for the matrix A from Example 3.8, and others. The group $\tau(\mathbb{Z}^N)$ is contained in $\tau(G) = \mathrm{ran}(\tau_A)$ and the following proposition indicates their relationship.

PROPOSITION 3.4. *Assume that the Perron–Frobenius row eigenvector v is chosen to lie in $\mathbb{Z}^N[\lambda]$. Then the map induced by the inclusion $\mathbb{Z}[\lambda]/\tau(\mathbb{Z}^N) \to \mathbb{Z}[1/\lambda]/\tau(G)$ is an epimorphism with kernel precisely*

$$\{x \in \mathbb{Z}[\lambda]/\tau(\mathbb{Z}^N) \mid \exists\, m \in \mathbb{Z}, \lambda^m x = 0\}.$$

Proof. Assuming, as we will show, that the image consists of torsion elements relatively prime to λ, the inclusion gives a natural mapping. If we multiply any element in $\mathbb{Z}[1/\lambda]$ by a power of λ, we can get an element of $\mathbb{Z}[\lambda]$, so this mapping is an epimorphism. We also claim that if we multiply any element of $\tau(G)$, say $vA^{-n}x$, $x \in \mathbb{Z}^N$, by a power of λ, we will get an element of $\tau(\mathbb{Z}^N)$. This is because $vA^{-n}x = v\lambda^{-n}x$ using the left two factors.

Note that since $\tau(G)$ and $\mathbb{Z}[1/\lambda]$ are both torsion-free and λ-divisible, their quotient has no λ-torsion. Hence every element annihilated by a power of λ lies in the kernel. (We will show that the image consists of torsion elements relatively prime to λ.)

Let y be in the kernel of this mapping. Then $y \in \tau(G)$, so that for some $n \in \mathbb{Z}_+$, $\lambda^n y \in \tau(\mathbb{Z}^N)$ and is zero in the original group. This identifies the quotient. The left-hand group, the quotient of a free abelian group by a full-rank subgroup, is finite, so some fixed n works for the whole kernel. □

Remark 3.5. (Rational λ) Even if $N = 2$, the dimension group $G(A)$ is not yet completely understood [**BJO99**] (perhaps far from it; see, however, [**Han87**]). If $\lambda = \lambda_{(A)}$ is rational, and therefore an integer, we can have non-isomorphic $G(A_1)$ and $G(A_2)$ even when A_1 and A_2 have the same characteristic polynomial and thus the same Perron–Frobenius eigenvalue λ, as different extensions, $i = 1, 2$,

$$0 \longrightarrow \mathbb{Z}[1/\mu] \hookrightarrow G(A_i) \xrightarrow{\tau} \mathbb{Z}[1/\lambda] \longrightarrow 0, \tag{3.7}$$

i.e. as different elements of the group $\mathrm{Ext}(\mathbb{Z}[1/\lambda], \mathbb{Z}[1/\mu])$. Here μ is the other root of the characteristic polynomial, so μ is a non-zero integer with $|\mu| < \lambda$. See (2.3) and (2.4). This may even happen when A_2 is the transpose of the matrix A_1, by Example 3.6 below. Since here λ is rational one may arrange that $\tau_A(G(A)) = \mathbb{Z}[1/\lambda]$ by choosing v with $\gcd(v) = 1$, and $\ker(\tau_A)$ is a rank-one non-zero group isomorphic to $\mathbb{Z}[1/\mu]$; see also below. There are specimens of (2×2) primitive matrices A, even with integral Perron–Frobenius eigenvalue such that $\lambda_{(A)} < |\det A|$, and yet the two groups $G(A)$ and $G(A^{\mathrm{tr}})$

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are order isomorphic. For example, $A = \left(\begin{smallmatrix} 1 & 5 \\ 3 & 3 \end{smallmatrix}\right)$ has that property. To see this, we may use [**BJKR99a**, equations (1.16) and (1.17)], i.e.

$$v(B)J = \mu v(A) \quad \text{for some } \mu > 0$$

and

$$\text{for all } n \in \mathbb{Z}_+, \text{ there is an } m \in \mathbb{Z}_+ \text{ such that } B^m J A^{-n} \text{ and } A^m J^{-1} B^{-n}$$
$$\text{both have integer matrix entries.}$$

Since $J = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right)$ satisfies $JA = A^{\mathrm{tr}}J$, the two conditions hold, and hence the matrix A is C^*-symmetric. So for this particular pair A, A^{tr}, the respective groups $G(A)$ and $G(A^{\mathrm{tr}})$ from the middle term in the diagram (3.7) will then in fact represent the same zero element of $\mathrm{Ext}(\mathbb{Z}[1/6], \mathbb{Z}[1/2])$. For this particular A,

$$G(A) \cong \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/6] \quad (\ker(\tau) \cong \mathbb{Z}[1/2]) \tag{3.8}$$

as a direct sum of abelian groups. For this, note that the integral column eigenvectors for A are $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 5 \\ -3 \end{smallmatrix}\right)$. Since $\det\left(\begin{smallmatrix} 1 & 5 \\ 1 & -3 \end{smallmatrix}\right) = -8 = -2^3$ and the eigenvalues of A are $6 = 2 \cdot 3$ and -2, we have $\mathbb{Z}[1/2]^2 \subset G(A)$. Thus $G(A) = \bigcup_{n=0}^{\infty} A^{-n}(\mathbb{Z}[1/2]^2)$, and (3.8) follows. Specifically, the representation (3.8) may be derived from [**BJKR99a**, equation (1.20)], i.e. $\mathbb{C}^N = V(A) \oplus \mathbb{C}w(A)$, and (2.1) and the two identities

$$\ker(\tau_A) = V(A) \cap G(A) = \mathbb{Z}[1/2] \begin{pmatrix} 5 \\ -3 \end{pmatrix} \tag{3.9}$$

and

$$G(A) \cap \mathbb{C}w(A) = \mathbb{Z}[1/6]w(A), \tag{3.10}$$

where $w(A) = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. The present computation of $G(A)$ is simplified by the fact that the orthogonal complement of the trace vector $v(A) = (3, 5)$ is spanned by the non-maximal column eigenvector. Here $\lambda_{(A)} = 6$, and so $\mathbb{Z}[\lambda_{(A)}] = \mathbb{Z}$. That $\tau(G(A)) = \mathbb{Z}[1/6]$ in this case follows from Remark 3.3 and the general observation that, with our choice of $v(A)$, we will have $\tau(G(A)) = \mathbb{Z}[1/\lambda_{(A)}]$ provided the ideal in $\mathbb{Z}[\lambda_{(A)}]$ generated by the $v_i(A)$ entries is principal. Ideals in $\mathbb{Z}$ are principal, of course. Here in this case the Ext-element corresponding to $G(A)$ is trivial. (Looking at prime factors in $\det A$, one could also get a $G(A)$ which is non-split. For example, taking $A = \left(\begin{smallmatrix} 1 & 6 \\ 2 & 2 \end{smallmatrix}\right)$, we get the spectrum $\{5, -2\}$ and that the corresponding dimension group $G(A)$ is represented here by a non-zero element of $\mathrm{Ext}(\mathbb{Z}[1/5], \mathbb{Z}[1/2])$. The analysis here is analogous to that presented above: we get $\ker(\tau_A) \cong \mathbb{Z}[1/2]$, $\mathrm{ran}(\tau_A) \cong \mathbb{Z}[1/5]$, and the corresponding short exact sequence

$$0 \longrightarrow \mathbb{Z}[1/2] \longrightarrow G(A) \longrightarrow \mathbb{Z}[1/5] \longrightarrow 0 \tag{3.11}$$

is now non-split. The example $A = \left(\begin{smallmatrix} 1 & 6 \\ 2 & 2 \end{smallmatrix}\right)$ is C^*-symmetric, as A and A^{tr} are in fact shift equivalent. Take $R = \left(\begin{smallmatrix} -1 & 2 \\ 2 & -2 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 7 & 4 \\ 4 & 3 \end{smallmatrix}\right)$. Then $RS = A^{\mathrm{tr}}$ and $SR = A$. It follows from [**BJO99**] that $G(A)$, when represented in the Ext-group, is generally not the zero element.)

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Example 3.6. Here we will exhibit a primitive non-singular (2×2) matrix A with rational eigenvalues such that A is not C^*-equivalent to A^{tr} (and thus is not shift equivalent to A^{tr}). The respective dimension groups $G(A)$ and $G(A^{\mathrm{tr}})$ are not even isomorphic as groups, let alone order isomorphic, and hence this A in (3.12) is 'more' non-symmetric than the corresponding specimen (3.15) in Example 3.8. The example is

$$A = \begin{pmatrix} 65 & 7 \\ 24 & 67 \end{pmatrix}. \tag{3.12}$$

Putting

$$E_A = \begin{pmatrix} -7 & 1 \\ 12 & 2 \end{pmatrix}, \quad E_B = \begin{pmatrix} -2 & 12 \\ 1 & 7 \end{pmatrix}, \quad D = \begin{pmatrix} 53 & 0 \\ 0 & 79 \end{pmatrix}, \tag{3.13}$$

we have

$$A = E_A D E_A^{-1}, \quad B = A^{\mathrm{tr}} = E_B D E_B^{-1}. \tag{3.14}$$

The eigenvalues of A and B are 53 and 79, which are both prime and congruent to -1 mod 13. Using Theorem 2.5 it follows that if A and B were C^*-equivalent there would exist some diagonal matrix $D_0 = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ where x, y are congruent to ± 1 mod 13 such that $E_A D_0 E_B^{-1}$ would have integral entries. But the (1, 1) entry of this matrix is $(49x + y)/26$. If this is an integer, and $x = \varepsilon_1 + n_1 \cdot 13$, $y = \varepsilon_2 + n_2 \cdot 13$, where n_1, n_2 are integers and $\varepsilon_i = \pm 1$, then $\frac{1}{13}(49x + y) = \frac{1}{13}((4 \cdot 13 - 3)x + y) = -\varepsilon_1 \frac{3}{13} + \varepsilon_2 \frac{1}{13}$ mod 1, but this can never be an integer. Thus A is not C^*-symmetric.

Remark 3.7. (Irrational λ) The assumption in Proposition 3.2 that the range of the respective traces τ_A and $\tau_{A^{\mathrm{tr}}}$ be the same (viewed as subgroups of $\mathbb{Z}[1/\lambda_{(A)}]$) cannot be omitted. It is true in general that $\mathrm{ran}(\tau_A)$ is an ideal in $\mathbb{Z}[1/\lambda_{(A)}]$, but the ideal may be proper, and it may be different from one to the other. An example showing this to be the case can be found in [**BMT87**, p. 104, **PaTu82**, pp. 79–83]. The example is a matrix A such that A and its transpose $B = A^{\mathrm{tr}}$ are not shift equivalent. We will give another example of this, and then apply Theorem 2.3 to show that they are not C^*-equivalent either.

Example 3.8. The example is $A = \left(\begin{smallmatrix} 19 & 5 \\ 4 & 1 \end{smallmatrix}\right)$. Here $\lambda = 10 + \sqrt{101}$, so the characteristic polynomial is irreducible and therefore $\ker(\tau_A) = 0$. Since $\det A = -1$, the unordered dimension groups $G(A)$ and $G(A^{\mathrm{tr}})$ are both $\mathbb{Z}^2$. However, we will show that they are not order isomorphic. We have

$$A = \begin{pmatrix} 19 & 5 \\ 4 & 1 \end{pmatrix}, \quad B = A^{\mathrm{tr}} = \begin{pmatrix} 19 & 4 \\ 5 & 1 \end{pmatrix}. \tag{3.15}$$

We prove that the two ideals $\mathrm{ran}(\tau_A)$ and $\mathrm{ran}(\tau_{A^{\mathrm{tr}}})$ are non-isomorphic. The eigenvalues are $10 \pm \sqrt{101}$. Let $\omega = (1 + \sqrt{101})/2$ so that $1, \omega$ form a $\mathbb{Z}$-basis for the algebraic integers in $\mathbb{Q}(\sqrt{101})$. (The fact that all algebraic integers in a quadratic field have this form is [**Wei98**, Theorem 6-1-1, p. 234]. One can check that $1, \omega$ are algebraic integers, then that the trace must be an algebraic integer, and see what happens when we subtract some $a + b\omega$ to simplify, in terms of the norm being an algebraic integer.) The respective (column) eigenvectors for A, A^{tr} are

$$\begin{pmatrix} 4 + \omega \\ 2 \end{pmatrix}, \begin{pmatrix} 5 - \omega \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ -5 + \omega \end{pmatrix}, \begin{pmatrix} -2 \\ 4 + \omega \end{pmatrix}. \tag{3.16}$$

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By transposing and interchanging the two, we get as Perron–Frobenius row eigenvectors for A, A^{tr}

$$\big(2, \quad \omega - 5\big), \quad \big(\omega + 4, \quad 2\big). \tag{3.17}$$

Let I_1, I_2 denote the ideals they generate. We note that $\omega - 5 = (-9 + \sqrt{101})/2$ and that the norm of this number is $(81 - 101)/4 = -5$. Hence, over the algebraic number ring, which is $\mathbb{Z}[\omega]$ and properly contains $\mathbb{Z}[1/\lambda]$, both ideals are the entire ring $(1) = \mathbb{Z}[\omega]$, since the two have isomorphic spans over the algebraic number ring. Thus, we need to see whether some element in the quotient field will multiply one ideal to the other, as additive groups, or modules over $\mathbb{Z}[1/\lambda] = \mathbb{Z}[\lambda] = \{a + b\sqrt{101} \mid a, b \in \mathbb{Z}\}$.

Note that the two generators listed in (3.17) will actually generate each ideal over $\mathbb{Z}$ additively, not just as modules over $\mathbb{Z}[\lambda]$, since multiplication by $\sqrt{101} = 2\omega - 1$ sends

$$\big(1, \quad \omega\big) \longrightarrow \big(2\omega - 1, \quad 2(\omega + 25) - \omega\big) = \big(2\omega - 1, \quad \omega + 50\big), \tag{3.18}$$

$$\big(2, \quad \omega - 5\big) \longrightarrow \big(4\omega - 2, \quad (\omega + 50) - 5(2\omega - 1)\big) = \big(4\omega - 2, \quad 55 - 9\omega\big), \tag{3.19}$$

$$\big(4 + \omega, \quad 2\big) \longrightarrow \big((\omega + 50) + 4(2\omega - 1), \quad 4\omega - 2\big) = \big(9\omega + 46, \quad 4\omega - 2\big), \tag{3.20}$$

which are still in the same additive subgroups.

The additive spans of the two pairs of generators in (3.17) are, respectively

$$2\mathbb{Z} + (\omega - 5)\mathbb{Z} = \{a + b\,\omega \mid a, b \in \mathbb{Z} \text{ such that } a - b \equiv 0 \pmod 2\}, \tag{3.21}$$

$$2\mathbb{Z} + (\omega + 4)\mathbb{Z} = \{a + b\,\omega \mid a, b \in \mathbb{Z} \text{ such that } a - b \equiv 1 \pmod 2\}. \tag{3.22}$$

These are preserved by multiplication by $\sqrt{101} = 2\omega - 1 \equiv 1 \pmod 2$, so that each span over $\mathbb{Z}$ is a $\mathbb{Z}[\lambda]$-module.

We will now complete the proof. If the ideals were isomorphic under multiplication by some $f \in \mathbb{Q}[\lambda]$, then f cannot involve primes of the algebraic number ring, since both ideals span the complete algebraic number ring as modules over it. Therefore f is a unit. Thus f is, up to a sign, a power of $\lambda = 9 + 2\omega \equiv 1 \pmod 2$. Hence multiplication by f preserves the congruence conditions defining the two additive spans, and thus it preserves each ideal separately. So it is impossible for a unit to send one ideal to the other.

Example 3.9. The following is an example of integer matrices A, B which have isomorphic dimension groups (unordered) but such that the corresponding transposed matrices A^{tr}, B^{tr} do not have isomorphic dimension groups. Informally, for matrices over $\mathbb{Z}$ with eigenvalues 1, p, q, the isomorphism type of the dimension group is determined by the way the p-divisible and q-divisible integer vectors lie, as subspaces, within all integer vectors:

$$\begin{pmatrix} 1 & 0 & 0 \\ x & p & 0 \\ y & 0 & q \end{pmatrix}.$$

The p, q-divisible spaces split off as spanned by $(0, 1, 0)$, $(0, 0, 1)$, hence the column dimension groups are the same no matter what x, y are. We can use the diagonal as a comparison. The row dimension groups depend on how the p and q eigenvectors lie in

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their sum. Row eigenvectors are in the row kernels of

$$\begin{pmatrix} 1-p & 0 & 0 \\ x & 0 & 0 \\ y & 0 & q-p \end{pmatrix}, \quad \begin{pmatrix} 1-q & 0 & 0 \\ x & p-q & 0 \\ y & 0 & 0 \end{pmatrix}.$$

These vectors are spanned by $(x, p-1, 0)$ and $(y, 0, q-1)$, respectively. If m divides $p-1$, $q-1$ and $x-y$ but not x, y then we have sum vectors which are divisible. This will give non-isomorphism of row dimension groups.

More formally, the column dimension groups consist of all vectors of the form $(a, b/p^n, c/q^n)$ (we may consider the form of the inverse and its columns). Any isomorphism on row dimension groups will be an integer matrix which preserves the row and column divisible eigenspaces. Take the case $p = 3$, $q = 5$, $x = y = 1$ as compared with $x = y = 0$. Then the row vectors $(0, 1, 0)$, $(0, 0, 1)$ for the diagonal case must be mapped to multiples by powers of 3, 5, -1 of vectors $(1, 2, 0)$, $(1, 0, 4)$, and the total determinant of the matrix must be odd. This should be a 2-adic isomorphism, but it cannot be since $(0, 1, 0) - (0, 0, 1)$ is not divisible by 2, whereas $(1, 2, 0) - (1, 0, 4)$ is divisible by 2, and the same is true if they are replaced by any of their odd multiples.

Remark 3.10. It follows from Theorem 3.1 of Boyle and Handelman [**BoHa93**] that there are non-negative integer matrices that are shift equivalent to the pair in Example 3.9 and hence have the same dimension groups. We will now construct an example where the ordered dimension groups are isomorphic for two matrices, but for the transpose matrices, the ordered dimension groups are not isomorphic.

We modify Example 3.9 a little bit so as to get an example of two non-negative integer matrices having identical ordered column dimension groups but non-isomorphic row dimension groups. We start with the same matrices as before except we take 3 and 7 as the two main diagonal primes. Then we add a large odd prime eigenvalue 101 which will enable a conjugate to be positive. This gives matrices A, B:

$$A = \begin{pmatrix} 101 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 101 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 7 \end{pmatrix}.$$

We multiply these matrices by 5 and then conjugate by the matrix C:

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 2 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix},$$

whose determinant is 5 and which approximately moves the row and column eigenvectors to $(1, 1, 1, 1)$ to get non-negative matrices

$$\begin{pmatrix} 205 & 200 & 200 & 200 \\ 104 & 109 & 94 & 94 \\ 110 & 110 & 105 & 70 \\ 86 & 86 & 106 & 141 \end{pmatrix}, \quad \begin{pmatrix} 205 & 200 & 200 & 200 \\ 103 & 108 & 93 & 93 \\ 111 & 111 & 106 & 71 \\ 86 & 86 & 106 & 141 \end{pmatrix}.$$

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These have identical dimension groups: the original unordered dimension groups are the same, by the above, just adding the eigenspace for the eigenvalue 101 in both cases. The order structure is determined by the Perron–Frobenius row eigenvalue, which is $(1, 0, 0, 0)C^{-1}$ in both cases.

However, for the transposes, row dimension groups, we have the direct sum of the 101 eigenspace with $\mathbb{Z}[1/5]$ times the previous examples. Making the prime 5 invertible will not affect the above argument that the row dimension groups are not isomorphic because this was a 2-adic non-isomorphism, given that the row eigenspaces for $5 \cdot 3$ and $5 \cdot 7$ eigenvalues must be preserved by an isomorphism. These are the spaces of vectors in the dimension group divisible by arbitrarily high powers of 3 and 7.

4. *The quotient $G/\mathbb{Z}^N$ is an invariant*

Recall that $G = G(A) = \bigcup_{n=0}^{\infty} A^{-n}\mathbb{Z}^N$. In this section we will consider the quotient group $G/\mathbb{Z}^N$. Here $\mathbb{Z}^N$ can be replaced with any free abelian subgroup L of G such that

$$G(A) = \bigcup_{n=0}^{\infty} A^{-n}L \tag{4.1}$$

and

$$AL \subset L. \tag{4.2}$$

We used the quotient group in [**BJO99**], but at the time we did not know if it was an invariant, and what the isomorphism classes were (in the category of abelian torsion groups). These issues are now resolved in the next proposition, which implies that the quotient is indeed an isomorphism invariant, i.e. that a given C^*-isomorphism implies that the corresponding two quotients are isomorphic groups.

Abelian torsion groups are classified in general by the so-called Ulm invariant [**Kap69** pp. 26–27, **KaMa51** and references given there]. The Ulm invariant in general is a sequence of natural numbers suitably indexed by ordinals. These numbers are calculated as dimensions of certain vector spaces over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. First, any given torsion group decomposes over its p-subgroups, and the Ulm dimensions are then calculated for each ordinal, when p is fixed. In the present application, the Ulm invariant is, as we show, very simple and concrete.

Example 4.1. (The quotient group $G/\mathbb{Z}^N$ for the special case of §1) It is easy to understand concretely the torsion group quotient for the special case of §1 when it is assumed that $|\det A| = \lambda_{(A)}$. Of course then $\lambda_{(A)}$ is an integer, and we may therefore form $\mathbb{Z}[1/\lambda_{(A)}]$ the usual way as an inductive limit $\bigcup_{n=1}^{\infty} \mathbb{Z}_{\lambda_{(A)}^n}$ as described in [**BJKR99a**, equation (4.3)] with natural embeddings $\mathbb{Z}_{\lambda_{(A)}^n} \hookrightarrow \mathbb{Z}_{\lambda_{(A)}^{n+1}}$, and it follows from the discussion in [**BJKR99a**, §4] and §1 above (Figure 1) that there is then a natural isomorphism between the two groups $G(A)/\mathbb{Z}^N$ and $\mathbb{Z}[1/\lambda_{(A)}]/\mathbb{Z}$. Hence, in this very special case, $\mathrm{Prim}(\lambda_{(A)})$ is a complete invariant for the corresponding torsion group quotient. See also [**BJO99**] for more details. For dimension groups $G(A)$ more general than those of §1 we need a non-trivial localization. The next two propositions deal with the general case, a modified construction and the appropriate localizations.

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One method of localizing at a prime p is to take the tensor product of an abelian group with $\mathbb{Z}[1/2, 1/3, \ldots, \widehat{1/p}, \ldots]$, inverting all primes except p; another is to tensor with the p-adic integers. Both agree for all torsion groups; the latter localization factors through the former. These tensor products are exact functors of abelian groups A which are subgroups of $\mathbb{Q}^N$, that is, they preserve exact sequences; this follows from [**CaEi56**, Proposition 7.2, p. 138], since the group $D(A)$ is a direct sum of copies of $\mathbb{R}/\mathbb{Z}$ which has no non-trivial continuous homomorphisms into the totally disconnected p-adics. Thus $\mathrm{Tor}^1(A, C)$ is zero. Unless otherwise specified we will mean the former, smaller tensor product when we localize.

PROPOSITION 4.2. *Assume that A is a non-singular $N \times N$ integer matrix and $G = \bigcup_n A^{-n}\mathbb{Z}^N$ the associated dimension group. Suppose L is any rank-N lattice in G such that $\bigcup_n A^{-n}L = G$. Then G/L is isomorphic to the product over primes dividing* $\det A$ *of a certain number $n(p)$ copies of $\mathbb{Z}_{p^\infty}$. The number $n(p)$ is the largest j such that if we write the characteristic polynomial of A as $x^N + c_1x^{N-1} + \cdots + c_{N-1}x + c_N$, p does not divide c_j. In addition, G is dual to the eventual row space in the sense*

$$G \otimes \mathbb{Z}[1/2, \ldots, \widehat{1/p}, \ldots] = \{v \in \mathbb{Q}^N \mid \langle w \mid v\rangle \in \mathbb{Z}_{(p)} \ \forall\, w \in G_{(p)}(A)\}. \tag{4.3}$$

Proof. G written as $\bigcup_{n=0}^{\infty} A^{-n}(\mathbb{Z}^N)$ will have as denominators only primes dividing $\det A$. If L includes $\mathbb{Z}^N$ then we have a torsion group whose torsion involves only primes in $\det A$.

We first argue that locally at each prime p in it, G consists of those vectors dual to the eventual p-adic row space $G_{(p)}(A)$ of A (see [**BJKR99a**, equations (4.12) and (4.13)]). That is, (4.3) holds. The dimension group is the group of vectors x such that for some n, $A^n x \in \mathbb{Z}^N$. This is the group of vectors such that $\exists\, n \in \mathbb{Z}_+$ such that for $\forall\, w \in \mathbb{Z}^N$, we have $wA^n x \in \mathbb{Z}$. This is the group of rational vectors whose products with the row space of A^n integer. This construction also goes through if we localize at any prime. To say that a vector has p-integer product with the row space of A^n for some n then implies that it has p-integer product with the idempotent p-adic limit $E_{(p)}(A)$ of powers of A, defined in [**BJKR99a**, equation (4.8)] and mentioned in [**BJKR98**, Theorem 7]. Conversely suppose it has p-integer product with the idempotent p-adic limit, then by p-adic continuity, it must have p-integer product with some finite power. This gives the claim.

Now to show that the quotient group at the prime p is p-divisible, take a p-adic dual basis to $G_{(p)}(A)$, which, like any p-adic torsion-free module, must be a free module (the p-adic integers are a principal ideal domain, and argue as with the ordinary integers). Approximate these vectors p-adically by rational vectors b_i using a p-adic approximation theorem such as [**Wei98**, 1-2-3, p. 8], choosing these rational vectors so that they give a p-adic basis. Take the free abelian group L_1 generated by b_i. As soon as we have a lattice L including L_1 and $\mathbb{Z}^N$, the p-adic dimension group consists of a sum of copies of the p-adic integers corresponding to L_1 and a sum of copies of the p-adic field corresponding to the remaining vectors (in the null space of $E_{(p)}(A)$—we can take additional basis vectors for it). When we divide by L, we are dividing out by all the L_1 part p-adically, and by something isomorphic inside a p-adic field in the rest, and the result will be p-divisible.

In fact, for any lattice L such that $\bigcup_n A^{-n}L$ is the dimension group, the quotient

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will be isomorphic to this, since multiplication by A^{-1} gives an isomorphism of pairs $\left(\bigcup_n A^{-n}L, L\right) \to \left(\bigcup_n A^{-n}L, A^{-1}L\right)$, and eventually this lattice must be large enough.

Now consider the p-adic rank, in relation to the characteristic polynomial. By Newton's method [**Wei98**, 3-1-1, p. 74], if the characteristic polynomial has the given form, we can factor it over the p-adics as a product of two polynomials, one of which is x^{N-j} modulo p, and the other of which has invertible constant term over the p-adics. We can put the matrix into corresponding block form. The former part will be p-adically nilpotent, and the null space will be its row space. □

COROLLARY 4.3. *Let A, B be matrices as in Proposition 4.2. For any choice of lattices L, L' in their dimension groups satisfying the hypotheses in Proposition 4.2, $G(A) \cong G(B)$ implies $G(A)/L \cong G(B)/L'$.*

Remark 4.4. As noted, our groups $G(A)$ are contained in $\mathbb{R}^N$ (even in $\mathbb{Q}^N$) where N is the rank of $G(A)$. However, it is clear that general lattices L in $\mathbb{R}^N$ are given by a choice of basis in $\mathbb{R}^N$ as a vector space. Writing the vectors in a basis, equivalently the generators for L, as column vectors, we note that the lattices L may be viewed as, or identified with, non-singular real matrices. Making this identification, and fixing the rank N, we further note that the containment $L \subset L'$, for two given lattices, holds if and only if there is some $C \in M_N(\mathbb{Z})$ such that we have the following matrix factorization:

$$L = L'C. \tag{4.4}$$

There is a similar version of this for row spaces (or lattices defined from row vectors), as well as a p-adic variation, *mutatis mutandis;* and we have already seen an instance of the latter in [**BJKR99a**, equations (4.11)–(4.13)].

Remark 4.5. We now show, using (4.4), that the conditions on L from Proposition 4.2 are all integrality conditions. There are three in all, and we proceed to spell them out. If A is given as usual, and if $G(A)$ is the corresponding group, i.e. $\bigcup_{n=0}^{\infty} A^{-n}(\mathbb{Z}^N)$, then a lattice L is a subgroup, i.e. $L \subset G(A)$, if and only if there is a natural number n such that

$$A^n L \in M_N(\mathbb{Z}). \tag{4.5}$$

Some given lattice L will satisfy the invariance property $A(L) \subset L$ if and only if the conjugate matrix $L^{-1}AL$ satisfies

$$L^{-1}AL \in M_N(\mathbb{Z}). \tag{4.6}$$

The further condition on L that it is generating, i.e. that $\bigcup_{n=0}^{\infty} A^{-n}(L) = G(A)$, holds if and only if for some natural number n we have

$$L^{-1}A^n \in M_N(\mathbb{Z}). \tag{4.7}$$

The three conditions should also be compared with [**BJKR99a**, equation (1.17)].

Now Proposition 4.2 applies when $G(A)$ is given and some lattice satisfies all three conditions (4.5)–(4.7), and we get as a corollary that if two lattices L and L' both satisfy the conditions, then the two torsion groups $G(A)/L$ and $G(A)/L'$ are isomorphic groups.

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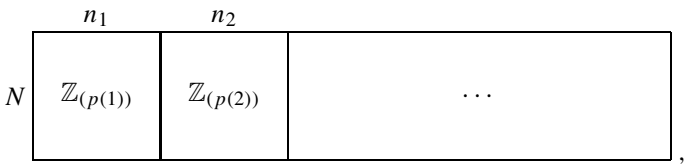
In the study of dimension groups, it is convenient to explicitly compute certain extensions. Let $\mathbb{Z}_{p^\infty}$ denote the union of $\mathbb{Z}_{p^n}$ under inclusion, a divisible p-torsion group whose order p subgroup has rank 1. By standard theory [**CaEi56**], the extension group $\mathrm{Ext}(\mathbb{Z}_{p^\infty}, \mathbb{Z})$ can be computed using the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \tag{4.8}$$

as the cokernel of the map $\mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{Q}) \to \mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{Q}/\mathbb{Z})$; the former group is zero and the latter group is $\mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty})$. Every p-adic integer gives a mapping in this group; we check this mapping is one-to-one and onto, so that the Ext group is the p-adic integers. (To check 'onto', note that we get every mapping $\mathbb{Z}_{p^n} \to \mathbb{Z}_{p^n}$, and take limits.) In general, we are dealing with a direct sum of copies of these Ext groups.

Next we look at the problem of isomorphism of dimension groups in a somewhat different way, by showing that dimension groups can easily be computed as extensions. In some cases this leads to a quick decision about whether two dimension groups are isomorphic. However, in the most general case, the problem of deciding isomorphism given this extension structure seems to still require the methods of [**BJKR99a**, §5]. In view of Remark 4.4 we need only state the result for the case when the lattice L is $\mathbb{Z}^N$.

COROLLARY 4.6. *As in Proposition 4.2, consider an unordered dimension group as an extension of $\mathbb{Z}^N$ by a divisible torsion group $G/\mathbb{Z}^N$ whose structure, computed as in Proposition 4.2, is a direct sum over i of n_i copies of $\mathbb{Z}_{p(i)^\infty} = \mathbb{Z}[1/p(i)]/\mathbb{Z}$. The extension class in $\mathrm{Ext}^1(G/\mathbb{Z}^N, \mathbb{Z}^N)$ is an element of $\bigoplus_i \bigoplus_{j=1}^{n_i} \mathbb{Z}_{p(i)}$. We write this as an $N \times \sum_i n_i$ matrix whose entries are $p(i)$-adic integers:*



where $p(i)$ runs over all elements in $\mathrm{Prim}(\det(A))$. *Its columns consist precisely of a basis for the null-space of the matrices $E(A)$ taken at each prime $p(i)$. Two such matrices M_1, M_2 represent isomorphic unordered dimension groups if and only if there is a matrix $C \in \mathrm{GL}(N, \mathbb{Z}[1/\det(A)])$ and an invertible direct sum of $p(i)$-adic integer matrices D such that $CM_1D = M_2$.*

Proof. The given structure (that is, M_1 up to its equivalence with any CM_1D) is an isomorphism invariant because the p-adic row spaces $G_{(p)}(A)$ defined in [**BJKR99a**, equations (4.12) and (4.13)] are invariants. It is shown in [**BJKR99a**, Corollary 4.2] that a rational matrix over $\mathbb{Z}[1/\det(A)]$ giving an isomorphism on dimension groups must give an isomorphism on the p-adic row spaces, hence the dual p-adic null spaces of $E_{(p)}(A)$. In fact [**BJKR99a**, Corollary 4.2] gives, as necessary and sufficient conditions for the existence of an unordered dimension group isomorphism, in effect, the existence of C and D: the p-adic symmetries just mean we are considering the row spaces up to isomorphism, and the $\mathrm{GL}(N, \mathbb{Z}[1/\det A])$ symmetry means that we have a rational map which is an isomorphism at all primes other than the ones considered here.

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The extension class of any extension of $\mathbb{Z}^N$ by a group $G/\mathbb{Z}^N$ may be computed by extending the map $\mathbb{Z}^N \subset \mathbb{Q}^N$ to a mapping $G \to \mathbb{Q}^N$, and letting this give a map in $\mathrm{Hom}(G/\mathbb{Z}^N, (\mathbb{Q}/\mathbb{Z})^N) \cong \mathrm{Ext}^1(G/\mathbb{Z}^N, \mathbb{Z}^N)$. This is the remark of Cartan–Eilenberg [**CaEi56**, p. 292]. To identify this class it suffices to look at the p-torsion subgroup of $G/\mathbb{Z}^N$ for each prime p since the group is the direct sum of its p-torsion subgroups. To identify this class, take the tensor product of G with the p-adic integers, getting a localized extension of $\mathbb{Z}^N_{(p)}$ by the p-torsion subgroup of $G/\mathbb{Z}^N$, which is $G \otimes \mathbb{Z}^N_{(p)}$. However, if we write all p-adic vectors as the direct sum $K \oplus R$ of the p-adic null space of R_A and a complementary space R, by Proposition 4.2,

$$G \otimes \mathbb{Z}^N_{(p)} = (K \otimes \mathbb{Q}_{(p)}) \oplus (R \otimes \mathbb{Z}_{(p)}). \tag{4.9}$$

Thus the extension class is represented taking

$$(K \otimes \mathbb{Q}_{(p)}) \oplus (R \otimes \mathbb{Z}_{(p)}) \to (K + R) \otimes \mathbb{Q}_{(p)} \tag{4.10}$$

and collapsing by $\mathbb{Z}^N_{(p)}$ to give the inclusion

$$(K \otimes \mathbb{Q}_{(p)})/(K \otimes \mathbb{Z}_{(p)}) \to (K + R) \otimes (\mathbb{Q}_{(p)}/\mathbb{Z}_{(p)}) = (\mathbb{Q}_{(p)}/\mathbb{Z}_{(p)})^N. \tag{4.11}$$

This map is induced by the map

$$(K \otimes \mathbb{Q}_{(p)}) \to (K + R) \otimes \mathbb{Q}_{(p)} = \mathbb{Z}^N \otimes \mathbb{Q}_{(p)} \tag{4.12}$$

which can be taken to send the ith unit vector on the left to the ith vector in a basis for K on the right. This means taking basis vectors for the null space of E_A as forming the columns of the matrix giving the extension. □

In Example 6.3 we will give an example where the groups G/L are the same, but the extensions are different.

5. *Remarks on the singular case*

Except for [**BJKR99a**, §§2 and 6 and Theorem 5.9], we have considered AF-algebras defined by non-singular primitive matrices $A, B, \ldots$. Let us comment that the class of dimension groups arising from primitive matrices does change if one dispenses with the non-singularity assumption. George Elliott has given an example of a dimension group which in his terminology is not ultrasimplicial. For example, the group arising from the stationary diagram associated with the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$$

cannot arise from a diagram associated with a non-singular matrix [**Ell79**]. Elliott actually proves the following: if G is the dimension group defined by the matrix via Bratteli diagrams, then it is not possible to span a given finite set of positive elements of G by another finite set of positive linearly independent (over $\mathbb{Z}$) elements, where the span uses only positive coefficients. However, it is not hard to see that this is equivalent to the

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impossibility of writing the group as a direct limit with injective maps, that is, with non-singular matrices. In the case at stake, Elliott gives the following three positive elements of G which cannot be spanned in the above manner: $(1, 0)$, $(1, -1)$, $(1, 1)$. In this case G is isomorphic to the direct product of $\mathbb{Z}[1/3]$ and $\mathbb{Z}$, with order defined by the condition that the positive elements are those elements which have strictly positive first coordinate—the one from $\mathbb{Z}[1/3]$. The matrix is primitive.

Since C^*-equivalence is weaker than shift equivalence, the matrix above is also an example of a matrix which is primitive, but not shift equivalent to a primitive non-singular matrix. Other such examples may be found in [**BoHa91**].

We do not expect that the analogue of our Theorem 5.3 is true in the context of shift equivalence. (For example, it looks like the example [**BoHa91**, Appendix 3, p. 310] can be modified as follows (multiply approximately by 10). There is no (4×4) non-negative matrix whose spectrum is $\{14, 10i, -10i, 3\}$. However, we can readily construct an integer (4×4) matrix with this spectrum (block diagonal), with the eigenvector at 14 being $(1, 0, 0, 0)$. Then we take a unimodular matrix mapping the vector $(1, 0, 0, 0)$ to $(1, 1, 1, 1)$. Then some power of this conjugated matrix is non-negative, giving us the situation of Theorem 5.3. Moreover it should follow from [**BoHa93**, Theorem 3.1] that this matrix over $\mathbb{Z}$ is shift equivalent to a primitive non-negative matrix of larger dimension.) For our weaker C^*-equivalence we have the added flexibility of replacing matrices by powers of themselves.

We will prove in Theorem 5.3 below that, in some special circumstances, the condition of non-singularity of the matrix A can be removed, and A merely assumed to be primitive, without changing the class of C^*-algebras. In the general case when A is not assumed invertible, we may introduce the eventual range of A,

$$\mathcal{W}(A) := \bigcap_{i=0}^{\infty} A^i \mathbb{Q}^N = A^N \mathbb{Q}^N. \tag{5.1}$$

Note that A is bijective as a map $\mathcal{W}(A) \to \mathcal{W}(A)$. We may now introduce an additive group $G(A)$ by

$$G(A) := \{g \in \mathcal{W}(A) \mid A^k g \in \mathbb{Z}^N \text{ for some } k \in \mathbb{Z}_+\}, \tag{5.2}$$

and one notes that this group $G(A)$ identifies with the inductive limit of the sequence

$$\mathbb{Z}^N \xrightarrow{A} \mathbb{Z}^N \xrightarrow{A} \mathbb{Z}^N \longrightarrow \cdots, \tag{5.3}$$

i.e. $G(A)$ is the dimension group when it is equipped with the obvious order. (This version of $G(A)$ was used, but not defined, in [**BJKR99a**, §2].) Let us give some details. An element of the inductive limit can be represented by a sequence $\{g_m, g_{m+1}, \dots\}$ in $\mathbb{Z}^N$ with $Ag_n = g_{n+1}$ for $n = m, m+1, \dots$. Two such sequences represent the same element if they coincide from a certain step n onward. Given such a sequence, there is a unique sequence $\{h_1, h_2, \dots\}$ in $\mathcal{W}(A)$ such that $Ah_n = h_{n+1}$ for $n = 1, 2, \dots$ and such that $h_n = g_n$ for all large n. Then h_1 is the element of $G(A) \subset \mathcal{W}(A)$ representing the dimension group element in (5.2), so this shows the equivalence between the two definitions (5.2) and (5.3) of $G(A)$. The definition (5.2) is the definition used in [**BMT87**, p. 49]. If A is non-singular

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and, as in (5.2), $A^k g = m \in \mathbb{Z}^N$, then $g = A^{-k}m$ is a typical element of the [**BJKR99a**, equations (1.8)–(1.10)] version of $G(A)$, and *vice versa*. If A is primitive, we still have the Perron–Frobenius data, and the order can be defined as before, *mutatis mutandis.*

LEMMA 5.1. *Given a vector $u \in \mathbb{R}^r$ there exist r vectors $w_i \in \mathbb{Z}^r$ such that the convex cone generated by the w_i contains an open neighborhood of u, and the determinant of the matrix the w_i form is ± 1.*

Proof. The standard unit vectors do this for any vector in the subsemigroup of strictly positive integer vectors. We claim transforms of these by integer row and column operations, permutations, and reversals of sign, take any vector to the interior of this subsemigroup—then just reverse those operations on the standard basis vectors. In fact, we get all coordinates non-zero by certain linear combinations, then reverse their signs. □

Remark 5.2. It is not in general possible to get a determinant-1 system of matrices which approximate multiples by some positive constant C of a given set of non-negative vectors w_i in Lemma 5.1. This is easiest to see when the vectors w_i are chosen diagonally dominant. However, Lemma 5.1 can probably be strengthened a little.

THEOREM 5.3. *Let A be an integer primitive matrix. Suppose that when the vector u in Lemma 5.1 is the Perron–Frobenius eigenvector of A, the vectors $w_i \in \mathbb{Z}^r$ can be chosen to be positive in terms of the order structure of the dimension group of A. Then the ordered dimension group arising from the primitive integer matrix A is order isomorphic to one arising from a non-singular primitive integer matrix B.*

Proof. Let the dimension of A be d and the rank of all sufficiently large powers A^s be r. By Lemma 5.1, we find a set of r vectors w_i in the eventual row space R, that is, the row space of A^N, or some specific higher power, a rank-r subspace of $\mathbb{Z}^d$ such that the cone over $\mathbb{Q}_+$ generated by this set includes a neighborhood of the maximum eigenvector v within R. This is sufficient to establish that all sufficiently large powers of A have their rows expressed as (unique) non-negative linear combinations of w_i, since all rows of A^s divided by their lengths converge to fixed multiples of v and hence are eventually in the convex cone; but to be in the convex cone means that we have these convex combinations.

However, we also need to have that it can be chosen, so these convex combinations are eventually integers. For this, it suffices that the determinant of the w_i expressed as combinations of a basis for the integral vectors in the eventual row space, i.e. $R \cap \mathbb{Z}^d$, a rank-r free abelian group, is 1 or -1. This follows from the lemma and the extra assumption.

Now let B be the matrix of A^s expressed as acting on the vectors w_i, which will be non-negative, and positive. Then B is shift equivalent to A^s over the integers (maybe with negative entries), just by the inclusion mapping given by the vectors w_i. By a theorem of Parry and Williams [**PaWi77**] (reproved in our 1979 paper [**KiRo79**]), any shift equivalence over $\mathbb{Z}$ of primitive matrices can be realized by a shift equivalence over $\mathbb{Z}_+$. This shift equivalence will induce an isomorphism of ordered dimension groups. □

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6. *Strong local isomorphism*

Definition 6.1. We will say that two dimension groups G, G' are strongly locally isomorphic at the prime p if and only if there is an isomorphism $G \otimes \mathbb{Z}_{(p)} \to G' \otimes \mathbb{Z}_{(p)}$ induced by a matrix of integers. (The first paragraph of the proof below shows that this is equivalent to requiring that the isomorphism be induced by a matrix of rational numbers.) (Recall that G and G' are locally isomorphic at prime p if there merely is an isomorphism $G \otimes \mathbb{Z}_{(p)} \to G' \otimes \mathbb{Z}_{(p)}$.)

In the next theorem we show that strong local isomorphism is described by a condition similar to that in Corollary 4.6 if we just take the submatrix corresponding to the prime in question. This condition is rather strong and can be decided by a simpler algorithm than the general algorithm in [**BJKR99a**, §5]. When we speak of realizing some p-adic construct over an algebraic number field K, we mean in terms of an inclusion $K \subset \mathbb{Z}_{(p)}$ corresponding to a non-archimedean completion of K. Note in connection with the following theorem that if the ranks of the p-adic eventual row spaces of A and B are different, local isomorphism cannot hold. Also note that the equivalent conditions imply that the smallest fields over which the eventual p-adic row spaces can be realized are the same for A and B.

THEOREM 6.2. *Given two non-negative matrices A, B, form a matrix whose rows are a basis for the p-adic eventual row spaces of A, B whose ranks are n_p. Their dimension groups are strongly locally isomorphic at the prime p if and only if the corresponding two matrices A, B for each p admit some matrices $C \in \mathrm{GL}(n, \mathbb{Q})$, $D \in \mathrm{GL}(n_p, \mathbb{Z}_{(p)})$ such that $AC = DB$. This condition is decidable.*

Proof. The proof of [**BJKR99a**, Corollary 4.2], or alternatively the proof of Proposition 4.2 above, shows that having a rational mapping which induces isomorphism of p-adic dimension groups is equivalent to having a rational mapping which induces an isomorphism of p-adic eventual row spaces. To make such a mapping integer, multiply by all denominators relatively prime to p. At p we must have integrality on the eventual row space $G_{(p)}(A)$ which is a summand of the space of all p-adic integer vectors. The use of a projection $E_{(p)}(A)$ to this subspace enables us to get a map equal to the given integer mapping defined over some algebraic number field, p-adic integer, and equal to the rational mapping on the eventual row space. The irrational part of this map will also consist of algebraic integers, since $\mathbb{Z}$ will be an additive summand of the algebraic number ring, and can be discarded, since it must be zero on the summand. This gives a matrix of integers inducing the p-adic isomorphism, and thus the local isomorphism is strong.

A rational mapping inducing an isomorphism of p-adic row spaces is equivalent to having a matrix C (giving the rational mapping) and a matrix D such that $DAC = B$ (where D expresses the image of the row basis vectors of A as linear combinations of a basis for B); it must be p-adic integer and invertible since we can also map backwards by the isomorphism of row spaces.

Next we show that this criterion is decidable. The field generated by all eigenvalues of A or B will be sufficient to realize the p-adic eventual row spaces: we take this field and some prime π giving an embedding in an extension of $\mathbb{Z}_{(p)}$. Then the p-adic eventual row

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space is spanned by the generalized eigenspaces for eigenvalues which are relatively prime to π, since multiplication by powers of A will not send them to zero, but will annihilate all other generalized eigenspaces, modulo any power of π. Some linear combinations of basis vectors for these generalized eigenspaces must give a p-adic basis.

The required p-adic matrix D then must lie in this field K, since C, A, B do. Being p-adic integer means that its denominators are relatively prime to p. It can then be expanded as a larger matrix over $\mathbb{Z}_{ip} = \mathbb{Z}[1/2, 1/3, \ldots, \widehat{1/p}, \ldots]$, using a basis for the p-adic integers of K over $\mathbb{Z}_{ip}$. Given the matrix D, the condition that a corresponding C exists over $\mathbb{Q}$ can be stated by linear equations in D. Existence of rational C means that all columns of BD are rational combinations of the columns of A. For the expanded matrices over $\mathbb{Z}_{ip}$, that means that we have any linear column combinations of the columns of A yielding the desired columns of BD^{-1}. In turn that means that the columns of BD^{-1} have zero inner product with all vectors which have zero inner product with the columns of A, which is a linear condition.

We can now determine a p-adic basis for this linear space of matrices D, write the determinant of D as a polynomial in the coefficients of a general linear combination of basis elements, and determine whether or not it is possible for the determinant to be non-zero modulo p by testing each congruence class of entries modulo p. □

Example 6.3. Let

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.1}$$

The respective characteristic polynomials are $x^2 - 6x + 7$ and $x^2 - 8x + 7$, with determinant 7, and we consider the local dimension groups at 7. Since 7 does not divide 8, only one root of the former polynomial is divisible by 7. Thus only the identity element of the $\mathbb{Z}_2$ Galois group fixes the eigenvalue not divisible by 7. This implies that the 7-adic row space is irrational, and the minimal fields over which eventual row spaces are defined are respectively $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}$, so the dimension groups are not locally isomorphic. Note that, even so, the two quotient groups $G(A)/\mathbb{Z}^2$ and $G(B)/\mathbb{Z}^2$ are isomorphic. This follows from Proposition 4.2: recall, to verify this we need only compute the respective Ulm numbers from the characteristic polynomials, and there is only the prime $p = 7$ to check. So the 7-reduced rank is 1 for each of the two quotient torsion groups calculated from A and B.

Example 6.4. Our next example illustrates C^*-symmetry, as well as the calculation of the p-adic eventual row spaces. Consider

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix} \tag{6.2}$$

and its transpose $B = A^{\mathrm{tr}}$; for this particular choice of the pair A, B, the minimal fields over which eventual row spaces are defined are isomorphic. We can arbitrarily choose which root of the characteristic polynomial $x^2 - 6x + 7$, the same as for A in Example 6.3, is divisible by 7 (representing the unique p-adic root which is divisible by 7), say $3 - \rho$ where ρ is a square root of 2 in $\mathbb{Z}_{(7)}$ (see next paragraph). The eventual row eigenspaces are spanned by the other row eigenvectors, which are $(1, \rho)$ for B and $(\rho, 1)$ for A. A mapping

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of eigenspaces must map one to a multiple c times the other. If it commutes with the Galois action, then it must do the same for their conjugates, so that it has the form

$$\begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}. \tag{6.3}$$

The determinant restricted to the eventual 7-adic row space is c, so the congruences are $c \equiv 0 \pmod 7$, which are solvable. The dimension groups of this matrix and its transpose are locally isomorphic at the prime 7. Since 7 is the only prime involved, this implies global isomorphism of the unordered dimension groups.

Here we think of the field $\mathbb{Q}[\sqrt{2}]$ as embedded in the field of 7-adics $\mathbb{Q}_{(7)}$ via $1 \mapsto 1$ and $\sqrt{2} \mapsto \rho$, where $\rho \in \mathbb{Z}_{(7)} \subset \mathbb{Q}_{(7)}$. The polynomial $x^2 - 2$ may be considered in $\mathbb{Z}_{(7)}[x]$, and it is reducible there. To find the two roots $\pm\rho$ in $\mathbb{Z}_{(7)}$, calculate the terms $t_0, t_1, \ldots \in \{0, 1, 2, \ldots, 6\}$ in $\rho = t_0 + t_1 \cdot 7 + t_2 \cdot 7^2 + \cdots \in \mathbb{Z}_{(7)}$ recursively, starting with $t_0 = 3$ or $t_0 = 4$ (see, for example, [**BoSh66**, p. 18]). They can be found with Maple or the PARI program. The results are $\rho = 3 + 1 \cdot 7 + 2 \cdot 7^2 + 6 \cdot 7^3 + 1 \cdot 7^4 + 2 \cdot 7^5 + \cdots$ and $-\rho = 4 + 5 \cdot 7 + 4 \cdot 7^2 + 0 \cdot 7^3 + 5 \cdot 7^4 + 4 \cdot 7^5 + \cdots$. Since the root $3 - \rho$ of $x^2 - 6x + 7$ is divisible by 7, the 7-adic eventual row spaces in $(\mathbb{Z}_{(7)})^2$ are the respective $\mathbb{Z}_{(7)}$-modules

$$G_{(7)}(A) = \mathbb{Z}_{(7)}(\rho, 1)$$

and

$$G_{(7)}(B) = \mathbb{Z}_{(7)}(1, \rho),$$

by [**BJKR99a**, Remark 4.3], i.e. they are generated over $\mathbb{Z}_{(7)}$ by the respective row eigenvectors corresponding to the second eigenvalue $3 + \rho$, the one not divisible by 7.

Note also in this case that A and A^{tr} are conjugate by the unimodular matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; and hence it even follows directly as in Example 3.1 that A and A^{tr} are (elementary) shift equivalent.

Note finally that if one denotes the A matrices in (6.1) and (6.2) by A_1, A_2, respectively, and one defines

$$J = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \tag{6.4}$$

then $A_2 J = J A_1$, and thus if $K = A_1 J^{-1}$ we have the system

$$A_2 = JK, \quad A_1 = KJ. \tag{6.5}$$

But K does not have positive matrix entries, so this does not imply elementary shift equivalence. However, if we redefine

$$K = A_1^2 J^{-1} = \begin{pmatrix} 11 & 6 \\ 1 & 3 \end{pmatrix}, \tag{6.6}$$

then we have the pair of shift relations for the squares,

$$A_1^2 = KJ, \quad A_2^2 = JK, \tag{6.7}$$

which is the assertion that A_1^2 and A_2^2 are elementary shift equivalent. In particular, A_1 and A_2 are C^*-equivalent. This latter conclusion and the one in Example 3.1 also follow the next general observation.

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Observation 6.5. If A, B are non-singular primitive $(N \times N)$ matrices and there exists a unimodular matrix J in $M_N(\mathbb{Z})$ such that

$$v(B)J = \mu v(A) \tag{6.8}$$

for a positive number μ, and

$$BJ = JA, \tag{6.9}$$

then A and B are C^*-equivalent.

Proof. Since J is unimodular, we have

$$\begin{cases} B^n J A^{-n} = J A^n A^{-n} = J \in M_N(\mathbb{Z}), \\ A^n J^{-1} B^{-n} = A^n A^{-n} J^{-1} = J^{-1} \in M_N(\mathbb{Z}), \end{cases} \tag{6.10}$$

and the observation follows from [**BJKR99a**, equations (1.16) and (1.17)]. (The condition (6.8) may be replaced by the strictly stronger requirement that J and J^{-1} have only non-negative matrix entries.) □

Remark 6.6. In fact we have the 'partial' implication (6.9) ⇒ (6.8), but (6.8) for some real scalar μ, while the positivity restriction on μ is not a consequence of (6.9) alone. We further stress that (6.8)–(6.9) are more restrictive than C^*-equivalence, even more restrictive than shift equivalence: take, for example, $A = \left(\begin{smallmatrix} 2 & 1 \\ 4 & 4 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 2 & 2 \\ 2 & 4 \end{smallmatrix}\right)$, which are shift equivalent by [**Bak83**], but do not satisfy (6.9).

To summarize, the two examples have four matrices in all, and the first one in Example 6.3 is C^*-equivalent to the two in Example 6.4, but $\left(\begin{smallmatrix} 7 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ from Example 6.3 is not C^*-equivalent to the other three. The first one, $\left(\begin{smallmatrix} 4 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ in Example 6.3, is symmetric, and A from Example 6.4 is C^*-symmetric in that it is C^*-equivalent to its own transpose.

Remark 6.7. Note that the two matrices A_1, A_2 in (6.1), (6.2) considered above are elementary shift equivalent over $\mathbb{Z}$ since they are conjugate over $\mathbb{Z}$. But while A_1^2, A_2^2 are elementary shift equivalent over $\mathbb{Z}_+$, A_1 and A_2 are not! (These types of (2×2) examples have been considered earlier by Kirby Baker [**Bak83, Bak87**].) This is seen as follows. Suppose $A_1 = CD$ where C, D are non-negative integer 2×2 matrices. Then C expresses the rows of A_1 as non-negative integer combinations of the rows of D. The entries 1 in the rows of A_1 can only come from entries 1 in the rows of D. Moreover, these 1's can only be in the same row. Furthermore, in the linear combinations these 1's can only be multiplied by 1's. So the product CD looks like, up to symmetry,

$$\begin{pmatrix} 1 \cdot c_{12} & d_{11} \cdot 1 \\ c_{21} \cdot 1 & 1 \cdot d_{22} \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}. \tag{6.11}$$

But if we write out the equations, there are no solutions unless one of C, D is a permutation matrix, and thus DC cannot be equal to A_2.

7. *Concluding remarks*

In the paper we addressed the interplay between the local and the global versions of the isomorphism problem. There are different, but related, decidability results in the literature. Ax and Kochen [**AxKo65a, AxKo65b, AxKo66**] and Grunewald and Segal [**GrSe82**] address decidability in a p-adic setting.

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