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Big mapping class groups and rigidity of the simple circle

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Abstract. Let Γ denote the mapping class group of the plane minus a Cantor set. We show that every action of Γ on the circle is either trivial or semiconjugate to a unique minimal action on the so-called *simple circle*.

Key words: group actions, low-dimensional dynamics, big mapping class groups, rigidity
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1. *Introduction*

Let Γ denote the mapping class group of the plane minus a Cantor set. This group and its subgroups arise naturally in dynamics (see e.g. [**5**, **8**]) and has been studied in detail by Bavard [**2**] and Bavard and Walker [**3**, **4**]. It is the best understood and most studied example of a so-called 'big' mapping class group; see e.g. [**1**, **10**, **15**].

Calegari [**5**] constructed a faithful action of Γ on the circle by homeomorphisms. Bavard and Walker [**3**] constructed a different action which is more closely related to the action of Γ on the Gromov boundary of the *ray graph*; see [**2**]. These actions are both constructed

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via hyperbolic geometry but they are in fact different. It is natural to ask if they are related and more generally to try to classify actions of Γ on the circle. In this paper we give such a classification: there is a unique minimal action on the so-called *simple circle* and every non-trivial action is semiconjugate to this one. In other words for every non-trivial action of Γ on the circle there is a unique minimal compact non-empty invariant subset and, after collapsing complementary intervals to points, the action on the quotient circle is conjugate to the action on the simple circle (possibly up to a change of orientation).

The description of the simple circle is closely related to Bavard and Walker's *conical circle*. Our analysis of the dynamics in this case sharpens some of the results of [**3**, **4**] and gives new proofs of others.

It is worth remarking that the analogue of our main result is known for certain mapping class groups of finite type. Let $\Gamma_{g,1}$ denote the mapping class group of a surface of genus g with one marked point. This group acts (by point-pushing) on the ideal boundary of the fundamental group of the surface; one calls this the *geometric action*. Mann and Wolff [**13**] showed that every non-trivial action of $\Gamma_{g,1}$ on the circle is semiconjugate to the geometric action when $g > 2$. Their proof and ours are quite dissimilar, but there is some conceptual overlap.

1.1. *Statement of results.* In §§2–3 we introduce, following Bavard and Walker [**3**], the action of Γ on the conical circle.

Let Ω denote the plane minus a Cantor set. The *conical cover* Ω_C is the covering space associated to the $\mathbb{Z}$ subgroup of $\pi_1(\Omega)$ generated by a loop around infinity. Ω_C is conformal to $\mathbb{D} - 0$ and the *conical circle* S^1_C is identified with $\partial\mathbb{D}$. The group Γ acts naturally on this circle by homeomorphisms.

Points in S^1_C, thought of as end points, correspond to bi-infinite geodesics in a hyperbolic structure on Ω_C 'starting' at ∞. Distinguished amongst these are the subset of *simple* geodesics, those whose image under the covering projection is embedded in Ω. The simple geodesics fall into three classes: the *short rays* R, the *lassos* L and the *long rays* X.

The main theorem we prove in this section relates the dynamics on S^1_C to the topology of these geodesics.

THEOREM 3.4. (Conical circle) *The set $R \cup X$ is a Cantor set in S^1_C and is the unique minimal set for the action of Γ on S^1_C. Every complementary interval contains a unique lasso and all lassos arise in this way. The set of boundary points of $R \cup X$ is exactly the set of long rays that spiral around some simple geodesic loop in Ω.*

In §4 we define the simple circle S^1_S as the quotient of S^1_C obtained by collapsing complementary gaps to $R \cup X$. The action of Γ on S^1_S is minimal and has both countable and uncountable orbits. The countable orbits have the following characterization.

THEOREM 4.1. (Countable orbit) *A point in S^1_C has a countable orbit if and only if the associated geodesic γ stays inside some subsurface $\Sigma \subset \Omega$ of finite type.*

In §5 we give some examples and non-examples of subgroups of Γ. A countable group is a subgroup of Γ if and only if it is circularly orderable. The situation for uncountable

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groups is more complicated: the abstract group S^1 embeds in Γ, whereas $\mathrm{PSL}_2(\mathbb{R})$ does not.

Section 6 is the heart of the paper and contains the main result.

THEOREM 6.1. (Rigidity) *Any homomorphism* $\Gamma \to \mathrm{Homeo}^+(S^1)$ *is either trivial or is semiconjugate (possibly up to a change of orientation) to the action on the simple circle* S^1_S.

The argument depends on an analysis of certain special subgroups of Γ. If $r \in R$ is a simple ray in Ω, we denote by Γ_r the stabilizer of r in Γ and by $\Gamma_{(r)}$ the subgroup of Γ_r represented by homeomorphisms which are the identity in a neighborhood of the end point of r.

The subgroups Γ_r, $\Gamma_{(r)}$ and their properties are part of the essential structure of Γ and these structural results should be of independent interest.

The skeleton of the argument proceeds as follows:

(1) for any action of Γ on the circle, each Γ_r has a global fixed point;
(2) for any two rays r, s with distinct end points, Γ_r and Γ_s together generate Γ;
(3) thus (with some work), one shows that either there is a global fixed point for Γ or there is a Γ-equivariant injective map P from R to S^1; and finally
(4) the circular order on $P(R)$ is rigid.

Identifying $P(R)$ in a given circle with R in the simple circle gives the semiconjugacy. The proof of (1) uses bounded cohomology and a homological vanishing argument due to Mather.

Low-dimensional (co-)homology, especially of certain subgroups of Γ, plays a significant role in the proof of our main theorem and it is evidently important to try to understand the homology of Γ and some related groups. In an appendix we compute the (co-)homology in dimension two of $\widehat{\Gamma}$, the mapping class group of the sphere minus a Cantor set (which we denote $\widehat{\Omega}$). The group $\widehat{\Gamma}$ is (uniformly) perfect, so $H_1(\widehat{\Gamma}; \mathbb{Z}) = H^1(\widehat{\Gamma}; \mathbb{Z}) = 0$. The main result in the appendix is as follows.

THEOREM A.2. (Homology) $H_2(\widehat{\Gamma}; \mathbb{Z}) = \mathbb{Z}/2$ *and* $H^2(\widehat{\Gamma}; \mathbb{Z}) = 0$.

2. *The conical circle*

Let Ω denote the plane minus a Cantor set. This surface admits many different hyperbolic structures. The notions of conical cover and conical circle introduced below do not depend on the choice of hyperbolic structures. For concreteness, choose a Schottky subgroup of $\mathrm{PSL}_2(\mathbb{C})$ and consider the conformal structure on the sphere minus its limit set; then remove some point and hyperbolize the result. This hyperbolic structure has the nice property that away from any neighborhood of the puncture ('infinity') the geometry is precompact in the space of pointed metric spaces, though we do not make use of this fact in what follows.

The *conical cover* Ω_C is the covering space of Ω associated to the $\mathbb{Z}$ subgroup of $\pi_1(\Omega)$ generated by a loop around infinity; see Figure 1. Geometrically, Ω_C is conformally equivalent to the punctured unit disk. The *conical circle* S^1_C is the boundary of the unit disk, and it may be identified in a natural way with the space of proper bi-infinite geodesics in Ω_C with one end going out of the puncture. Informally, we call these *geodesic rays*. The identification with S^1_C topologizes the space of such geodesic rays.

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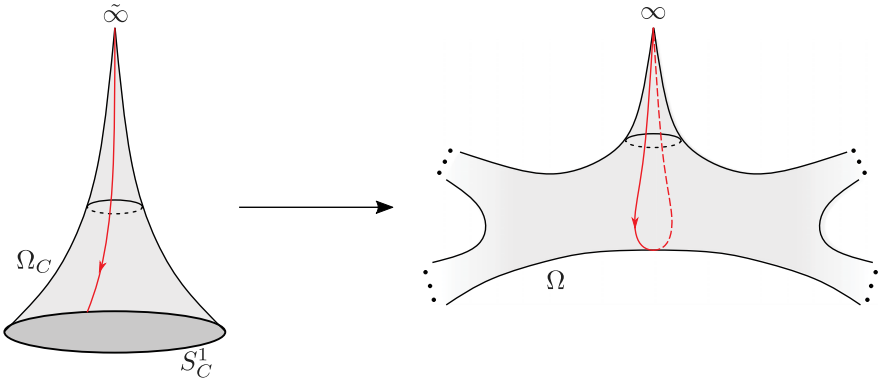


FIGURE 1. The conical cover with a geodesic ray projecting to a lasso.

The mapping class group Γ acts on Ω by isotopy classes of homeomorphisms, and such homeomorphisms all have canonical lifts to Ω_C, by lifting the action in the obvious way near infinity. Although these maps are not quasi-isometries in the hyperbolic metric, they nevertheless extend *continuously* to homeomorphisms of the conical circle, and these extensions depend only on the mapping class. We thus obtain a representation $\Gamma \to \mathrm{Homeo}^+(S^1_C)$.

The conical circle and this action were introduced in [**3**], although another closely related action was introduced earlier in [**5**].

3. *The simple set*

A geodesic ray γ is *simple* if it is embedded in Ω. Such simple geodesics fall into three natural classes.

(1) Proper simple geodesics that run from ∞ to a point in the Cantor set; these are called *short rays*.
(2) Proper simple geodesics that run from ∞ to ∞; these are called *lassos*.
(3) Simple geodesics that start at ∞ and enter Ω but are not proper (and therefore recur somewhere); these are called *long rays*.

We denote the set of short rays by R, the set of lassos by L and the set of long rays by X. All our geodesics are *oriented*; for a short or long ray this is unambiguous since ∞ is the 'initial point'. But for a lasso we need to choose which end of the geodesic is 'initial' and which end is 'terminal'. We think of R, L, X as rays in Ω and at the same time as points in S^1_C. The *simple set* is the union $R \cup L \cup X \subset S^1_C$.

Note that Γ acts transitively on the set of short rays and also on the set of lassos; i.e. R and L are two orbits of Γ. Also note that R is uncountable, whereas L is countable.

We first prove some elementary facts about the dynamics of Γ.

LEMMA 3.1. [**3**] *The simple set is closed.*

Proof. Let γ be a geodesic ray from infinity. If it intersects itself, the first point of intersection is stable under a small perturbation; thus, the complement of the simple set is open. □

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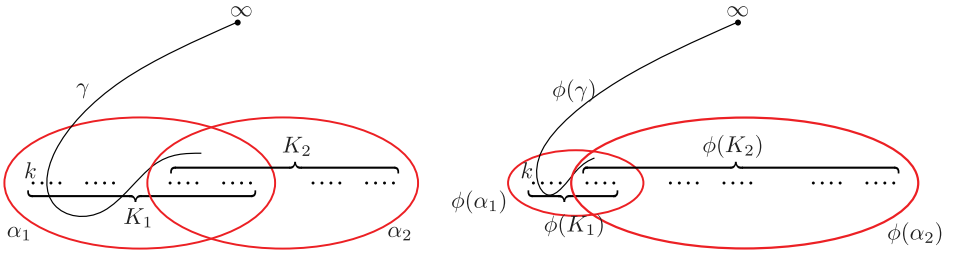


FIGURE 2. A homeomorphism ϕ shrinking a Cantor subset K_1 and expanding K_2 such that ϕ^n shrinks K_1 to k and $\phi^n(\gamma)$ converges to a short ray to k.

LEMMA 3.2. *R is contained in the closure of every orbit. Thus, the closure of R is the unique minimal set.*

Proof. We can separate the Cantor set from infinity by two simple closed geodesics α_1, α_2 in many different ways. For instance, let each α_j separate a proper subset K_j of the Cantor set from infinity, so that $K_1 \cap K_2$ is non-empty. Any geodesic ray γ must be simple up to the first time it crosses one of the α_j; without loss of generality let it cross α_1. Now we can take a sequence of mapping classes ϕ_n which shrink K_1 homothetically down to some point k in the Cantor set and the images $\phi_n(\gamma)$ converge to a short ray from ∞ to k. See Figure 2. □

LEMMA 3.3. *The simple set is nowhere dense.*

Proof. Any proper simple geodesic is a short ray or a lasso. We have seen that every short ray is a limit of every orbit; in particular, a short ray can be approximated by non-simple geodesics. A small perturbation of a lasso is a geodesic running deep into the cusp near infinity; any such geodesic which does not limit to infinity but goes sufficiently deep must self-intersect. It follows that lassos are *isolated* in the simple set. Finally, a non-proper geodesic γ must recur somewhere and therefore comes arbitrarily close to itself infinitely often; thus, it may be perturbed very slightly to produce a self-intersection. □

We are now able to prove the main theorem of this section.

THEOREM 3.4. (Conical circle) *The set $R \cup X$ is a Cantor set in S^1_C and is the unique minimal set for the action of Γ on S^1_C. Every complementary interval contains a unique lasso and all lassos arise in this way. The set of boundary points of $R \cup X$ is exactly the set of long rays that spiral around some simple geodesic loop in Ω.*

Proof. Let γ be a non-simple geodesic. Since it begins as an embedded geodesic ray from ∞ it has a first point p such that the subgeodesic from ∞ to p self-intersects. The geodesic up to the point p looks like the letter ρ embedded in Ω; see Figure 3. As we perturb γ, this first point of intersection slides up and down the 'stem' of the ρ. We can push the intersection point in one direction all the way to ∞ and obtain a lasso as the limit. Or we can push in the other direction around and around so that it spirals onto the unique simple geodesic which is in the isotopy class of the 'loop' of the ρ. See the right-hand part of

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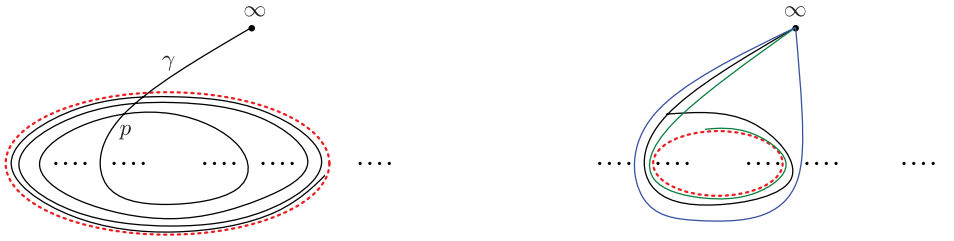


FIGURE 3. The left-hand figure depicts a non-simple geodesic γ and its first self-intersection p. The right-hand figure shows the perturbations of γ in two directions to a lasso and a long ray spiraling around the unique geodesic loop isotopic to the 'loop' part of γ up to p.

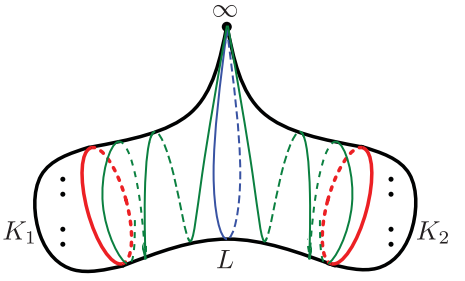


FIGURE 4. A lasso L separating the Cantor set into $K_1 \sqcup K_2$; the two long spirals correspond to the end points of the complementary interval of $S \cup X$ containing L.

Figures 3 and 4. Thus, every lasso is a boundary point on both sides of a complementary interval to the simple set and all complementary intervals arise in this way; and the other boundary point of a complementary interval is a long ray which spirals around a simple geodesic. Let us call this kind of long ray a *long spiral*.

Formally, one can see this pushing on the universal cover, depicted in Figure 5. In Ω, there is an annular region between the 'loop' of the ρ and the geodesic loop homotopic to it. Pick an arbitrary geodesic arc α to cut this region open and lift it to the universal cover. Then the lifted region D witnesses a lift τ of the geodesic loop and two lifts γ_1, γ_2 of γ. Moreover, the hyperbolic element g_τ representing the image of τ in Ω oriented as in the figure takes γ_1 to γ_2. Then the geodesics in blue and green are obtained by pushing γ_1 to its right and left, projecting to the lasso and the long spiral, respectively. To see that any smaller perturbation γ_1' of γ_1 has non-simple image in Ω, note that g_τ takes the sector between γ_1 and γ_1' to the one between γ_2 and $\gamma_2' := g_\tau \gamma_1'$. The monotonicity of the g_τ-action on the boundary $\partial\mathbb{H}^2$ implies that γ_1' and γ_2' must intersect as in the figure and thus the projection of γ_1' in Ω is not simple.

A long spiral γ spirals around a simple geodesic α and this geodesic α separates ∞ from some proper subset K of the Cantor set. Pick $k \in K$ and a simple arc from k to α, and spiral this arc around α in the opposite direction to the spiralling of γ; call the result δ. We may truncate γ after some finite amount of spiralling around α and then join it to δ to produce the isotopy class of a short ray. Such short rays approximate γ in $R \cup X$. Consequently every long spiral is in the closure of R and is not isolated in the simple set.

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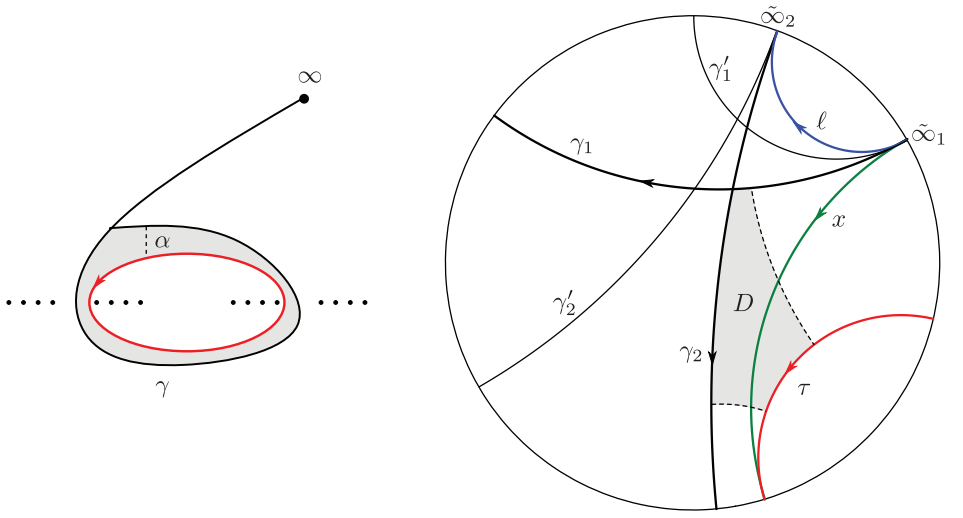


FIGURE 5. Pushing γ_1 to its right and left produces geodesics ℓ and x projecting to a lasso and a long spiral in Ω. All smaller perturbations are not simple in Ω.

Since no other long ray is a boundary point of a complementary interval to the simple set, it follows that $R \cup X$ is a Cantor set, whose complementary intervals each contain a unique lasso and all lassos arise in this way.

It remains to show that $R \cup X$ is minimal or equivalently that it is equal to the closure of R. We have already seen that boundary points of $R \cup X$ are in the closure of R; these are the long spirals. But a long ray γ which is not a boundary point is a limit on *both sides* of boundary points, which are all limits (on one side) of short rays. Thus, every long ray which is not a long spiral is a limit of short rays from both sides and is therefore contained in the closure of R. This completes the proof. □

4. *Countable orbits and the simple circle*

Although Γ is uncountable, it has several countable orbits; the lasso set for instance. The next theorem characterizes these countable orbits.

THEOREM 4.1. (Countable orbit) *A point in S^1_C has a countable orbit if and only if the associated geodesic γ stays inside some subsurface $\Sigma \subset \Omega$ of finite type.*

Proof. Suppose that γ stays in Σ. There are only countably many subsurfaces of Ω of the topological type of Σ. Also, the mapping class group of Σ is countable. Thus, γ has a countable orbit.

Conversely suppose that γ goes essentially through infinitely many disjoint essential subsurfaces Σ_i of Ω. Choose a subsequence converging to some point k in the Cantor set and choose in each subsurface a simple loop α_i intersecting γ essentially. Since for any given initial subarc of γ there is some Σ_i disjoint from it, Dehn twists in the α_i perturb γ in S^1_C an arbitrarily small amount for i large. Choose a sufficiently sparse subsequence of indices, which we relabel α_i; associated to a sequence $\sigma : \mathbb{N} \to \{0, 1\}$ we can form the

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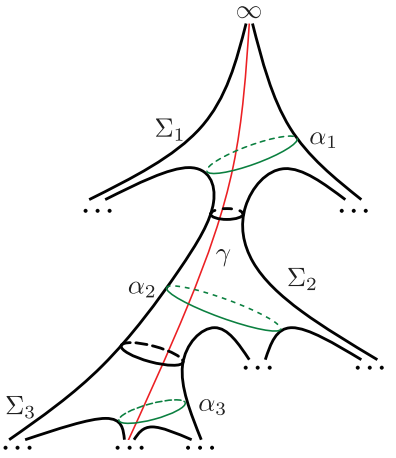


FIGURE 6. Infinitely many simple essential loops α_i in infinitely many disjoint subsurfaces Σ_i that a ray γ visits, with only the first three depicted.

mapping class τ_σ which is the product of a (positive) Dehn twist in all α_i with $\sigma(i) = 1$. If for a fixed metric d on S^1_C and for each $i > 0$ we choose α_{i+1} so that $d(\tau_{\bar\sigma}(\gamma), \tau_\sigma(\gamma))$ is much smaller than all $d(\tau_{\sigma'}(\gamma), \tau_\sigma(\gamma))$ for any $\sigma \neq \sigma'$ supported on $\{1, \dots, i\}$ and $\bar\sigma(k) = \sigma(k)$ if and only if $k \neq i + 1$, then the map $\{0, 1\}^{\mathbb{N}} \to S^1_C$ sending σ to $\tau_\sigma(\gamma)$ is injective. Figure 6 gives an illustration. □

Example 4.2. There are uncountably many long rays with countable orbits. Pick a subsurface Σ of finite type and let Λ be a minimal geodesic lamination in Σ. Since Λ is minimal, it avoids a neighborhood of ∞. Let U be the complementary region to Λ in Σ containing infinity and let λ be a boundary leaf of U. Then we can take γ to be a geodesic ray from ∞ to one of the end points of λ.

The *simple circle* S^1_S is the quotient of the conical circle S^1_C where we collapse every complementary interval to $R \cup X$ to a point. The action of Γ on S^1_S is minimal by Theorem 3.4 and has uncountably many countable orbits by Example 4.2. The existence of these orbits can be used to obstruct the existence of injective homomorphisms from certain (necessarily uncountable) circularly orderable groups to Γ.

5. *Subgroups and non-subgroups*

THEOREM 5.1. *A countable group G is isomorphic to a subgroup of Γ if and only if it is circularly orderable.*

Proof. Since Γ acts faithfully on S^1, it is circularly orderable and so is every subgroup, countable or not. If G is circularly orderable, then it acts faithfully on S^1 by homeomorphisms. Take a countable dense set of orbits of G and blow them up to complementary intervals. This defines an action of G on the circle preserving a Cantor set, so that the action on the Cantor set is already faithful. Suspend this to an action on S^2 where the Cantor set is contained in the equator, and puncture S^2 at the north pole. This

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defines a map from G to Γ. This map is injective; in fact it is already injective when we pass to the natural quotient $\Gamma \to \text{Homeo(Cantor)}$. □

The argument actually shows that any action of G on S^1 is semiconjugate to some embedding $G \to \Gamma \to \text{Homeo}^+(S^1_S)$. In other words, we can realize *every semiconjugacy class* of action of any countable group on S^1 as a subgroup of Γ acting on S^1_S.

THEOREM 5.2. *The discrete group S^1 embeds in Γ.*

Proof. By thinking of $\mathbb{R}$ as a vector space over $\mathbb{Q}$, we obtain $S^1 \cong \mathbb{Q}/\mathbb{Z} \oplus (\oplus_{\mathbb{R}/\mathbb{Q}}\mathbb{Q})$. It follows that S^1 is a subgroup of $\mathbb{Q}/\mathbb{Z} \oplus \prod_{\mathbb{N}} \mathbb{Q}$. The $\mathbb{Q}/\mathbb{Z}$ factor embeds into Γ by the construction above, preserving a Cantor set contained in the equator.

For this action, choose distinct $\mathbb{Q}/\mathbb{Z}$-orbits Z_i corresponding to $i \in \mathbb{N}$ on the equator minus a Cantor set such that their union only accumulates to points in the Cantor set. For each $i \in \mathbb{N}$, blow up each point $z \in Z_i$ to a small closed disk D_z identified with the unit disk so that $\mathbb{Q}/\mathbb{Z}$ permutes these disks and acts by the identity map under the identifications.

Fix a Cantor set in the interior of the unit disk, which provides a Cantor set in each D_z by the given identification. Then the union of these Cantor sets over all $z \in Z_i$ and $i \in \mathbb{N}$ together with the original Cantor set on the equator is still a Cantor set.

Let Γ_D denote the mapping class group of the disk minus a Cantor set (fixed on the boundary). Now Γ_D is a central $\mathbb{Z}$ extension of Γ, where the center is generated by a Dehn twist around the boundary. The analog of this fact in the world of small mapping class groups is well known and the same proof works here; see e.g. [**11**, equation (4.2)]. Thus, we may embed $\mathbb{Q}$ into the Γ_D by taking the preimage of some $\mathbb{Q}/\mathbb{Z}$ in Γ. Using such an embedding, we let the $\mathbb{Q}$ factor corresponding to $i \in \mathbb{N}$ act on each D_z minus its Cantor set for each $z \in Z_i$ simultaneously under the given identification. Then this action commutes with the $\mathbb{Q}/\mathbb{Z}$ action and different $\mathbb{Q}$ factors have disjoint supports for their actions. This embeds S^1 in Γ. □

In the next theorem and in § 6 we will need to make use of a well-known homological argument, whose rudiments we now explain. For a reference, see Ghys [**12**] or see [**6**] for more on quasimorphisms and stable commutator length.

Let G be any group and $\rho : G \to \text{Homeo}^+(S^1)$ any action on the circle. Associated to this action there is an *Euler class* $e_\rho \in H^2(G; \mathbb{Z})$ or just e if ρ is understood, which is the pullback under ρ of the generator of $H^2(\text{Homeo}^+(S^1); \mathbb{Z}) = \mathbb{Z}$. The class e is the obstruction to lifting the action to $\text{Homeo}^+(\mathbb{R})$ under the covering $\mathbb{R} \to S^1$.

For any $g \in \text{Homeo}^+(S^1)$, the Poincaré's *rotation number* $\text{rot}(g) \in \mathbb{R}/\mathbb{Z}$ is defined by choosing any lift $\tilde{g} \in \text{Homeo}^+(\mathbb{R})$ and computing

$$\text{rot}^\sim(\tilde{g}) := \lim_{n\to\infty} \frac{\tilde{g}^n(0)}{n}$$

and then defining $\text{rot}(g)$ to be the reduction of $\text{rot}^\sim$ mod $\mathbb{Z}$.

If $\tilde{\rho} : G \to \text{Homeo}^+(\mathbb{R})$ is obtained by lifting $\rho : G \to \text{Homeo}^+(S^1)$, the composition $\text{rot}^\sim \circ \tilde{\rho} : G \to \mathbb{R}$ is a *homogeneous quasimorphism* with defect ≤ 1; thinking of it as a 1-cocycle, its coboundary determines an element in bounded cohomology $\text{eu}_\mathbb{R} \in H^2_b(G; \mathbb{R})$, the *bounded Euler class*. In fact, $\text{eu}_\mathbb{R}$ comes from a class $\text{eu}_\mathbb{Z} \in H^2_b(G; \mathbb{Z})$ by change of

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coefficient, unique if G is perfect (and otherwise determined by the values of the rotation numbers on generators for $H_1(G)$).

For any group, Ghys showed that the class $\mathrm{eu}_{\mathbb{Z}} \in H_b^2(G;\mathbb{Z})$ determines the action of G up to monotone equivalence—i.e. the relation generated by semiconjugacy. In particular, the rotation numbers are determined by the class $\mathrm{eu}_{\mathbb{Z}}$. For a perfect group, the rotation numbers are determined by $\mathrm{eu}_{\mathbb{R}}$; otherwise they are determined up to an element of $\mathrm{Hom}(G, S^1)$.

Now suppose that G is uniformly perfect, i.e. that every element is the product of a uniformly bounded number of commutators. Bavard duality (see [**6**]) implies that every homogeneous quasimorphism must vanish identically on G. In particular, this implies that the comparison maps $H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$ and $H_b^2(G;\mathbb{Z}) \to H^2(G;\mathbb{Z})$ are injective. So, for a uniformly perfect group G acting on the circle, the rotation numbers of every element are determined by the Euler class e. If G is a uniformly perfect group with $H^2(G;\mathbb{Z}) = 0$, one can say further that for the lifted action to $\mathbb{R}$ one has $\mathrm{rot}^{\sim} \circ \tilde{\rho}(g) = 0$ for all g. In particular, every orbit on $\mathbb{R}$ is bounded (in fact, it has diameter ≤ 1) and consequently there is a global fixed point in $\mathbb{R}$ which projects to a global fixed point for the original action in S^1. We summarize these observations in a proposition, which we emphasize is essentially due to Ghys.

PROPOSITION 5.3. (Ghys) *Let G be a uniformly perfect group. If $\rho : G \to \mathrm{Homeo}^+(S^1)$ is any action, the rotation numbers of every element are determined by the Euler class $e \in H^2(G;\mathbb{Z})$. If further $H^2(G;\mathbb{Z}) = 0$, then every $\rho : G \to \mathrm{Homeo}^+(S^1)$ has a global fixed point.*

We are now in a position to prove the following result.

THEOREM 5.4. $\mathrm{PSL}_2(\mathbb{R})$ *is not isomorphic to a subgroup of* Γ.

Proof. We will show that any faithful action of $\mathrm{PSL}_2(\mathbb{R})$ on the circle is transitive. Since the action of Γ on the simple circle has countable orbits, this will prove the theorem.

First, observe that $\mathrm{PSL}_2(\mathbb{R})$ is uniformly perfect. In fact, every element is a commutator. This can be seen by hyperbolic geometry: there is a hyperbolic structure on a once-punctured torus with a boundary geodesic of any length (including zero) and there is also a hyperbolic cone structure on a torus with one cone point of any angle $< 2\pi$. It follows from Proposition 5.3 that for any $\rho : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{Homeo}^+(S^1)$ the Euler class e determines the rotation numbers of every element.

Second, it is well known that $H^2(\mathrm{PSL}_2(\mathbb{R});\mathbb{Z}) = \mathbb{Z}$. Since $\mathrm{PSL}_2(\mathbb{R})$ has torsion, no faithful action on the circle can have trivial Euler class since otherwise the action would lift to $\mathbb{R}$. Thus, $e_\rho = \lambda e$ for some $\lambda \neq 0$, where e_ρ and e are the Euler classes associated to the ρ-action and the standard action, respectively. Certain subgroups (e.g. triangle subgroups) of $\mathrm{PSL}_2(\mathbb{R})$ are rigid—the only non-trivial homomorphisms to $\mathrm{Homeo}^+(S^1)$ are standard. Thus, $\lambda = \pm 1$, i.e. there is only one Euler class e (up to sign) associated to a faithful action of $\mathrm{PSL}_2(\mathbb{R})$ on the circle, namely the one coming from the standard action on the boundary of the hyperbolic plane.

This implies that the rotation number of $\rho(g)$ is equal to the (usual) rotation number for every $g \in \mathrm{PSL}_2(\mathbb{R})$. In particular, for θ in the S^1 subgroup, the rotation number of

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any $\rho(\theta)$ is θ. It follows that S^1 acts *freely* on S^1 since any homeomorphism with a fixed point has zero rotation number. Thus, if we pick any $p \in S^1$, the orbit map $S^1 \to S^1$ is injective and order-preserving. We claim that this orbit map is surjective. First, its image is dense; for if not, the closure has countably many gaps, and these gaps correspond to a distinguished countable subset of S^1 invariant under the action of S^1 on itself, which is absurd. Second, its image is everything. To see this, let q be a point in S^1; by considering points to the left and right of q in the image of the orbit map, we see that q determines a unique Dedekind cut and therefore a unique point in S^1, whose image under the orbit map necessarily is to the right of the left-hand points and to the left of the right-hand points, and is therefore equal to q. Since the orbit map $S^1 \to S^1$ is both order-preserving and a bijection, it is a homeomorphism; it follows that the S^1 subgroup of $\mathrm{PSL}_2(\mathbb{R})$ must act on S^1 conjugate to the standard action. In particular, the action is *transitive* on S^1, as claimed. □

Proposition 5.3 is used also in the proof of Lemma 6.5.

6. *Rigidity of* Γ *actions on* S^1

We now arrive at the main result of the paper, the classification of Γ actions on the circle.

THEOREM 6.1. (Rigidity) *Any homomorphism* $\Gamma \to \mathrm{Homeo}^+(S^1)$ *is either trivial or is semiconjugate (possibly up to a change of orientation) to the action on the simple circle* S^1_S.

COROLLARY 6.2. *Any non-trivial action of* Γ *on* S^1 *must contain a countable orbit and an uncountable orbit.*

Recall the notation R for the set of short rays, i.e. proper isotopy classes of simple lines in Ω from infinity to some point on the Cantor set. If $r \in R$ is a ray, denote by $\mathrm{end}(r)$ the end point of r in the Cantor set.

We define two subgroups of Γ associated to a ray. For any ray $r \in R$, let Γ_r be the stabilizer of r in Γ and let $\Gamma_{(r)} \leq \Gamma_r$ be the subgroup of mapping classes that can be represented by homeomorphisms that are the identity in a neighborhood of $\mathrm{end}(r)$ (equivalently: in a neighborhood of r).

Definition 6.3. We say that a disk D in S^2 is a *proper dividing disk* if:

(1) ∞ is in the exterior of D;
(2) both the interior and exterior of D intersect the Cantor set while its boundary does not.

LEMMA 6.4. *Every element* $g \in \Gamma_{(r)}$ *is a commutator. In addition,* $\Gamma_{(r)}$ *is acyclic, i.e.* $\widetilde{H}_k(\Gamma_{(r)}; \mathbb{Z}) = 0$ *for all* k.

Proof. This is proved by Mather's suspension argument [**14**]. We explain.

Any $g \in \Gamma_{(r)}$ is supported in some proper dividing disk D_0 whose complement contains r. Then there is some $h \in \Gamma_{(r)}$ such that the sequence of disks $D_n := h^n(D_0)$ are pairwise disjoint and converging to a point. See Figure 7. It follows that $g_n = h^n g h^{-n}$ is supported in D_n and any g_i, g_j commute due to disjoint supports. Moreover, the infinite product

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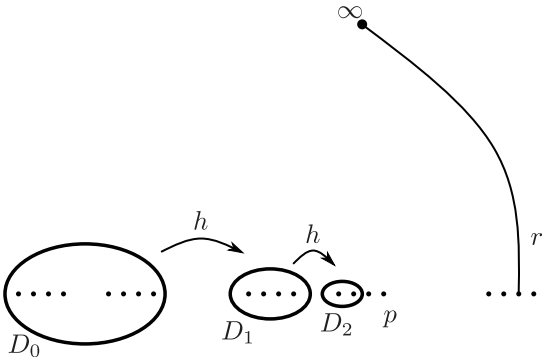


FIGURE 7. A sequence of disjoint disks D_n converging to some point p in the Cantor set such that $h(D_n) = D_{n+1}$ for some element $h \in \Gamma_{(r)}$.

$a(g) := \prod_{n=0}^{\infty} g_n$ is a well-defined element in $\Gamma_{(r)}$. Then $g = a(g)ha(g)^{-1}h^{-1}$ is a commutator.

Mather's suspension argument [**14**] shows that $\Gamma_{(r)}$ is acyclic. We give a sketch. Each homology class $[\sigma] \in H_k(\Gamma_{(r)}; \mathbb{Z})$ is represented by some cycle σ in the bar complex for $B\Gamma_{(r)}$, which only involves finitely many elements in $\Gamma_{(r)}$. Hence, there is some disk D_0 as above containing the support of all these elements. Replacing each element g involved in the cycle σ by $a(g)$ defined using some $h \in \Gamma_{(r)}$ as above results in another cycle $a(\sigma)$. Then one can easily check that $[\sigma] = [a(\sigma)] - h_*[a(\sigma)]h_*^{-1}$, which must vanish since any inner automorphism induces the identity map on group homology. □

LEMMA 6.5. *Any action of* $\Gamma_{(r)}$ *on a circle has a global fixed point.*

Proof. We have shown that $H^2(\Gamma_{(r)}) = 0$ and every element of $\Gamma_{(r)}$ is a commutator; thus, this is a special case of Proposition 5.3. □

LEMMA 6.6. Γ_r *is generated by* $\Gamma_{(r)}$ *and* $\Gamma_{(s)} \cap \Gamma_r$ *for any ray* s *with* $\mathrm{end}(s) \neq \mathrm{end}(r)$.

Proof. Simply note that any $g \in \Gamma_r$ can be decomposed as $g = g_1 g_2$ with $g_1 \in \Gamma_{(r)}$ and g_2 supported in any small neighborhood of the end point of r. □

LEMMA 6.7. *For any action of* Γ *on the circle, every* Γ_r *has a global fixed point.*

Proof. First, note that $\Gamma_{(r)}$ is normal in Γ_r (actually, Γ_r is the normalizer of $\Gamma_{(r)}$). In fact, if $g \in \Gamma_{(r)}$ is the identity in some neighborhood U of $\mathrm{end}(r)$ and $h \in \Gamma_r$, then hgh^{-1} is the identity on $h(U)$, which is also a neighborhood of $\mathrm{end}(r)$.

Fix any ray s with $\mathrm{end}(s) \neq \mathrm{end}(r)$. Let F_r and F_s be the sets of points on the circle fixed by $\Gamma_{(r)}$ and $\Gamma_{(s)}$, respectively. These are non-empty closed sets by Lemma 6.5. Lemma 6.6 implies that Γ_r fixes any point in $F_r \cap F_s$. So, we are left with the case where F_r and F_s are disjoint. Γ_r permutes F_r and thus its complementary intervals since $\Gamma_{(r)}$ is normal in Γ_r. It follows that $\Gamma_{(s)} \cap \Gamma_r$ fixes the boundary points of any complementary interval of F_r that intersects F_s. Hence, such boundary points would be fixed by Γ_r. □

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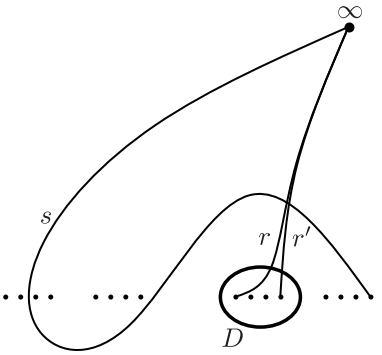


FIGURE 8. A mapping class supported in a small disk D taking r to a disjoint ray r'.

LEMMA 6.8. *If $r, s \in R$ have distinct end points, then Γ_r and Γ_s generate Γ.*

Proof. Denote by $\Gamma(r, s)$ the subgroup generated by Γ_r and Γ_s. It suffices to show that $\Gamma(r, s)$ acts transitively on R. Since r and s have distinct end points, there is a mapping class g supported in a small neighborhood of end(r) so that $g \in \Gamma_s$ and $r' := gr$ is disjoint from r (see Figure 8). Then $\Gamma_{r'} = g\Gamma_r g^{-1} \subset \Gamma(r, s)$ and thus $\Gamma(r, r') \subset \Gamma(r, s)$. So, it suffices to prove the lemma with the additional assumption that r and s are disjoint.

Now we show that any short ray x can be taken to s by elements in $\Gamma(r, s)$. This will evidently complete the proof. We show this by induction on the length of a minimal sequence $x = x_0, x_1, \ldots, x_n = s$ where adjacent rays are disjoint in the interior (i.e. they may share the end point). Such a minimal number exists by the connectedness of the ray graph; see e.g. [**7**] or [**2**] for details.

Note that every Γ_r is transitive on the set of rays disjoint from r. Thus, if x is disjoint from s, we can apply an element of Γ_s to make it disjoint from both r and s, and then apply an element of Γ_r to move it to s. This proves the base case since any x disjoint from s in the interior can be made disjoint from s by applying an element of Γ_r supported in a small neighborhood of end(s) similar to the one in Figure 8.

Now we suppose that we have a sequence $x = x_0, x_1, \ldots, x_n = s$ and suppose that $g \in \Gamma(r, s)$ takes x_1 to s. Then gx is disjoint from s in the interior, so by the base case there is an $h \in \Gamma(r, s)$ with $hgx = s$. This completes the proof. □

PROPOSITION 6.9. Γ *is generated by* 2*-torsion.*

Proof. Let Γ_2 be the subgroup generated by 2-torsion. Then Γ_2 is normal in Γ. In particular, any commutator $[a, b] = a(ba^{-1}b^{-1}) \in \Gamma_2$ if $a \in \Gamma_2$.

Construct a Cantor set K in the plane in the following manner.

Take a closed disk D_0 with small radius centered at a point on the positive side of the x-axis. Let h_0 be a shrinking dilation centered at the origin such that $D_1 := h_0 D_0$ is disjoint from D_0. Put a Cantor set K_0 in the interior of D_0 and let $D_n := h_0^n D_0$, $K_n := h_0^n K_0$. Let r be the rotation by π centered at the origin and $D_n' := rD_n$, $K_n' := rK_n$. Then the union of all K_n, K_n' and the origin is a Cantor set K. See Figure 9.

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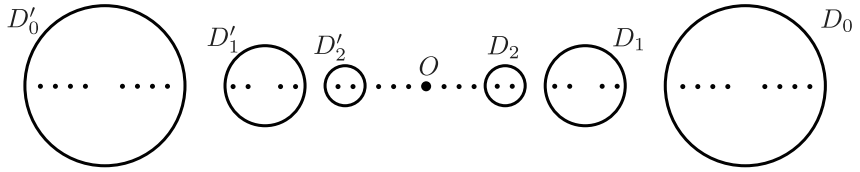


FIGURE 9. A specific symmetric configuration of the Cantor set and disks D_n, D_n'.

In this way, the homeomorphisms h_0 and r commute and $r^2 = id$. One can modify h_0 to a homeomorphism h that preserves K, still commutes with r and agrees with h_0 on D_n, D_n' for all $n \geq 1$. Consider r and h as elements in Γ. There is another $b \in \Gamma$ satisfying the following properties:

(1) $b|_{D_n} = r$ for all $n \geq 1$;
(2) $b|_{D_n'} = rh = hr$ for all $n \geq 1$;
(3) $b|_{D_0} = \mathrm{id}$;
(4) $b(K_0') = K_0' \cup K_1$,

where (3) and (4) ensure that b preserves the Cantor set K.

Let $g \in \Gamma$ be any element supported in a proper dividing disk D, which we may assume to be D_1 by conjugation. Then $h^n g h^{-n}$ is supported in D_{n+1} and the infinite product $x_g = \prod_{n\geq 0} h^n g h^{-n} \in \Gamma$ is well defined. Now

$$a_g := [x_g, r] = \prod_{n\geq 0} h^n g h^{-n} \cdot \prod_{n\geq 0} r h^n g^{-1} h^{-n} r^{-1}$$

is an element in Γ_2 since r is 2-torsion.

Note that $bh^n g h^{-n} b^{-1} = r h^n g h^{-n} r^{-1}$ since $b|_{D_{n+1}} = r$ for all $n \geq 0$. Similarly $brh^n g h^{-n} r^{-1} b^{-1} = h^{n+1} g h^{-(n+1)}$ since $b|_{D_{n+1}'} = rh$ for all $n \geq 0$. Thus,

$$\begin{aligned} b a_g b^{-1} &= \prod_{n\geq 0} bh^n g h^{-n} b^{-1} \cdot \prod_{n\geq 0} brh^n g^{-1} h^{-n} r^{-1} b^{-1} \\ &= \prod_{n\geq 0} r h^n g h^{-n} r^{-1} \cdot \prod_{n\geq 0} h^{n+1} g^{-1} h^{-(n+1)} \end{aligned}$$

and $g = a_g (b a_g b^{-1})$ lies in Γ_2 since a_g does.

In particular, any $g \in \Gamma_{(r)}$ for any ray $r \in R$ is of the form above. Hence, $\Gamma_{(r)}$ is contained in Γ_2. Applying this to r and s with distinct end points, Lemma 6.6 implies that $\Gamma_r \subset \Gamma_2$. Then we conclude that $\Gamma_2 = \Gamma$ by Lemma 6.8. □

In the proof above, we actually proved the stronger statement that Γ is *normally* generated by a single 2-torsion. In an upcoming paper [**9**], we show that Γ is normally generated by any element that does not fix the Cantor set pointwise.

COROLLARY 6.10. *Any homomorphism $\rho : \Gamma \to \mathrm{Homeo}^+([0, 1])$ is trivial. Thus, a non-trivial homomorphism $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$ has no global fixed point.*

Proof. The first assertion immediately follows from Proposition 6.9 by noticing that the group $\mathrm{Homeo}^+([0, 1])$ is torsion-free. Any action on the circle with a global fixed point gives rise to an action on the interval by cutting at some fixed point. □

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From now on, fix an arbitrary action $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$ without global fixed points. Fix some $r_0 \in R$ and let $P(r_0) \in S^1$ be a fixed point of Γ_{r_0}. Any $r \in R$ can be written as $r = gr_0$ for some $g \in \Gamma$. Let $P(r) := gP(r_0)$, which does not depend on the choice of g and is stabilized by Γ_r. Then $P : R \to S^1$ is Γ-equivariant.

LEMMA 6.11. *If $r, s \in R$ have distinct end points, then $P(r)$ and $P(s)$ are distinct.*

Proof. If $P(r) = P(s)$, then its stabilizer contains both Γ_r and Γ_s, which generate Γ by Lemma 6.8. This contradicts our assumption that ρ has no global fixed points. □

LEMMA 6.12. *By appropriately choosing the orientation of S^1 on which $\rho(\Gamma)$ acts, the triple $(P(r_1), P(r_2), P(r_3))$ is positively oriented whenever (r_1, r_2, r_3) is for any disjoint $r_1, r_2, r_3 \in R$ with distinct end points.*

Proof. Pick some positively oriented triple (r_1, r_2, r_3) as above. Then $P(r_1), P(r_2), P(r_3)$ are distinct by Lemma 6.11. Choose the orientation on S^1 to make $(P(r_1), P(r_2), P(r_3))$ positively oriented. For any other such triple (r_1', r_2', r_3') with positive orientation, there is some $g \in \Gamma$ such that $gr_i = r_i'$ for $i = 1, 2, 3$. Then

$$(P(r_1'), P(r_2'), P(r_3')) = (gP(r_1), gP(r_2), gP(r_3))$$

is also positively oriented. □

In what follows, fix the orientation provided by Lemma 6.12.

We now describe an increasing filtration of R by Cantor sets. This filtration depends on the choice of an embedded circle γ in the plane containing the Cantor set; we call such a γ an *equator*. Observe that an equator inherits a canonical orientation as the boundary of the complementary region containing infinity.

For any equator γ, we define the *γ-length* of a short ray r to be the minimal number of transverse intersections of r with γ. Note that we do not count $\mathrm{end}(r) \in \gamma$ as an intersection and that the γ-length might be infinite. Let $R_n(\gamma) \subset R$ denote the set of short rays with γ-length $\leq n$. Thus, for example, $R_0(\gamma)$ consists of rays going straight from infinity to some point on the Cantor set without crossing γ. The canonical orientation on γ induces a cyclic ordering on $R_0(\gamma)$, which agrees with the circular order on R induced by its inclusion in the simple circle.

COROLLARY 6.13. *For any set T of mutually disjoint rays with distinct end points, the cyclic order on T agrees with the cyclic order on $P(T)$. In particular, this holds with $T = R_0(\gamma)$ for the equator γ as above.*

Proof. A cyclic order on a set is determined by its restriction to all triples. By Lemma 6.12, the cyclic orders on T and $P(T)$ agree on triples, so they are equal. □

LEMMA 6.14. *For any equator γ and for any finite n, the set $R_n(\gamma)$ is a Cantor set in both the conical circle and the simple circle.*

Proof. We prove the lemma in the conical circle by showing that $R_n(\gamma)$ is a perfect subset of the Cantor set $R \cup X$. First, note that $R_n(\gamma)$ is closed since long rays intersect γ infinitely many times and since having at least $n + 1$ transverse intersections with γ is

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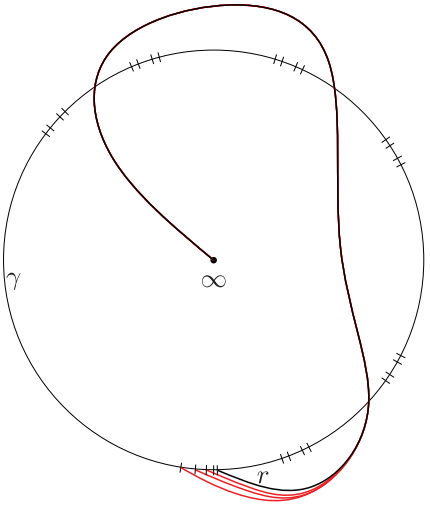


FIGURE 10. A sequence of short rays of the same γ-length limiting to r.

an open condition. Second, a short ray r of γ-length n is a limit of short rays of the same length. To see this, note that the intersections with γ cut r into $n+1$ essential arcs on alternating sides of γ. Perturbing the last arc by moving end(r) to nearby points in the Cantor set along γ produces a sequence of short rays with the same γ-length converging to r; see Figure 10. Now the quotient map from the conical circle to the simple circle is injective on R and therefore the image of the compact subset $R_n(\gamma)$ in the simple circle is also a Cantor set. □

LEMMA 6.15. *Any ray of γ-length n is the limit on both sides of rays of γ-length $n+1$.*

Proof. Let r be any ray of γ-length n. Let α_n be the segment of r connecting its last transverse intersection with γ to end(r). The rays we present below can be straightened to geodesics up to isotopy, which does not affect the intersection patterns since geodesics minimize intersection numbers.

If end(r) is not the boundary of any complementary interval of the Cantor set in γ, then there are sequences of complementary intervals limiting to end(r) from both sides. In this case, we can replace α_n by an arc that runs through a nearby complementary interval I and then limits to end(r) without further intersecting γ; see the blue rays in Figure 11. Letting I limit to end(r) from the left or right creates sequences of rays of γ-length $n+1$ limiting to r on both sides.

If end(r) is the boundary of some complementary interval I, then the construction above can be done only from the side not witnessing I. For the other side, we can replace α_n by an arc running through I and then ending on a point p near end(r) in the Cantor set without further intersecting γ; see the red rays in Figure 11. Letting p converge to end(r), we obtain the desired sequence of rays of γ-length $n+1$. □

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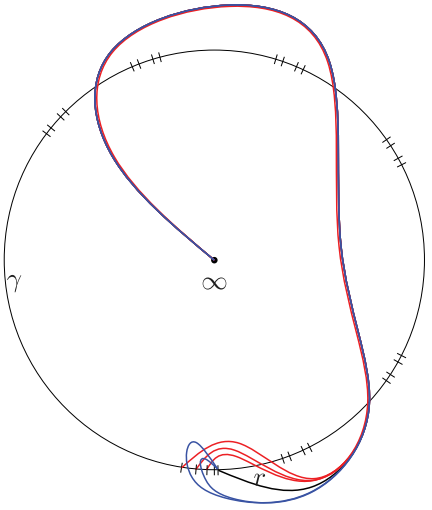


FIGURE 11. Two sequences of short rays, in blue and red, of γ-length four limiting to r, which has γ-length three, from two sides.

COROLLARY 6.16. *For any complementary interval (r_-, r_+) of $R_n(\gamma)$ in the simple circle, the rays r_- and r_+ have γ-length exactly n.*

It follows that if r has γ-length $> n$, then there are two rays r_-^n, r_+^n of γ-length n associated to r so that (r_-^n, r_+^n) is the complementary interval of $R_n(\gamma)$ containing r and which satisfy the following properties:

(1) (r_-^n, r, r_+^n) is positively oriented on the simple circle; and
(2) (r_-^n, s, r_+^n) is negatively oriented for any $s \in R_n(\gamma)$ other than $r_\pm^n$.

LEMMA 6.17. *For any $n \geq 0$ and any complementary interval (r_-, r_+) of $R_n(\gamma)$, there is a complementary interval I of the Cantor set in γ such that all rays $r \in (r_-, r_+)$ first intersect with γ at some point in I.*

Proof. When $n = 0$, we know from our concrete description of $R_0(\gamma)$ that $\text{end}(r_-)$ and $\text{end}(r_+)$ are the boundary points of some complementary interval I of the Cantor set in γ, and the rays in (r_-, r_+) are exactly those that first intersect with γ at some point in I.

If $n > 0$, by Corollary 6.16, $r_\pm$ has length $n \geq 1$ and intersects γ transversely. We claim that r_- and r_+ first intersect γ in the same complementary interval I of the Cantor set in γ. If not, let p_- and p_+ be the first intersections of r_- and r_+ with γ, respectively. Then the arc on γ from p_- to p_+ in the positive orientation contains some point in the Cantor set, which determines a short ray $r_m \in R_0(\gamma)$. The initial arcs on r_+, r_- up to p_+, p_- together with the arc on γ from p_- to p_+ in the positive orientation form a sector, which contains r_m. This shows that $r_m \in (r_-, r_+)$, contradicting the fact that (r_-, r_+) is a complementary interval of $R_n(\gamma)$. This proves the claim, from which it follows that all rays in (r_-, r_+) are forced to first intersect the interval I as well. □

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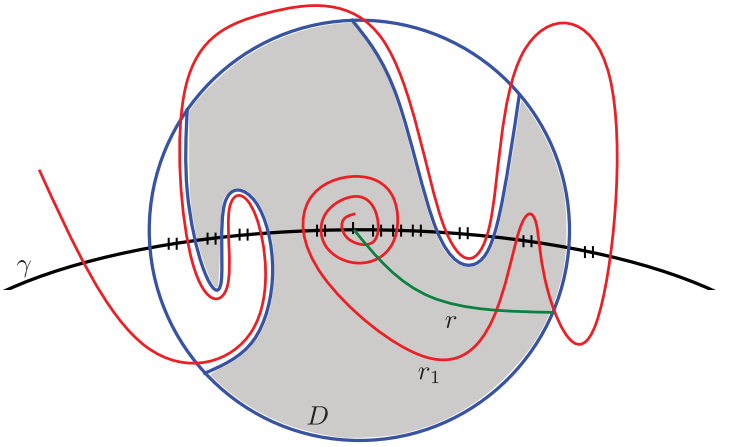


FIGURE 12. We shrink the proper dividing (round) disk to the smaller shaded disk D such that $D \cap r_1$ has only one component. There is a ray r in D starting from the same point in ∂D as $D \cap r_1$ and (whose interior is) disjoint from γ.

LEMMA 6.18. *For any finite collection of short rays $r_1, \ldots, r_n$ with distinct end points, there is an equator γ containing the Cantor set such that all r_i simultaneously have finite γ-length.*

Proof. Start with an arbitrary equator γ. We will modify γ so that r_1 has finite γ-length without changing the γ-lengths of r_i for all $i > 1$. Let D be a small proper dividing disk that contains a neighborhood of $\mathrm{end}(r_1)$ and is disjoint from r_i for all $i > 1$. Put r_1 in general position with γ and ∂D. Then $r_1 \cap D$ has finitely many components, only one of which limits to $\mathrm{end}(r_1)$. Shrinking D by removing finitely many bigons, one at a time, we may assume that there are no other components of $r_1 \cap D$. Then $r_1 \cap D$ is a ray in D minus a Cantor set starting from a marked point on the boundary. The mapping class group of the disk minus a Cantor set acts transitively on the set of such rays for the same reason that Γ acts transitively on R; see e.g. [**3**, Lemma 2.6.2]. Now $\gamma \cap D$ has finitely many components, one of which contains $\mathrm{end}(r_1)$; see Figure 12. Thus, there are some ray r in D disjoint from $\gamma \cap D$ and some $g \in \Gamma$ supported in D such that $g(r) = r_1 \cap D$. Then $r_1 \cap D$ is disjoint from $g(\gamma) \cap D$. Replacing each component of $\gamma \cap D$ by its corresponding component in $g(\gamma) \cap D$, we obtain the desired modification of γ that makes the γ-length of r_1 finite. □

LEMMA 6.19. *The map $P : R \to S^1$ preserves the circular order. In particular, it is injective.*

Proof. We first induct on $n \geq 0$ to prove that P is order-preserving on $R_n(\gamma)$ for all γ as above. The base case is covered by Corollary 6.13. Suppose that $n \geq 0$ and P is order-preserving on $R_n(\gamma)$ for all γ. For any γ, it suffices to show that P is order-preserving on $R_{n+1}(\gamma) \cap [r_-, r_+]$ for each complementary interval (r_-, r_+) of $R_n(\gamma)$. Let I be the complementary interval of the Cantor set in γ provided by Lemma 6.17. Let α_- and α_+ be the geodesic short rays in $R_0(\gamma)$ going directly from infinity to the boundary points

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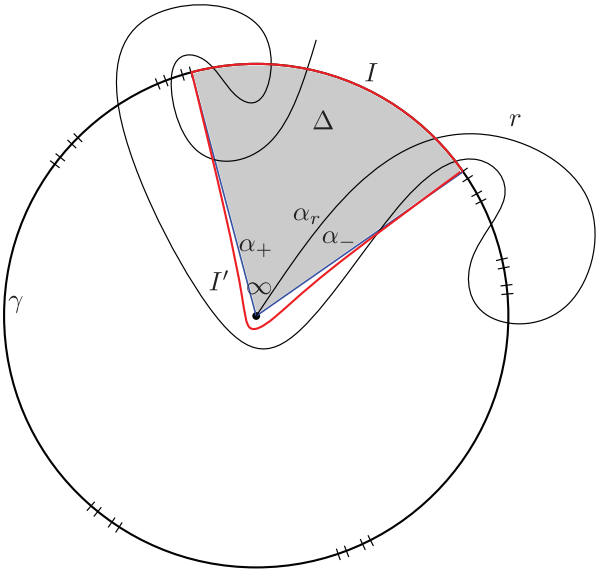


FIGURE 13. Pushing I across the triangle Δ changes the equator γ to γ' reducing the length of r.

of I. Consider the triangular region Δ bounded by sides I, α_- and α_+. For any $r \in R_{n+1}(\gamma) \cap [r_-, r_+]$ represented by a geodesic, let α_r be the subarc of r from infinity to its first intersection with γ. Then r never enters Δ from α_- and leaves from α_+, or vice versa, since otherwise this would result in an intersection with α_r and thus a self-intersection of r. Thus, every component of $r \cap \Delta$ has at least one end on I. Isotoping I through Δ to $\alpha_- \cup \alpha_+$ and slightly further across infinity to an arc I', we obtain a new equator γ' such that any ray $r \in R_{n+1}(\gamma) \cap [r_-, r_+]$ has γ'-length at most n by resolving the first intersection with γ without creating new intersections; see Figure 13. Thus, by the induction hypothesis, P is order-preserving on this set of short rays, which completes our induction.

Next, by Lemma 6.18, P preserves the order of any triple of short rays with distinct end points.

Finally, consider an arbitrary positively oriented triple of rays (r_1, r_2, r_3). Evidently any ray $r \in R_0(\gamma)$ is the limit of disjoint rays in $R_1(\gamma)$ with distinct end points on both sides. Thus, by transitivity of the Γ action on R, every ray $r \in R$ is the limit of disjoint rays with distinct end points on both sides in the simple circle. In particular, there are rays r_{i+}, r_{i-} such that $(r_{1\pm}, r_{2\pm}, r_{3\pm})$ is a positively oriented triple of rays with distinct end points for any choice of the $\pm$ signs, and that (r_{i-}, r_i, r_{i+}) is a positively oriented triple of disjoint rays for any $i = 1, 2, 3$. By what we have shown and Corollary 6.13, $(P(r_{1\pm}), P(r_{2\pm}), P(r_{3\pm}))$ is positively oriented independent of the choice of signs and $(P(r_{i-}), P(r_i), P(r_{i+}))$ is positively oriented for any $i = 1, 2, 3$. This implies that $(P(r_1), P(r_2), P(r_3))$ is positively oriented. □

Now we prove Theorem 6.1.

Proof of Theorem 6.1. Let $\hat{P}$ be the closure of the image of P. Then $\hat{P}$ is Γ-invariant. Collapse all complementary intervals of $\hat{P}$ to obtain another circle S^1_P where Γ acts.

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Then the quotient map $\pi : S^1 \to S^1_P$ semiconjugates this action to ρ. By Theorem 3.4, there is always some ray in the interval $(r, s) \subset S^1_S$ for any two $r, s \in R$. Thus, the image of $\pi \circ P$ is still injective and in the correct circular order by Lemma 6.19. This correspondence of dense Γ-orbits extends to a homeomorphism $S^1_S \cong S^1_P$ that conjugates the two actions. □

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A. Appendix. Homology of big mapping class groups

Let $\widehat{\Gamma}$ denote the mapping class group of the sphere minus a Cantor set. It is closely related to Γ by the following point-pushing Birman exact sequence:

$$1 \to \pi_1(\widehat{\Omega}) \to \Gamma \to \widehat{\Gamma} \to 1,$$

where $\widehat{\Omega} := \Omega \cup \infty$ is the sphere minus a Cantor set.

It is shown in [**7**] that $\widehat{\Gamma}$ is uniformly perfect, which implies the following result.

LEMMA A.1. $H_1(\widehat{\Gamma}; \mathbb{Z}) = 0$.

The goal is to compute $H_2(\widehat{\Gamma})$ and $H^2(\widehat{\Gamma})$. All (co)homology groups in this appendix are singular (co)homology with $\mathbb{Z}$ coefficients. The main theorem we prove is the following.

THEOREM A.2. (Homology) $H_2(\widehat{\Gamma}) = \mathbb{Z}/2$ *and* $H^2(\widehat{\Gamma}) = 0$.

The strategy is to break the classifying space $B\widehat{\Gamma}$ into simpler ones by a fragmentation argument which dates back to Segal [**16**]. See also [**17**]. Here we realize $B\widehat{\Gamma}$ as $G\backslash|E\widehat{\Gamma}|$, where $E\widehat{\Gamma}$ is the simplicial complex whose k-simplices are $(k+1)$-tuples $(g_0, \ldots, g_k) \in \widehat{\Gamma}^{k+1}$ and $|E\widehat{\Gamma}|$ is the underlying topological space. For any finite subset P of the Cantor set, let $\widehat{\Gamma}_P$ be the subgroup of elements represented by homeomorphisms that are the identity in some neighborhood of P. Then $E\widehat{\Gamma}_P$ naturally sits inside $E\widehat{\Gamma}$ as a subcomplex in a $\widehat{\Gamma}_P$-equivariant way. Note that $E_P := \widehat{\Gamma} \cdot (E\widehat{\Gamma}_P)$ is the subcomplex of $E\widehat{\Gamma}$ consisting of simplices whose components $g_0, \ldots, g_k$ coincide in some neighborhood of P. Then E_P is $\widehat{\Gamma}$-invariant and $B\widehat{\Gamma}_P = \widehat{\Gamma}_P\backslash|E\widehat{\Gamma}_P| = \widehat{\Gamma}\backslash|E_P| \subset B\widehat{\Gamma}$. In what follows, we always use this preferred realization of $B\widehat{\Gamma}_P$ and denote it simply by B_P.

Let $B_* := \bigcup_P B_P$, where P runs over all finite subsets of the Cantor set. Note that $B_P \cap B_Q = B_{P\cup Q}$ and $B_* = \bigcup_p B_{\{p\}}$, where p runs over the Cantor set.

LEMMA A.3. *The inclusion* $B_* \to B\widehat{\Gamma}$ *is a homotopy equivalence and thus* $H_k(B\widehat{\Gamma}) \cong H_k(B_*)$ *for any* $k \geq 0$.

Proof. This follows the argument in [**17**, Proposition 4.1]. Let $E_* := \bigcup_P E_P$, where P runs over all finite subsets of the Cantor set. It suffices to show that $|E_*|$ is contractible since it has a free $\widehat{\Gamma}$ action with quotient B_*. We will actually show that any cycle

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consisting of simplices $\sigma_i = (g_0^{(i)}, \dots, g_k^{(i)}) \subset E_{P_i}$ can be coned off in E_*. Note by definition that the components $g_j^{(i)}$ of σ_i agree in a neighborhood of P_i and thus $\sigma_i \subset E_{P_i'}$ for any set P_i' sufficiently close to P_i. Up to replacing P_i by nearby sets, we can assume that all P_i are disjoint and that all $g_j^{(i)}(P_i)$ are also disjoint for any fixed j (note that $g_j^{(i)}(P_i)$ does not depend on j).

Then there are small neighborhoods V_i of P_i such that $V_i \cap V_j = \emptyset$ for all $i \neq j$ and the components $g_j^{(i)}$ of σ_i restrict to the same homeomorphism h_i on V_i with $h_i(V_i) \cap h_j(V_j) = \emptyset$. We may choose each V_i to be a disjoint union of open disks whose intersections with the Cantor set are clopen subsets of the Cantor set. Then there is a homeomorphism $h : S^2 \to S^2$ preserving the Cantor set such that $h|_{V_i} = h_i$. Now $\bar{\sigma_i} := (h, g_0^{(i)}, \dots, g_k^{(i)})$ (i.e. the cone over σ_i with vertex h) is in E_{P_i}, which cones off the cycle in E_* as desired, that is, $\partial \sum_i \bar{\sigma_i} = \sum_i \sigma_i$. □

To compute the homology of B_*, we need to understand the homology of the subspaces B_P and their unions.

LEMMA A.4. *$B_{\{p\}}$ is acyclic for any p, that is, $\widetilde{H}_k(B_{\{p\}}) = 0$ for all k.*

Proof. We omit the proof since it is almost the same as the proof of Lemma 6.4 using Mather's suspension argument. □

LEMMA A.5. *We have $H_1(\bigcup_{p\in P} B_{\{p\}}) = 0$ for any non-empty finite subset P of the Cantor set.*

Proof. We induct on $|P|$. The base case $|P| = 1$ is proved by Lemma A.4. Suppose that the conclusion holds for P'; we consider $P = P' \cup \{q\}$ for some $q \notin P'$. Applying Mayer–Vietoris to the pair $(\bigcup_{p\in P'} B_{\{p\}}, B_{\{q\}})$, we have a surjection $H_1(\bigcup_{p\in P'} B_{\{p\}}) \oplus H_1(B_{\{q\}}) \to H_1(\bigcup_{p\in P} B_{\{p\}})$. The domain is trivial by the induction hypothesis, so the conclusion follows. □

For each $g \in \widehat{\Gamma}_P$, there are $|P|$ disjoint disks each containing one element of P such that g is the identity in these disks. Thus, g projects to a mapping class in $\mathrm{Mod}(S^2 - |P|D^2)$. The disks depend on g but the groups $\mathrm{Mod}(S^2 - |P|D^2)$ are canonically isomorphic by restricting homeomorphisms. Thus, this gives rise to a well-defined homomorphism $\pi : \widehat{\Gamma}_P \to \mathrm{Mod}(S^2 - |P|D^2)$. Denote the kernel by $\widehat{\Gamma}_P^0$.

We say that a disk in S^2 is a *dividing disk* if both its interior and exterior intersect the Cantor set while its boundary does not.

LEMMA A.6. *Any $g \in \widehat{\Gamma}_P^0$ can be factored in $\widehat{\Gamma}_P^0$ as a product of finitely many elements each supported in some dividing disk.*

Proof. We induct on $|P|$. When $|P| = 1$, every $g \in \widehat{\Gamma}_P^0$ is itself supported in a dividing disk. Suppose that $|P| = n \geq 2$ and we have proved the lemma when $|P| < n$. Fix any $g \in \widehat{\Gamma}_P^0$ supported in $S := S^2 - \bigcup_{i=1}^n D_i$, where D_i are disjoint dividing disks each containing an element of P. Let K be the part of the Cantor set in S, which we may assume to be non-empty and thus is itself a Cantor set.

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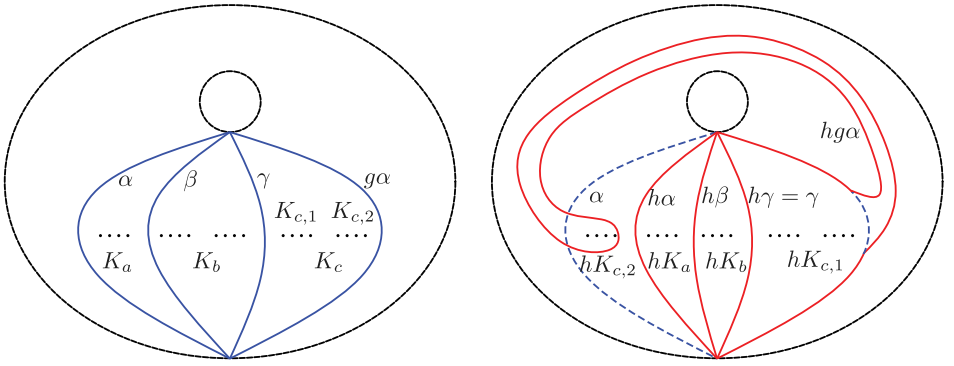


FIGURE A.1. The figure on the left depicts the arcs β, γ dividing the bigon into three subdisks. The figure on the right shows a homeomorphism h preserving γ and mapping α to β.

If g preserves some simple arc $\alpha \subset S - K$ connecting two distinct boundary components ∂D_i and ∂D_j, then up to homotopy g is supported in $S - N(\alpha)$ for some tubular neighborhood $N(\alpha)$ of α disjoint from K. Note that g is homotopic to the identity in $S - N(\alpha) \cong S^2 - (n-1)D^2$. The induction hypothesis gives the desired factorization for g.

In the general case, it suffices to factor g as the product of some $g_i \in \widehat{\Gamma}^0_P$, where each g_i preserves some simple arc connecting two distinct boundary components of S. Fix a simple arc $\alpha \subset S - K$ connecting ∂D_1 and ∂D_2. Then $g\alpha$ and α are homotopic in S since g is trivial in $\mathrm{Mod}(S)$.

Suppose that $g\alpha$ and α are disjoint in $S - K$. Then they cobound a disk D in S. If $D \cap K = \emptyset$, then g preserves α and we are done. Otherwise, $D \cap K$ is itself a Cantor set. Pick two disjoint arcs β and γ in $D - K$ that cut D into three subdisks D_a, D_b, D_c as shown in Figure A.1. Let $K_* := K \cap D_*$ with $* = a, b, c$. Then there is $h \in \widehat{\Gamma}^0_P$ preserving γ such that $h(\alpha) = \beta$. Indeed, such a homeomorphism h can be constructed by squeezing $D_a \cup D_b$ (respectively $K_a \cup K_b$) to D_b (respectively K_b) and expanding part of K_c to the whole K_c and pushing the rest of K_c to K_a along a path P that connects D_c to D_a disjoint from γ. Such a path exists since γ is homotopic to the non-separating arc α in S. By symmetry, there is also $h' \in \widehat{\Gamma}^0_P$ preserving some simple arc homotopic to α such that $h'(g\alpha) = \beta$. Then $h'' = h^{-1}h'g \in \widehat{\Gamma}^0_P$ preserves the arc α. Hence, $g = h'^{-1}hh''$ factors g into elements preserving arcs, as desired.

Now suppose that $g\alpha$ and α intersect. Since $g\alpha$ and α are homotopic in S, there is some innermost bigon D in S whose boundary consists of two arcs $\alpha_0 \subset \alpha$ and $\alpha_g \subset g\alpha$. We assume that $D \cap K$ is non-empty since otherwise the intersection number of $g\alpha$ and α in $S - K$ can be reduced. Similar to the argument above, there are $h, h' \in \widehat{\Gamma}^0_P$ preserving certain arcs such that $h^{-1}h'g$ maps α to α', where α' is obtained from $g\alpha$ by substituting the subarc α_g by α_0. Now α' and α have a smaller intersection number. Repeating this process finitely many times, we obtain a factorization of g into elements in $\widehat{\Gamma}^0_P$ each preserving some simple arc connecting ∂D_1 and ∂D_2. □

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COROLLARY A.7. *$H_1(\widehat{\Gamma}^0_P) = 0$ and the induced map $\pi_* : H_1(B_P) \to H_1\mathrm{Mod}(S^2 - |P|D^2)$ is an isomorphism.*

Proof. Each $h \in \widehat{\Gamma}^0_P$ supported in a dividing disk is a commutator by the suspension trick. Thus, $H_1(\widehat{\Gamma}^0_P) = 0$ by Lemma A.6. The exact sequence

$$0 = H_1(\widehat{\Gamma}^0_P) \to H_1(B_P) \xrightarrow{\pi_*} H_1\mathrm{Mod}(S^2 - |P|D^2) \to 0$$

shows that π_* is an isomorphism. □

For $P = P_1 \sqcup P_2$, we have an inclusion $f_{P_2} : B_P \to B_{P_1}$ by forgetting the information about P_2. When $P_2 = \{p\}$, we simply denote $f_{\{p\}}$ by f_p.

LEMMA A.8. *For any subset P of the Cantor set, we have $H_1(B_P) \cong \mathbb{Z}$ if $|P| = 2$ and $H_1(B_P) \cong \mathbb{Z}^3$ if $|P| = 3$. Moreover, the image of*

$$(f_{p*}, f_{q*}, f_{r*}) : H_1(B_{\{p,q,r\}}) \to H_1(B_{\{q,r\}}) \oplus H_1(B_{\{p,r\}}) \oplus H_1(B_{\{p,q\}}) \cong \mathbb{Z}^3$$

has index two with basis $e_1 + e_2$, $e_2 + e_3$, $e_3 + e_1$, where e_1, e_2, e_3 are the generators of the factors. In particular, any pair of maps, say (f_{p*}, f_{q*}), is surjective.

Proof. We have $H_1(B_P) \cong H_1\mathrm{Mod}(S^2 - |P|D^2)$ by Corollary A.7. When $|P| = 2$, the computation follows from the fact that $\mathrm{Mod}(S^2 - |P|D^2) \cong \mathbb{Z}$ is generated by a Dehn twist.

For $S = S^2 - D_1 \cup D_2 \cup D_3$, the Dehn twists around ∂D_1 and ∂D_2 generate a $\mathbb{Z}^2$ subgroup which gives rise to a central extension

$$1 \to \mathbb{Z}^2 \to \mathrm{Mod}(S) \to \mathrm{PB}_2 \to 1,$$

where $\mathrm{PB}_2 \cong \mathbb{Z}$ is the pure braid group with two strands keeping track of the configuration of D_1 and D_2 in $S^2 - D_3$; see e.g. [**11**] for details. Then the Dehn twist around ∂D_3 hits the generator of PB_2. It follows that these three Dehn twists generate $\mathrm{Mod}(S)$ and their images under (f_{p*}, f_{q*}, f_{r*}) are exactly the basis elements, as claimed. This implies that $H_1(B_P) \cong \mathrm{Mod}(S) \cong \mathbb{Z}^3$. □

LEMMA A.9. *Let P and $\{q\}$ be disjoint subsets of the Cantor set, where $2 \le |P| < \infty$. Then $H_1(\bigcup_{p\in P} B_{\{p,q\}}) = 0$.*

Proof. Fix $r \in P$ and let $P' = P - \{r\}$. We obtain the following exact sequence by applying Mayer–Vietoris to the pair $(\bigcup_{p\in P'} B_{\{p,q\}}, B_{\{r,q\}})$ and noticing that $(\bigcup_{p\in P'} B_{\{p,q\}}) \cap B_{\{r,q\}} = \bigcup_{p\in P'} B_{\{p,q,r\}}$:

$$H_1(\cup_{p\in P'} B_{\{p,q,r\}}) \xrightarrow{\alpha} H_1(\cup_{p\in P'} B_{\{p,q\}}) \oplus H_1(B_{\{r,q\}}) \to H_1(\cup_{p\in P} B_{\{p,q\}}) \to 0.$$

It suffices to show that $\alpha = (i_1, i_2)$ is surjective, which we now prove by induction on $|P|$. The base case $|P| = 2$ follows from Lemma A.8. When $|P| \ge 3$, we have $H_1(\bigcup_{p\in P'} B_{\{p,q\}}) = 0$ by the induction hypothesis. For any specific choice of $p_0 \in P'$, the composition

$$H_1(B_{\{p_0,q,r\}}) \to H_1(\cup_{p\in P'} B_{\{p,q,r\}}) \xrightarrow{i_2} H_1(B_{\{r,q\}})$$

is surjective. It follows that i_2 and hence α are surjective. □

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LEMMA A.10. *Let P and $\{r\}$ be disjoint finite subsets of the Cantor set. Consider the map $(i_P)_*: H_2(\bigcup_{p\in P} B_{\{p,r\}}) \to H_2(\bigcup_{p\in P} B_{\{p\}})$ induced by inclusion. Then either* $\operatorname{coker}(i_P)_* = \mathbb{Z}/2$ *for all P with $2 \le |P| < \infty$, or there is some $N \ge 3$ such that* $\operatorname{coker}(i_P)_* = 0$ *for all P with $N \le |P| < \infty$.*

Proof. Fix $q \in P$ and let $P = P' \cup \{q\}$. Consider the inclusion of pairs $(\bigcup_{p\in P} B_{\{p,r\}}, B_{\{q,r\}}) \to (\bigcup_{p\in P} B_{\{p\}}, B_{\{q\}})$. By naturality of Mayer–Vietoris, we have the following commutative diagram, where the columns are exact.

$$\begin{array}{ccc}
H_2(\cup_{p\in P'} B_{\{p,r\}}) \oplus H_2(B_{\{q,r\}}) & \xrightarrow{((i_{P'})_*,(i_{\{q\}})_*)} & H_2(\cup_{p\in P'} B_{\{p\}}) \oplus H_2(B_{\{q\}}) \\
\downarrow & & \downarrow \\
H_2(\cup_{p\in P} B_{\{p,r\}}) & \xrightarrow{(i_P)_*} & H_2(\cup_{p\in P} B_{\{p\}}) \\
\downarrow \partial & & \downarrow \\
H_1(\cup_{p\in P'} B_{\{p,q,r\}}) & \xrightarrow{f_r} & H_1(\cup_{p\in P'} B_{\{p,q\}}) \\
\downarrow (f_q, f_{P'}) & & \downarrow \\
H_1(\cup_{p\in P'} B_{\{p,r\}}) \oplus H_1(B_{\{q,r\}}) & \longrightarrow & H_1(\cup_{p\in P'} B_{\{p\}}) \oplus H_1(B_{\{q\}})
\end{array}$$

The last term on the right vanishes by Lemma A.5.

When $|P| = 2$, let p be the unique element of P'. Then the first term on the right vanishes by Lemma A.5 and thus $H_2(\bigcup_{p\in P} B_{\{p\}}) \to H_1(B_{\{p,q\}}) \cong \mathbb{Z}$ is an isomorphism. By Lemma A.8, the restriction $f_r : \operatorname{Im}\partial \to H_1(B_{\{p,q\}})$ has image $2\mathbb{Z} \subset \mathbb{Z}$. It follows that $\operatorname{coker}(i_P)_* = \mathbb{Z}/2$.

Suppose that $|P| \ge 3$. Then Lemma A.9 applies to $|P'| \ge 2$, which together with Lemma A.4 reduces the diagram above to

$$\begin{array}{ccc}
H_2(\cup_{p\in P'} B_{\{p,r\}}) & \xrightarrow{(i_{P'})_*} & H_2(\cup_{p\in P'} B_{\{p\}}) \\
\downarrow & & \downarrow \\
H_2(\cup_{p\in P} B_{\{p,r\}}) & \xrightarrow{(i_P)_*} & H_2(\cup_{p\in P} B_{\{p\}}) \\
\downarrow \partial & & \downarrow \\
\operatorname{Im}\partial & \xrightarrow{f_r} & 0
\end{array}$$

From the diagram, we see that $\operatorname{coker}(i_{P'})_*$ surjects $\operatorname{coker}(i_P)_*$. Thus, the conclusion follows by induction on $|P|$. □

LEMMA A.11. *Let $P \subset Q$ be two finite subsets of the Cantor set. Then there is some $N \ge 3$ such that the inclusion $H_2(\bigcup_{p\in P} B_{\{p\}}) \to H_2(\bigcup_{p\in Q} B_{\{p\}})$ is isomorphic whenever $|P| > N$, and $H_2(\bigcup_{p\in P} B_{\{p\}})$ stabilizes to either 0 or $\mathbb{Z}/2$.*

Proof. Consider the two possibilities in Lemma A.10. Suppose that $\operatorname{coker}(i_{P'})_* = \mathbb{Z}/2$ for all P' with $2 \le |P'| < \infty$. Let $|P| > 2$ and $P = P' \sqcup \{q\}$. Note that $(\bigcup_{p\in P'} B_{\{p\}}) \cap B_{\{q\}} = \bigcup_{p\in P'} B_{\{p,q\}}$; by Mayer–Vietoris, we have the following exact sequence.

$$H_2(\cup_{p\in P'} B_{\{p,q\}}) \xrightarrow{j} H_2(\cup_{p\in P'} B_{\{p\}}) \oplus H_2(B_{\{q\}}) \to H_2(\cup_{p\in P} B_{\{p\}}) \to H_1(\cup_{p\in P'} B_{\{p,q\}}).$$

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Since $|P'| = |P| - 1 \geq 2$, the last term in the exact sequence above vanishes by Lemma A.9. On the other hand, Lemma A.4 and $\operatorname{coker}(i_{P'})_* = \mathbb{Z}/2$ imply that the map j has index two. Thus, $H_2(\bigcup_{p\in P} B_{\{p\}}) \cong \mathbb{Z}/2$. Moreover, if we further have $|P'| > 2$, the exact sequence shows that the inclusion $H_2(\bigcup_{p\in P'} B_{\{p\}}) \to H_2(\bigcup_{p\in P} B_{\{p\}})$ is an isomorphism. This verifies the $\mathbb{Z}/2$ case of our assertion with $N = 2$.

Now suppose that there is some $N \geq 3$ such that $\operatorname{coker}(i_{P'})_* = 0$ for all P' with $N \leq |P'| < \infty$. Again let $|P| > N$ and $P = P' \sqcup \{q\}$. The map j in the same exact sequence above is now surjective. Thus, $H_2(\bigcup_{p\in P} B_{\{p\}}) = 0$ for all such P, which establishes the other case of our assertion. □

Remark A.12. In contrast, a similar Mayer–Vietoris computation shows that $H_2(\bigcup_{p\in P} B_{\{p\}}) = \mathbb{Z}$ when $|P| = 2$.

The lemmas above narrow down $H_2(B\widehat{\Gamma})$ to two possibilities, either $\mathbb{Z}/2$ or 0. Lemma A.13 below gives the extra information we need. Denote the Cantor subset of S^2 by K. Let $\mathrm{Homeo}^+(S^2, K)$ be the group of orientation-preserving homeomorphisms of S^2 preserving K. One can show that any $f \in \mathrm{Homeo}^+(S^2 - K)$ uniquely extends to $\bar{f} \in \mathrm{Homeo}^+(S^2, K)$ and thus the restriction map gives an isomorphism $\mathrm{Homeo}^+(S^2, K) \cong \mathrm{Homeo}^+(S^2 - K)$.

LEMMA A.13. *$H_2(\widehat{\Gamma})$ is isomorphic to $H_2(B\mathrm{Homeo}^+(S^2, K))$, which surjects $\mathbb{Z}/2$ via the map $i_* : H_2(B\mathrm{Homeo}^+(S^2, K)) \to H_2(B\mathrm{Homeo}^+(S^2)) \cong \mathbb{Z}/2$ induced by inclusion.*

Proof. The identity component of $\mathrm{Homeo}^+(S^2 - K)$ is contractible by [**18**, Theorem 1.1]. Thus, $B\widehat{\Gamma}$ and $B\mathrm{Homeo}^+(S^2, K)$ are weakly homotopy equivalent, so the first assertion immediately follows.

Let $K_4 = \mathbb{Z}/2 \times \mathbb{Z}/2$ be the Klein 4-group. Realize BK_4 as $\mathbb{R}P^\infty \times \mathbb{R}P^\infty$ and note that $H_2(K_4) \cong \mathbb{Z}/2$ by the Künneth formula, where the generator is represented by $[\mathbb{R}P^1 \times \mathbb{R}P^1]$. We claim that any inclusion of a K_4 subgroup into $\mathrm{Homeo}^+(S^2)$ induces an isomorphism $H_2(K_4) \cong H_2(B\mathrm{Homeo}^+(S^2))$. This implies the second assertion by considering any K_4 subgroup of $\mathrm{Homeo}^+(S^2, K)$ and the composition $K_4 \to \mathrm{Homeo}^+(S^2, K) \to \mathrm{Homeo}^+(S^2)$.

To prove the claim, we may assume K_4 to be generated by two rigid rotations by angle π with perpendicular axes, for all K_4 subgroups of $\mathrm{Homeo}^+(S^2)$ conjugate into $\mathrm{SO}(3)$ [**19**]. Let $f : BK_4 \to B\mathrm{Homeo}^+(S^2)$ be the map induced by the inclusion. Recall that $B\mathrm{Homeo}^+(S^2) \simeq B\mathrm{SO}(3)$ by reduction of the structure group, and that $H_2(B\mathrm{SO}(3)) \cong \pi_2(B\mathrm{SO}(3)) \cong \pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2$ by the Hurewicz theorem and the long exact sequence of fibration. Thus, the map $g : S^2 \to B\mathrm{SO}(3)$ representing the non-trivial element in π_2 classifies the unique non-trivial oriented S^2 bundle over S^2. We need to show that $f_*[\mathbb{R}P^1 \times \mathbb{R}P^1] = g_*[S^2]$. This is actually the case since the following diagram commutes up to homotopy, i.e. the two pull back oriented S^2 bundles on T^2 are isomorphic, where p is a degree-one map.

$$\begin{array}{ccc} \mathbb{R}P^1 \times \mathbb{R}P^1 \cong T^2 & \xrightarrow{\ p\ } & S^2 \\ \downarrow & & \downarrow g \\ \mathbb{R}P^\infty \times \mathbb{R}P^\infty & \xrightarrow{\ f\ } & B\mathrm{Homeo}^+(S^2) \end{array}$$

□

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Finally, we are in a place to prove Theorem A.2.

Proof of Theorem A.2. Recall that $B_* = \bigcup_P B_{\{p\}}$, where p runs over the Cantor set. Then Lemma A.11 implies that $H_2(B_*)$ is either 0 or $\mathbb{Z}/2$. Combining with Lemmas A.3 and A.13, we conclude that $H_2(\widehat{\Gamma}) \cong \mathbb{Z}/2$. Then the desired result follows from Lemma A.1 and the universal coefficient theorem. □

Remark A.14. The fact that $H_2(\widehat{\Gamma}) \cong \mathbb{Z}/2$ in turn implies that $\operatorname{coker}(i_P)_* = \mathbb{Z}/2$ for all P with $2 \leq |P| < \infty$ in Lemma A.10, and that $H_2(\bigcup_{p\in P} B_{\{p\}})$ actually stabilizes to $\mathbb{Z}/2$ in Lemma A.11.

The Birman exact sequence gives rise to a Leray–Serre spectral sequence relating the (co)-homology of Γ and $\widehat{\Gamma}$. Since $H^2(\widehat{\Gamma}) = H^2(\widehat{\Omega}) = 0$, it follows that there is a left-exact sequence

$$0 \to H^2(\Gamma) \to H^1(\widehat{\Gamma}, H^1(\widehat{\Omega})) \to H^3(\widehat{\Gamma}),$$

where the last map is the d_2 differential. We know that $H^2(\Gamma)$ contains a $\mathbb{Z}$ subgroup, generated by the Euler class of the action on the simple circle. This class is not torsion since Γ contains torsion of all orders, so no finite-index subgroup lifts to an action on $\mathbb{R}$.

Question A.15. Does $H^2(\Gamma) = \mathbb{Z}$?

Question A.16. What class in $H^1(\widehat{\Gamma}, H^1(\widehat{\Omega}))$ is the image of the Euler class in $H^2(\Gamma)$?

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