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CONTINUOUS FLOWS GENERATE FEW HOMEOMORPHISMS

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Abstract We describe topological obstructions (involving periodic points, topological entropy and rotation sets) for a homeomorphism on a compact manifold to embed in a continuous flow. We prove that homeomorphisms in a C^0-open and dense set of homeomorphisms isotopic to the identity in compact manifolds of dimension at least two are not the time-1 map of a continuous flow. Such property is also true for volume-preserving homeomorphisms in compact manifolds of dimension at least five. In the case of conservative homeomorphisms of the torus $\mathbb{T}^d$ $(d \geq 2)$ isotopic to identity, we describe necessary conditions for a homeomorphism to be flowable in terms of the rotation sets.

Keywords: Embedding problem; flowable homeomorphisms; topological entropy; rotation sets

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1. Introduction

The embedding problem aims to describe the class of maps (homeomorphisms, diffeomorphisms, symplectomorphisms, etc.) on a topological space M which can be embedded in a flow on M displaying the same regularity as the original map. Such class of maps, which can be obtained as time-1 map of a flow with the same regularity, are often referred to as flowable. Most results considering this topic describe obstructions for a dynamical system to be flowable but do not describe the set of flowable homeomorphisms. In particular, the solutions to the embedding problem depend intrinsically on the space of dynamical systems and their topology, and this is reasonably well understood in the differentiable setting.

Historically, the embedding problem for homeomorphisms is a problem that soon attracted the interest of the mathematics community, with fundamental contributions since the late fifties in the case of the line and the plane. Zdun [34] presented a complete solution for the embedding problem for circle homeomorphisms: a homeomorphism of

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the circle $\mathbb{S}^1$ can be embedded in a continuous flow if and only if it is orientation preserving and the homeomorphism is either minimal, the set of fixed points is non-empty and it does not coincide with the whole circle or, alternatively, if the homeomorphism is a root of the identity. There are partial solutions for the embedding problem known on the interval, on a simple closed curve, on a closed 2-cell and for orientation preserving contractions on the plane (see [3, 10, 11]). Utz presented an excellent overview of conditions (fundamentally concerning orientability and contractability of regions of the real line and the plane) which ensure a positive answer for the embedding problem (see [31] and references therein). More recently, Guzik [16] established necessary and sufficient conditions for skew-products to be embedded in flows while Le Roux *et al.* [19] gave necessary and sufficient conditions for a fixed-point free planar homeomorphism that preserves the standard Reeb foliation to embed in a planar flow that leaves the foliation invariant. However, as we will see below there are almost no results that examine whether the space of flowable homeomorphisms is significant (e.g. open and dense, or Baire generic).

The situation is quite different in the case of smooth maps acting on compact manifolds, where numerous contributions (making use of the concept of hyperbolicity and geometric methods) started to appear since the late sixties. Kopell [18] proved that there exists a C^r $(r \geq 2)$ open and dense subset of circle diffeomorphisms that are not embedded in a smooth flow. In the case of higher dimensional manifolds, Brin [8] and Palis [24] proved independently that C^1-generic diffeomorphisms do not embed in a topological flow, concluding that vector fields generate few diffeomorphisms. The previous analysis need only considers Morse–Smale diffeomorphisms (i.e. diffeomorphisms whose non-wandering set consists of finitely many periodic orbits). Indeed, a generic diffeomorphism in the C^r-topology $(r \geq 1)$ that embeds in a flow is necessarily a Morse–Smale diffeomorphism having only fixed points, as a consequence of Kupka–Smale theorem (cf. [24]). The previous discussion suggests to approach the embedding problem for C^r-generic diffeomorphisms $(r \geq 1)$ by proving that most smooth dynamics have few symmetries or, more precisely, that the C^1-centralizer of a Morse–Smale diffeomorphism f, defined by $\mathcal{Z}^1(f) = \{g \in \mathrm{Diff}^1(M)\colon g \circ f = f \circ g\}$, is a discrete subgroup of $\mathrm{Diff}^1(M)$. Indeed, the centralizer of a C^1-generic Morse–Smale diffeomorphism f is formed only by its integer powers [30] and there exists a C^3-open and C^∞-dense subset of C^∞ Morse–Smale diffeomorphisms with discrete centralizer [2]. In particular, diffeomorphisms in the previous classes are not embedded as time-1 maps of smooth flows. More recently, Grines *et al.* [14] solved the problem of finding necessary and sufficient conditions for a Morse–Smale to embed in a topological flow in the case of three-dimensional manifolds, and Polterovich and Shelukhin [26] obtained robust obstructions for a C^∞-Hamiltonian diffeomorphism to be flowable.

Here, we propose a new approach to the embedding problem for homeomorphisms on compact manifolds of arbitrary dimension $n \geq 2$ involving periodic points and well-known topological invariants in dynamical systems, namely topological entropy and rotation sets. At this point, we emphasize that, while allowing more flexible perturbations, the coarseness of the C^0-topology makes obstructions to flowability much harder to be proved. Among the natural questions considering flowability of homeomorphisms, we highlight the following:

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(1) Is the set of homeomorphisms that embed in continuous flows a first category subset?

(2) Is the set of homeomorphisms that embed in continuous flows C^0-(locally) dense among homeomorphisms isotopic to the identity?

(3) Does there exist C^0-open sets of homeomorphisms that are not embedded in continuous flows?

These questions have been addressed for one-dimensional manifolds. As the sufficient and necessary conditions for flowability established by Zdun [34] hold for a C^0-open and dense set of circle homeomorphisms, question (1) is answered negatively in the case of circle homeomorphisms. The situation is substantially different in the higher-dimensional setting. We give an affirmative answer to the question (1) for general homeomorphisms on compact topological manifolds of dimension $n \geq 2$ by obtaining a mechanism, which combines ideas involving topological entropy together and the notion of persistent periodic points, to create obstructions for the embedding problem (cf. Theorem A). Topological entropy itself is not sufficient to create obstructions for the embedding problem. While question (2) remains open in general, we answer it negatively for homeomorphisms on surfaces, providing simultaneously a positive solution to question (3) in this context (cf. Remark 2.1). Finally, we consider the space of homeomorphisms on the torus $\mathbb{T}^d$ ($d \geq 2$), in which case the space $\mathrm{Homeo}_0(\mathbb{T}^d)$ of homeomorphisms isotopic to the identity forms a C^0-open neighbourhood of the identity, and establish a bridge between the concept of rotation sets and the embedding problem. We consider also the space $\mathrm{Homeo}_{0,\lambda}(\mathbb{T}^d) \subset \mathrm{Homeo}_0(\mathbb{T}^d)$ of volume-preserving homeomorphisms. We prove that flowable homeomorphisms in $\mathrm{Homeo}_{0,\lambda}(\mathbb{T}^d)$ and in $\mathrm{Homeo}_0(\mathbb{T}^d)$ are contained in a C^0-closed subset with empty interior (cf. Proposition B). In the case of the torus, we actually give a characterization of flowable homeomorphisms which may be of independent interest (cf. Proposition B).

This paper is organized as follows. Section 2 is devoted to the statement of the main results. In § 3, we recall the concepts and preliminary results on topological entropy and rotation sets. In § 4, we prove the main results, while § 5 is devoted to examples.

2. Statement of the main results

Our first result asserts that the set of homeomorphisms on compact manifolds of dimension larger than one which can be embedded in continuous flows forms a meagre subset in the space of all homeomorphisms.

Theorem A. *Let M be a compact smooth manifold of dimension n. If $n \geq 2$ then there exists a C^0-open and dense subset $\mathcal{O}_1 \subset \mathrm{Homeo}_0(M)$ such that every $f \in \mathcal{O}_1$ is not the time-1 map of a continuous flow in M. If, in addition $n \geq 5$ then there exists a C^0-open and dense subset $\mathcal{O}_2 \subset \mathrm{Homeo}_{0,\lambda}(M)$ so that each $g \in \mathcal{O}_2$ is not the time-1 map of a continuous flow in M.*

The previous result gives a positive answer to questions (1)–(3) in great generality. In the dissipative setting, it is easy to check that a C^0-dense set of homeomorphisms have

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a (non-fixed) periodic attractor, which is a clear obstruction for a homeomorphism to be flowable. However, this mechanism is not persistent, as C^0-generic homeomorphisms have no periodic attractors or repellers (see [1]). Moreover, although the existence of attracting domains is a C^0-open condition, it is not clear that it can be used to create obstructions for a homeomorphism to be flowable in the context of dissipative homeomorphisms. Hence, the stronger statement above concerns the case of volume-preserving homeomorphisms, which do not admit attracting domains and for which there is no clear obstruction for a homeomorphism to be flowable.

Remark 2.1. In the case that M is a surface, Theorem A is a simple consequence of the fact that continuous flows on compact surfaces have zero topological entropy [33]. Indeed, a C^0-open and dense set of homeomorphisms have positive topological entropy (this is a consequence of [32, Proposition 1], stated originally for dissipative homeomorphisms, which also holds for conservative homeomorphisms [15]). In general, topological entropy and periodic points itself are not sufficient to create obstructions to the embedding of a homeomorphism in a continuous flow. Indeed, the proof of Theorem A does not follow directly from the existence of periodic points of the arbitrarily large period (e.g. time-1 map of an Anosov flow).

In the special case of volume-preserving homeomorphisms of the torus $\mathbb{T}^d$, those isotopic to the identity form an open neighbourhood of the identity in $\mathrm{Homeo}_{0,\lambda}(\mathbb{T}^d)$, and we obtain a counterpart of Theorem A using simpler alternative arguments, of independent interest, which bridge between the embedding problem and (we refer the reader to § 3.2 for the definitions).

Proposition B. *There exists a C^0-open and dense subset $\mathcal{A}_2 \subset \mathrm{Homeo}_{0,\lambda}(\mathbb{T}^d)$ such that every $f \in \mathcal{A}_2$ is not the time-1 map of a continuous volume-preserving flow.*

The idea underlying the proof of the previous result is a characterization of rotation sets for continuous flows in the torus $\mathbb{T}^d$ and the fact that the rotation set of their time-t maps change homothetically with the time t (see § 3.2).

We finish this section with some comments on the formulation of the problem in terms of flows instead of vector fields. In rough terms, a priori the embedding problem can also be posed by asking which homeomorphisms are generated by vector fields. In this respect, in view of Picard's theorem, it is natural to consider the space of Lipschitz vector fields endowed with the C^0-topology (see [6]). However, since the set of bi-Lipschitz homeomorphisms form a first category subset in the space of homeomorphisms, it seldom occurs for a homeomorphisms to be the time-1 map of a flow generated by a Lipschitz vector field. The latter justifies that the use of vector fields seems not suitable in a formulation of the embedding problem for homeomorphisms.

3. Preliminaries

3.1. Topological entropy

In this subsection, we recall the concept of topological entropy of a continuous map f on a compact metric space (M, d).

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Given $\varepsilon > 0$ and $n \in \mathbb{N}$, we say that $E \subset M$ is (n, ε)-separated set if for every $x \neq y \in E$ it holds that $d_n(x, y) > \varepsilon$, where d_n denotes Bowen's distance defined by $d_n(x, y) = \max\{d(f^j(x), f^j(y)); j = 0, \ldots, n-1\}$. By compactness, the maximal cardinality $s_f(n, \varepsilon)$ of a (n, ε)-separated subset of M is finite. The *topological entropy* of f is defined by

$$h_{\text{top}}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_f(n, \varepsilon),$$

The previous notion does not depend on the metric d and is a topological invariant (i.e. it is invariant by topological conjugacy). Moreover, the classical variational principle ensures that $h_{\text{top}}(f) = \sup\{h_\mu(f) : \mu \in \mathcal{M}(f)\}$, where $h_\mu(f)$ is the usual Kolmogorov–Sinai entropy of the invariant probability μ. While a C^1-diffeomorphism f on a compact manifold M always has finite topological entropy (bounded by $\dim M \times \log \|Df\|_\infty$), one has that C^0-generic homeomorphisms on a closed manifold of dimension at least two have infinite topological entropy (cf. [32]).

3.2. Rotation sets of maps and flows

In this subsection, we recall the definitions and collect some auxiliary results on the rotation sets of continuous maps and continuous flows on the torus. First, given $d \geq 2$, recall that a homeomorphism $f \in Homeo(\mathbb{T}^d)$ is volume preserving if $\text{Leb}(f^{-1}(E)) = \text{Leb}(E)$ for every measurable $E \subset \mathbb{T}^d$, and we denote the set of volume-preserving homeomorphisms in $\mathbb{T}^d$ isotopic to the identity by $\text{Homeo}_{0,\lambda}(\mathbb{T}^d)$. A flow is called volume preserving if every time-t map is a volume-preserving homeomorphism.

3.2.1. Homeomorphisms on the torus

The rotation number for circle homeomorphisms, introduced by Poincaré, provides a rather complete topological characterization of circle dynamics. Given a homeomorphism on the circle f, its *rotation number* is

$$\rho(f) = \lim_{n \to \infty} \frac{F^n(x) - x}{n} (\text{mod} 1), \tag{3.1}$$

where F is any lift of the circle homeomorphism f. The previous limit is independent of the lift F and of the point $x \in \mathbb{S}^1$. In the case of homeomorphisms on the torus $\mathbb{T}^d$ $(n \geq 2)$, the situation is more complex as the limit in the right-hand side of (3.1) may fail to exist and, in general, the set of accumulation vectors depends on the base point $x \in \mathbb{T}^d$.

Let us recall the concept of rotation sets introduced for homeomorphisms isotopic to the identity on the torus by Misiurewicz and Ziemian [20]. If $f \in \text{Homeo}_0(\mathbb{T}^d)$ and $F : \mathbb{R}^d \to \mathbb{R}^d$ is a lift (i.e. $\pi \circ F = f \circ \pi$ where $\pi : \mathbb{R}^d \to \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the natural projection) then the *rotation set* $\rho(F)$ is the set of accumulation vectors of sequences of the form

$$\left\{ \frac{F^{n_i}(x_i) - x_i}{n_i} \right\}_{i=1}^{\infty},$$

where $n_i \to \infty$ and $x_i \in \mathbb{R}^d$. The *pointwise rotation set* is defined as $\rho_p(F) = \bigcup_{x \in \mathbb{T}^d} \rho(F, x)$, where, for every $x \in \mathbb{T}^d$, $\rho(F, x) := \lim_{n \to \infty}(F^n(\tilde{x}) - \tilde{x}/n)$, for some $\tilde{x} \in$

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$\pi^{-1}(x)$. Finally, the *measure rotation set* is $\rho_{\text{meas}}(F) := \{\int \varphi_F d\mu : \mu \in \mathcal{M}_e(f)\}$, where $\mathcal{M}_e(f)$ denote the set of f-invariant and ergodic probability measures and $\varphi_F : \mathbb{R}^d \to \mathbb{R}^d$ is the displacement function defined by $\varphi_F(x) = F(x) - x$, $x \in \mathbb{R}^d$. It is clear from their definition and Birkhoff's ergodic theorem that

$$\rho_{\text{meas}}(F) \subseteq \rho_p(F) \subseteq \rho(F). \tag{3.2}$$

In the torus $\mathbb{T}^d$, the closure of the convex hull of the three sets above coincides [20]. In the case of homeomorphisms in $\text{Homeo}_0(\mathbb{T}^d)$, the rotation set is always convex if $d = 2$ and may fail to be convex if $d \geq 3$ [20]. Nevertheless, rotation sets are convex for an open and dense set of homeomorphisms in $\text{Homeo}_{0,\lambda}(\mathbb{T}^d)$ [7]. Finally, recall that a lift F of $f \in \text{Homeo}_0(\mathbb{T}^d)$ is *irrotational* if $\rho(F) = \{\vec{0}\}$ (by some abuse of notation, we say f is irrotational).

3.2.2. Continuous flows and time-t maps on the torus

Since all time-t maps of a continuous flow are isotopic to identity, the concepts of rotation set for flows adapt naturally. Franks and Misiurewicz [13] introduced the concept of rotation set for a continuous flow $(X_t)_t$ on the torus, considering lifts $(Y_t)_t$ of the flow $(X_t)_t$ to $\mathbb{R}^2$: the rotation set of $(Y_t)_t$, denoted by $\rho((Y_t)_t)$, is the set accumulation points of sequences of the form

$$\left\{\frac{Y_{t_i}(x_i) - x_i}{t_i}\right\}_{i=1}^{\infty},$$

with $x_i \in \mathbb{R}^2$ and $t_i \in \mathbb{R}$. In other words, $v \in \rho((Y_t)_t)$ if and only if there are $x_i \in \mathbb{R}^2$ and $t_i \to +\infty$ such that $\lim_{i\to+\infty}((Y_{t_i}(x_i) - x_i)/t_i) = v$. The rotation vector of $x \in \mathbb{R}^2$ is defined by $\rho((Y_t)_t, x) = \lim_{t\to+\infty}((Y_t(x) - x)/t)$, if the limit exists. It is not hard to check that $\rho((Y_t)_t) = \rho(Y_1)$ and inclusions (3.2) hold, so that the rotation sets are completely determined by the time-1 map of the flow. The following instrumental lemma shows that the rotation set of every time-t map can be recovered by an homothety from the rotation set of the time-1 map.

Lemma 3.1. *Given $d \geq 2$, let $(Y_t)_t$ be a continuous flow on $\mathbb{R}^d$ which is a lift of a continuous flow $(X_t)_t$ on $\mathbb{T}^d$. For any $s \in \mathbb{R}$ it holds $\rho(Y_s) = \{sv : v \in \rho(Y_1)\}$.*

Proof. If $s = 0$ then both sets coincide with $\{(0,0)\}$ and the equality holds. Otherwise, assume $s \in \mathbb{R} \setminus \{0\}$ is fixed and consider the continuous flow $(Z_t)_t$ on $\mathbb{R}^d$ obtained by linear reparametrization of $(Y_t)_t$ as $Z_t = Y_{s\cdot t}$ for all $t \in \mathbb{R}$. Observe that $\rho((Z_t)_t) = \rho(Z_1) = \rho(Y_s)$. Moreover, if $v \in \rho((Z_t)_t) = \rho(Y_s)$, there exists sequences $(x_i)_i \in \mathbb{R}^d$ and $t_i \to +\infty$ such that

$$v = \lim_{t\to+\infty} \frac{Z_{t_i}(x_i) - x_i}{t_i} = s \cdot \lim_{t\to+\infty} \frac{Y_{s\cdot t_i}(x_i) - x_i}{s \cdot t_i}.$$

This proves that $1/s \cdot v \in \rho((Y_t)_t) = \rho(Y_1)$ and therefore, $\rho(Y_s) \subseteq \{sv : v \in \rho(Y_1)\}$. The proof of the other inclusion is identical. □

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In the special case that the continuous flow $(Y_t)_t$ is integrable to a vector field $Y \in \mathfrak{X}^0(\mathbb{R}^2)$ then it is clear from the integral formulation

$$Y_t(x) = x + \int_0^t Y(Y_s(x))\,\mathrm{d}s \quad \text{for every } t \in \mathbb{R}$$

that $\rho((Y_t)_t)$ measures the variation of the vector field along the orbits by the flow. Franks and Misiurewicz [13] proved that the rotation set of a continuous flow on $\mathbb{T}^2$ is either a vector or a line segment in $\mathbb{R}^2$ whose supporting line goes through the origin. In the higher dimensional setting, the characterization of rotation sets for continuous flows on $\mathbb{T}^d$ $(d \geq 2)$ still remains an open question.

4. Proofs

4.1. Proof of Theorem A

Let M be a compact topological manifold of dimension $n \geq 2$, with or without boundary. Since all periodic orbits of a continuous flow form a continuum, i.e. a compact and connected metric space that contains at least two points, in order to prove Theorem A, it is enough to prove that a C^0-open and dense set of homeomorphisms have some (non-fixed) periodic point whose orbit is not contained in a continuum. Given $f \in \mathrm{Homeo}(M)$ and $x \in M$, we denote by $O_f(x) = \{f^n(x) \colon n \in \mathbb{Z}\}$ the orbit of the point x by f. As mentioned in § 2, we treat the cases of dissipative and volume-preserving homeomorphisms separately.

4.1.1. Dissipative homeomorphisms

The proof relies on the following:

Proposition 4.1. *Let M be a compact smooth manifold of dimension at least two. There exists a C^0-open and dense subset $\mathcal{O}_1 \subset \mathrm{Homeo}_0(M)$ such that every $f \in \mathcal{O}_1$ there exists $k > 1$ and a periodic point $p \in \mathrm{Per}_k(f)$ such that $O_f(p)$ is not contained in a continuum in $\mathrm{Per}_k(f)$.*

Proof. The idea here is to explore that typical homeomorphisms in $\mathrm{Homeo}_0(M)$ are locally dissipative in order to consider attracting domains, which will be useful to consider periodic points persistent by C^0-perturbations.

Fix $f \in \mathrm{Homeo}_0(M)$. Given $\varepsilon > 0$, by uniform continuity of f, there exists let $\delta > 0$ (depending only on f) such that $d_{C^0}(h \circ f, f) < \varepsilon$ for every homeomorphism h that coincides with the identity except for a ball of radius δ.

Assume first that there exists $x \in R(f) \setminus \mathrm{Fix}(f)$, where $R(f)$ denotes the set of recurrent points for f and $\mathrm{Fix}(f)$ denotes the set of fixed points for f. Let $k \geq 1$ be the smallest positive integer such that $d(f^k(x), x) < \delta$. Diminishing δ, if necessary, we guarantee that $k > 1$. By the C^0-closing lemma (cf. [17]), there exists a ε-C^0-perturbation $f_1 = h \circ f$ of f, supported on $B(x, \delta)$ with h isotopic to the identity such that $f_1^k(x) = x$. Reduce δ as necessary, so that $\mathcal{B} = \{B(f_1^i(x), \delta) : 1 \leq i \leq k\}$ become a pairwise disjoint collection of balls.

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Using bump-functions, we perform k disjoint C^0-perturbations of f_1, each of which supported in a ball in the collection $\mathcal{B}$, in such a way that the resulting homeomorphism f_2 is C^1-smooth in a neighbourhood of $\mathcal{O}_{f_2}(x)$ and $f_2^i(x)$ becomes a periodic attractor for f_2^k, for every $1 \leq i \leq k$. Since the supports of the perturbations are disjoint, we have that $d_{C^0}(f_2, f_1) < \varepsilon$. Since $\varepsilon > 0$ was chosen arbitrary we conclude that f is arbitrary C^0-approximated by homeomorphisms having (non-fixed) periodic points of period $k > 1$ such that $O_f(p)$ is not contained in a continuum in $\mathrm{Per}_k(f)$ (as the fixed points for f_2^k do not intersect the boundary of U). Hence, such homeomorphisms are C^0-dense.

We claim that the previous set of homeomorphisms is also C^0-open. For each homeomorphism f_2 as above, there exists a small open neighbourhood U of x so that $f_2^k|_U$ has no fixed points on the boundary of U, $\partial U \simeq \mathbb{S}^{n-1}$, and the Hopf degree of the continuous map

$$\gamma_{f_2} : \partial U \simeq \mathbb{S}^{n-1} \to \mathbb{S}^{n-1} \quad \text{given by} \quad \gamma_{f_2^k}(x) = \frac{f_2^k(x) - x}{\|f_2^k(x) - x\|}$$

is non-zero. Moreover, using that $\deg(\gamma_{f_2}^k) = \deg(g^k)$ for every homeomorphism g that is C^0-close to f_2 and that $\deg(\gamma_g^k) \neq 0$ ensures the existence of a fixed point for g^k in U. Since there exist no fixed points in the boundary of U for every homeomorphism C^0-close to f_2, we conclude that there exists a C^0-open neighbourhood V_{f_2} of f_2 such that every homeomorphism $g \in V_{f_2}$ has a periodic point of period $k > 1$ that is not contained in a continuum subset of the set of k-periodic points.

It remains to consider the case that $R(f) = \mathrm{Fix}(f)$. Indeed, take $x \in \mathrm{Fix}(f)$ and let $0 < \delta < \varepsilon$. The idea, inspired by the proof of Franks' lemma [12], is to consider a C^1-diffeomorphism f_1 that is a C^0-perturbation of f, is conjugated to a linear map and has period two periodic points in a small neighbourhood of x.

Reduce $\delta > 0$, if necessary, so that the exponential map $\exp : T_zM(\delta) \to B(z, \delta)$ is a diffeomorphism for every $z \in M$, where $T_zM(\delta) := \{v \in T_zM : \|v\| < \delta\}$ and $B(z, \delta)$ denotes the ball of radius δ around z (where the distance in M is induced by the Riemannian metric). In particular, since $f(x) = x$, it will be useful to write

$$f(z) = \exp_x \circ F \circ \exp_x^{-1}(z)$$

for all $z \in B(x, \delta)$, where $F(v) = \exp_x^{-1} \circ f \circ \exp_x(v)$ for every $v \in T_xM(\delta)$. Indeed, consider a C^1-bump function $0 \leq \rho \leq 1$ such that $\rho|_{B(x,\delta/2)} \equiv 1$ and $\rho|_{M \setminus B(x,\delta)} \equiv 0$, and consider the following perturbation of the diffeomorphism f defined as

$$f_1(z) = \exp_x \circ [\rho \cdot R + (1 - \rho) \cdot F] \circ \exp_x^{-1}(z)$$

for every $z \in B(x, \delta)$, and $f_1(z) = f(z)$, for every $z \in M \setminus B(x, \delta)$, where $R \neq Id$ is an involution of $T_xM(\delta)$. As $R^2 = Id$, there are period two periodic points for f_1 in $B(x, \delta/2)$. Moreover, using that $D\exp_x(0) = Id$, there exists $L > 1$ (tending to 1 as δ tends to zero) so that

$$d(f_1(z), f(z)) \leq L\|(F - R) \circ \exp_x^{-1}(z)\| \leq L\|F - R\|\delta \leq L\|F - R\|\varepsilon.$$

Since ε is chosen arbitrary (and R may be chosen independently of ε), we conclude that f is arbitrarily C^0-approximated by a C^1-diffeomorphism that exhibits a period

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two periodic point. The remaining steps follow the argument described in the previous situation. This finishes the proof of the proposition. □

We observe that the previous proof demands no restriction on the dimension of the manifold M.

4.1.2. Volume-preserving homeomorphisms

We now prove the theorem in the case of volume-preserving homeomorphisms. In the dissipative context, the proof explores the possibility of creation of sinks, an argument incompatible with the present volume-preserving setting. Here, we use instead the uniform approximation of homeomorphisms isotopic to identity by diffeomorphisms (it is here that the restriction on the dimension appears).

Note that every homeomorphisms is uniformly approximated (meaning the C^0-topology) by diffeomorphisms in smooth manifolds of dimension 2 and 3 (we refer the reader e.g. to [17, 22]). Even though there are exotic structures, hence there exist homeomorphisms on smooth manifolds which are not C^0-aproximated by diffeomorphisms, this is not the case for homeomorphisms isotopic to some diffeomorphism as a consequence of [21, 23, 28]. Indeed, Müller [21] proved that if $\dim M \geq 5$ then a homeomorphism $f \in \mathrm{Homeo}(M)$ can be C^0-approximated by diffeomorphisms if and only if f is isotopic to a diffeomorphism and, if this holds, the diffeomorphisms can be chosen volume preserving whenever this is the case for the homeomorphism f. We recall a specific statement which will be used later on.

Proposition 4.2 (Müller [21, Proposition 3]). *If M is a closed smooth manifold of dimension $n \geq 5$ then every $f \in Homeo(M)$ isotopic to the identity is C^0-approximated by diffeomorphisms isotopic to the identity.*

The previous proposition says that $\overline{(\mathrm{Diff}_0^1(M), C^0)} = \mathrm{Homeo}_0(M)$. Although Proposition 3 in [21] does not mention that the approximating diffeomorphisms are isotopic to the identity, this follows from the locally contractability of the space $\mathrm{Homeo}(M)$ [9]. Moreover, any volume-preserving homeomorphism f that can be C^0-approximated by diffeomorphisms can be C^0-approximated by volume-preserving diffeomorphisms [23, 28]. Together with Proposition 4.2, this yields the following:

Corollary 4.3 (Müller [21, p. 319]). *If M is a closed smooth manifold of dimension $n \geq 5$ then every $f \in Homeo_{0,\lambda}(M)$ is C^0-approximated by volume-preserving diffeomorphisms isotopic to the identity.*

The proof of the statement concerning volume-preserving homeomorphisms in Theorem A relies on the following result, whose proof is substantially different from the one of Proposition 4.1.

Proposition 4.4. *There exists a C^0-open and dense subset $\mathcal{O}_2 \subset Homeo_0, \lambda(M)$ such that every $f \in \mathcal{O}_1$ there exists $k > 1$ and a periodic point $p \in Per_k(f)$ such that $O_f(p)$ is not contained in a continuum in $Per_k(f)$.*

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Proof. Fix an arbitrary $f \in \mathrm{Homeo}_{0,\lambda}(M)$ which is not the identity map. Given $\varepsilon > 0$, let $f_1 \in \mathrm{Diffeo}^1_\lambda(M) \cap \mathrm{Homeo}_{0,\lambda}(M)$ be given by Corollary 4.3 such that $d_{C^0}(f_1, f) < \varepsilon$.

Poincaré recurrence theorem ensures that Lebesgue almost every point is recurrent. Since $f \neq id$, we conclude that there exists a recurrent point $x \in M$ such that $f(x) \neq x$. By the C^1-closing lemma for conservative homeomorphisms, there exists a ε-C^1-diffeomorphism $f_2 = h \circ f_1$, where h is isotopic to the identity, and such that f_2 has a periodic point p of some period $k > 1$ and close to x (cf. [5, 27]). By the conservative pasting lemma (see [29, Theorem 7]), up to an arbitrary small C^1-perturbation, we may assume that p is a periodic saddle (here we exploit that elliptic points periodic points are not robust when $\dim M > 2$).

If p is isolated in $\mathrm{Per}_k(f_2)$ we may proceed as in Proposition 4.1: there exists a small open neighbourhood U of x such that $f_2^k \mid_U$ has no fixed points on the boundary of U, $\partial U \simeq \mathbb{S}^{n-1}$, and the Hopf degree of $\gamma_{f_2} : \partial U \simeq \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is non-zero. Hence, there exists a C^0-open neighbourhood of f_2 formed by homeomorphisms that exhibit periodic points of period k and that are not contained in a continuum in $\mathrm{Per}_k(f)$.

It remains to deal with the case when p is not isolated in $\mathrm{Per}_k(f_2)$. By the conservative pasting lemma (see [29, Theorem 7] and also [4, Theorem 3.6]), there exist $\delta > 0$ and open neighbourhoods $V_1 \subsetneq V_2$ of $f_2^{k-1}(p) \in M$ so that the following holds: given $A \in \mathrm{SL}(n, \mathbb{R})$ satisfies $\|Df_2(f_2^{k-1}(p)) - A\| < \delta$ for every $1 \leq i \leq k$, there exists an ε-C^1-perturbation f_3 of f_2 such that $f_3 \mid_{M \setminus V_2} = f_2$ and $f_3^k \mid_{V_1} \equiv A \cdot Df_2(f_2^{k-1}(p))$ (in local charts). In particular, $f_3^k(p) = p$ and

$$Df_3^k(p) = A \cdot Df_2^k(p).$$

Therefore, if in addition one chooses $A \in \mathrm{SL}(n, \mathbb{R})$ such that 1 is not an eigenvalue for $Df_3^k(p)$ then p is an isolated periodic point for the volume-preserving diffeomorphism f_3. Since ε was chosen arbitrary, the proof follows as before. This finishes the proof of the lemma. □

Remark 4.5. In the dissipative setting, we may take the C^0-small perturbation $\tilde{g}$ of g such that $\tilde{g}^j(x) = g^j(x)$ and $\tilde{g}^j(x)$ is an attracting fixed point for g^k, for every $0 \leq j \leq k-1$. Since x is periodic attractor, there is a small open neighbourhood U of x such that $g^k \mid_U$ has no fixed points on the boundary of U and the argument in the proof of Proposition 4.1 implies on the existence of a periodic point for all C^0-close homeomorphisms.

4.2. Proof of Proposition B

There exists a C^0-open and dense subset $\mathcal{A}_2 \subset \mathrm{Homeo}_{0,\lambda}(\mathbb{T}^d)$ formed by volume-preserving homeomorphisms whose rotation set is a stable rational polyhedron with non-empty interior [7]. More precisely, there exists a C^0-dense set $\mathcal{D} \subset \mathrm{Homeo}_{0,\lambda}(M)$ such that every $f \in \mathcal{D}$ admits a C^0-open neighbourhood $\mathcal{V}_f$ satisfying that for every $g \in \mathcal{V}_f$ one has

(a) $\rho(G) = \rho(F)$, and

(b) $\rho(G)$ is a compact and convex polyhedron (possibly degenerated) with rational vertices,

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where $F : \mathbb{R}^d \to \mathbb{R}^d$ denotes a continuous lifts of f and G is the continuous lift for g which is closest to F (recall that all continuous lifts for g are of the form $G + v$, for some $v \in \mathbb{Z}^d$). By construction, the set

$$\mathcal{O} = \bigcup_{f \in \mathcal{D}} \mathcal{V}_f$$

is C^0-open and dense in $\mathrm{Homeo}_{0,\lambda}(\mathbb{T}^d)$.

In order to complete the proof of the proposition, it is enough to prove that every $g \in \mathcal{O}$ is not flowable. Indeed, if otherwise $g = X_1$ for some continuous flow $(X_t)_t$ and $(Y_t)_t$ is a continuous lift of the flow $(X_t)_t$ then g is C^0-accumulated by the family of homeomorphisms $g_s := X_s$ as $s \to 1$. Then, the stability of rotation sets on $\mathcal{O}$ implies that $\rho(X_s) = \rho(X_1)$, which contradicts Lemma 3.1. The previous argument ensures that there are exists a C^0-open and dense set $\mathcal{O} \subset \mathrm{Homeo}_{0,\lambda}(\mathbb{T}^d)$ formed by homeomorphisms that do not embed in a continuous flow, and proves the proposition.

Remark 4.6. In the case of dissipative homeomorphisms homotopic to the identity on the torus $\mathbb{T}^2$, the C^0-dense stability of rotation sets has been proved in [25]. An argument identical to the one used in the proof of Proposition B ensures that, for a C^0-open and dense set of homeomorphisms, either the homeomorphism is irrotational, or it is not flowable. This raises the question about the existence of C^0-open sets of irrotational homeomorphisms.

5. Examples

In this section, we provide some examples that illustrate the relation between infinite topological entropy, existence of periodic points of arbitrarily large period and rotation sets.

Example 5.1. *Given $\alpha \notin \mathbb{Q}$, the irrational translation on the torus $f : \mathbb{T}^2 \to \mathbb{T}^2$ given by $f(x, y) = (x, y) + (1, \alpha)(mod\ \mathbb{Z}^2)$ is minimal (i.e. all orbits are dense in $\mathbb{T}^2$) and not irrotational, but it is flowable as it is the time-1 map of the smooth flow $X_t(x, y) = (x + t, y + t\alpha)(mod\ \mathbb{Z}^2)$. Since the rotation set of f is clearly not stable by C^0-perturbations then f does not belong to the open and dense set in Proposition B.*

The next simple examples show that topological entropy and the property of having periodic points with large period do not characterize irrotational homeomorphisms.

Example 5.2. *There exists an irrotational homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ with infinite topological entropy and having periodic points of arbitrarily large period. Indeed, fix a small ball B on the flat torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ that does not contain $(0, 0)$ and let $\mathcal{S}$ be the set of homeomorphisms g isotopic to the identity so that $g|_{\mathbb{T}^2 \setminus B} = id$. If $g \in \mathcal{S}$ and $G : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of G so that $G(0, 0) = (0, 0)$ then one has $\|G^n(x, y) - (x, y)\| \leq 1$ for every $(x, y) \in \mathbb{R}^2$ and every $n \geq 1$, which ensures that G is irrotational. The set $\mathcal{S}$ is a closed subset of $\mathrm{Homeo}_0(\mathbb{T}^2)$ (which has empty interior) and, in particular, $\mathcal{S}$ is a complete metric space (hence a Baire space). Using the local perturbations of Yano [32], by isotopy and supported in B we conclude that for every $K > 0$ there exists a C^0-open and dense subset of $\mathcal{S}$ formed by homeomorphisms which have pseudo-horseshoes (meaning,*

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invariant sets semi-conjugated to subshifts of finite type) with topological entropy larger than K. Thus, there exists a C^0-Baire residual $\mathcal{R} \subset \mathcal{S}$ formed by homeomorphisms with infinite topological entropy and periodic points of arbitrarily large period. In particular, these homeomorphisms are not time-1 maps of continuous flows.

Example 5.3. *There exists an irrotational homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ with periodic points of arbitrarily large period and zero topological entropy. Let $\mathcal{S}$ be the class of homeomorphisms as defined in Example 5.2 and we consider a perturbation of the identity on B to obtain the desired homeomorphism f. Since $\overline{B}$ is homeomorphic to $\overline{B(0,1)}$, we may deal instead with homeomorphisms in Homeo$(\overline{B(0,1)})$ which coincide with the identity at the boundary. Given $\alpha : [0,1] \to [0,1]$ continuous, not identically zero and such that $\alpha(0) = \alpha(1) = 0$, consider the flow $(Y_t)_t$ on $\overline{B(0,1)}$ given by $Y_t(x,y) = X_{\alpha(\|(x,y)\|)\,t}$ where $X_t(x,y) = (x\cos(t) - y\sin(t), x\sin(t) + y\cos(t))$. By construction Y_1 is flowable, it has a singularity at $(0,0)$ and coincides with the identity at the boundary of $\overline{B(0,1)}$. Moreover, all points in $\{(x,y) \in \overline{B(0,1)} \colon \|(x,y)\| = 1/k\}$ with $k \in \mathbb{N}$ are periodic with prime period $2\pi k$. The homeomorphism $f \in \mathcal{S}$, obtained via the homeomorphism $\overline{B} \simeq \overline{B(0,1)}$ is irrotational, has periodic points of arbitrarily large period and is flowable on $\mathbb{T}^2$ (hence it has zero topological entropy).*

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