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On Gibbs measures for almost additive sequences associated to some relative pressure functions

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Abstract. Given a weakly almost additive sequence of continuous functions with bounded variation $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on a subshift X over finitely many symbols, we study properties of a function f on X such that $\lim_{n\to\infty}(1/n)\int \log f_n \, d\mu = \int f \, d\mu$ for every invariant measure μ on X. Under some conditions, we construct a function f on X explicitly, and study a relation between the property of $\mathcal{F}$ and some particular types of f. As applications, we study images of Gibbs measures for continuous functions under one-block factor maps. We investigate a relation between the almost additivity of the sequences associated to relative pressure functions and the fiber-wise sub-positive mixing property of a factor map. For a special type of one-block factor maps between shifts of finite type, we study necessary and sufficient conditions for the image of a one-step Markov measure to be a Gibbs measure for a continuous function.

Key words: symbolic dynamical systems, thermodynamic formalism, relative pressure, almost additive potentials, Gibbs measures
2020 Mathematics Subject Classification: 37D35 (Primary); 37B10 (Secondary)

1. *Introduction*

The thermodynamic formalism for sequences of continuous functions generalizes that for continuous functions and it has been developed in connection to the study of dimension problems. In particular, Barreira and Mummert independently studied the sequences called almost additive sequences in [1, 13], where the variational principle and definition of the Gibbs measure for continuous functions are generalized for such sequences. Let (X, σ_X) be a subshift over finitely many symbols. In [1, 8, 13], it was shown that if the sequence

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$\mathcal{F}$ of continuous functions is almost additive on X with the weak specification property and has bounded variation, then it has a unique equilibrium state and this equilibrium state is a unique invariant Gibbs measure for $\mathcal{F}$. Later, Feng and Huang [9] considered a more general class of sequences of continuous functions called asymptotically additive sequences and established the variational principle for such sequences.

Recently, Cuneo [6] gave a result which relates the thermodynamic formalism for continuous functions and that for asymptotically additive sequences of continuous functions. He showed that if a sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on a subshift X is asymptotically additive, then there exists a function $\hat{f} \in C(X)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \|\log f_n - S_n \hat{f}\|_\infty = 0, \tag{1.1}$$

where $(S_n \hat{f})(x) := \sum_{i=0}^{n-1} \hat{f}(\sigma_X^i(x))$ and $\|\cdot\|_\infty$ is the supremum norm. Such a function $\hat{f}$ is not unique. Equation (1.1) implies

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n \, dm = \int \hat{f} \, dm \quad \text{for every } m \in M(X, \sigma_X), \tag{1.2}$$

where $M(X, \sigma_X)$ is the set of all σ_X-invariant Borel probability measures on X. Therefore, the sequence $\mathcal{F}$ and function $\hat{f}$ satisfying (1.1) have the same equilibrium states and finding equilibrium states for asymptotically additive sequences is equivalent to finding equilibrium states for continuous functions. However, no explicit form for $\hat{f}$ is known and many questions concerning its properties remain open. For example, it is not known under which conditions of $\mathcal{F}$ we can find a function $\hat{f}$ which belongs to the Bowen class. It is easy to see that if the sequence $\mathcal{F}$ has a unique invariant Gibbs measure μ on a subshift X with the weak specification property and $\hat{f}$ satisfying (1.1) belongs to the Bowen class (equivalently, μ is an invariant Gibbs measure for $\hat{f}$ satisfying (1.1)), then $\mathcal{F}$ is almost additive with bounded variation on X. However, we do not know if the reverse implication holds. In other words, we do not know if μ being a unique invariant Gibbs measure for an almost additive sequence $\mathcal{F}$ with bounded variation implies that there exists a function $\hat{f}$ in (1.1) for which μ is Gibbs. This question is addressed in [6]. Hence, in this paper, we study the following questions.

[Q] What form does $\hat{f}$ take when $\mathcal{F}$ is (weakly) almost additive? What are the properties of $\hat{f}$?

For this purpose, we first investigate a natural form of a Borel measurable function $\hat{f}$ on X satisfying (1.1). We extend the notion of the (weak) Gibbs measure for a continuous function to a Borel measurable function (see §2). In Theorem 3.2, for an almost additive sequence $\mathcal{F}$ on a subshift with the weak specification property, under certain conditions, we find a function $\hat{f}$ satisfying (1.1) explicitly by using the sequence $\mathcal{F}$. If, in addition, $\mathcal{F}$ has bounded variation, then the unique equilibrium state for $\mathcal{F}$, which is the unique invariant Gibbs measure for $\mathcal{F}$, is the unique invariant Gibbs measure for $\hat{f}$. We also consider the case in which $\mathcal{F}$ is merely weakly almost additive. Such a sequence $\mathcal{F}$ is asymptotically additive. Defining $\hat{f}$ satisfying (1.1) in the same manner, we show that if $\mathcal{F}$ has an equilibrium state μ which is Gibbs for $\mathcal{F}$ (such a case does exist, see Remark 3.3), then it is an invariant weak Gibbs measure for $\hat{f}$. In Theorem 3.5, assuming that there is an

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invariant Gibbs measure for a weakly almost additive sequence $\mathcal{F}$, we construct a function $\hat{f}$ satisfying (1.1) by using the Gibbs measure and obtain similar results.

In §4, by applying Theorem 3.2, we study the following questions related to [Q] concerning factors of Gibbs measures. Given a one-block factor map $\pi : X \to Y$ between subshifts and a Gibbs measure $\mu \in M(X, \sigma_X)$ for $f \in C(X)$, we are interested in studying some properties of the push-forward measure $\pi\mu$ (which we call an image or a factor of μ). These questions originate from statistical mechanics and many studies have been also conducted in the area of dynamical systems (see for example, [**4**, **5**, **11**, **15–17**, **19**, **24**]). In particular, Yoo [**24**] showed that the fiber-wise sub-positive mixing property of factor maps (Definition 4.5) is a sufficient condition for a factor map to send all Markov measures on a topologically mixing shift of finite type to Gibbs measures and the result was extended by Piraino [**16**] to the factors of Gibbs measures associated to continuous functions. However, the fiber-wise sub-positive mixing property is not a necessary condition (Proposition 4.7). Hence, we are interested in finding an equivalent condition for $\pi\mu$ to be Gibbs for a continuous (or, more generally, Borel measurable) function. It is not known and we investigate it under a special setting. To this end, we study relative pressure functions by using the non-additive thermodynamic formalism.

Let (X, σ_X), (Y, σ_Y) be subshifts and $\pi : X \to Y$ be a factor map. Let $f \in C(X)$, $n \in \mathbb{N}$, and $\delta > 0$. For each $y \in Y$, define

$$P_n(\sigma_X, \pi, f, \delta)(y) = \sup\left\{\sum_{x\in E} e^{(S_n f)(x)} : E \text{ is an } (n, \delta) \text{ separated subset of } \pi^{-1}(\{y\})\right\},$$

$$P(\sigma_X, \pi, f, \delta)(y) = \limsup_{n\to\infty} \frac{1}{n} \log P_n(\sigma_X, \pi, f, \delta)(y), \tag{1.3}$$

$$P(\sigma_X, \pi, f)(y) = \lim_{\delta\to 0} P(\sigma_X, \pi, f, \delta)(y).$$

The function $P(\sigma_X, \pi, f) : Y \to \mathbb{R}$, introduced by Ledrappier and Walters [**12**], is the *relative pressure* function of $f \in C(X)$ with respect to π. In general, it is merely Borel measurable. We have $P(\sigma_X, \pi, f) = \lim_{n\to\infty}(1/n) \log g_n$ almost everywhere with respect to every $\mu \in M(Y, \sigma_Y)$, where g_n is defined in (4.1) in §4. The sequence $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ on Y is subadditive in general and is the *sequence associated to the relative pressure function* $P(\sigma_X, \pi, f)$.

Since the factor of the Gibbs measure for $f \in C(X)$ is a unique equilibrium state for the sequence $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ on Y associated to $P(\sigma_X, \pi, f)$ and the factor is also a unique invariant Gibbs measure for $\mathcal{G}$ (see for example, [**8**, **21**, **22**]), our approach is to find an explicit form of $\hat{f}$ satisfying (1.1) when $\mathcal{F}$ is replaced by $\mathcal{G}$. In Theorem 4.8 and in the rest of the paper, we consider this factor under the following particular setting. A map $\pi : X \to Y$ is a one-block factor map between irreducible shifts of finite type where $X \subseteq \{1, 2, 3\}^{\mathbb{N}}$ and $Y \subseteq \{1, 2\}^{\mathbb{N}}$ such that Y is the full shift or has the transition matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, satisfying $\pi^{-1}\{1\} = \{1\}$, and f is a function of of two coordinates on X. The images of measures of maximal entropy under the factor maps of this type were originally studied in [**18**, **20**]. Finding an explicit form of a function for which the factor of the Gibbs measure is (weak) Gibbs and studying the almost additivity of the sequence $\mathcal{G}$ associated to $P(\sigma_X, \pi, f)$, we investigate an equivalent condition for the factor of a Gibbs

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measure μ associated to f to be a Gibbs measure for a continuous (or, more generally, Borel measurable) function $\hat{g}$. (Theorem 4.8 and Corollary 4.10). Section 5 consists of the proof of Theorem 4.8. To find $\hat{g}$, we apply some ideas from [**4**, **24**] and connect $\mathcal{G}$ with the sum of all entries of products of certain matrices. In §6, we give two examples of factor maps from [**18**, **20**] which illustrate Theorem 3.2 (Corollary 4.1) and Theorem 4.8. Finally, we are left with a question: can we generalize the results (Theorem 4.8) to the factors of Gibbs measures for continuous functions under a more general setting?

We remark that in [**10**], given an almost additive $\mathcal{F}$ with bounded variation, some properties of a function $\hat{f}$ satisfying (1.1) were studied. An explicit form for $\hat{f}$ was not studied.

2. *Background*

2.1. *Shift spaces.* We give a brief summary of the basic definitions in symbolic dynamics. Here, (X, σ_X) is a *one-sided subshift* over $\{1, \ldots, k\}$ if X is a closed shift-invariant subset of $\{1, \ldots, k\}^{\mathbb{N}}$ for some $k \geq 1$, that is, $\sigma_X(X) \subseteq X$, where the shift $\sigma_X : X \to X$ is defined by $(\sigma_X(x))_i = x_{i+1}$ for all $i \in \mathbb{N}$, $x = (x_n)_{n=1}^{\infty} \in X$. Define a metric d on X by $d(x, x') = 1/2^k$ if $x_i = x'_i$ for all $1 \leq i \leq k$ and $x_{k+1} \neq x'_{k+1}$, $d(x, x') = 1$ if $x_1 \neq x'_1$, and $d(x, x') = 0$ otherwise. Define a cylinder set $[x_1 \cdots x_n]$ of length n in X by $[x_1 \cdots x_n] = \{(z_i)_{i=1}^{\infty} \in X : z_i = x_i \text{ for all } 1 \leq i \leq n\}$. For each $n \in \mathbb{N}$, denote by $B_n(X)$ the set of all n-blocks that appear in points in X. Define $B_0(X) = \{\epsilon\}$, where ϵ is the empty word of length 0. The language of X is the set $B(X) = \bigcup_{n=0}^{\infty} B_n(X)$. A subshift (X, σ_X) is *irreducible* if for any allowable words $u, v \in B(X)$, there exists $w \in B(X)$ such that $uwv \in B(X)$, and has the *weak specification property* if there exists $p \in \mathbb{N}$ such that for any allowable words $u, v \in B(X)$, there exist $0 \leq k \leq p$ and $w \in B_k(X)$ such that $uwv \in B(X)$. A point $x \in X$ is a periodic point of σ_X if there exists $p \in \mathbb{N}$ such that $\sigma_X^p(x) = x$. Let $A = (a_{ij})$ be a $k \times k$ matrix of zeros and ones. Define X_A by

$$X_A = \{(x_n)_{n=1}^{\infty} \in \{1, \ldots, k\}^{\mathbb{N}} : a_{x_n, x_{n+1}} = 1 \text{ for every } n \in \mathbb{N}\}.$$

Then, (X_A, σ_{X_A}) is a *one-sided shift of finite type* and it is a subshift over $\{1, \ldots, k\}$. It is topologically mixing if there exists $p \in \mathbb{N}$ such that $A^p > 0$.

Let (X, σ_X) be a subshift over a finite set S_1 and (Y, σ_Y) be a subshift over a finite set S_2. A map $\pi : X \to Y$ is a *factor map* if it is continuous, surjective, and satisfies $\pi \circ \sigma_X = \sigma_Y \circ \pi$. A one-block code is a map $\pi : X \to Y$ for which there exists a function $\tilde{\pi} : S_1(X) \to S_2(Y)$ such that $(\pi(x))_i = \tilde{\pi}(x_i)$ for all $i \in \mathbb{N}$.

2.2. *Sequences of continuous functions.* Given a subshift (X, σ_X), for each $n \in \mathbb{N}$, let $f_n : X \to \mathbb{R}^+$ be a continuous function. Then, $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions on X. A sequence $\mathcal{F}$ is *almost additive* on X if there exists a constant $C > 0$ such that

$$f_{n+m}(x) \leq e^C f_n(x) f_m(\sigma_X^n x) \tag{2.1}$$

and

$$f_{n+m}(x) \geq e^{-C} f_n(x) f_m(\sigma_X^n x) \tag{2.2}$$

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for every $x \in X$, $n, m \in \mathbb{N}$. More generally, a sequence $\mathcal{F}$ is *weakly almost additive* if there exists a sequence of positive real numbers $\{C_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty}(1/n)C_n = 0$ such that (2.1) and (2.2) hold if we replace C by C_n [7]. A sequence $\mathcal{F}$ is *subadditive* if $\mathcal{F}$ satisfies (2.1) with $C = 0$ [3]. Asymptotically additive sequences were introduced in [9] and we use the following definition (see [6]). A sequence $\mathcal{F}$ is *asymptotically additive* on X if there exists $\hat{f} \in C(X)$ such that (1.1) holds. Weakly almost additive sequences are asymptotically additive [7, Lemma 6.2]. A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ has *bounded variation* if there exists $M \in \mathbb{R}^+$ such that $\sup\{M_n : n \in \mathbb{N}\} \le M$, where $M_n = \sup\{f_n(x)/f_n(y) : x, y \in X, x_i = y_i \text{ for } 1 \le i \le n\}$. Given a sequence of continuous functions $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on X, a measure $\mu \in M(X, \sigma_X)$ is an *equilibrium state* for $\mathcal{F}$ if

$$h_\mu(\sigma_X) + \lim_{n\to\infty} \frac{1}{n} \int \log f_n \, d\mu = \sup_{m \in M(X,\sigma_X)} \left\{ h_m(\sigma_X) + \lim_{n\to\infty} \frac{1}{n} \int \log f_n \, dm \right\}. \tag{2.3}$$

A measure $\mu \in M(X, \sigma_X)$ is a *weak Gibbs measure* for $\mathcal{F}$ if there exist $P \in \mathbb{R}$ and $C_n > 0$ satisfying $\lim_{n\to\infty}(1/n) \log C_n = 0$ such that

$$\frac{1}{C_n} \le \frac{\mu[x_1 \cdots x_n]}{e^{-nP} f_n(x)} \le C_n \tag{2.4}$$

for every $x \in X$ and $n \in \mathbb{N}$. If there exists $C > 0$ such that $C_n = C$ for all $n \in \mathbb{N}$, then μ is a *Gibbs measure* for $\mathcal{F}$. If $\mathcal{F}$ is almost additive with bounded variation on a subshift with the weak specification property, then $\mathcal{F}$ has a unique equilibrium state and it is the unique invariant Gibbs measure for $\mathcal{F}$ (see [1, 8, 13]). We note that if μ is a weak Gibbs measure for an asymptotically additive sequence $\mathcal{F}$, then the variational principle of topological pressure holds and P in (2.4) is given by the topological pressure of $\mathcal{F}$ (see for example [9]). Similarly, given a Borel measurable function g on X, a measure $\mu \in M(X, \sigma_X)$ is an *equilibrium state* for g if $h_\mu(X) + \int g \, d\mu = \sup_{m \in M(X,\sigma_X)}\{h_m(\sigma_X) + \int g \, dm\}$. A measure $\mu \in M(X, \sigma_X)$ is a *weak Gibbs measure* for g if there exist $P \in \mathbb{R}$ and $C_n > 0$ such that (2.4) holds if we replace $f_n(x)$ by $e^{(S_n g)(x)}$ and it is a *Gibbs measure* for g if $C_n = C$ for all $n \in \mathbb{N}$ for some $C > 0$.

3. *Gibbs measures for weakly almost additive sequences*

In this section, given an almost additive sequence $\mathcal{F}$ of continuous functions on a subshift with the weak specification property, we construct a Borel measurable function $\hat{f}$ satisfying (1.1) under certain conditions and study the Gibbs property of the unique equilibrium state for $\hat{f}$ when $\mathcal{F}$ has bounded variation. We also consider the case when $\mathcal{F}$ is weakly almost additive. We first start with a simple lemma.

LEMMA 3.1. *Let (X, σ_X) be a subshift with the weak specification property and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on X. Suppose that there exist a Borel measurable function f on X and a sequence of positive real numbers $\{A_k\}_{k=1}^{\infty}$ satisfying $\lim_{k\to\infty}(1/k) \log A_k = 0$ such that*

$$\frac{1}{A_k} \le \frac{f_k(x)}{e^{(S_k f)(x)}} \le A_k \quad \text{for every } x \in X, k \in \mathbb{N}. \tag{3.1}$$

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Then, the following hold.

(i) *If μ is a Gibbs measure for $\mathcal{F}$, then it is a weak Gibbs measure for f.*

(ii) *If there exists a constant $A > 0$ such that $A_k = A$ for every $k \in \mathbb{N}$, then μ is a Gibbs measure for $\mathcal{F}$ if and only if it is a Gibbs measure for f.*

Proof. If μ is a Gibbs measure for $\mathcal{F}$, then there exist $P \in \mathbb{R}$, $C_0 > 0$ such that

$$\frac{1}{C_0} \le \frac{\mu[x_1 \cdots x_n]}{e^{-nP} f_n(x)} \le C_0 \quad \text{for every } x \in X,\ n \in \mathbb{N}.$$

Replacing C_n and f_n in (2.4) by $A_n C_0$ and $e^{S_n f}$, respectively, μ is weak Gibbs for f. The second statement follows by similar arguments, replacing A_k by A in (3.1). □

Setting (A). Let (X, σ_X) be a subshift with the weak specification property and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on X. Define for $x \in X$,

$$\underline{f}(x) := \liminf_{n\to\infty} \log\left(\frac{f_n(x)}{f_{n-1}(\sigma_X(x))}\right) \quad \text{and} \quad \overline{f}(x) := \limsup_{n\to\infty} \log\left(\frac{f_n(x)}{f_{n-1}(\sigma_X(x))}\right).$$

Then, $\underline{f}$ and $\overline{f}$ are Borel measurable functions on X. In the next theorem, we construct a Borel measurable function $\hat{f}$ satisfying (1.1) under certain conditions and study the properties of equilibrium states of $\hat{f}$.

THEOREM 3.2. *Under Setting (A), suppose that $\underline{f}(x) = \overline{f}(x)$ for every $x \in X$ and define $\hat{f}(x) := \lim_{n\to\infty} \log(f_n(x)/f_{n-1}(\sigma_X(x)))$. Then, the following hold.*

(i) *If $\mathcal{F}$ is almost additive on X, then $\hat{f}$ is a bounded Borel measurable function satisfying (1.1) and (1.2). If, in addition, $\mathcal{F}$ has bounded variation on X, then the unique equilibrium state for $\mathcal{F}$ which is also the unique invariant Gibbs measure for $\mathcal{F}$ is the unique equilibrium state for $\hat{f}$ and it is the unique invariant Gibbs measure for $\hat{f}$.*

(ii) *If $\mathcal{F}$ is merely weakly almost additive on X, then $\hat{f}$ is a bounded Borel measurable function satisfying (1.1) and (1.2). If an equilibrium state for $\mathcal{F}$ is Gibbs for $\mathcal{F}$, then it is an equilibrium state for $\hat{f}$ and it is also an invariant weak Gibbs measure for $\hat{f}$.*

Remark 3.3. A weakly almost additive sequence $\mathcal{F}$ on X has an equilibrium state. See Example 6.2 for the case when such an $\mathcal{F}$ has a unique equilibrium state with the Gibbs property.

We apply the following lemma to show Theorem 3.2.

LEMMA 3.4. *Under Setting (A), the following hold.*

(i) *If $\mathcal{F}$ satisfies (2.1), then $\underline{f}$ is bounded from above and $f_k(x)/e^{(S_k \underline{f})(x)} \ge e^{-C}$ for every $x \in X$ and $k \in \mathbb{N}$.*

(ii) *If $\mathcal{F}$ satisfies (2.2), then $\overline{f}$ is bounded from below and $f_k(x)/e^{(S_k \overline{f})(x)} \le e^{C}$ for every $x \in X$ and $k \in \mathbb{N}$.*

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Proof. We first we show statement (i). Since $\mathcal{F}$ satisfies (2.1), $\underline{f}$ is bounded from above by $C + \max_{x\in X} \log f_1(x)$. Fix $k \in \mathbb{N}$. Then,

$$\begin{aligned}(S_k \underline{f})(x) &= \liminf_{n\to\infty} \log\left(\frac{f_n(x)}{f_{n-1}(\sigma_X(x))}\right) + \liminf_{n\to\infty} \log\left(\frac{f_{n-1}(\sigma_X(x))}{f_{n-2}(\sigma_X^2(x))}\right)\\ &\quad + \cdots + \liminf_{n\to\infty} \log\left(\frac{f_{n-k+1}(\sigma_X^{k-1}(x))}{f_{n-k}(\sigma_X^k(x))}\right)\\ &\le \liminf_{n\to\infty} \log\left(\frac{f_n(x) f_{n-1}(\sigma_X(x)) \cdots f_{n-k+1}(\sigma_X^{k-1}(x))}{f_{n-1}(\sigma_X(x)) f_{n-2}(\sigma_X^2(x)) \cdots f_{n-k}(\sigma_X^k(x))}\right)\\ &= \liminf_{n\to\infty} \log\left(\frac{f_n(x)}{f_{n-k}(\sigma_X^k(x))}\right) \le \log(e^C f_k(x)),\end{aligned}$$

where the last inequality holds by (2.1). We next show statement (ii). Since $\mathcal{F}$ satisfies (2.2), $\overline{f}$ is bounded from below by $-C + \min_{x\in X} \log f_1(x)$. Fix $k \in \mathbb{N}$. Then,

$$\begin{aligned}(S_k \overline{f})(x) &= \limsup_{n\to\infty} \log\left(\frac{f_n(x)}{f_{n-1}(\sigma_X(x))}\right) + \limsup_{n\to\infty} \log\left(\frac{f_{n-1}(\sigma_X(x))}{f_{n-2}(\sigma_X^2(x))}\right)\\ &\quad + \cdots + \limsup_{n\to\infty} \log\left(\frac{f_{n-k+1}(\sigma_X^{k-1}(x))}{f_{n-k}(\sigma_X^k(x))}\right)\\ &\ge \limsup_{n\to\infty} \log\left(\frac{f_n(x) f_{n-1}(\sigma_X(x)) \cdots f_{n-k+1}(\sigma_X^{k-1}(x))}{f_{n-1}(\sigma_X(x)) f_{n-2}(\sigma_X^2(x)) \cdots f_{n-k}(\sigma_X^k(x))}\right)\\ &= \limsup_{n\to\infty} \log\left(\frac{f_n(x)}{f_{n-k}(\sigma_X^k(x))}\right) \ge \log(e^{-C} f_k(x)),\end{aligned}$$

where the last inequality holds by (2.2). This proves statement (ii). □

Proof of Theorem 3.2. We first show statement (i). Lemma 3.4 implies that $\hat{f}$ is a measurable bounded function satisfying (1.1) and hence (1.2) holds. By (1.2), the unique equilibrium state μ for $\mathcal{F}$ which is also a unique invariant Gibbs measure for $\mathcal{F}$ is the unique equilibrium state for $\hat{f}$. By Lemma 3.1(ii), μ is an invariant Gibbs measure for $\hat{f}$. To show that it is a unique invariant Gibbs measure for $\hat{f}$, suppose that there is another invariant Gibbs measure $\tilde{\mu} \neq \mu$ for $\hat{f}$. Then, by Lemma 3.1(ii), $\tilde{\mu}$ is also an invariant Gibbs measure for $\mathcal{F}$, which is a contradiction. Next we show statement (ii). Since $\mathcal{F}$ is weakly almost additive, we replace C of (2.1) and (2.2) by $C_n > 0$ such that $\lim_{n\to\infty}(1/n)C_n = 0$. Then, $-C_1 + \min_{x\in X} \log f_1(x) \le \hat{f}(x) \le C_1 + \max_{x\in X} \log f_1(x)$ for all $x \in X$. The inequalities in Lemma 3.4(i) and (ii) replacing C by C_k hold. Hence, (1.1) and (1.2) hold. Since an equilibrium state for $\mathcal{F}$ is an equilibrium state for $\hat{f}$, Lemma 3.1(i) implies the last statement.

Next, assuming the existence of an invariant Gibbs measure for a sequence $\mathcal{F}$, we construct a Borel measurable function $\hat{f}$ satisfying (1.1). □

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Setting (B). Let (X, σ_X) be a subshift with the weak specification property and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on X. Suppose that there exists an invariant Gibbs measure ν for $\mathcal{F}$, that is, there exist $P \in \mathbb{R}$, $C_0 > 0$ such that

$$\frac{1}{C_0} \leq \frac{\nu[x_1 \cdots x_n]}{e^{-nP} f_n(x)} \leq C_0 \quad \text{for every } x \in X, n \in \mathbb{N}. \tag{3.2}$$

Note that (3.2) implies that $\mathcal{F}$ has bounded variation. Define for $x \in X$

$$\underline{r}(x) := \liminf_{n\to\infty} \log\left(\frac{\nu[x_1 \cdots x_n]}{\nu[x_2 \cdots x_n]}\right) + P \quad \text{and} \quad \overline{r}(x) := \limsup_{n\to\infty} \log\left(\frac{\nu[x_1 \cdots x_n]}{\nu[x_2 \cdots x_n]}\right) + P.$$

Then, $\underline{r}$ and $\overline{r}$ are Borel measurable functions on X.

THEOREM 3.5. *Under Setting (B), suppose that $\underline{r}(x) = \overline{r}(x)$ for every $x \in X$ and define $r(x) := \lim_{n\to\infty} \log(\nu[x_1 \cdots x_n]/\nu[x_2 \cdots x_n]) + P$.*

(i) *If $\mathcal{F}$ is almost additive on X, then r is bounded and Borel measurable on X and*

$$\lim_{n\to\infty} \frac{1}{n} \int \log f_n \, d\mu = \lim_{n\to\infty} \frac{1}{n} \int \log(\nu[x_1 \cdots x_n] e^{np}) \, d\mu = \int r \, d\mu \tag{3.3}$$

for every $\mu \in M(X, \sigma_X)$. Then, the unique equilibrium state ν for $\mathcal{F}$ is the unique equilibrium state for r which is also the unique invariant Gibbs measure for r.

(ii) *If $\mathcal{F}$ is merely weakly additive, then r is bounded and Borel measurable, and (3.3) holds. The measure ν is an equilibrium state for r which is an invariant weak Gibbs measure for r.*

Proof. We first notice that the first equality of (3.3) follows from (3.2). Now we show statement (i). For $x \in X$, $n \in \mathbb{N}$, define $r_n(x) := \nu[x_1 \cdots x_n] e^{nP}$. If $\mathcal{F}$ is almost additive satisfying (2.1) and (2.2), then using (3.2), a simple computation shows that

$$(C_0^3 e^C)^{-1} r_n(x) r_m(\sigma^n x) \leq r_{n+m}(x) \leq r_n(x) r_m(\sigma^n x) C_0^3 e^C$$

for every $x \in X$, $n, m \in \mathbb{N}$. Hence, $\{\log r_n\}_{n=1}^{\infty}$ is almost additive and has bounded variation. Since $\underline{r} = \overline{r}$, applying Theorem 3.2(i) to $\{\log r_n\}_{n=1}^{\infty}$, we obtain the second equality of (3.3) and the first part of statement (i). Notice that ν is the unique equilibrium state for $\mathcal{F}$ and ν is Gibbs for $\{\log r_n\}_{n=1}^{\infty}$ because $\nu[x_1 \cdots x_n]/(e^{-nP} r_n(x)) = 1$. Hence, it is the unique equilibrium state for $\{\log r_n\}_{n=1}^{\infty}$. Applying Theorem 3.2(i), we obtain statement (i). We next show statement (ii) in the similar manner. If $\mathcal{F}$ is weakly almost additive, then (3.2) implies that there exists $C_n > 0$ such that $\lim_{n\to\infty} C_n/n = 0$ satisfying

$$(C_0^3 e^{C_n})^{-1} r_n(x) r_m(\sigma^n x) \leq r_{n+m}(x) \leq r_n(x) r_m(\sigma^n x) C_0^3 e^{C_n}$$

for every $x \in X$, $n, m \in \mathbb{N}$. Hence, $\{\log r_n\}_{n=1}^{\infty}$ is weakly almost additive. Theorem 3.2(ii) implies the second equality of (3.3) and the first part of statement (ii). Since ν satisfies the Gibbs property where P is given by the topological pressure of $\{\log r_n\}_{n=1}^{\infty}$, it is an equilibrium state for $\{\log r_n\}_{n=1}^{\infty}$ and Theorem 3.2(ii) implies statement (ii). □

4. *Applications*

4.1. *Relative pressure functions and images of Gibbs measures for continuous functions.*

We apply Theorem 3.2 to the sequences $\mathcal{G}$ associated to relative pressure functions

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(see (1.3)). Corollaries 4.1 and 4.4 connect $\mathcal{G}$ with images of Gibbs measures. For a survey of the study of images of Gibbs measures, see the paper by Boyle and Petersen [2]. Given a one-block factor map $\pi : X \to Y$ between subshifts and an invariant measure μ on X, define the image $\pi\mu \in M(Y, \sigma_Y)$ by $\pi\mu(B) = \mu(\pi^{-1}B)$ for a Borel set B of Y. Let $f \in C(X)$. For $y = (y_1, \ldots, y_n, \ldots) \in Y$, define $E_n(y)$ to be a set consisting of exactly one point from each cylinder $[x_1 \cdots x_n]$ in X such that $\pi(x_1 \cdots x_n) = y_1 \cdots y_n$. For $n \in \mathbb{N}$,

$$g_n(y) := \sup_{E_n(y)} \left\{ \sum_{x \in E_n(y)} e^{(S_n f)(x)} \right\}. \tag{4.1}$$

Then,

$$P(\sigma_X, \pi, f)(y) = \limsup_{n\to\infty} \frac{1}{n} \log g_n(y), \quad \mu\text{-almost everywhere for any } \mu \in M(Y, \sigma_Y)$$

(see [8, 14]). It is clear that $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ has bounded variation.

COROLLARY 4.1. *Let $\pi : X \to Y$ be a one-block factor map between subshifts, where X is a subshift with the weak specification property. Given $f \in C(X)$, let $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ be the sequence on Y associated to $P(\sigma_X, \pi, f)$. For $y \in Y$, define a function $g : Y \to \mathbb{R}$ by*

$$g(y) = \lim_{n\to\infty} \log \left(\frac{g_n(y)}{g_{n-1}(\sigma_Y y)} \right) \tag{4.2}$$

if the limit exists at every point $y \in Y$. When g is defined, the following hold.

(i) *If the sequence $\mathcal{G}$ is almost additive on Y, then g is bounded and Borel measurable on Y and*

$$\int P(\sigma_X, \pi, f)\, d\mu = \int g\, d\mu \quad \text{for every } \mu \in M(Y, \sigma_Y). \tag{4.3}$$

Then, the unique equilibrium state for $\mathcal{G}$ which is also the unique invariant Gibbs measure for $\mathcal{G}$ is the unique equilibrium state for g and it is the unique invariant Gibbs measure for g.

(ii) *If $\mathcal{G}$ is merely weakly almost additive on Y, then g is bounded and Borel measurable on Y and (4.3) holds. If, in addition, $\mathcal{G}$ has an equilibrium state which is also an invariant Gibbs measure for $\mathcal{G}$, then it is an equilibrium state for g which is also an invariant weak Gibbs measure for g.*

Remark 4.2. The function g is similar to the function u defined in [11, 17].

Proof. Set $f_n = g_n$ in Theorem 3.2. □

The next lemma connects relative pressure functions with images of Gibbs measures.

LEMMA 4.3. (See [8, 22]) *Let $\pi : X \to Y$ be a one-block factor map between subshifts, where X is a subshift with the weak specification property. If μ is an invariant Gibbs measure for $f \in C(X)$, then $\pi\mu$ is the unique equilibrium state for the sequence $\mathcal{G}$ on Y associated to $P(\sigma_X, \pi, f)$ and it is the unique invariant Gibbs measure for $\mathcal{G}$.*

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The next corollary is useful to study the images of Gibbs measures under factor maps and applied in §5.

COROLLARY 4.4. *Under the assumption of Corollary 4.1, suppose $f \in C(X)$ has an invariant Gibbs measure μ and g in Corollary 4.1 is defined. If $\mathcal{G}$ is almost additive (respectively weakly almost additive) on Y, then $\pi\mu$ is the unique invariant Gibbs (respectively weak Gibbs) measure for g and it is the unique equilibrium state for g.*

Proof. Corollary 4.1 and Lemma 4.3 imply the result. □

4.2. *The fiber-wise sub-positive mixing property of factor maps and almost additivity of subadditive sequences associated to relative pressure functions.* The fiber-wise sub-positive mixing property of factor maps is a sufficient condition for a factor map to send all Markov measures to Gibbs measures [**24**]. This property is assumed in various papers to study images of Gibbs measures (for the most general results, see [**16**]).

In this section, noticing that the fiber-wise sub-positive mixing property is not a necessary condition for the image to be Gibbs (see Proposition 4.7), we characterize the Gibbs property of the images of Markov measures in terms of almost additivity of the sequences associated to relative pressure functions for a special type of factor maps (Theorem 4.8 and Corollary 4.10). Theorem 4.8 gives an answer to [Q] from §1 under a particular setting (Corollary 4.10).

Definition 4.5. [**16**, **24**] Let $\pi : X \to Y$ be a one-block factor map between subshifts, where X is a shift of finite type. Then, $\pi : X \to Y$ is *fiber-wise sub-positive mixing* if there exists $k \in \mathbb{N}$ such that for any $w \in B_k(Y)$, $u_1 \cdots u_k \in B_k(X)$, $v_1 \cdots v_k \in B_k(X)$ satisfying $\pi(u_1 \cdots u_k) = \pi(v_1 \cdots v_k) = w$, there exists $a_1 \cdots a_k \in B_k(X)$ with $a_1 = u_1$ and $a_k = v_k$ such that $\pi(a_1 \cdots a_k) = w$.

THEOREM 4.6. [**16**] *Let $\pi : X \to Y$ be a one-block factor map between subshifts, where X is a topologically mixing shift of finite type, satisfying the fiber-wise sub-positive mixing property. If μ is an invariant Gibbs measure for $f \in C(X)$, then $\pi\mu$ is an invariant Gibbs measure for $\psi \in C(Y)$, where ψ is defined by $\psi(y) = \lim_{n\to\infty} \log(\pi\mu[y_1 \cdots y_n]/\pi\mu[y_2 \cdots y_n])$.*

By Corollary 4.4, we can consider conditions of images of Gibbs measures to be Gibbs for continuous functions regardless of the fiber-wise sub-positive mixing property.

PROPOSITION 4.7. *Let $\pi : X \to Y$ be a one-block factor map, where X is a topologically mixing shift of finite type and Y is a subshift. Let μ be the unique invariant Gibbs measure for $f \in C(X)$ and $\mathcal{G}$ be the subadditive sequence associated to the relative pressure function $P(\sigma_X, \pi, f)$.*

(i) *If π is fiber-wise sub-positive mixing, then $\mathcal{G}$ is almost additive on Y.*

(ii) *The almost additivity of $\mathcal{G}$ does not imply the fiber-wise sub-positive mixing property of a factor map. There exist a one-block factor map not fiber-wise sub-positive mixing and $f \in C(X)$ such that the image $\pi\mu$ is an invariant Gibbs measure for some $g \in C(Y)$ where $\mathcal{G}$ is almost additive on Y.*

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Proof. If π is fiber-wise sub-positive mixing, by Theorem 4.6, there exists $g \in C(Y)$ for which $\pi\mu$ is Gibbs. It is easy to see that $\mathcal{G}$ is almost additive because $\pi\mu$ is Gibbs for g and $\mathcal{G}$ [**23**, Corollary 6.11(iii)]. Example 6.1 in §6 proves statement (ii). □

In the rest of the paper, we study equivalent conditions for factors of the Gibbs measures for continuous functions to be Gibbs, under the setting below.

Setting (C). Let $\pi : X \to Y$ be a one-block factor map such that $\pi(1) = 1, \pi(2) = \pi(3) = 2$, where $X \subseteq \{1, 2, 3\}^{\mathbb{N}}$ and $Y \subseteq \{1, 2\}^{\mathbb{N}}$ are irreducible shifts of finite type such that Y is the full shift or has the transition matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Let f be a function of two coordinates on X and define $f[ij] := f(x)$ if $x \in [ij]$.

Under Setting (C), let $A = (a_{ij})_{3\times 3}$ be the transition matrix of X and define the 3×3 non-negative matrix M with entries $M(i, j)$ given by $M(i, j) = e^{f[ij]}a_{ij}$ for $i, j = 1, 2, 3$. Given each $b_1b_2 \in B_2(Y)$, let $M_{b_1b_2}$ be the 3×3 non-negative matrix with entries $M_{b_1b_2}(i, j)$ defined by

$$M_{b_1b_2}(i, j) = \begin{cases} e^{f[ij]} & \text{if } ij \in B_2(X) \text{ and } \pi(ij) = b_1b_2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Define the submatrix $M_{22}|_{\pi^{-1}(22)} := (M_{22}(i, j))_{i,j\in\{2,3\}}$. Recall that $\mathcal{G}$ is the sequence associated to the relative pressure function $P(\sigma_X, \pi, f)$.

THEOREM 4.8. *Under Setting (C), let μ be an invariant Gibbs measure for f. Then, the following hold.*

(i) *The sequence $\mathcal{G}$ is almost additive on Y if and only if there exists a Borel measurable function $\hat{g}$ on Y for which $\pi\mu$ is a unique invariant Gibbs measure. If $\mathcal{G}$ is merely weakly almost additive on Y, then there exists a Borel measurable function $\hat{g}$ on Y for which $\pi\mu$ is a unique invariant weak Gibbs measure.*

(ii) *If $M_{22}|_{\pi^{-1}(22)} \neq \left(\begin{smallmatrix} 0 & a_1 \\ a_2 & 0 \end{smallmatrix}\right)$, $a_1, a_2 > 0$, then statement (i) holds for $\hat{g} \in C(Y)$.*

(iii) *If $M_{22}|_{\pi^{-1}(22)} = \left(\begin{smallmatrix} 0 & a_1 \\ a_1 & 0 \end{smallmatrix}\right)$, $a_1 > 0$, and M satisfies $M(2, 1) = M(3, 1)$ or $M(1, 2) = M(1, 3)$, then statement (i) holds for $\hat{g} \in C(Y)$.*

Proof. See the proof in §5. □

Remark 4.9. Theorem 4.8 is valid for any one-step Markov measure on X. Let $A = (a_{ij})_{3\times 3}$ be a transition matrix of an irreducible shift of finite type X. Let $P = (P_{ij})$ be a stochastic matrix where $P_{ij} > 0$ exactly when $a_{ij} > 0$ and let $p = (p_1, \ldots p_k)$ be the unique probability vector with $pP = p$. A one-step Markov measure $\mu \in M(X, \sigma_X)$ is defined by $\mu[x_1 \cdots x_k] = p_1P_{12} \cdots P_{k-1k}$. Set $f(x) := \log P_{ij}$ for $x \in [ij]$. Then, μ is an equilibrium state for f with the Gibbs property.

COROLLARY 4.10. *Under Setting (C), let μ be the measure of maximal entropy for σ_X, that is, $h_\mu(\sigma_X) = \sup\{h_\nu(\sigma_X) : \nu \in M(X, \sigma_X)\}$. Let Φ be the sequence associated to the relative pressure function $P(\sigma_X, \pi, 0)$. Then, Φ is almost additive on Y if and only if there exists a function $\hat{g} \in C(Y)$ for which $\pi\mu$ is a unique invariant Gibbs measure. If Φ is merely weakly almost additive on Y, then there exists $\hat{g} \in C(Y)$ for which $\pi\mu$ is a unique invariant weak Gibbs measure.*

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Proof. If $f = 0$, then we obtain $M_{22}|_{\pi^{-1}(22)} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in Theorem 4.8(iii). Now, we apply Lemma 5.6. □

COROLLARY 4.11. *Under Setting (C), the following hold.*

(i) *The sequence $\mathcal{G}$ is almost additive if and only if there exists a Borel measurable function $\hat{g}$ on Y for which there exists a unique invariant Gibbs measure with the following property:*

$$\lim_{n\to\infty} \frac{1}{n} \|\log g_n - S_n \hat{g}\|_\infty = 0. \tag{4.5}$$

If $\mathcal{G}$ is merely weakly almost additive, then there exists a Borel measurable function $\hat{g}$ satisfying (4.5) for which there is a unique invariant weak Gibbs measure.

(ii) *Under the assumption of Theorem 4.8(ii) or (iii), statement (i) holds for $\hat{g} \in C(Y)$.*

Proof. Using Lemma 5.7 and the Proof of Theorem 4.8 from §5, it is easy to see that $\hat{g}$ constructed in Theorem 4.8 when $\mathcal{G}$ is (weakly) almost additive satisfies (4.5). The rest of the results follows easily using Proposition 5.1. □

5. *Proof of Theorem 4.8*

This section is devoted to the proof of Theorem 4.8. Given a factor map $\pi : X \to Y$ between shifts of finite type, define for $n \in \mathbb{N}$, $y \in Y$,

$$h_n(y) := \sup_{E_n(y)} \left\{ \sum_{x \in E_n(y)} e^{(S_{n-1} f)(x)} \right\}, \tag{5.1}$$

where $S_0 f := f$ and $E_n(y)$ is defined in (4.1). Let $\mathcal{H} := \{\log h_n\}_{n=1}^\infty$ on Y.

PROPOSITION 5.1. *Let $\pi : X \to Y$ be a factor map between shifts of finite type and $\mathcal{G} = \{\log g_n\}_{n=1}^\infty$ be the sequence in (4.1). Then, the following hold.*

(i) *There exists a constant $A > 0$ such that for every $n \in \mathbb{N}$, $y \in Y$, $1/A \le g_n(y)/h_n(y) \le A$. If $f = 0$, then $g_n(y) = h_n(y)$ for every $n \in \mathbb{N}$.*

(ii) *We have $P(\sigma_X, \pi, f)(y) = \limsup_{n\to\infty}(1/n) \log g_n(y) = \limsup_{n\to\infty}(1/n) \log h_n(y)$, μ-almost everywhere for any $\mu \in M(Y, \sigma_Y)$. The sequence $\mathcal{G}$ is almost additive (respectively weakly almost additive) if and only if $\mathcal{H}$ is almost additive (respectively weakly almost additive).*

(iii) *Define $\hat{h}(y) := \lim_{n\to\infty} \log(h_n(y)/h_{n-1}(\sigma_Y y))$ for each $y \in Y$ if the limit exists. If $\hat{h}$ is defined and $\mathcal{H}$ is weakly almost additive, there exists $C_n > 0$, $\lim_{n\to\infty}(1/n)C_n = 0$, such that for every $n \in \mathbb{N}$, $y \in Y$,*

$$e^{-C_n} \le \frac{h_n(y)}{e^{(S_n \hat{h})(y)}} \le e^{C_n}. \tag{5.2}$$

If $\mathcal{H}$ is almost additive, then there exists $C > 0$ such that $C = C_n$ for every $n \in \mathbb{N}$.

(iv) *Corollary 4.1(i), (ii), and Corollary 4.4 hold if we replace $\mathcal{G}$ and g by $\mathcal{H}$ and by $\hat{h}$, respectively.*

Proof. The results are easily obtained by the definitions of g_n and h_n. □

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From now on, assume that f is a function of two coordinates on X. Then,

$$h_n(y) = \sum_{x_1\cdots x_n \in B_n(X), \pi(x_1\cdots x_n)=y_1\cdots y_n} e^{f[x_1x_2]+f[x_2x_3]+\cdots+f[x_{n-1}x_n]} \tag{5.3}$$

for each $n \geq 2$, $y \in Y$, and $h_1 = g_1$. Define $h[b_1 \cdots b_n] := h_n(y)$ for $y \in [b_1 \cdots b_n]$, $n \in \mathbb{N}$.

LEMMA 5.2. *Let $X \subseteq \{1, \ldots, l\}^{\mathbb{N}}$, $l > 2$, and $Y \subseteq \{1, 2\}^{\mathbb{N}}$ be irreducible subshifts. Let $\pi : X \to Y$ be a one-block factor map such that $\pi(1) = 1$, $\pi(i) = 2$ for $i = 2, \ldots, l$. Let f be a function on X of two coordinates. Suppose that*

$$\lim_{n\to\infty} \frac{h[2^n]}{h[2^{n-1}]} \quad \textit{and} \quad \lim_{n\to\infty} \frac{h[12^n]}{h[2^n]} \textit{ exist.}$$

Then, $\hat{h}$ defined in Proposition 5.1(iii) is given by

$$\hat{h}(y) = \begin{cases} \log h[21], & y \in [21], \\ \log\left(\dfrac{h[2^n1]}{h[2^{n-1}1]|}\right), & y \in [2^n1],\ n \geq 2, \\ \lim\limits_{n\to\infty} \log\left(\dfrac{h[2^n]}{h[2^{n-1}]}\right), & y = 2^\infty, \\ f[11], & y \in [1^n2],\ n \geq 2,\ \textit{or } y = 1^\infty, \\ \log\left(\dfrac{h[12^n1]}{h[2^n1]}\right), & y \in [12^n1],\ n \geq 1, \\ \lim\limits_{n\to\infty} \log\left(\dfrac{h[12^n]}{h[2^n]}\right), & y = 12^\infty. \end{cases} \tag{5.4}$$

If $\hat{h}$ is continuous at 2^∞ and 21^∞, then $\hat{h}$ is continuous on Y.

Proof. If $y \in [2^k1]$ for some $k \geq 2$, then for any $n \geq k+2$, it is easy to see that $h_n(y) = h_{k+1}(y)h_{n-k}(\sigma_Y^k y)$ and $h_{n-1}(\sigma_Y y) = h_k(\sigma_Y y)h_{n-k}(\sigma_Y^k y)$. Hence, $h_n(y)/h_{n-1}(\sigma_Y y) = h[2^k1]/h[2^{k-1}1]$. Other cases are obtained similarly. □

We apply some ideas found in [4, 24] to study the existence of the limits in (5.4) and continuity of $\hat{h}$. Given $y \in Y$, we identify $h_n(y)$ with the sum of entries of products of matrices described below. In the rest of the section, we assume setting (C). Recall the definitions of the matrix $M_{b_1b_2}$ with $b_1b_2 \in B_2(Y)$. Clearly, $M_{12} = \begin{pmatrix} 0 & z & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for some $z, w \geq 0$, $(z, w) \neq (0, 0)$, and $M_{21} = \begin{pmatrix} 0 & 0 & 0 \\ \bar{x} & 0 & 0 \\ \bar{y} & 0 & 0 \end{pmatrix}$ for some $\bar{x}, \bar{y} \geq 0$, $(\bar{x}, \bar{y}) \neq (0, 0)$. For each $b_1 \cdots b_n \in B_n(Y)$, $n \geq 2$, define $M_{b_1\cdots b_n} := M_{b_1b_2}M_{b_2b_3} \cdots M_{b_{n-1}b_n}$. Then, for $y \in [b_1 \cdots b_n]$, $h_n(y) =$ sum of all entries of $M_{b_1\cdots b_n}$. Since $M_{22}|_{\pi^{-1}(22)}$ is a non-negative real 2×2 matrix, it has real eigenvalues and eigenvectors $\subset \mathbb{R}^2$ exist. Hence, $M_{22}|_{\pi^{-1}(22)}$ is similar to a matrix of real Jordan form J and there exists a matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{R}$, $ad - bc \neq 0$ such that $J = P^{-1}M_{22}|_{\pi^{-1}(22)}P$. If $M_{22}|_{\pi^{-1}(22)}$ is similar to J, we write $M_{22}|_{\pi^{-1}(22)} \sim J$. In the following lemmas, we continue to use the notation and assume that $\mathcal{H}$ is weakly almost additive.

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LEMMA 5.3.

(1) *If* $M_{22}|_{\pi^{-1}(22)} \sim \left(\begin{smallmatrix} \alpha & 1 \\ 0 & \alpha \end{smallmatrix}\right)$, $\alpha \in \mathbb{R}$, *then* $\alpha > 0$ *and* $\lim_{n\to\infty} \log(h[2^{n+1}]/h[2^n]) = \log \alpha$.

(2) *If* $M_{22}|_{\pi^{-1}(22)} \sim \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$, *where* $\beta \neq -\alpha$, *then* $(\alpha, \beta) \neq (0, 0)$ *and the limit* $\lim_{n\to\infty} \log(h[2^{n+1}]/h[2^n])$ *exists.*

(3) *If* $M_{22}|_{\pi^{-1}(22)} \nsim \left(\begin{smallmatrix} \alpha & 0 \\ 0 & -\alpha \end{smallmatrix}\right)$, $\alpha \neq 0$, *then* $\hat{h}$ *is continuous at* $2^\infty \in Y$.

Proof. We first show statement (1). Let $n \geq 4$. Since $(M_{22}|_{\pi^{-1}(22)})^n = P \left(\begin{smallmatrix} \alpha^n & n\alpha^{n-1} \\ 0 & \alpha^n \end{smallmatrix}\right) P^{-1}$, for an invertible matrix P, a straight forward calculation shows that

$$\begin{aligned} h[2^{n+2}] &= \text{sum of all entries of the matrix } M_{22}^{n+1} \\ &= \frac{2(ad-bc)\alpha^{n+1} + (n+1)(a^2-c^2)\alpha^n}{ad-bc} > 0. \end{aligned} \tag{5.5}$$

Then, $\alpha \neq 0$ and statement (1) holds. Next, we show statement (2). Since $(M_{22}|_{\pi^{-1}(22)})^n = P \left(\begin{smallmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{smallmatrix}\right) P^{-1}$,

$$h[2^{n+2}] = \frac{(a+c)(d-b)\alpha^{n+1} + (a-c)(d+b)\beta^{n+1}}{ad-bc} > 0. \tag{5.6}$$

Clearly, $(\alpha, \beta) \neq (0, 0)$. If α or β is zero, assume $\beta = 0$. Then, $\lim_{n\to\infty} h[2^{n+2}]/h[2^{n+1}] = \alpha > 0$. If $\alpha \neq 0$ and $\beta \neq 0$, first suppose $|\alpha| = |\beta|$. Then, $\lim_{n\to\infty} h[2^{n+2}]/h[2^{n+1}] = \alpha = \beta > 0$. If $|\alpha| \neq |\beta|$, assume that $|\alpha| > |\beta|$. Then, $\alpha > 0$ and, if $(a+c)(d-b) \neq 0$, we obtain $\lim_{n\to\infty} h[2^{n+2}]/h[2^{n+1}] = \alpha$. If $(a+c)(d-b) = 0$, then $(a-c)(d+b) \neq 0$ and $\lim_{n\to\infty} h[2^{n+2}]/h[2^{n+1}] = \beta > 0$. For statement (3), it is enough to show that $\lim_{n\to\infty} \log(h[2^{n+2}]/h[2^{n+1}]) = \lim_{n\to\infty} \log(h[2^{n+2}1]/h[2^{n+1}1])$. If $M_{22}|_{\pi^{-1}(22)} \sim \left(\begin{smallmatrix} \alpha & 1 \\ 0 & \alpha \end{smallmatrix}\right)$, then we obtain that

$$\begin{aligned} h[2^{n+2}1] &= \text{sum of all entries of the matrix } M_{22}^{n+1} M_{21} \\ &= \frac{(ad-bc)(\bar{x}+\bar{y})\alpha^{n+1} + (n+1)(a+c)(-c\bar{x}+a\bar{y})\alpha^n}{ad-bc}. \end{aligned} \tag{5.7}$$

Then, $\lim_{n\to\infty} h[2^{n+2}1]/h[2^{n+1}1] = \alpha$. If $M_{22}|_{\pi^{-1}(22)} \sim \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$, then

$$h[2^{n+2}1] = \frac{(d\bar{x}-b\bar{y})(a+c)\alpha^{n+1} + (a\bar{y}-c\bar{x})(b+d)\beta^{n+1}}{ad-bc}. \tag{5.8}$$

If $\beta = 0$, then $\alpha \neq 0$ and the continuity of $\hat{h}$ at 2^∞ follows combining the proof above. The same result holds for $\alpha = \beta \neq 0$. If $|\alpha| \neq |\beta|$, $|\alpha| > |\beta|$, then we have the following:

(i) $(d\bar{x}-b\bar{y})(a+c) = 0$. Then, $(a\bar{y}-c\bar{x})(b+d) \neq 0$ and $\lim_{n\to\infty} h[2^{n+1}1]/h[2^n1] = \beta$. The weak almost additivity of $\mathcal{H}$ implies that $(a+c)(d-b) = 0$.

(ii) $(d\bar{x}-b\bar{y})(a+c) \neq 0$. Then, $\lim_{n\to\infty} h[2^{n+1}1]/h[2^n1] = \alpha$. The weak almost additivity of $\mathcal{H}$ implies that $(a+c)(d-b) \neq 0$.

Hence, the results follow by the proof above. □

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LEMMA 5.4.

(1) *If* $M_{22}|_{\pi^{-1}(22)} \sim \left(\begin{smallmatrix}\alpha & 1\\ 0 & \alpha\end{smallmatrix}\right)$, *then* $\lim_{n\to\infty} \log(h[12^{n+1}]/h[2^{n+1}])$ *exists.*

(2) *If* $M_{22}|_{\pi^{-1}(22)} \sim \left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)$, $\beta \neq -\alpha$, *then* $\lim_{n\to\infty} \log(h[12^{n+1}]/h[2^{n+1}])$ *exists.*

Proof. We show statement (1). Using the real Jordan form of $M_{22}|_{\pi^{-1}(22)}$,

$$\begin{aligned} h[12^{n+2}] &= \text{sum of all entries of the matrix } M_{12}M_{22}^{n+1} \\ &= \frac{(ad-bc)(z+w)\alpha^{n+1} + (n+1)(az+cw)(a-c)\alpha^n}{ad-bc}. \end{aligned} \tag{5.9}$$

Case 1: $(az+cw)(a-c) \neq 0$. The weak almost additivity of $\mathcal{H}$ and (5.5) imply $a+c \neq 0$. Hence, $\lim_{n\to\infty} h[12^{n+2}]/h[2^{n+2}] = (az+cw)/(a+c)$.

Case 2: $(az+cw)(a-c) = 0$. The weak almost additivity of $\mathcal{H}$ and (5.5) imply that $a+c=0$. Then, $z+w \neq 0$ and $\lim_{n\to\infty} h[12^{n+1}]/h[2^n] = (z+w)/2$.

Next, we show statement (2). Since

$$h[12^{n+2}] = \frac{\alpha^{n+1}(d-b)(az+cw) + \beta^{n+1}(a-c)(bz+dw)}{ad-bc}. \tag{5.10}$$

Case 1: $\beta = 0$. Then, $\lim_{n\to\infty} h[12^{n+1}]/h[2^n] = (az+cw)/(a+c)$.

Case 2: $\alpha \neq 0$ and $\beta \neq 0$. We apply (5.6) and (5.10):

(1) $\alpha = \beta$. We obtain $\lim_{n\to\infty} h[12^{n+1}]/h[2^n] = (z+w)/2$.

(2) $|\alpha| \neq |\beta|$. Assume that $|\alpha| > |\beta|$:

  (i) $(d-b)(az+cw) \neq 0$. The weak almost additivity of $\mathcal{H}$ implies that $(d-b)(az+cw) \neq 0 \iff (a+c)(d-b) \neq 0$. Hence, $\lim_{n\to\infty} h[12^{n+1}]/h[2^n] = (az+cw)/(a+c)$.

  (ii) $(d-b)(az+cw) = 0$. Then, $(d-b)(az+cw) = 0 \iff (a+c)(d-b) = 0$. Thus, $\lim_{n\to\infty} h[12^{n+1}]/h[2^n] = (bz+dw)/(b+d)$. □

LEMMA 5.5.

(1) *If* $M_{22}|_{\pi^{-1}(22)} \sim \left(\begin{smallmatrix}\alpha & 1\\ 0 & \alpha\end{smallmatrix}\right)$, *then* $\alpha > 0$ *and the following hold.*

  (a) *If* $(wc+az)(-c\bar{x}+a\bar{y}) \neq 0$, *then* $\hat{h}$ *is continuous at* $12^\infty \in Y$.

  (b) *If* $(wc+az)(-c\bar{x}+a\bar{y}) = 0$, *then* $\hat{h}$ *is continuous at* $12^\infty \in Y$ *if and only if* $(\bar{x}-\bar{y})(w-z) = 0$.

(2) *If* $M_{22}|_{\pi^{-1}(22)} \sim \left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)$, *then* $(\alpha,\beta) \neq (0,0)$ *and the following hold.*

  (a) *If* $\alpha \neq \beta$, *where* $\beta \neq -\alpha$, *then* $\hat{h}$ *is continuous at* $12^\infty \in Y$.

  (b) *If* $\alpha = \beta$, *then* $\hat{h}$ *is continuous at* $12^\infty \in Y$ *if and only if* $(\bar{x}-\bar{y})(w-z) = 0$.

Proof. We apply the proof of Lemma 5.4. We show that

$$\lim_{n\to\infty} \log \frac{h[12^{n+2}1]}{h[2^{n+2}1]} = \lim_{n\to\infty} \log \frac{h[12^{n+2}]}{h[2^{n+2}]}. \tag{5.11}$$

$$\begin{aligned} h[12^{n+2}1] &= \text{sum of all entries of the matrix } M_{12}M_{22}^{n+1}M_{21} \\ &= \frac{(ad-bc)(\bar{x}z+\bar{y}w)\alpha^{n+1} + (n+1)\alpha^n(wc+az)(-c\bar{x}+a\bar{y})}{ad-bc}. \end{aligned} \tag{5.12}$$

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If $(wc + az)(-c\bar{x} + a\bar{y}) \neq 0$, then the weak almost additivity of $\mathcal{H}$ implies $a + c \neq 0$ and $\lim_{n\to\infty} h[12^{n+2}1]/h[2^{n+2}1] = (wc + az)/(a + c)$. In (5.5), if $a - c = 0$, then $\lim_{n\to\infty} h[12^{n+2}1]/h[2^{n+2}] = \infty$. Hence, $a - c \neq 0$ and the result follows by the proof of Lemma 5.4(1), Case 1. If $(wc + az)(-c\bar{x} + a\bar{y}) = 0$, then $(ad - bc)(\bar{x}z + \bar{y}w) \neq 0$. If $wc + az = 0$, then we obtain $\lim_{n\to\infty} h[12^{n+2}1]/h[2^{n+2}1] = (\bar{x}z + \bar{y}w)/(\bar{x} + \bar{y})$. If $wc + az \neq 0$ and $-c\bar{x} + a\bar{y} = 0$, then (5.9) implies that $a - c = 0$. Then, $\lim_{n\to\infty} h[12^{n+2}1]/h[2^{n+2}1] = (\bar{x}z + \bar{y}w)/(\bar{x} + \bar{y})$. The proof of Lemma 5.4(1), Case 2, yields the results. To show statement (2), observe that

$$h[12^{n+2}1] = \frac{(az + cw)(d\bar{x} - b\bar{y})\alpha^{n+1} + (a\bar{y} - c\bar{x})(bz + dw)\beta^{n+1}}{ad - bc}. \tag{5.13}$$

If $\beta = 0$, then $\alpha \neq 0$ and, by (5.8), $(a + c)(d\bar{x} - b\bar{y}) \neq 0$. Then, $\lim_{n\to\infty} h[12^{n+2}1]/h[2^{n+2}1] = (az + cw)/(a + c)$ and (5.11) holds. Next, suppose that $\alpha \neq 0$ and $\beta \neq 0$. If $\alpha = \beta$, a simple computation shows that $\lim_{n\to\infty} h[12^{n+1}1]/h[2^{n+1}1] = (\bar{x}z + \bar{y}w)/(\bar{x} + \bar{y})$. Hence, h is continuous at 12^∞ if and only if $(z + w)/2 = (\bar{x}z + \bar{y}w)/(\bar{x} + \bar{y})$, that is, $(\bar{x} - \bar{y})(w - z) = 0$. If $|\alpha| \neq |\beta|$, assume that $|\alpha| > |\beta|$. Then,

$$\frac{h[12^{n+2}1]}{h[2^{n+2}1]} = \frac{(az + cw)(d\bar{x} - b\bar{y})\alpha^{n+1} + (a\bar{y} - c\bar{x})(bz + dw)\beta^{n+1}}{(d\bar{x} - b\bar{y})(a + c)\alpha^{n+1} + (a\bar{y} - c\bar{x})(b + d)\beta^{n+1}}. \tag{5.14}$$

If $(az + cw)(d\bar{x} - b\bar{y}) = 0$, then by the weak almost additivity of $\mathcal{H}$, we have $(d\bar{x} - b\bar{y})(a + c) = 0$ and hence, $h[12^{n+2}1]/h[2^{n+2}1] = (bz + dw)/(b + d)$. The weak almost additivity of $\mathcal{H}$ implies $(a + c)(d - b) = 0$. By the proof of Lemma 5.4(2), the result follows. If $(az + cw)(d\bar{x} - b\bar{y}) \neq 0$, then by the weak almost additivity of $\mathcal{H}$, $(d\bar{x} - b\bar{y})(a + c) \neq 0$ and hence, $\lim_{n\to\infty} h[12^{n+2}1]/h[2^{n+2}1] = (az + cw)/(a + c)$. We have $(a + c)(d - b) \neq 0$ by (5.6) and the weak almost additivity of $\mathcal{H}$. The proof of Lemma 5.4(2) implies the result. □

LEMMA 5.6. *Suppose that* $M_{22}|_{\pi^{-1}(22)} \sim \left(\begin{smallmatrix} \alpha & 0 \\ 0 & -\alpha \end{smallmatrix}\right)$. *If* $M(2, 3) = M(3, 2)$, *then* $\hat{h}$ *is continuous on* Y *except possibly at the point* 12^∞. *If, in addition,* $\bar{x} = \bar{y}$ *or* $z = w$, *then* $\hat{h} \in C(Y)$. *In particular, if* $f = 0$, *then* $\hat{h} \in C(Y)$.

Proof. Clearly, $\alpha \neq 0$, and $M_{22}|_{\pi^{-1}(22)}$ has eigenvalues $\pm\alpha$ if and only if $M_{22}|_{\pi^{-1}(22)} = \left(\begin{smallmatrix} 0 & a_1 \\ a_2 & 0 \end{smallmatrix}\right)$, $a_1, a_2 > 0$. For $k \in \mathbb{N}$, $h[2^{2k+1}] = 2a_1{}^k a_2{}^k$ and $h[2^{2k+2}] = a_1{}^k a_2{}^k (a_1 + a_2)$. Hence, $h[2^{2k+1}]/h[2^{2k}] = 2a_1a_2/(a_1 + a_2)$ and $h[2^{2k+2}]/h[2^{2k+1}] = (a_1 + a_2)/2$. Thus, $\hat{h}$ is defined at 2^∞ if and only if $a_1 = a_2$. Observe that $\lim_{k\to\infty} h[2^{2k+1}1]/h[2^{2k}1] = a_1a_2(\bar{x} + \bar{y})/(a_2\bar{x} + a_1\bar{y})$ and $\lim_{k\to\infty} h[2^{2k+2}1]/h[2^{2k+1}1] = (a_2\bar{x} + a_1\bar{y})/(\bar{x} + \bar{y})$. Hence, $a_1 = a_2$ implies the continuity at 2^∞. Since $\lim_{k\to\infty} h[12^{2k+1}]/h[2^{2k+1}] = (z + w)/2$ and $\lim_{k\to\infty} h[12^{2k+2}]/h[2^{2k+2}] = (za_1 + wa_2)/(a_1 + a_2)$, $\hat{h}$ is defined at 12^∞ if $a_1 = a_2$ or $z = w$. Since $\lim_{k\to\infty} h[12^{2k+1}1]/h[2^{2k+1}1] = (z\bar{x} + w\bar{y})/(\bar{x} + \bar{y})$ and $\lim_{k\to\infty} h[12^{2k}1]/h[2^{2k}1] = (\bar{y}za_1 + \bar{x}wa_2)/(a_1\bar{y} + a_2\bar{x})$, $\hat{h}$ is continuous at 12^∞ if $a_2\bar{x}^2 = a_1\bar{y}^2$ or $z = w$. If $f = 0$, then $a_1 = a_2 = 1$. By Setting (C), $\bar{x} = \bar{y}$ or $z = w$ holds. □

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LEMMA 5.7. *Suppose that $M_{22}|_{\pi^{-1}(22)}$ satisfies one of the following conditions:*
(i) $M_{22}|_{\pi^{-1}(22)} \sim \left(\begin{smallmatrix}\alpha & 1\\ 0 & \alpha\end{smallmatrix}\right)$ *and* $(wc + az)(-c\bar{x} + a\bar{y}) = 0$;
(ii) $M_{22}|_{\pi^{-1}(22)} \sim \left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right)$;
(iii) $M_{22}|_{\pi^{-1}(22)} \sim \left(\begin{smallmatrix}\alpha & 0\\ 0 & -\alpha\end{smallmatrix}\right)$.
If condition (i) or (ii) holds, define for $y \in Y$,

$$\hat{h}_1(y) = \begin{cases} \log\left(\dfrac{\bar{x}z + \bar{y}w}{\bar{x} + \bar{y}}\right), & y = 12^\infty, \\ \hat{h}(y) & \text{otherwise.} \end{cases} \tag{5.15}$$

If condition (iii) holds, define for $y \in Y$,

$$\hat{h}_2(y) = \begin{cases} \frac{1}{2}\log(a_1 a_2), & y = 12^\infty, 2^\infty, \\ \hat{h}(y) & \text{otherwise.} \end{cases} \tag{5.16}$$

Then, $\hat{h}_1 \in C(Y)$ and $\hat{h}_2$ is Borel measurable on Y. If $\mathcal{H}$ is almost additive (respectively weakly almost additive), then the unique invariant Gibbs measure for $\mathcal{H}$ is Gibbs (respectively weak Gibbs) for $\hat{h}_i$, $i = 1, 2$.

Proof. If $M_{22}|_{\pi^{-1}(22)}$ satisfies condition (i) or (ii), then $\hat{h}_1 \in C(Y)$ by the proof of Lemma 5.5. Since $\hat{h}(12^\infty) = (z + w)/2$, it is easy to see that $\sup_{n\in\mathbb{N}}\{e^{(S_n\hat{h})(y)}/e^{(S_n\hat{h}_1)(y)} : y \in Y\} < \infty$. By Proposition 5.1(iii), we obtain the result. If $M_{22}|_{\pi^{-1}(22)}$ satisfies condition (iii), we show (5.2) replacing $\hat{h}$ by $\hat{h}_2$ for $y \in P$, where $P := \{y \in Y :$ there exists $p \in \mathbb{N} \cup \{0\}$ such that $\sigma_Y^p y = 2^\infty\}$. If $y = 2^\infty$, then for $k \in \mathbb{N}$, we obtain $h_{2k}(y)/e^{(S_{2k}\hat{h}_2)(y)} = (a_1 + a_2)/a_1 a_2$ and $h_{2k+1}(y)/e^{(S_{2k+1}\hat{h}_2)(y)} = 2$ by the proof of Lemma 5.6. Noticing that (5.2) holds every $y \notin P$ and using the weak almost additivity of $\mathcal{H}$, it is not difficult to see that $\sup_{n\in\mathbb{N}}\{h_n(y)/e^{(S_n\hat{h}_2)(y)} : y \in P\} < \infty$. □

Proof of Theorem 4.8. If $\pi\mu$ is Gibbs for $\hat{g}$ and $\mathcal{G}$, then clearly, $\mathcal{G}$ is almost additive. Notice that $M_{22}|_{\pi^{-1}(22)}$ has eigenvalues $\pm\alpha$, $\alpha \neq 0$, if and only if $M_{22}|_{\pi^{-1}(22)} = \left(\begin{smallmatrix}0 & a_1\\ a_2 & 0\end{smallmatrix}\right)$, a_1, $a_2 > 0$. If $M_{22}|_{\pi^{-1}(22)} \neq \left(\begin{smallmatrix}0 & a_1\\ a_2 & 0\end{smallmatrix}\right)$, $a_1, a_2 > 0$, then by Lemmas 5.3 and 5.4, set $\hat{g} = \hat{h}$. Otherwise, set $\hat{g} = \hat{h}_2$ by Lemma 5.7. This proves statement (i). For statement (ii), if $M_{22}|_{\pi^{-1}(22)}$ satisfies Lemma 5.7(i) or (ii), then set $\hat{g} = \hat{h}_1$. Otherwise, set $\hat{g} = \hat{h}$. Lemma 5.6 proves statement (iii). □

6. *Examples*

In this section, we give examples that illustrate Theorem 3.2 (Corollary 4.1) and Theorem 4.8. Example 6.1 proves Proposition 4.7(ii). In both examples, we consider a one-block factor map $\pi : X \subset \{1, 2, 3\}^{\mathbb{N}} \to Y \subset \{1, 2\}^{\mathbb{N}}$ determined by $\pi(1) = 1$ and $\pi(2) = \pi(3) = 2$, and let μ be the Gibbs measure for $f = 0$ on X. See §5 for the notation $\mathcal{H}$, h, and $\hat{h}$.

Example 6.1. [**18**, Example 3.2] Let $X \subset \{1, 2, 3\}^{\mathbb{N}}$ be the topologically mixing shift of finite type with the transition matrix A given by

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$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then, Y is the topologically mixing shift of finite type with the transition matrix B,

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then, π is not fiber-wise sub-positive mixing because $23 \notin B_2(X)$. We study the factor $\pi\mu$ using the sequence $\mathcal{G}$ associated to $P(\sigma_X, \pi, 0)$. Then, $\mathcal{G}{=}\mathcal{H}$. Since $h[12^n1] = h[2^n] = h[2^n1] = h[12^n] = 2$ for each $n \in \mathbb{N}$, $\mathcal{H}$ is almost additive. By Corollary 4.10, there exists a continuous function for which $\pi\mu$ is Gibbs for a continuous function and it is given by $\hat{h}(= g$ in (4.2)).

Example 6.2. [**20**, Example 5.1] Let $X \subset \{1, 2, 3\}^{\mathbb{N}}$ be the topologically mixing shift of finite type with the transition matrix A given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then, Y is the topologically mixing shift of finite type with the transition matrix B from Example 6.1. Then, π is not fiber-wise sub-positive mixing. It is not difficult to show that $\mathcal{H}$ is weakly almost additive by using that $h[2^n] = h[2^n1] = n + 1$ and $h[12^n] = h[12^n1] = n$ for $n \geq 1$. Hence, by Corollary 4.10, $\pi\mu$ is a unique weak Gibbs for a continuous function and it is given by $\hat{h}(= g$ in (4.2)).

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