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ON TOTALLY COMPLEX QUARTIC FIELDS WITH SMALL DISCRIMINANTS

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A table of totally complex quartic fields, with small discriminants, which possess quadratic subfields was given by Delone and Faddeev (1); the purpose of the present paper is to give a table of such fields which have no subfield. The method used is based on an idea used recently (2) in the case of totally real quartic fields and depends on a special case of the following theorem, which is stated in full generality.

THEOREM. *If the field K of degree $n = r + 2s$ has r real conjugates and s pairs of complex conjugates and has no subfield, then there exists a polynomial, irreducible in any field of degree less than n, with zeros $\theta_1, \ldots, \theta_r, \theta_{r+1} \pm i\psi_1, \ldots, \theta_{r+s} \pm i\psi_s$ (the θ's and ψ's being real) such that K is generated by one of these zeros and*

$$\sum_i \sum_j (\theta_i - \theta_j)^2 + (r+s) \sum_i \psi_i^2 \leqslant (r+s)^{(n-2)/(n-1)} 2^{-2s/(n-1)} |\Delta|^{1/(n-1)} \gamma_{n-1},$$

where Δ is the discriminant of K and γ_{n-1} is Hermite's constant of order $n-1$.*

Proof. The integers of K, with their conjugates, form an n-dimensional lattice $\mathscr{L}_1$, with determinant $\sqrt{\Delta}$. We may suppose that the base-vectors of $\mathscr{L}_1$ are ordered so that all the real ones come first and the complex ones follow in conjugate pairs, and we apply to $\mathscr{L}_1$ the transformation defined by the matrix

$$\begin{pmatrix} I_r & & & O \\ & J & & \\ & & \ddots & \\ O & & & J \end{pmatrix},$$

where I_r is the r-rowed unit matrix and J (which is repeated s times) is $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}i & \frac{1}{2}i \end{pmatrix}$, thus producing a real lattice $\mathscr{L}_2$. $\mathscr{L}_2$ contains the points

$$O(0, 0, \ldots, 0) \text{ and } I(1, \ldots, 1, 1, 0, \ldots, 1, 0)$$

* That is, the critical determinant of an $(n-1)$-dimensional hypersphere of radius r is $r^{n-1}/\gamma_{n-1}^{\frac{1}{2}(n-1)}$.

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and projects into an $(n-1)$-dimensional lattice $\mathscr{L}_3$ in the hyperplane perpendicular to OI. Now there is a point of $\mathscr{L}_3$ whose distance ρ from OI is such that

$$0 < \rho^{n-1} \leqslant \gamma_{n-1}^{\frac{1}{2}(n-1)} \det \mathscr{L}_3.$$

If $(\theta_1, \ldots, \theta_r, \theta_{r+1}+i\psi_1, \theta_{r+1}-i\psi_1, \ldots, \theta_{r+s}-i\psi_s)$ is the corresponding point of $\mathscr{L}_1$ then

$$\rho^2 = \sum_i \theta_i^2 - \frac{1}{(r+s)} (\sum_i \theta_i)^2 + \sum_i \psi_i^2$$

and

$$\det \mathscr{L}_3 = \frac{1}{\sqrt{(r+s)}} \det \mathscr{L}_2 = \frac{1}{\sqrt{(r+s)}} \frac{1}{2^s} \sqrt{|\Delta|},$$

whence the result follows.

We now take $r = 0$, $s = 2$, $\Delta = 1458 = 2 \times 9^3$ and, since $\gamma_3 = 2^{\frac{1}{3}}$, we consider the polynomials with zeros $\alpha \pm i\beta$, $\gamma \pm i\delta$ for which

$$S = (\alpha-\gamma)^2 + 2(\beta^2+\delta^2) \leqslant 9. \tag{1}$$

Since we do not alter K by adding the same rational integer to each zero, or by changing the signs of the zeros, we may suppose that

$$0 \leqslant \alpha < 1, \quad \alpha \leqslant \gamma \quad \text{and} \quad (\gamma-\alpha)(\delta^2-\beta^2) \geqslant 0. \tag{2}$$

We take as the standard form of a quartic polynomial

$$f(x) = x^4 - ax^3 + bx^2 - cx + d;$$

and conditions (2) then imply

$$0 \leqslant a, b, c, d \quad \text{and} \quad c \leqslant \tfrac{1}{8}(4ab - a^3), \tag{3}$$

since $8c - 4ab + a^3 = 8(\alpha-\gamma)(\delta^2-\beta^2)$. Moreover, if $b = 0$ or $d = 0$ then $f(x) = x^4$, which is reducible in the rational field; if $a = 0$ or $c = 0$ then $f(x) = (x^2+\beta^2)(x^2+\delta^2)$ which is reducible in a quadratic field; while if $c = \frac{1}{8}(4ab - a^3)$ then

$$f(x) = (x^2 - \tfrac{1}{2}ax + \tfrac{1}{8}(4b-a^2))^2 + (d - \tfrac{1}{64}(4b-a^2)^2),$$

which is again reducible in a quadratic field. Consequently strict inequality obtains in (3).

Now

$$a = 2\alpha + 2\gamma = 4\alpha + 2(\gamma-\alpha) < 4 + 2\sqrt{9} = 10, \tag{4}$$

and from (3),

$$\tfrac{1}{4}a^2 < b. \tag{5}$$

Also

$$\begin{aligned} b &= \alpha^2 + 4\alpha\gamma + \gamma^2 + \beta^2 + \delta^2 \\ &\leqslant \tfrac{9}{2} + \tfrac{3}{2}(\alpha+\gamma)^2 - (\alpha-\gamma)^2 \quad \text{(from (1))} \\ &< \tfrac{9}{2} + \tfrac{3}{8}a^2, \end{aligned} \tag{6}$$

while if $a \geqslant 4$ we can sharpen this to

$$b < \tfrac{9}{2} + \tfrac{1}{8}a^2 + 2a - 4, \tag{7}$$

since $\alpha < 1$.

The equation

$$\tfrac{1}{64}(4u+a^2)^3 - \tfrac{1}{8}b(4u+a^2)^2 + ac(4u+a^2) - 8c^2 + d(32b - 12a^2 - 16u) = 0 \tag{8}$$

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has roots S, $-(\alpha-\gamma)^2+4\beta\delta$ and $-(\alpha-\gamma)^2-4\beta\delta$. Thus S is the largest root of (8) and so if $S \leqslant 9$ we have

$$\tfrac{1}{64}(a^2+36)^3-\tfrac{1}{8}b(a^2+36)^2+ac(a^2+36)-8c^2+d(32b-12a^2-144) \geqslant 0,$$

which, using (6), can be written as

$$d \leqslant \left(\frac{a^2+36}{16}\right)^2-\frac{2(\frac{1}{16}a(a^2+36)-c)^2}{3a^2+36-8b}. \tag{9}$$

The equation

$$\phi(u) = u^6-3au^5+(3a^2+2b)u^4-(4ab+a^3)u^3 + (b^2+2a^2b+ac-4d)u^2-(ab^2+a^2c-4ad)u+(abc-c^2-a^2d) = 0$$

has roots 2α, 2γ, $(\alpha+\gamma)\pm i(\beta\pm\delta)$, whence

$$abc-c^2-a^2d > 0. \tag{10}$$

Also, if $\alpha < 1$ and $a = 2(\alpha+\gamma) \geqslant 4$ we require $\phi(2) < 0$, which gives

$$d > \tfrac{1}{4}(4-2a+b)^2-\frac{(\frac{1}{2}a(4-2a+b)-c)^2}{(a-4)^2}. \tag{11}$$

Finally we have the trivial bounds

$$f(1) > 0, \quad f(2) > 0, \ldots \tag{12}$$

which taken in conjunction with (3), (9), (10) and (11) give lower bounds for c.

Using the bounds in (3)–(12) we now compile a table of possible polynomials and evaluate their discriminants.

The processes of evaluating discriminants of fields, and finding whether or not two fields with the same discriminant are identical (as they are in every case obtained here) now follow as in (2).

We obtain finally the following table of discriminants, together with polynomials whose zeros θ define the fields; there are no other discriminants less than 1458. In every case $1, \theta, \theta^2, \theta^3$, constitute a basis of the field.

Δ	a	b	c	d	Δ	a	b	c	d	Δ	a	b	c	d
229	3	6	4	1	892	5	11	8	2	1257	2	5	3	3
257	2	3	1	1	985	2	5	3	1	1264	4	9	8	3
592	2	4	2	2	1016	3	7	6	2	1384	5	8	3	1
697	3	5	4	3	1076	3	6	4	2	1396	3	4	2	2
761	4	9	9	5	1129	4	9	7	2	1436	5	12	13	7
788	3	5	3	2	1229	2	4	1	1					

At the suggestion of Dr Cassels, and for the benefit of those to whom (1) is inaccessible, I include also a table of the fields possessing a subfield. This may be calculated in two ways: we may examine the integers with small norms in fields with small discriminants and determine their quadratic character with respect to 4 and its divisors (see (3), pp. 306–8 for details), or we may examine the polynomials obtained above which are reducible in some quadratic field (so that we have to consider the cases with equality

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in (3)). In addition to these we must, as in (2), Theorem I (ii), consider the quadratic fields generated by zeros of x^2-px+q, where $(x^2-px+q)^2$ satisfies (1) above, i.e. with $S \leqslant 9$, the fields $k(i)$, $k(\sqrt{-2})$, $k(\sqrt{-3})$, $k(\sqrt{-7})$, $k(\sqrt{2})$ and $k(\sqrt{5})$.

We obtain the following table of fields with discriminant Δ generated by $\sqrt{\mu}$.

Δ	μ	Δ	μ	Δ	μ
117	$\frac{1}{2}(7+\sqrt{-3})$	549	$\frac{1}{2}(14+4\sqrt{-3})$	1040	$1+8i$
125	$\frac{1}{2}(-5+\sqrt{5})$	576	$(\sqrt{2}+\sqrt{-3})^2$	1088	$3+2\sqrt{-2}$
144	$\frac{1}{2}(1+\sqrt{-3})$	576	$(\sqrt{-2}+\sqrt{-3})^2$	1088	$-5+2\sqrt{2}$
189	$3+2\sqrt{-3}$	605	$\frac{1}{2}(3+\sqrt{-11})$	1089	$\frac{1}{2}(11+11\sqrt{-3})$
225	$(\sqrt{5}+\sqrt{-3})^2$	656	$5+4i$	1161	$9+4\sqrt{-3}$
256	i	657	$\frac{1}{2}(7+9\sqrt{-3})$	1168	$3+8i$
272	$1+4i$	784	$(\sqrt{7}+i)^2$	1197	$11+2\sqrt{-3}$
320	$1+2i$	832	$3+2i$	1197	$\frac{1}{2}(23+\sqrt{-3})$
333	$\frac{1}{2}(11+3\sqrt{-3})$	837	$\frac{1}{2}(15+7\sqrt{-3})$	1225	$(\sqrt{-7}+\sqrt{5})^2$
392	$\frac{1}{2}(-1+\sqrt{-7})$	873	$\frac{1}{2}(19+3\sqrt{-3})$	1280	$2+i$
400	$(\sqrt{5}+i)^2$	981	$\frac{1}{2}(19+5\sqrt{-3})$	1372	$-\sqrt{-7}$
432	$\sqrt{-3}$	1008	$\frac{1}{2}(5+\sqrt{-3})$	1413	$7+6\sqrt{-3}$
441	$\frac{1}{2}(7+7\sqrt{-3})$	1008	$\frac{1}{2}(-5+\sqrt{-3})$	1421	$-1+2\sqrt{-7}$
512	$1+i$	1025	$\frac{1}{2}(-13+\sqrt{5})$	1424	$5+8i$
513	$\frac{1}{2}(15+\sqrt{-3})$	1040	$7+4i$		

In Delone and Faddeev's table 1161 is misprinted as 1141.

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