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Augmented dimension groups and ordered cohomology

MARCY BARGE and MEGAN SMITH
Department of Mathematics, Montana State University, Bozeman, MT 59717, USA
(e-mail: umsfmbar@math.montana.edu)

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Abstract. We show that the ordered Matsumoto K_0-group of a substitutive system is derivable from the ordered Čech cohomology of a simple 'augmentation' of the tiling space associated with the substitution. Besides supplying a topological interpretation of the Matsumoto K_0-group, this leads to finer invariants of the flow equivalence class of a substitutive system.

1. *Introduction*

In their papers [**CE1**] and [**CE2**], Carlsen and Eilers demonstrate the use of the Matsumoto K_0-groups as an invariant of flow equivalence of substitutive systems. The present article was motivated by our desire to understand the topological underpinnings of their invariant. We have found that the ordered Matsumoto K_0-group of a substitutive system is closely related to (derivable from) the ordered Čech cohomology group of a relatively easy to understand quotient of the suspension of the substitutive system. Besides topological transparency and comparative simplicity, our approach has the virtue that it leads to a finer invariant.

Associated with any (primitive, aperiodic) substitution, there is a Cantor set consisting of allowed bi-infinite words. This Cantor set, with the shift dynamics, is called a substitutive system. There has been considerable recent progress in the problem of classifying these systems up to various notions of equivalence; most notably, [**GPS**] have proved that the dimension group, with order unit, of the incidence matrix associated with the substitution is a complete invariant of strong orbit equivalence for substitutive systems.

Each (primitive, aperiodic) substitution gives rise to a space of tilings of the real line. This tiling space is a one-dimensional continuum that supports two dynamics: inflation and substitution, by which the tiles of a tiling are stretched and replaced by a pattern of tiles determined by the substitution, and translation flow.

The inflation and substitution dynamics were investigated by Williams in the 1960s and 1970s [**Wi1, Wi2**] in the guise of hyperbolic one-dimensional attractors. Williams

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obtained complete invariants of pointed conjugacy classes of these systems in terms of shift equivalence of π_1-actions in elementary presentations. The translation flow on the tiling space is nothing more than the suspension flow of the substitutive system (with the appropriate roof function). Thus, the problem of classifying substitutive systems up to flow equivalence is just the problem of classifying the corresponding tiling spaces up to orientation-preserving homeomorphism. The latter problem was reduced in [**BD**] to the calculation of weak equivalence (a non-stationary version of shift equivalence) classes of certain derived substitutions. Although this gives a complete invariant, there is no known way to decide if two substitutions are weakly equivalent. The K-theoretical augmented dimension group developed by Carlsen and Eilers provides a means that can be effective in distinguishing between flow equivalence classes, even for substitutions with identical incidence matrices.

The topology behind the augmented dimension group is quite interesting and not so difficult to describe, at least in fairly rough terms (a detailed analysis is presented in §5). Each one-dimensional substitution tiling space has a positive and finite number of 'forward asymptotic composants'—arc components that contain tilings that asymptotically approach each other under the positive flow. These correspond to 'left special words' in the substitutive system. On each such asymptotic composant there is a point (tiling) that is periodic under inflation and substitution. As the property of arc components being asymptotic is a topological property, any orientation-preserving homeomorphism from one tiling space to another must take forward asymptotic composants to forward asymptotic composants. After adjustment by an isotopy, such a homeomorphism will take the periodic points on the forward asymptotic composants of the one tiling space onto those of the other; that is, an orientation-preserving homeomorphism between tiling spaces implies an orientation-preserving homeomorphism between the quotient spaces obtained by identifying all periodic points on forward asymptotic composants to a single point. The ordered Čech cohomology group is an invariant of orientation-preserving homeomorphism. For the tiling space, this group is just the dimension group of the incidence matrix. Sticking the periodic points, say m in number, together in the above described quotient creates $m - 1$ additional, independent generators in cohomology. The ordered Čech cohomology of the quotient is (essentially) the augmented dimension group of Carlsen and Eilers. This extension of the ordinary dimension group is not, in general, split; the pattern in which the asymptotic composants wrap on the cohomology generators of the tiling space is the distinctive, non-abelian information carried by the augmented dimension group.

Of course one can also consider backward asymptotic composants or forward and backward asymptotic composants simultaneously. This yields the strongest invariant presented in the present paper (Theorem 3.4). We give an example to show that substitutions may have order isomorphic forward augmented cohomology groups and order isomorphic backward augmented cohomology groups yet have non-order isomorphic two-sided augmented cohomology groups. (The forward and backward versions of the Matsumoto K_0-groups require the notions of past equivalence and future equivalence. It is unclear how to combine these into an analogous two-sided relation.)

In §2 we will lay out the terminology and necessary background. Section 3 contains the description of augmented tiling spaces, augmented cohomology, and the main theorems;

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the fourth section is devoted to examples. The final §5 examines the relationship between augmented cohomology and the Matsumoto K_0-group.

2. *Notation, terminology and background*

A *substitution* is a map $\phi : \mathcal{A} \to \mathcal{A}^\star$ from a finite set $\mathcal{A}$, called the *alphabet*, to the set $\mathcal{A}^\star$ of finite, non-empty words with letters in $\mathcal{A}$. The substitution ϕ extends in a natural way to $\phi : \mathcal{A}^\star \to \mathcal{A}^\star$. If $\mathcal{A} = \{1, \ldots, d\}$, then associated with a substitution $\phi : \mathcal{A} \to \mathcal{A}^\star$ is a $d \times d$ *incidence matrix* $A = A(\phi)$ whose ijth entry is the number of occurrences of the letter i in the word $\phi(j)$; ϕ is *primitive* if there is an $m > 0$ such that A^m is strictly positive. Note that $A(\phi^n) = (A(\phi))^n$ so that ϕ is primitive if and only if there is an $m > 0$ so that i occurs in $\phi^m(j)$ for all $i, j \in \mathcal{A}$.

For primitive ϕ there is a simple Perron–Frobenius eigenvalue $\lambda = \lambda_\phi$ of $A(\phi)$ (λ is real and larger than the modulus of all other eigenvalues) with associated strictly positive left and right eigenvectors $\vec{v}_l(\phi)$ and $\vec{v}_r(\phi)$. We will take these to be normalized so that $\langle \vec{v}_r \mid \vec{v}_r \rangle = 1$ and $\langle \vec{v}_l \mid \vec{v}_r \rangle = 1$, where $\langle \cdot \mid \cdot \rangle$ is the Euclidean inner product.

We denote by W_ϕ the set of all *allowed bi-infinite words* for the substitution ϕ:

$$\underline{w} = \ldots w_{-1}.w_0 w_1 \ldots \in W_\phi$$

(the period indicates the location of the 0th letter) if and only if for each finite subword $w_n w_{n+1} \ldots w_m$ of $\underline{w}$ there are $i \in \mathcal{A}$ and $k \in \mathbb{N}$ such that $w_n w_{n+1} \ldots w_m$ is a subword of $\phi^k(i)$. Equipped with the product topology W_ϕ is compact and, if ϕ is primitive, the *shift*

$$\sigma = \sigma_\phi : W_\phi \to W_\phi,$$

given by

$$\sigma(\ldots w_{-1}.w_0 w_1 \ldots) = (\ldots w_{-1} w_0.w_1 \ldots),$$

is a minimal homeomorphism. The substitution ϕ is called *aperiodic* in the case where W_ϕ consists of more than just a single shift-periodic orbit. If ϕ is primitive and aperiodic then W_ϕ is a Cantor set.

The *tiling space*, $\mathcal{T}_\phi$, associated with the (primitive and aperiodic) substitution ϕ is a collection of tilings of the real line by tiles corresponding to the letters of the alphabet $\mathcal{A}$, the tilings following patterns specified by the words in W_ϕ. More precisely, let

$$\vec{v}_l(\phi) = (v_l^1, \ldots, v_l^d)$$

be the left Perron–Frobenius eigenvector for $A(\phi)$ and let P_i be the *prototile* $P_i := [0, v_l^i]$. If for some $i \neq j$ it happens that $v_l^i = v_l^j$, we consider P_i and P_j to be 'marked' so that these prototiles are distinguishable. Now given $\underline{w} \in W_\phi$ we form the tiling

$$T = T(\underline{w}) = \{T_i\}_{i=-\infty}^{\infty}$$

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with

$$\begin{aligned} T_0 &= P_{w_0}, \\ T_1 &= P_{w_1} + |T_0|, \\ T_2 &= P_{w_2} + |T_0| + |T_1|, \\ &\dots, \\ T_{-1} &= P_{w_{-1}} - |P_{w_{-1}}|, \\ T_{-2} &= P_{w_{-2}} - |P_{w_{-2}}| - |T_{-1}|, \\ &\dots. \end{aligned}$$

Given such a tiling T, its *translate*

$$T - t = \{T_i - t\}_{i=-\infty}^{\infty}$$

for each $t \in \mathbb{R}$ is also a tiling of $\mathbb{R}$ following the pattern $\underline{w}$ (with the location of the origin changed). The *tiling space* is

$$\mathcal{T}_\phi := \{T(\underline{w}) - t \mid \underline{w} \in W_\phi, t \in \mathbb{R}\}$$

with the metric topology: T and T' are close provided there is a small $|t|$ so that $T - t$ and T' are identical tilings in a large neighborhood of the origin. (See, for example, [**AP**] for a more careful definition of the metric.)

If ϕ is primitive and aperiodic, $\mathcal{T}_\phi$ is a *continuum* (a compact, connected metric space) that is everywhere locally homeomorphic with the product of a Cantor set and an arc. For such ϕ, the *flow* on $\mathcal{T}_\phi$ defined by $T \mapsto T - t$ is *minimal* (every orbit is dense). There is also an *inflation and substitution* homeomorphism

$$\Phi : \mathcal{T}_\phi \to \mathcal{T}_\phi$$

defined as follows: for

$$T = \{T_i\}_{i=-\infty}^{\infty} \in \mathcal{T}_\phi,$$

form $\Phi(T)$ by multiplying (inflating) each tiled interval $T_i = [a_i, b_i]$ by the Perron–Frobenius eigenvalue λ to get the interval $[\lambda a_i, \lambda b_i]$. Now, if T_i is of type j (that is, $T_i = P_j + a_i = [0, v_l^j] + a_i$), tile the interval λT_i by tiles following the pattern of the word $\phi(j)$. That this determines a tiling of each λT_i, and hence of $\mathbb{R}$, follows from $\vec{v}_l A = \lambda \vec{v}_l$. The so defined Φ is indeed a homeomorphism that interacts with the flow by

$$\Phi(T - t) = \Phi(T) - \lambda t.$$

Given a homeomorphism $f : X \to X$, the *suspension* of f is the space

$$\mathrm{susp}(f) := X \times [0, 1]/\sim$$

with $(x, 1) \sim (f(x), 0)$ and the quotient topology, together with the *suspension flow*,

$$f_t : X \times [0, 1]/\sim \to X \times [0, 1]/\sim$$

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by

$$f_t([(x, s)]) := [(f^n(x), s + t - n)]$$

with

$$n = \lfloor s + t \rfloor = \text{greatest integer} \leq s + t.$$

The homeomorphisms $f : X \to X$ and $g : Y \to Y$ are *flow equivalent* provided the suspension flows f_t and g_t are equivalent flows (that is, there is a homeomorphism $h : X \times [0, 1]/\sim \to Y \times [0, 1]/\sim$ that takes flow orbits to flow orbits, preserving the positive direction).

If ϕ and ψ are primitive and aperiodic substitutions, the arc components of $\mathcal{T}_\phi$ and $\mathcal{T}_\psi$ have natural orientations induced by the flows. A homeomorphism h from $\mathcal{T}_\phi$ to $\mathcal{T}_\psi$ is then either orientation-preserving or orientation-reversing (this dichotomy is guaranteed by the minimality of the flows). It is not difficult to see that the suspension $\text{susp}(\sigma_\phi)$ of the shift $\sigma_\phi : W_\phi \to W_\phi$ is orientation-preserving homeomorphic with the tiling space $\mathcal{T}_\phi$. Thus, the shifts σ_ϕ and σ_ψ are flow equivalent if and only if the tiling spaces $\mathcal{T}_\phi$ and $\mathcal{T}_\psi$ are orientation-preserving homeomorphic.

Given a substitution ϕ, the *reverse substitution* $\bar{\phi}$ is defined by

$$\bar{\phi}(i) = i_n \ldots i_1,$$

provided

$$\phi(i) = i_1 \ldots i_n.$$

The map

$$T = \{T_i\}_{i=-\infty}^{\infty} \mapsto -T = \{-T_{-i}\}_{i=-\infty}^{\infty}$$

defines an orientation-reversing homeomorphism from $\mathcal{T}_\phi$ to $\mathcal{T}_{\bar{\phi}}$. Thus, the tiling spaces $\mathcal{T}_\phi$ and $\mathcal{T}_\psi$ are orientation-reversing homeomorphic if and only if the shifts σ_ϕ and $\sigma_{\bar{\psi}}$ are flow equivalent.

Our objective of finding flow equivalence invariants for the shift σ_ϕ is now transferred to the problem of finding (orientation-preserving) topological invariants of the tiling space $\mathcal{T}_\phi$. The advantage offered by $\mathcal{T}_\phi$ over $\text{susp}(\sigma_\phi)$ lies in geometry. In particular, the inflation and substitution homeomorphism Φ will lead to presentations of $\mathcal{T}_\phi$ and related spaces as inverse limits with single bonding maps that can, in turn, be used to compute various cohomology groups.

For a primitive and aperiodic $\phi : \mathcal{A} = \{1, \ldots, d\} \to \mathcal{A}^\star$ we now construct the *rose* $\mathcal{R}_\phi$ (Figure 1) and the *rose map* $f_\phi : \mathcal{R}_\phi \to \mathcal{R}_\phi$. Topologically, $\mathcal{R}_\phi$ is just a wedge of d circles:

$$\mathcal{R}_\phi := \left(\bigcup_{i=1}^{d} ([0, v_l^i] \times \{i\}) \right) / \sim$$

with petals

$$P_i := ([0, v_l^i] \times \{i\}) / \sim,$$

where $\sim$ identifies all the points $(0, i)$, (v_l^i, i), $i = 1 \ldots d$, into a single center point c ($\vec{v}_l = (v_l^1, \ldots, v_l^d)$ is the left Perron–Frobenius eigenvector of ϕ).

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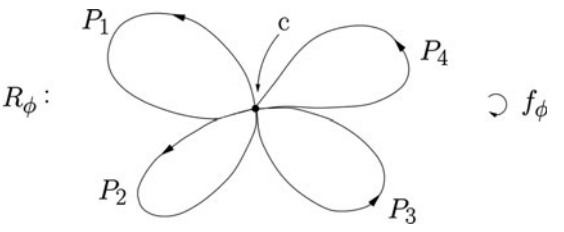


FIGURE 1. The rose R_ϕ.

The petals are naturally oriented and $f_\phi : \mathcal{R}_\phi \to \mathcal{R}_\phi$ is defined by $f_\phi(c) = c$ and $f_\phi|_{P_i}$ wraps the petal P_i around the petals P_{i_1}, then P_{i_2}, then $\ldots$, then P_{i_n}, where $\phi(i) = i_1, \ldots, i_n$; f_ϕ is uniquely specified by also requiring that it locally stretch arc length along petals by the constant factor $\lambda = \lambda_\phi$ and that f_ϕ locally preserves orientation. (Note that we are using P_i to stand for both a petal of the rose and the prototile from which the petal is built.)

Now let $P : \mathcal{T}_\phi \to \mathcal{R}_\phi$ be defined by $P(T) = [(t, j)] \in \mathcal{R}_\phi$ provided $T = \{T_i\}_{i=-\infty}^{\infty}$ with $0 \in T_0 = P_j - t$ (recall that $P_j = [0, v_l^j]$ is the prototile corresponding to letter j). It is clear that P is continuous and that $P \circ \Phi = f_\phi \circ P$.

If $f : X \to X$ is a map of the space X, the *inverse limit space* associated with f is the collection of sequences

$$\varprojlim f := \{(x_0, x_1, \ldots) \mid x_n \in X,\ f(x_{n+1}) = x_n,\ n = 0, 1, \ldots\}$$

endowed with the product topology. The *shift homeomorphism*

$$\hat{f} : \varprojlim f \to \varprojlim f$$

is given by

$$\hat{f}((x_0, x_1, \ldots)) := (f(x_0), x_0, x_1, \ldots).$$

The property $P \circ \Phi = f_\phi \circ P$ noted in the preceding paragraph allows us to define

$$\hat{P} : \mathcal{T}_\phi \to \varprojlim f_\phi$$

by

$$\hat{P}(T) := (P(T),\ P(\Phi^{-1}(T)),\ P(\Phi^{-2}(T)), \ldots).$$

The substitution ϕ is said to be *proper* if there are letters $b, e \in \mathcal{A}(\phi)$ and $n \in \mathbb{N}$ so that $\phi^n(i) = b \ldots e$ for each $i \in \mathcal{A}(\phi)$.

THEOREM 2.1. *If the substitution ϕ is primitive, aperiodic and proper, then $\hat{P} : \mathcal{T}_\phi \to \varprojlim f_\phi$ is a homeomorphism that conjugates the inflation and substitution homeomorphism with the shift homeomorphism ($\hat{f}_\phi \circ \hat{P} = \hat{P} \circ \Phi$).*

Proof. See [**AP**] and note that a proper substitution forces its border. □

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The tiling space $\mathcal{T}_\phi$ (for ϕ primitive and aperiodic), being orientable and locally the product of a Cantor set with an arc, looks much like a one-dimensional solenoid. A key difference is that $\mathcal{T}_\phi$ has a finite, non-zero, number of *asymptotic composants*: composants (arc components) $\mathcal{C}$ and $\mathcal{C}'$ are forward (backward) asymptotic provided there are $T \in \mathcal{C}$ and $T' \in \mathcal{C}'$ so that

$$d(T - t, T' - t) \to 0 \quad \text{as } t \to \infty \text{ (as } t \to -\infty).$$

(To say that $d(T - t, T' - t) \to 0$ as $t \to \infty$ is just to say that the tilings T and T' are identical on some interval $[a, \infty)$.)

THEOREM 2.2. *If the substitution ϕ on d letters is primitive and aperiodic, then the tiling space $\mathcal{T}_\phi$ has composants that are forward asymptotic to other composants and composants that are backward asymptotic to other composants. The total number of such composants is at most d^2.*

Proof. See [**BD**] for the first assertion and [**BDH**] for the second. □

THEOREM 2.3. *If ϕ and ψ are primitive and aperiodic substitutions and $h : \mathcal{T}_\phi \to \mathcal{T}_\psi$ is an orientation-preserving homeomorphism, then h takes forward (backward) asymptotic composants of $\mathcal{T}_\phi$ to forward (backward) asymptotic composants of $\mathcal{T}_\psi$. If h is orientation-reversing, it takes forward (backward) asymptotic composants to backward (forward) asymptotic composants.*

Proof. See [**BD**]. □

Thus, the 'asymptotic composant structure' of $\mathcal{T}_\phi$ is a topological invariant (and hence a flow invariant of σ_ϕ). We will use this structure to find a deeper invariant. Since $\Phi : \mathcal{T}_\phi \to \mathcal{T}_\phi$ takes asymptotic composants to asymptotic composants, and there are only finitely many asymptotic composants in $\mathcal{T}_\phi$, it must be the case that each asymptotic composant is periodic under Φ. From this, and the arc-length stretching property of Φ on composants, it follows that on each asymptotic composant there is a unique *asymptotic tiling* T that is periodic under Φ. Let $\{\mathcal{C}_1^F, \ldots, \mathcal{C}_{n_F(\phi)}^F\}$ and $\{\mathcal{C}_1^B, \ldots, \mathcal{C}_{n_B(\phi)}^B\}$ be the collections of forward and backward asymptotic composants in $\mathcal{T}_\phi$ (so, for each i, there is a j so that $\mathcal{C}_i^F$ is forward asymptotic to $\mathcal{C}_j^F$, etc) and let T_F^i, $i = 1, \ldots, n_F(\phi)$, and T_B^j, $j = 1, \ldots, n_B(\phi)$, be the Φ periodic tilings on the $\mathcal{C}_i^F$ and $\mathcal{C}_j^B$, respectively. (Note: it can happen that $\mathcal{C}_i^F = \mathcal{C}_j^B$ for some i, j so that $T_F^i = T_B^j$.) There is an $n \in \mathbb{N}$ so that $\Phi^n(T_F^i) = T_F^i$ and $\Phi^n(T_B^j) = T_B^j$ for all $i = 1, \ldots, n_F(\phi)$, $j = 1, \ldots, n_B(\phi)$ (a crude bound is $n \leq (d^2)!$). We say that a substitution ϕ is *prepared* if ϕ is primitive, aperiodic, proper, and $\Phi(T_F^i) = T_F^i$, $\Phi(T_B^j) = T_B^j$ for all the asymptotic tilings T_F^i, $T_B^j \in \mathcal{T}_\phi$.

PROPOSITION 2.4. *Given a primitive, aperiodic substitution ϕ_1, there is an algorithm that produces a prepared substitution ϕ_2 with σ_{ϕ_1} flow equivalent with σ_{ϕ_2} (so $\mathcal{T}_{\phi_1}$ is orientation-preserving homeomorphic with $\mathcal{T}_{\phi_2}$).*

Proof. See [**BD**] and [**CE3**]. □

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A homeomorphism from $\mathcal{T}_\phi$ to $\mathcal{T}_\psi$ must take asymptotic composants to asymptotic composants but it may not take asymptotic tilings to asymptotic tilings. This can be remedied by isotopy.

PROPOSITION 2.5. *If ϕ and ψ are primitive and aperiodic substitutions and $h : \mathcal{T}_\phi \to \mathcal{T}_\psi$ is an orientation-preserving homeomorphism, then there is an orientation-preserving homeomorphism h' isotopic to h so that $h'(T_F^i(\phi)) = T_F^i(\psi)$ and $h'(T_B^j(\phi)) = T_B^j(\psi)$ for $i = 1, \ldots, n_F(\phi) = n_F(\psi)$ and $j = 1, \ldots, n_B(\phi) = n_B(\psi)$ (perhaps after re-indexing if necessary).*

Proof. See [**BD**]. □

Suppose that $\Phi(T) = T$ and that T follows the pattern of the word $\underline{w} \in W_\phi$. There are then words

$$p, u, s \in \mathcal{A}^\star$$

with

$$|u| = \text{number of letters in } u = 1 \text{ or } 2,$$
$$\phi(u) = pus$$

and

$$\underline{w} = \ldots \phi^2(p)\phi(p)pus\phi(s)\phi^2(s) \ldots .$$

If ϕ is prepared then we may take $|u| = 1$ for all but one such T. Thus, if ϕ is prepared, and

$$\mathcal{E}_F^1, \ldots, \mathcal{E}_F^{m_F(\phi)}, \mathcal{E}_B^1, \ldots, \mathcal{E}_B^{m_B(\phi)}$$

are the equivalence classes of forward and backward asymptotic tilings (so T_F^i, $T_F^l \in \mathcal{E}_F^k$ if and only if $d(T_F^i - t, T_F^l - t) \to 0$ as $t \to \infty$, etc), we may select representatives

$$T_F^{i_k} \in \mathcal{E}_F^k, \quad k = 1, \ldots, m_F(\phi)$$

and

$$T_B^{j_k} \in \mathcal{E}_B^k, \quad k = 1, \ldots, m_B(\phi)$$

whose corresponding words $\underline{w}^{i_k}$, $\underline{w}^{j_k}$ satisfy

$$\underline{w}^{i_k} = \ldots \phi(p_{i_k})p_{i_k}u_{i_k}s_{i_k}\phi(s_{i_k}) \ldots ,$$
$$\underline{w}^{j_k} = \ldots \phi(p_{j_k})p_{j_k}u_{j_k}s_{j_k}\phi(s_{j_k}) \ldots ,$$

where $u_{i_k}, u_{j_k} \in \mathcal{A}$ are single letters with $\phi(u_{i_k}) = p_{i_k}u_{i_k}s_{i_k}$ and $\phi(u_{j_k}) = p_{j_k}u_{j_k}s_{j_k}$ for all k.

For the prepared substitution $\phi : \mathcal{A} = \{1, \ldots, d\} \to \mathcal{A}^\star$, we define the *augmented transition matrix* $\tilde{A}(\phi)$ by

$$\tilde{A}(\phi) := \left(\begin{array}{c|c} A(\phi) & E(\phi) \\ \hline 0 & I \end{array}\right),$$

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where $A(\phi)$ is the $d \times d$ transition matrix of ϕ, I is the $(m_F(\phi) + m_B(\phi) - 1) \times (m_F(\phi) + m_B(\phi) - 1)$ identity matrix ($m_F(\phi)$ and $m_B(\phi)$ are the numbers of forward and, respectively, backward equivalence classes of asymptotic tilings) and the lkth entry of $E(\phi)$ is

$$e_{lk} := \begin{cases} \text{number of occurrences of the letter } l \text{ in } p_{j_1} s_{i_k}, \\ \qquad k = 1, \ldots, m_F(\phi), \\ \text{number of occurrences of the letter } l \text{ in } p_{j_{k-m_F(\phi)}} s_{i_1}, \\ \qquad k = m_F(\phi) + 2, \ldots, m_F(\phi) + m_B(\phi). \end{cases}$$

Finally, if B is a non-negative $n \times n$ integer matrix with simple Perron–Frobenius eigenvalue, let

$$DG(B) := \{\vec{u} \in E(B) \mid B^k \vec{u} \in \mathbb{Z}^n \text{ for some } k \in \mathbb{N}\},$$

where

$$E(B) = B^n(\mathbb{Q}^n) \subset \mathbb{Q}^n$$

is the eventual range of

$$B : \mathbb{Q}^n \to \mathbb{Q}^n,$$

and let

$$DG^+(B) := \{\vec{u} \in DG(B) \mid \langle \vec{v} | \vec{v}_B^l \rangle \geq 0\},$$

where $\vec{v}_B^l$ is a non-negative left Perron–Frobenius eigenvector of B. The pair $(DG(B), DG^+(B))$ is called the (ordered) *dimension group* of B. In the case where ϕ is a prepared substitution with augmented transition matrix $\tilde{A} = \tilde{A}(\phi)$, the dimension group of the transpose, $(DG(\tilde{A}^t), DG^+(\tilde{A}^t))$, will be called the *augmented dimension group* of ϕ and denoted by $(DG(\phi), DG^+(\phi))$.

3. *The augmented tiling space and the augmented dimension group*

We construct an augmented tiling space whose ordered first Čech cohomology, it will turn out, is order isomorphic with the augmented dimension group defined in the previous section. This will lead to the main theorem of this paper: if two primitive and aperiodic substitutions have flow-equivalent shifts then the augmented dimension groups of corresponding prepared substitutions are order isomorphic.

Let $\phi : \{1, \ldots, d\} = \mathcal{A} \to \mathcal{A}^\star$ be a prepared (primitive, aperiodic and proper) substitution with tiling space $\mathcal{T}_\phi$, rose map $f_\phi : \mathcal{R}_\phi \to \mathcal{R}_\phi$, and representative asymptotic tilings $T_F^{i_1}, \ldots, T_F^{i_{m_F(\phi)}}, T_B^{j_1}, \ldots, T_B^{j_{m_B}(\phi)}$, as in §2. According to Theorem 2.1, the map $\hat{P} : \mathcal{T}_\phi \to \varprojlim f_\phi$ is a homeomorphism and it follows that the points $\bar{x}_F^{i_k} := P(T_F^{i_k}) \in \mathcal{R}_\phi$, $k = 1, \ldots, m_F(\phi)$, are all distinct and fixed by f_ϕ, as are the $\bar{x}_B^{j_k} := P(T_B^{j_k})$, $k = 1, \ldots, m_B(\phi)$. Each of these fixed points is interior to some petal (that is, is not the center c); if $u_{i_k} \in \mathcal{A}$ is the letter with the property that $\underline{w}^{i_k} = \ldots \phi(p_{i_k}) p_{i_k} u_{i_k} s_{i_k} \phi(s_{i_k}) \ldots$ is the word corresponding to $T_F^{i_k}$, then $\bar{x}_F^{i_k}$ is interior to petal $P_{u_{i_k}}$. Say $\bar{x}_F^{i_k} = [(x_F^{i_k}, u_{i_k})] \in [0, v_l^{u_{i_k}}] \times \{u_{i_k}\}/\sim = P_{u_{i_k}}$, $k = 1, \ldots, m_F(\phi)$. Similarly, define $\bar{x}_B^{j_k}$

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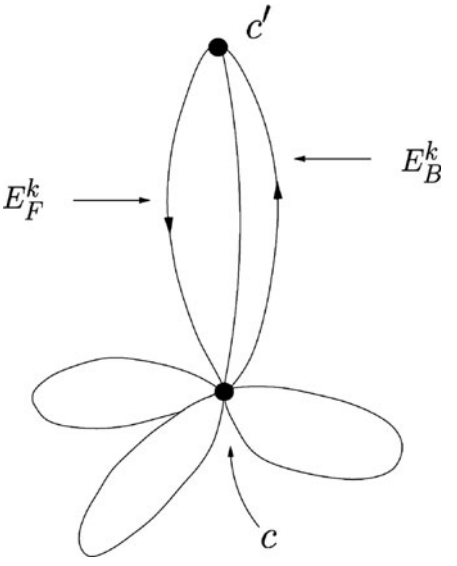


FIGURE 2. The space $\tilde{R}_\phi$.

by $\bar{x}_B^{j_k} = [(x_B^{j_k}, u_{j_k})] \in P_{u_{j_k}}$, $k = 1, \ldots, m_B(\phi)$. We augment the rose $\mathcal{R}_\phi$ by adding $m_F(\phi) + m_B(\phi)$ new edges and one new vertex. For each $k \in \{1, \ldots, m_F(\phi)\}$, attach an edge $E_F^k := [x_F^{i_k}, v_l^{u_{i_k}}] \times \{d + k\}$ to $\mathcal{R}_\phi$ with endpoint $(v_l^{u_{i_k}}, d + k)$ attached to c, the center of the rose, and for each $k \in \{1, \ldots, m_B(\phi)\}$, attach an edge $E_B^k := [0, x_B^{j_k}] \times \{d + m_F(\phi) + k\}$ to $\mathcal{R}_\phi$ with the endpoint $(0, d + m_F(\phi) + k)$ attached to c. Now identify all the remaining endpoints, $(x_F^{i_k}, d + k)$, $k = 1, \ldots, m_F(\phi)$, and $(x_B^{j_k}, d + m_F(\phi) + k)$, $k = 1, \ldots, m_B(\phi)$, to a new vertex c'. Denote the resulting space by $\tilde{\mathcal{R}}_\phi$ (see Figure 2).

Define $\tilde{f}_\phi : \tilde{\mathcal{R}}_\phi \to \tilde{\mathcal{R}}_\phi$, extending f_ϕ, by $\tilde{f}_\phi(c') = c'$, $\tilde{f}_\phi|_{E_F^k}$ maps E_F^k over E_F^k and then around $\mathcal{R}_\phi$ following the pattern of the suffix s_{i_k} of $\phi(u_{i_k})$, $k = 1, \ldots, m_F(\phi)$, and $\tilde{f}_\phi|_{E_B^k}$ maps E_B^k around $\mathcal{R}_\phi$ following the pattern of the prefix p_{j_k} of $\phi(u_{k_k})$, $k = 1, \ldots, m_B(\phi)$, and then over E_B^k. The above combinatorial description of $\tilde{f}_\phi$, together with the requirements that $\tilde{f}_\phi$ everywhere locally stretches arc length by a factor of $\lambda = \lambda(\phi)$ and preserves orientation, uniquely determines $\tilde{f}_\phi$. The *augmented tiling space* $\widetilde{\mathcal{T}}_\phi$ (Figure 3) is defined to be the inverse limit: $\widetilde{\mathcal{T}}_\phi := \varprojlim \tilde{f}_\phi$.

Note that $\varprojlim f_\phi$ includes into $\varprojlim \tilde{f}_\phi$ so that the tiling space $\mathcal{T}_\phi = (\hat{P})^{-1}(\varprojlim f_\phi) \simeq \varprojlim f_\phi$ sits in $\widetilde{\mathcal{T}}_\phi$ naturally as a subspace. The subset $\varprojlim \tilde{f}_\phi \setminus \varprojlim f_\phi$ of $\widetilde{\mathcal{T}}_\phi$ consists of $m_F(\phi) + m_B(\phi)$ rays joined at the point $(c', c', c', \ldots)$. The ray $\{(x_0, x_1, \ldots) \in \varprojlim \tilde{f}_\phi : x_n \in E_F^k$ for all $n \geq$ some $N\}$ winds densely onto $\varprojlim f_\phi \simeq \mathcal{T}_\phi$ in the forward direction and is asymptotic with all the composants that are forward asymptotic with the composant containing $T_F^{i_k}$, $k = 1, \ldots, m_F(\phi)$, while the ray $\{(x_0, x_1, \ldots) \in \varprojlim \tilde{f}_\phi : x_n \in E_B^k$ for all $n \geq$ some $N\}$ winds densely onto $\varprojlim f_\phi \simeq \mathcal{T}_\phi$ in the backward direction and is asymptotic with all the composants that are backward asymptotic with the composant containing $T_B^{j_k}$, $k = 1, \ldots, m_B(\phi)$.

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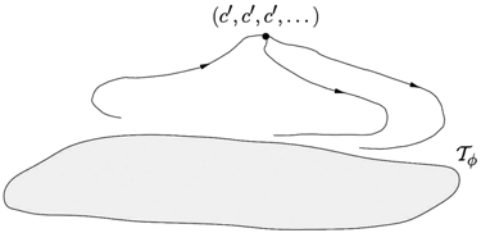


FIGURE 3. The space $\widetilde{\mathcal{T}}_\phi$.

THEOREM 3.1. *Suppose that ϕ and ψ are prepared substitutions. Then $\mathcal{T}_\phi$ is (orientation-preserving) homeomorphic with $\mathcal{T}_\psi$ if and only if $\widetilde{\mathcal{T}}_\phi$ is (orientation-preserving) homeomorphic with $\widetilde{\mathcal{T}}_\psi$.*

THEOREM 3.2. *Suppose that ϕ and ψ are prepared substitutions with transition matrices $A(\phi)$ and $A(\psi)$. An orientation-preserving homeomorphism from $\mathcal{T}_\phi$ to $\mathcal{T}_\psi$ induces an order isomorphism from $(DG((A(\psi))^{\mathrm{t}}), DG^+((A(\psi))^{\mathrm{t}}))$ onto $(DG((A(\phi))^{\mathrm{t}}), DG^+((A(\phi))^{\mathrm{t}})$.*

Remark 3.3. The meaning of the term *induces* in Theorem 3.2 (this will be important in the proof of Theorem 3.4) is as follows. Using the oriented petals as bases, the homomorphisms $(f_\phi)_* : H_1(\mathcal{R}_\phi) \to H_1(\mathcal{R}_\phi)$ and $(f_\psi)_* : H_1(\mathcal{R}_\psi) \to H_1(\mathcal{R}_\psi)$ are represented by $A(\phi)$ and $A(\psi)$, respectively. Dualizing, $(f_\phi)^* : H^1(\mathcal{R}_\phi) \to H^1(\mathcal{R}_\phi)$ and $(f_\psi)^* : H^1(\mathcal{R}_\psi) \to H^1(\mathcal{R}_\psi)$ are represented by $(A(\phi))^{\mathrm{t}}$ and $(A(\psi))^{\mathrm{t}}$ so that

$$\check{H}^1(\varprojlim f_\phi) \simeq \varinjlim (f_\phi)^* \simeq DG((A(\phi))^{\mathrm{t}})$$

and

$$\check{H}^1(\varprojlim f_\psi) \simeq \varinjlim (f_\psi)^* \simeq DG((A(\psi))^{\mathrm{t}}).$$

Also,

$$(\hat{P}(\phi))^* : \check{H}^1(\varprojlim f_\phi) \to \check{H}^1(\mathcal{T}_\phi) \quad \text{and} \quad (\hat{P}(\psi))^* : \check{H}^1(\varprojlim f_\psi) \to \check{H}^1(\mathcal{T}_\psi)$$

are isomorphisms. If $h : \mathcal{T}_\phi \to \mathcal{T}_\psi$ is a homeomorphism, the induced isomorphism from $DG((A(\psi))^{\mathrm{t}})$ to $DG((A(\phi))^{\mathrm{t}})$ is obtained from $h^* : \check{H}^1(\mathcal{T}_\psi) \to \check{H}^1(\mathcal{T}_\phi)$ by pre- and post-composition with the above isomorphisms.

THEOREM 3.4. *Suppose that ϕ_1 and ψ_1 are primitive and aperiodic substitutions whose shifts σ_{ϕ_1} and σ_{ψ_1} are flow equivalent, and suppose that ϕ and ψ are prepared from ϕ_1 and ψ_1. Then the augmented dimension groups $(DG(\phi), DG^+(\phi))$ and $(DG(\psi), DG^+(\psi))$ are order isomorphic.*

Remark 3.5. If $\mathcal{T}_{\phi_1}$ is homeomorphic with $\mathcal{T}_{\psi_1}$ by an orientation-reversing homeomorphism, then $\mathcal{T}_{\phi_1}$ and $\mathcal{T}_{\bar{\psi}_1}$ are orientation-preserving homeomorphic ($\bar{\psi}_1$ being the reverse of ψ_1). Then σ_{ϕ_1} and $\sigma_{\bar{\psi}_1}$ are flow equivalent and, since $\bar{\psi}$ is prepared for $\bar{\psi}_1$, Theorem 3.4 yields that $(DG(\phi), DG^+(\phi))$ and $(DG(\bar{\psi}), DG^+(\bar{\psi}))$ are order isomorphic. But the matrices $\tilde{A}(\bar{\psi})$ and $\tilde{A}(\psi)$ are conjugated by a permutation matrix

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so that $(DG(\bar{\psi}), DG^+(\bar{\psi}))$ is order isomorphic with $(DG(\psi), DG^+(\psi))$. Thus, if $\mathcal{T}_{\phi_1}$ is (order-preserving or order-reversing) homeomorphic with $\mathcal{T}_{\psi_1}$, the augmented dimension groups $(DG(\phi), DG^+(\phi))$ and $(DG(\psi), DG^+(\psi))$ must be order isomorphic. Thus, unlike the invariant of [**CE2**], the 'two-sided' augmentation we have presented is unable to detect reversal of orientation. We sketch below two 'one-sided' versions of the augmented dimension group that are sensitive to orientation. As we will see in §4, these carry the same information as the invariants of [**CE2**].

Let ϕ be a prepared substitution and suppose that (as in §2) representative tilings $T_F^{i_k}$, $k = 1, \ldots, m_F(\phi)$, and $T_B^{j_k}$, $k = 1, \ldots, m_B(\phi)$, have been selected whose words

$$\underline{w}^{i_k} = \ldots \phi(p_{i_k}) p_{i_k} u_{i_k} s_{i_k} \phi(s_{i_k}) \ldots$$

and

$$\underline{w}^{j_k} = \ldots \phi(p_{j_k}) p_{j_k} u_{j_k} s_{j_k} \phi(s_{j_k}) \ldots$$

satisfy $u_{i_k}, u_{j_k} \in \mathcal{A} = \{1, \ldots, d\}$ and $\phi(u_{i_k}) = p_{i_k} u_{i_k} s_{i_k}$, $\phi(u_{j_k}) = p_{j_k} u_{j_k} s_{j_k}$. Let

$$\tilde{A}_F(\phi) := \left(\begin{array}{c|c} A(\phi) & E_F(\phi) \\ \hline 0 & I \end{array}\right),$$

with I the $(m_F(\phi) - 1) \times (m_F(\phi) - 1)$ identity matrix and $E_F(\phi)$ the $d \times (m_F(\phi) - 1)$ matrix with lkth entry equal to the number of occurrences of the letter l in $p_{i_1} s_{i_{k+1}}$.

Let

$$\tilde{A}_B(\phi) := \left(\begin{array}{c|c} A(\phi) & E_B(\phi) \\ \hline 0 & I \end{array}\right),$$

with I the $(m_B(\phi) - 1) \times (m_B(\phi) - 1)$ identity matrix and $E_B(\phi)$ the $d \times (m_B(\phi) - 1)$ matrix with lkth entry equal to the number of occurrences of the letter l in $p_{j_{k+1}} s_{j_1}$.

Let

$$(DG_F(\phi), DG_F^+(\phi)) := (DG((\widetilde{A}_F(\phi))^{\mathrm{t}}), DG^+((\widetilde{A}_F(\phi))^{\mathrm{t}})) \quad \text{and}$$
$$(DG_B(\phi), DG_B^+(\phi)) := (DG((\widetilde{A}_B(\phi))^{\mathrm{t}}), DG^+((\widetilde{A}_B(\phi))^{\mathrm{t}})).$$

THEOREM 3.6. *Suppose that ϕ_1 and ψ_1 are primitive, aperiodic substitutions with σ_{ϕ_1} flow equivalent with σ_{ψ_1} and suppose that ϕ and ψ are prepared from ϕ_1 and ψ_1. Then $(DG_F(\phi), DG_F^+(\phi))$ is order isomorphic with $(DG_F(\psi), DG_F^+(\psi))$ and $(DG_B(\phi), DG_B^+(\phi))$ is order isomorphic with $(DG_B(\psi), DG_B^+(\psi))$.*

If $\bar{\psi}$ is the reverse of ψ, then $(DG_F(\bar{\psi}), DG_F^+(\bar{\psi})) \simeq (DG_B(\psi), DG_B^+(\psi))$.

COROLLARY 3.7. *Suppose that ϕ_1 and ψ_1 are primitive and aperiodic substitutions and that ϕ and ψ are prepared from ϕ_1 and ψ_1.*

(1) *If $\mathcal{T}_{\phi_1}$ is orientation-preserving homeomorphic with $\mathcal{T}_{\psi_1}$, then $(DG_F(\phi), DG_F^+(\phi)) \simeq (DG_F(\psi), DG_F^+(\psi))$ and $(DG_B(\phi), DG_B^+(\phi)) \simeq (DG_B(\psi), DG_B^+(\psi))$.*
(2) *If $\mathcal{T}_{\phi_1}$ is orientation-reversing homeomorphic with $\mathcal{T}_{\psi_1}$, then $(DG_F(\phi), DG_F^+(\phi)) \simeq (DG_B(\psi), DG_B^+(\psi))$ and $(DG_B(\phi), DG_B^+(\phi)) \simeq (DG_F(\psi), DG_F^+(\psi))$.*

An example in the next section will show that the two-sided groups may not be isomorphic even though both pairs of one-sided groups are isomorphic.

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Remark 3.8. (As in [**CE1**]) If $\widetilde{B}$ has the shape

$$\widetilde{B} = \left(\begin{array}{c|c} B_{d\times d} & 0 \\ \hline F_{m\times d} & I_{m\times m} \end{array}\right),$$

then the map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Q}^m & \longrightarrow & \mathbb{Q}^d \oplus \mathbb{Q}^m & \longrightarrow & \mathbb{Q}^d & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle \widetilde{B}} & & \downarrow{\scriptstyle B} & & \\ 0 & \longrightarrow & \mathbb{Q}^m & \longrightarrow & \mathbb{Q}^d \oplus \mathbb{Q}^m & \longrightarrow & \mathbb{Q}^d & \longrightarrow & 0 \end{array}$$

induces a short exact sequence of dimension group homomorphisms

$$0 \to \mathbb{Z}^m \to DG(\tilde{B}) \to DG(B) \to 0.$$

Thus, if B is invertible over the integers on its eventual range (that is, if the product of the non-zero eigenvalues of B is ± 1), then $DG(B)$ is free and the above sequence splits. It follows (in this case) that if

$$\widetilde{C} = \left(\begin{array}{c|c} C_{d'\times d'} & 0 \\ \hline E & I_{m\times m} \end{array}\right)$$

and $(DG(C), DG^+(C)) \simeq (DG(B), DG^+(B))$ then $(DG(\widetilde{C}), DG^+(\widetilde{C})) \simeq (DG(\widetilde{B}), DG^+(\widetilde{B}))$. That is, there is no additional information in the augmentation.

Proof of Theorem 3.1. Let $S_\phi = \varprojlim \tilde{f}_\phi \setminus \varprojlim f_\phi$ and $S_\psi = \varprojlim \tilde{f}_\psi \setminus \varprojlim f_\psi$. The subsets S_ϕ and S_ψ are connected and locally path connected. On the other hand, the subspaces $\varprojlim f_\phi$ and $\varprojlim f_\psi$ are nowhere locally path connected (being everywhere locally the product of a Cantor set with an arc; see for example [**BJV**, Lemma 1.10]). Thus, any homeomorphism from $\tilde{\mathcal{T}}_\phi$ to $\tilde{\mathcal{T}}_\psi$ must take $\varprojlim f_\phi \simeq \mathcal{T}_\phi$ homeomorphically onto $\varprojlim f_\psi \simeq \mathcal{T}_\psi$.

Conversely, if h is a homeomorphism from $\mathcal{T}_\phi$ to $\mathcal{T}_\psi$ (which we will assume to be orientation-preserving – the necessary adjustments in the other case will be obvious), then, according to Proposition 2.5, we may assume that h takes the collection of forward asymptotic tilings $\{T_F^1(\phi), \ldots, T_F^{n_F}(\phi)\}$ in $\mathcal{T}_\phi$ to the like collection $\{T_F^1(\psi), \ldots, T_F^{n_F}(\psi)\}$ in $\mathcal{T}_\psi$. Similarly, we may assume that h takes $\{T_B^1(\phi), \ldots, T_B^{n_B}(\phi)\}$ to $\{T_B^1(\psi), \ldots, T_B^{n_B}(\psi)\}$. Furthermore (Theorem 2.3), h must preserve the equivalence classes of the asymptotic tilings. Thus, after re-indexing if necessary,

$$d(h(T_F^{i_k}(\phi)) - t, T_F^{i_k}(\psi) - t) \to 0 \quad \text{as } t \to \infty$$

for $k = 1, \ldots, m_F$, and

$$d(h(T_B^{j_k}(\phi)) - t, T_B^{j_k}(\psi) - t) \to 0 \quad \text{as } t \to -\infty$$

for $k = 1, \ldots, m_B$. Let $\tilde{h} : \varprojlim f_\phi \to \varprojlim f_\psi$ be given by $\tilde{h} = \hat{P}_\psi \circ h \circ (\hat{P}_\phi)^{-1}$, $\hat{P}_\phi$ and $\hat{P}_\psi$ being the homeomorphisms of Theorem 2.1.

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We now extend $\tilde{h}$ to S_ϕ. Recall that S_ϕ and S_ψ each consist of $m_F + m_B$ rays joined at the point $(c', c', \ldots)$. We select one of those rays,

$$\text{ray}(\phi, F, k) := \{(x_0, x_1, \ldots) = \underline{x} \in \varprojlim \tilde{f}_\phi : x_n \in E_F^k(\phi) \text{ for all } n \geq \text{some } N\},$$

and map this ray onto the corresponding $\text{ray}(\psi, F, k)$ in $\varprojlim \tilde{f}_\psi$. All other rays are handled similarly. To define $\tilde{h}$ on $\text{ray}(\phi, F, k)$ we first define retractions $r_\phi : \mathcal{R}_\phi \cup E_F^k(\phi) \to \mathcal{R}_\phi$ and $r_\psi : \mathcal{R}_\psi \cup E_F^k(\psi) \to \mathcal{R}_\psi$. For r_ϕ, recall that $E_F^k(\phi) = [x_F^{i_k}, v_l^{n_{i_k}}] \times \{d + k\}$ (with $(v_l^{u_{i_k}}, d + k)$ attached to c), and let $r_\phi((x, d + k)) = [(x, u_{i_k})]$ in the petal $P_{u_{i_k}}$. Define r_ψ in like manner and note that, restricted to $\mathcal{R}_\phi \cup E_F^k(\phi)$ and $\mathcal{R}_\psi \cup E_F^k(\psi)$, respectively, $r_\phi \circ \tilde{f}_\phi = f_\phi \circ r_\phi$ and $r_\psi \circ \tilde{f}_\psi = f_\psi \circ r_\psi$. Thus, there are induced maps $\hat{r}_\phi : \varprojlim f_\phi \cup \text{ray}(\phi, F, k) \to \varprojlim f_\phi$ and $\hat{r}_\psi : \varprojlim f_\psi \cup \text{ray}(\psi, F, k) \to \varprojlim f_\psi$.

Now, since $h(T_F^{i_k}(\phi)) = T_F^l(\psi)$ for some l (with $T_F^l(\psi) \sim T_F^{i_k}(\psi)$), there is a homeomorphism $s : [0, \infty) \to [0, \infty)$ defined by $h(T_F^{i_k}(\phi) - t) = T_F^l(\psi) - s(t)$. (This is simply the statement that h takes arc components to arc components, preserving orientation.) For $\underline{x} \in \text{ray}(\phi, F, k)$,

$$\underline{x} = (\hat{r}_\phi|_{\text{ray}(\phi,F,k)})^{-1}(\hat{P}_\phi(T_F^{i_k}(\phi) - t)$$

for some $t \geq 0$ and we define

$$\tilde{h}(\underline{x}) = (\hat{r}_\psi|_{\text{ray}(\psi,F,k)})^{-1}(\hat{P}_\psi(T_F^{i_k}(\psi) - s(t))).$$

It is clear that $\tilde{h}$ takes $\text{ray}(\phi, F, k)$ homeomorphically onto $\text{ray}(\psi, F, k)$. We must also check that, if $\underline{y} \in \varprojlim f_\phi$ is close to $\underline{x} \in \text{ray}(\phi, F, k)$, then $\tilde{h}(\underline{y})$ is close to $\tilde{h}(\underline{x})$. Say $d(\underline{x}, \underline{y}) < \epsilon$. The closest point to $\underline{x}$ in $\varprojlim f_\phi$ is

$$\hat{r}_\phi(\underline{x}) = \hat{P}_\phi(T_F^{i_k}(\phi) - t).$$

Thus $d(\underline{x}, \hat{r}_\phi(\underline{x})) < \epsilon$ and $d(\hat{r}_\phi(\underline{x}), \underline{y}) < 2\epsilon$. Now $\tilde{h}(\hat{r}_\phi(\underline{x}))$ and $\tilde{h}(\underline{y})$ are close in $\varprojlim f_\psi$ since $\tilde{h}|_{\varprojlim f_\phi}$ is a homeomorphism. It remains to argue that $\tilde{h}(\underline{x})$ and $\tilde{h}(\hat{r}_\phi(\underline{x}))$ are close. For small ϵ, the t in $\hat{r}_\phi(\underline{x})$ is large so that $s = s(t)$ is also large. Consequently, the points $\tilde{h}(\underline{x}) \in \hat{r}_\psi^{-1}(\hat{P}_\psi(T_F^{i_k}(\psi) - s))$ and $\hat{P}_\psi(T_F^{i_k}(\psi) - s)$ are close. That is, for small ϵ, $\tilde{h}(\underline{x})$ and $\tilde{h}(\hat{r}_\phi(\underline{x}))$ are close. Thus $\tilde{h}$ is continuous. Symmetric arguments show that $\tilde{h}^{-1}$ is continuous. □

Proof of Theorem 3.2. Suppose that ϕ and ψ are prepared substitutions and that $h : \mathcal{T}_\phi \to \mathcal{T}_\psi$ is an orientation-preserving homeomorphism. The isomorphism $h^* : \check{H}^1(\mathcal{T}_\psi) \to \check{H}^1(\mathcal{T}_\phi)$ then induces an isomorphism $h_{DG}^* : DG((A(\psi))^t) \to DG((A(\phi))^t)$, as noted in Remark 3.3. Our task is to show that h_{DG}^* preserves non-negative parts. To accomplish this we will construct alternative presentations of the tiling spaces, in terms of which the orientation-preserving nature of h will have combinatorial consequences.

Let

$$\begin{aligned} X &:= \{\bar{x}_F^i \mid i = 1, \ldots, n_F(\phi)\} \cup \{\bar{x}_B^j \mid j = 1, \ldots, n_B(\phi)\} \\ &= P_\phi(\{T_F^i \mid i = 1, \ldots, n_F(\phi)\} \cup \{T_B^j \mid j = 1, \ldots, n_B(\phi)\}). \end{aligned}$$

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We will call a locally isometric, orientation-preserving immersion $\alpha : [0, l] \to \mathcal{R}_\phi$ *admissible* if $\alpha(0), \alpha(l) \in X$, $\alpha((0, l)) \cap X = \emptyset$, and there is an embedding $\beta : [0, l] \to \mathcal{R}_\phi$ so that $f_\phi^k \circ \beta = \alpha$ for some $k \in \mathbb{N}$ (note that β shrinks arc length by a factor of $1/\lambda^k$, $\lambda = \lambda(\phi)$). Let $\alpha_i : [0, l] \to \mathcal{R}_\phi$, $i = 1, \ldots, d_o$, be all distinct such admissible immersions. There is then a substitution $\phi_o : \mathcal{A}_o = \{1, \ldots, d_o\} \to \mathcal{A}_o^\star$ defined by $\phi_o(i) = i_1 \ldots i_{k(i)}$, where $f_\phi \circ \alpha_i$ is the concatenation of α_{i_1} through $\alpha_{i_{k(i)}}$. That is,

$$f_\phi \circ \alpha_i(t) = \begin{cases} \alpha_{i_1}(\lambda t), & 0 \le t < l_{i_1}/\lambda, \\ \alpha_{i_2}(\lambda t - l_{i_1}), & l_{i_1}/\lambda \le t < (l_{i_1} + l_{i_2})/\lambda, \\ \vdots \end{cases}$$

Let

$$\mathcal{R}_{\phi_0} := \left(\bigcup_{i=1}^{d_0} [0, l_i] \times \{i\} \right) / \sim$$

be the rose with branch point denoted c_o, and let $f_{\phi_o} : \mathcal{R}_{\phi_o} \to \mathcal{R}_{\phi_o}$ be the rose map associated with ϕ_o (ϕ_o corresponds to the symbolically defined $\tilde{\phi}$ in [**BD**]).

We now define a map $q_\phi^o : \mathcal{R}_\phi \to \mathcal{R}_{\phi_o}$. We would like to set $q_\phi^o(x) = [(t, i)]$ for t, i with $\alpha_i(t) = x$. But if $c \notin X$, there may be many such $[(t, i)]$. To eliminate this ambiguity, we will compose with an appropriate power of f_{ϕ_o}.

Since ϕ is proper, there are letters $b, e \in \mathcal{A}$ and an $n \in \mathbb{N}$ so that $\phi(b) = b \ldots$, $\phi(e) = \ldots e$, and $\phi^n(i) = b \ldots e$ for all $i \in \mathcal{A}$. Thus, for small ϵ, the $(2d)$-od consisting of the union of all the closed arcs of length ϵ/λ^n at the start and end of the petals in $\mathcal{R}_\phi$ is a closed neighborhood N of c with the property that $f_\phi^n(N) = [e(\epsilon), c] \cup [c, b(\epsilon)]$ is a closed arc. Here $[e(\epsilon), c]$ denotes the closed arc of length ϵ at the end of petal P_e and $[c, b(\epsilon)]$ denotes the closed arc of length ϵ at the beginning of petal P_b. Let $k \in \mathbb{N}$ be large enough so that $f_\phi^k([e(\epsilon), c]) = \mathcal{R}_\phi = f_\phi^k([c, b(\epsilon)])$ and define $e_1 \in [e(\epsilon), c]$, $b_1 \in [c, b(\epsilon)]$ by $f_\phi^k(e_1)$, $f_\phi^k(b_1) \in X$ and $f_\phi^k((e_1, c)) \cap X = \emptyset = f_\phi^k((c, b_1)) \cap X$. In each leg of N there is a unique point in $f_\phi^{-n}(\{e_1, b_1\})$ and these points are the endpoints of the $(2d)$-od N' ($N = f_\phi^n(N')$), a closed neighborhood of c with the property that $f_\phi^{n+k}(x) \in X$ if x is an endpoint of N' and $f_\phi^{n+k}(\mathring{N}') \cap (X \setminus \{c\}) = \emptyset$.

We claim that, if $x \in \mathcal{R}_\phi$ and $\alpha_i(t) = x = \alpha_j(s)$ for some $i, j \in \mathcal{A}_0$, $t \in [0, l_i]$, $s \in [0, l_j]$, then

$$f_{\phi_o}^{n+k}([(t, i)]) = f_{\phi_o}^{n+k}([(s, j)]).$$

This is clear in the case $f_\phi^{n+k}(x) \in X$ for then

$$f_{\phi_o}^{n+k}([(t, i)]) = c_o = f_{\phi_o}^{n+k}([(s, j)]).$$

Otherwise, let $\mathcal{U}$ be the arc component of $\mathcal{R}_\phi \setminus f_\phi^{-(n+k)}(X)$ that contains x. Either $\mathcal{U}$ is an open arc in the interior of a petal or $\mathcal{U} = \mathring{N}'$ (in which case $c \in X$). In the first case, if $\beta : [0, l] \to \bar{\mathcal{U}}$ is a positively oriented embedding that shrinks arc length by a factor of $1/\lambda^{n+k}$, then $l = l_r$ and $f_\phi^{n+k} \circ \beta = \alpha_r$ for some $r \in \mathcal{A}_o$. If $\mathcal{U} = \mathring{N}'$, then $f_\phi^n(\bar{\mathcal{U}}) = [e_1, c] \cup [c, b_1]$ and if $\beta : [0, l] \to [e_1, c] \cup [c, b_1]$ is the positively oriented embedding that shrinks arc length by a factor of $1/\lambda^k$, then $l = l_r$ and $f_\phi^k \circ \beta = \alpha_r$ for some $r \in \mathcal{A}_o$. In either case,

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$$f_{\phi_o}^{n+k}([(t,i)]) = f_{\phi_o}^{n+k}([(s,j)]) = [(t',r)],$$

with t' uniquely defined by

$$\alpha_r(t') = f_{\phi_o}^{n+k}(x).$$

Thus we may define the continuous $q_\phi^o : \mathcal{R}_\phi \to \mathcal{R}_{\phi_o}$ by

$$q_\phi^o(x) = f_{\phi_o}^{n+k}([(t,i)])$$

provided $\alpha_i(t) = x$. We have that

$$q_\phi^o \circ f_\phi = f_{\phi_o} \circ q_\phi^o$$

so that

$$\hat{q}_\phi^o : \varprojlim f_\phi \to \varprojlim f_{\phi_o}$$

is well defined.

Let $\mathcal{Q}_o(\phi)$ be the matrix representing $(q_\phi^o)_\star : H_1(\mathcal{R}_\phi) \to H_1(\mathcal{R}_{\phi_o})$ with respect to the ordered bases consisting of oriented petals. Then $\mathcal{Q}_o(\phi)$ is non-negative and $A(\phi_o)\mathcal{Q}_o(\phi) = \mathcal{Q}_o(\phi)A(\phi)$. Thus $(\mathcal{Q}_o(\phi))^{\mathrm{t}}$ induces a homomorphism from $DG((A(\phi_o))^{\mathrm{t}})$ to $DG((A(\phi))^{\mathrm{t}})$ and it is straightforward to show that this homomorphism takes $DG^+((A(\phi_o))^{\mathrm{t}})$ into $DG^+((A(\phi))^{\mathrm{t}})$. Indeed, if $\vec{v}_l^o$, $\vec{v}_r^o$ (respectively $\vec{v}_l$, $\vec{v}_r$) are positive left and right Perron–Frobenius eigenvectors (since ϕ is aperiodic, so is ϕ_o) for $(A(\phi_o))^{\mathrm{t}}$ (respectively $(A(\phi))^{\mathrm{t}}$) corresponding to the Perron–Frobenius eigenvalue $\lambda = \lambda(\phi_o) = \lambda(\phi)$, suppose that $\langle \vec{v}_l^o | \vec{w} \rangle \geq 0$ but that $\langle \vec{v}_l | (\mathcal{Q}_o(\phi))^{\mathrm{t}} \vec{w} \rangle < 0$. Write $\vec{w} = \vec{w}_1 + \vec{w}_2$ with $\vec{w}_1 \in (\vec{v}_l^o)^\perp$, $\vec{w}_2 = s\vec{v}_r^o$, $s \geq 0$, and write $(\mathcal{Q}_o(\phi))^{\mathrm{t}}\vec{w} = \vec{w}_1' + \vec{w}_2'$ with $\vec{w}_1' \in (\vec{v}_l)^\perp$ and $\vec{w}_2' = -s'\vec{v}_r$, $s' > 0$. Then

$$(\mathcal{Q}_o(\phi))^{\mathrm{t}}((A(\phi_o))^{\mathrm{t}})^k(\vec{w}_1 + \vec{w}_2) = (\mathcal{Q}_o(\phi))^{\mathrm{t}}((A(\phi_o))^{\mathrm{t}})^k\vec{w}_1 + s\lambda^k(\mathcal{Q}_o(\phi))^{\mathrm{t}}(\vec{v}_l^o),$$

whereas

$$((A(\phi))^{\mathrm{t}})^k(\mathcal{Q}_o(\phi))^{\mathrm{t}}(\vec{w}_1 + \vec{w}_2) = ((A(\phi))^{\mathrm{t}})^k\vec{w}_1' - s'\lambda^k\vec{v}_r.$$

Now $\|((A(\phi_o))^{\mathrm{t}})^k\vec{w}_1\| \leq c\eta^k\|\vec{w}_1\|$ and $\|((A(\phi))^{\mathrm{t}})^k\vec{w}_1'\| \leq c\eta^k\|\vec{w}_1'\|$ for some c, η with $0 < \eta < \lambda$, and we see that the equality

$$(\mathcal{Q}_o(\phi))^{\mathrm{t}}((A(\phi_o))^{\mathrm{t}})^k(\vec{w}_1 + \vec{w}_2) = ((A(\phi))^{\mathrm{t}})^k(\mathcal{Q}_o(\phi))^{\mathrm{t}}(\vec{w}_1 + \vec{w}_2)$$

must be violated for sufficiently large k (for $s \geq 0$, $s' > 0$).

Thus $(\mathcal{Q}_o(\phi))^{\mathrm{t}}$ respects non-negative parts, and we next argue that $(\mathcal{Q}_o(\phi))^{\mathrm{t}}$ induces a surjection from $DG((A(\phi_o))^{\mathrm{t}})$ onto $DG((A(\phi))^{\mathrm{t}})$.

Given a tiling $T = \{T_i\}_{i=-\infty}^{\infty} \in \mathcal{T}_\phi$ we define a tiling $T' \in \mathcal{T}_{\phi_o}$ as follows. The tile T_i corresponds to some petal P_j of $\mathcal{R}_\phi$; if T_i is of type j then $T_i = a + [0, v_j]$ for some $a \in \mathbb{R}$. If the element $[(t,j)]$ of P_j is in X, put a mark at the point $a + t \in \mathbb{R}$. The set of all such marks in $\mathbb{R}$ partitions $\mathbb{R}$ into a collection of translates of prototiles for ϕ_o. This collection defines the tiling $T' \in \mathcal{T}_{\phi_o}$. It is clear that this correspondence $T \mapsto T' := g_\phi(T)$ defines a homeomorphism $g_\phi : \mathcal{T}_\phi \to \mathcal{T}_{\phi_o}$. Furthermore,

$$\hat{q}_\phi^o = \hat{P}_{\phi_o} \circ g_\phi \circ (\hat{P}_\phi)^{-1} : \varprojlim f_\phi \to \varprojlim f_{\phi_o}.$$

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Although $\hat{P}_\phi$ is a homeomorphism (ϕ is proper), $\hat{P}_{\phi_o}$ is not. One easily checks that

$$\hat{P}_{\phi_o}^{-1}(\{(c_o, c_o, \ldots)\}) = \{g_\phi(T_F^i) \mid i = 1, \ldots, n_F(\phi)\} \cup \{g_\phi(T_B^j) \mid j = 1, \ldots, n_B(\phi)\} := \mathcal{S}_\phi$$

and that, for all other $\underline{x} \in \varprojlim f_{\phi_o}$, $\hat{P}_{\phi_o}^{-1}(\{\underline{x}\})$ is a singleton. Consideration of the exact sequence for the pair $(\mathcal{T}_{\phi_o}, \mathcal{S}_\phi)$ shows

$$(\hat{P}_{\phi_o})^\star : \check{H}^1(\varprojlim f_{\phi_o}) \to \check{H}^1(\mathcal{T}_\phi)$$

to be a surjection. Therefore,

$$(\hat{q}_\phi^o)^\star : \check{H}^1(\varprojlim f_{\phi_o}) \to \check{H}^1(\varprojlim f_\phi)$$

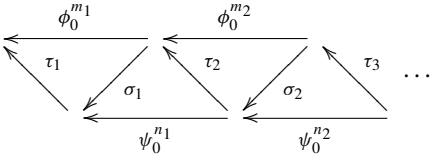
is also a surjection. Identifying $\check{H}^1(\varprojlim f_{\phi_o}) \simeq \varinjlim ((A(\phi_o))^t : \mathbb{Z}^{d_o} \to \mathbb{Z}^{d_o})$ with $DG((A(\phi_o))^t)$ and $\check{H}^1(\varprojlim f_\phi) \simeq \varinjlim ((A(\phi))^t : \mathbb{Z}^d \to \mathbb{Z}^d)$ with $DG((A(\phi))^t)$, $(\hat{q}_\phi^o)^\star$ is identified with the homomorphism induced by $(\mathcal{Q}_o(\phi))^t$.

Carrying out the same construction for ψ, we have an ordered surjection

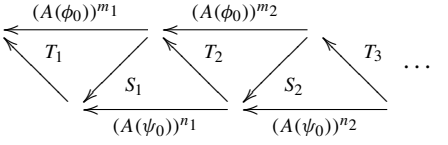
$$(\mathcal{Q}_o(\psi))^t : (DG((A(\psi_o))^t), DG^+((A(\psi_o))^t)) \to (DG((A(\psi))^t), DG^+((A(\psi))^t)).$$

The next step is to argue that the orientation-preserving homeomorphism $h : \mathcal{T}_\phi \to \mathcal{T}_\psi$ induces an isomorphism from $(DG((A(\psi_o))^t), DG^+((A(\psi_o))^t))$ to $(DG((A(\phi_o))^t), DG^+((A(\phi_o))^t))$. This amounts to a homological interpretation of a construction carried out in [**BJV**]. First, after an isotopy, we may assume that h takes the set $\mathcal{S}_\phi$ of asymptotic tilings in $\mathcal{T}_\phi$ to the set $\mathcal{S}_\psi$ of asymptotic tilings in $\mathcal{T}_\psi$ (Proposition 2.5). Now $\hat{P}_{\phi_o} \circ g_\phi : \mathcal{T}_\phi \to \varprojlim \phi_o$ and $\hat{P}_{\psi_o} \circ g_\psi : \mathcal{T}_\psi \to \varprojlim \psi_o$ are quotient maps that identify $\mathcal{S}_\phi$, respectively $\mathcal{S}_\psi$, to points, so h induces a homeomorphism $h_o : \varprojlim f_{\phi_o} \to \varprojlim f_{\psi_o}$ that takes the distinguished point $(c_o, c_o, \ldots) \in \varprojlim f_{\phi_o}$ to the identically labeled distinguished point in $\varprojlim f_{\psi_o}$.

According to [**BJV**, Theorem 1.16] there are sequences $m_i, n_i \in \mathbb{N}$ and morphisms $\sigma_i : \mathcal{A}_o(\phi_o) \to \mathcal{A}_o^\star(\psi_o)$, $\tau_i : \mathcal{A}_o(\psi_o) \to \mathcal{A}_o^\star(\phi_o)$ so that the diagram



commutes. A proof [**BJV**, proof of Lemma 1.15] shows that the homological role of h_o in this diagram is as follows: there are finite open covers $\mathcal{U}_i$ of $\mathcal{R}_{\phi_o}$ and $\mathcal{V}_i$ of $\mathcal{R}_{\psi_o}$ with nerves $\mathcal{N}(\mathcal{U}_i)$ and $\mathcal{N}(\mathcal{V}_i)$ isomorphic with simplicial subdivisions of $\mathcal{R}_{\phi_o}$ and $\mathcal{R}_{\psi_o}$ with the property that $f_{\phi_o}^{m_i}(\mathcal{U}_i)$, $f_{\psi_o}^{n_i}(\mathcal{V}_i)$, $\pi_{n_{i-1}} h_o \pi_{n_i}^{-1}(\mathcal{U}_i)$, and $\pi_{m_i} h_o^{-1} \pi_{n_i}^{-1}(\mathcal{V}_i)$ refine $\mathcal{U}_{i-1}$, $\mathcal{V}_{i-1}$, $\mathcal{V}_i$, and $\mathcal{U}_i$, respectively. Moreover, if ordered bases corresponding to the oriented petals of $\mathcal{R}_{\phi_o}$, $\mathcal{R}_{\psi_o}$ are chosen for $H_1(\mathcal{N}((U)))$ and $H_1(\mathcal{N}(\mathcal{V}_i))$, then the induced homomorphisms on the homologies of these nerves are represented by the matrices in the commuting diagram



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in which S_i is the incidence matrix for σ_i and T_i is the incidence matrix for τ_i. Dualizing, we have that the isomorphism

$$DG((A(\psi_o))^{\mathrm{t}}) \simeq \check{H}^1(\varprojlim \psi_o) \xrightarrow{h_o^\star} \check{H}^1(\varprojlim \phi_o) \simeq DG((A(\phi_o))^{\mathrm{t}})$$

is represented by

$$DG((A(\psi_o))^{\mathrm{t}}) \ni \vec{w} \longmapsto S_1^{\mathrm{t}}\vec{w} \in DG((A(\phi_o))^{\mathrm{t}}).$$

We claim that the isomorphism $\vec{w} \mapsto S_1^{\mathrm{t}}\vec{w}$ takes $DG^+((A(\psi_o))^{\mathrm{t}})$ onto $DG^+((A(\phi_o))^{\mathrm{t}})$. For suppose that $\langle \vec{v}_l(\psi_o)|\vec{w}\rangle \geq 0$, but that $\langle \vec{v}_l(\phi_o)|S_1^{\mathrm{t}}\vec{w}\rangle < 0$, where $\vec{v}_l(\psi_o)$ and $\vec{v}_l(\phi_o)$ are positive left Perron–Frobenius eigenvectors for $(A(\psi_o))^{\mathrm{t}}$ and $(A(\phi_o))^{\mathrm{t}}$, respectively. Then $S_1^{\mathrm{t}}\vec{w} = \vec{w}_1 - s\vec{v}_r(\phi_o)$ for some $\vec{w}_1 \in (\vec{v}_l(\phi_o))^{\perp}$, $s > 0$, and $\vec{v}_r(\phi_o)$ a positive right Perron–Frobenius eigenvector for $(A(\phi_o))^{\mathrm{t}}$. Thus, for sufficiently large k and

$$K = \sum_{i=2}^{k} m_i,$$
$$((A(\phi_o))^{\mathrm{t}})^K(S_1^{\mathrm{t}}(\vec{w})) = ((A(\phi_o))^{\mathrm{t}})^K\vec{w}_1 - s\lambda^K\vec{v}_r(\phi_o)$$

has all negative entries, since the Perron–Frobenius eigenvector $\lambda = \lambda(\phi_o)$ is larger in modulus than all eigenvalues of $(A(\phi_o))^{\mathrm{t}}$ on $(\vec{v}_l(\phi_o))^{\perp}$. Let

$$K' = \sum_{i=1}^{k} n_i.$$

Then

$$((A(\psi_o))^{\mathrm{t}})^{K'}(\vec{w}) = T_{k+1}^{\mathrm{t}}((A(\phi_o))^{\mathrm{t}})^K S_1^{\mathrm{t}}\vec{w}$$

also has all negative entries (T_{k+1}, being an incidence matrix, is non-negative). But this is not possible since $\vec{w} \in DG^+((A(\psi_o))^{\mathrm{t}})$ implies that

$$((A(\psi_o))^{\mathrm{t}})^{K'}\vec{w} \in DG^+((A(\psi_o))^{\mathrm{t}}).$$

We now have the commuting diagram

$$\begin{array}{ccc} DG((A(\psi))^{\mathrm{t}}) & \xleftarrow{(\mathcal{Q}_o(\psi))^{\mathrm{t}}} & DG((A(\psi_o))^{\mathrm{t}}) \\ \downarrow & & \downarrow \\ DG((A(\phi))^{\mathrm{t}}) & \xleftarrow{(\mathcal{Q}_o(\phi))^{\mathrm{t}}} & DG((A(\phi_o))^{\mathrm{t}}) \end{array}$$

in which $(\mathcal{Q}_o(\psi))^{\mathrm{t}}$ and $(\mathcal{Q}_o(\phi))^{\mathrm{t}}$ are surjective and take $DG^+((A(\psi_o))^{\mathrm{t}})$ and $DG^+((A(\phi_o))^{\mathrm{t}})$ into $DG^+((A(\psi))^{\mathrm{t}})$ and $DG^+((A(\phi))^{\mathrm{t}})$, the left and right vertical isomorphisms are induced by the homeomorphisms $\hat{P}_\psi \circ h \circ (\hat{P}_\phi)^{-1}$ and h_o, respectively, and the right vertical isomorphism takes $DG^+((A(\psi_o))^{\mathrm{t}})$ onto $DG^+((A(\phi_o))^{\mathrm{t}})$. It follows immediately that the left vertical isomorphism takes $DG^+((A(\psi))^{\mathrm{t}})$ onto $DG^+((A(\phi))^{\mathrm{t}})$, as desired. □

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Proof of Theorem 3.4. Flow equivalence of σ_{ϕ_1} and σ_{ψ_1} implies the existence of an orientation-preserving homeomorphism $h : \mathcal{T}_\phi \to \mathcal{T}_\psi$, ϕ and ψ prepared from ϕ_1 and ψ_1, which we may assume takes the collection of asymptotic tilings in $\mathcal{T}_\phi$ to the like collection in $\mathcal{T}_\psi$. By (the proof of) Theorem 3.1, $\hat{P}_\psi \circ h \circ \hat{P}_\phi^{-1}$ extends to an orientation-preserving homeomorphism $\tilde{h} : \widetilde{\mathcal{T}}_\phi \to \widetilde{\mathcal{T}}_\psi$.

If we choose the ordered basis for $H_1(\widetilde{\mathcal{R}}_\phi)$ consisting of the positively oriented petals and loops

$$\langle P_1, P_2, \ldots, P_d, E_B^1 E_F^1, E_B^1 E_F^2, \ldots, E_B^1 E_F^{m_F}, E_B^2 E_F^1, \ldots, E_B^{m_B} E_F^1 \rangle$$

and, for $H_1(\widetilde{\mathcal{R}}_\psi)$,

$$\langle P_1, \ldots, P_{d'}, E_B^1 E_F^1, \ldots, E_B^1 E_F^{m_F}, E_B^2 E_F^1, \ldots, E_B^{m_B} E_F^1 \rangle,$$

the homomorphisms $(\tilde{f}_\phi)_\star : H_1(\widetilde{\mathcal{R}}_\phi) \to H_1(\widetilde{\mathcal{R}}_\phi)$ and $(\tilde{f}_\psi)_\star : H_1(\widetilde{\mathcal{R}}_\psi) \to H_1(\widetilde{\mathcal{R}}_\psi)$ are represented by the augmented transition matrices $\widetilde{A}(\phi)$ and $\widetilde{A}(\psi)$, respectively. Note that d and d' may be different but that $m_F(\phi) = m_F(\psi)$ and $m_B(\phi) = m_B(\psi)$. Dualizing, we have

$$\begin{aligned} \check{H}^1(\widetilde{\mathcal{T}}_\phi) &\simeq \varinjlim (\tilde{f}_\phi)^\star \simeq \varinjlim ((\widetilde{A}(\phi))^{\mathrm{t}} : \mathbb{Z}^{d+m_F+m_B-1} \to \mathbb{Z}^{d+m_F+m_B-1}) \\ &\simeq DG((\widetilde{A}(\phi))^{\mathrm{t}}) = DG(\phi), \end{aligned}$$

and similarly

$$\check{H}^1(\widetilde{\mathcal{T}}_\psi) \simeq DG(\psi).$$

Choosing the bases dual to the ordered oriented petals for $H^1(\mathcal{R}_\phi)$ and $H^1(\mathcal{R}_\psi)$, the inclusion induced homomorphisms

$$i_\phi^\star : H^1(\widetilde{\mathcal{R}}_\phi) \to H^1(\mathcal{R}_\phi) \quad \text{and} \quad i_\psi^\star : H^1(\widetilde{\mathcal{R}}_\psi) \to H^1(\mathcal{R}_\psi)$$

are represented by

$$J_\phi = (I_{d\times d} \mid O) \quad \text{and} \quad J_\psi = (I_{d'\times d'} \mid O)$$

and these matrices in turn induce homomorphisms

$$J_\phi : DG((\widetilde{A}(\phi))^{\mathrm{t}}) = DG(\phi) \to DG((A(\phi))^{\mathrm{t}})$$

and

$$J_\psi : DG((\widetilde{A}(\psi))^{\mathrm{t}}) = DG(\psi) \to DG((A(\psi))^{\mathrm{t}}),$$

so that the diagram

$$\begin{array}{ccc} DG(\psi) & \xrightarrow{\widetilde{H}} & DG(\phi) \\ {\scriptstyle J_\psi}\downarrow & & \downarrow{\scriptstyle J_\phi} \\ DG((A(\psi))^{\mathrm{t}}) & \xrightarrow{H} & DG((A(\phi))^{\mathrm{t}}) \end{array}$$

commutes, H and $\widetilde{H}$ being induced by h and $\tilde{h}$. Let

$$\vec{w} = \begin{pmatrix} \vec{w}_1 \\ \vec{w}_2 \end{pmatrix} \in DG^+(\psi), \quad \vec{w}_1 \in \mathbb{Q}^{d'}.$$

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This means that $\langle \vec{w} | (\vec{v}_\psi^l, 0) \rangle \geq 0$ where $(\vec{v}_\psi^l, 0)$ is a non-negative left Perron–Frobenius eigenvector for

$$(\widetilde{A}(\psi))^{\mathrm{t}} = \begin{pmatrix} (A(\psi))^{\mathrm{t}} & 0 \\ (E(\psi))^{\mathrm{t}} & I \end{pmatrix}.$$

Thus $\langle \vec{w}_1 | \vec{v}_\psi^l \rangle \geq 0$ and $J_\psi(\vec{w}) = \vec{w}_1 \in DG^+((A(\psi))^{\mathrm{t}})$. From Theorem 3.2, $H(\vec{w}) \in DG^+((A(\phi))^{\mathrm{t}})$, so that $J_\phi \widetilde{H}(\vec{w}) \in DG^+((A(\phi))^{\mathrm{t}})$. Thus if

$$\widetilde{H}(\vec{w}) = \begin{pmatrix} \vec{w}_1' \\ \vec{w}_2' \end{pmatrix}, \quad \vec{w}_1' \in \mathbb{Q}^d,$$

then $\langle \vec{w}_1 | \vec{v}_\phi^l \rangle \geq 0$, $\vec{v}_\phi^l$ a positive left Perron–Frobenius eigenvector for $(A(\phi))^{\mathrm{t}}$. But then $(\vec{v}_\phi^l, 0)$ is a non-negative left Perron–Frobenius eigenvector for

$$(\widetilde{A}(\phi))^{\mathrm{t}} = \begin{pmatrix} (A(\phi))^{\mathrm{t}} & 0 \\ (E(\phi))^{\mathrm{t}} & I \end{pmatrix},$$

so that $\langle \widetilde{H}(\vec{w}) | (\vec{v}_\phi^l, 0) \rangle = \langle \vec{w}_1' | \vec{v}_\phi^l \rangle \geq 0$ and $\widetilde{H}$ is an order isomorphism. □

A proof of Theorem 3.6 can be obtained by mimicking that of Theorem 3.4: first construct forward and backward augmented tiling spaces; prove an analogue of Theorem 3.1; then use Theorem 3.2, as above. We omit the details, as Theorem 3.6 is essentially the result of [**CE2**], as we demonstrate in §5.

4. *Examples*

We will present examples of pairs of substitutions that give rise to non-isomorphic augmented dimension groups. The *crossing group* (see [**BS**]), which we introduce now, will greatly simplify computations.

Given non-negative integer matrices $A_{d\times d}$ and $E_{k\times d}$, let

$$(A, E) := \begin{pmatrix} A & 0 \\ E & I \end{pmatrix}.$$

In the case when $A - I$ is non-singular, define the crossing group of (A, E) by

$$\mathcal{CG}(A, E) := (E(A - I)^{-1}\mathbb{Z}^d + \mathbb{Z}^k)/\mathbb{Z}^k.$$

THEOREM 4.1. [**BS**, Theorem 5.10] *Suppose that* $(A - I)$ *and* $(B - I)$ *are nonsingular, that* $\det(A)$, $\det(B) \notin \{\pm 1\}$ *and that the characteristic polynomials of A, B are irreducible over* $\mathbb{Q}$. *Then* $(DG(A, E), DG^+(A, E)) \simeq (DG(B, F), DG^+(B, F))$ *implies that* $\mathcal{CG}(A, E) \simeq \mathcal{CG}(B, F)$.

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Example 4.2. This is the example of [**CE1**]:

$$\begin{aligned}\tau(a) &= accdadbb,\\ \tau(b) &= acdcbadb,\\ \tau(c) &= aacdcdbb,\\ \tau(d) &= accbdadb,\\ \nu(a) &= accbbadd,\\ \nu(b) &= accdbabd,\\ \nu(c) &= aacbbcdd,\\ \nu(d) &= acbcdabd.\end{aligned}$$

Then

$$A(\tau) = A(\nu) = \begin{pmatrix} 2 & 2 & 2 & 2\\ 2 & 2 & 2 & 2\\ 2 & 2 & 2 & 2\\ 2 & 2 & 2 & 2\end{pmatrix}.$$

Computing the forward augmented matrices for τ and ν (see the definition preceding Theorem 3.6) we obtain:

$$E_F(\tau) = \begin{pmatrix} 2\\ 1\\ 2\\ 2\end{pmatrix} \quad \text{and} \quad E_F(\nu) = \begin{pmatrix} 2\\ 2\\ 2\\ 1\end{pmatrix}.$$

Order-preserving shift equivalences (as in [**CE1**]) reduce $(A^{\mathrm{t}}(\tau),\, E_F^{\mathrm{t}}(\tau))$, respectively $(A^{\mathrm{t}}((\nu),\, E^{\mathrm{t}}(\nu))$, to

$$\begin{pmatrix} 8 & 0\\ 0 & 1\end{pmatrix}, \quad \text{respectively } \begin{pmatrix} 8 & 0\\ 1 & 1\end{pmatrix}.$$

Thus an order isomorphism between $DG(A^{\mathrm{t}}(\tau),\, E_F^{\mathrm{t}}(\tau))$ and $DG(A^{\mathrm{t}}(\nu),\, E_F^{\mathrm{t}}(\nu))$ would imply one between $DG((8, 0))$ and $DG((8, 1))$. The latter groups, forgetting order, are isomorphic. But $\mathcal{CG}(8, 0)$ is trivial while

$$\mathcal{CG}(8, 1) \simeq \mathbb{Z}/7\mathbb{Z}.$$

Thus, by Theorem 4.1, the forward augmented dimension groups for τ and ν are not order isomorphic. Thus, by Theorem 3.6 and Corollary 3.7, the shifts σ_τ and σ_ν are not flow equivalent and the tiling spaces $\mathcal{T}_\tau$ and $\mathcal{T}_\nu$ are not orientation-preserving homeomorphic. One can similarly show that $\mathcal{T}_\tau$ and $\mathcal{T}_\nu$ are also not orientation-reversing homeomorphic.

Example 4.3. Consider the example:

$$\begin{aligned}\phi(a) &= aaabbaa,\\ \phi(b) &= abbba,\\ \psi(a) &= aaababa,\\ \psi(b) &= abbba.\end{aligned}$$

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Each of ϕ and ψ gives rise to one pair of backward and one pair of forward asymptotic composants. Furthermore, for both ϕ and ψ, the pair of forward asymptotic composants is disjoint from the pair of backward asymptotic composants, so $\mathcal{T}_\phi$ and $\mathcal{T}_\psi$ cannot be distinguished on the basis of 'configuration data'. The (two-sided) augmented matrices are

$$\widetilde{A}(\phi) = \begin{pmatrix} 5 & 2 & 2 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \widetilde{A}(\psi) = \begin{pmatrix} 5 & 2 & 2 \\ 2 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

A simple computation gives

$$\mathcal{CG}\left(\begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix}, (2, 0)\right) \simeq \mathbb{Z}$$

and

$$\mathcal{CG}\left(\begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix}, (2, 1)\right) \simeq \mathbb{Z}/2\mathbb{Z}.$$

Thus the ordered dimension groups $DG(\widetilde{A}^{\mathrm{t}}(\phi))$, $DG(\widetilde{A}^{\mathrm{t}}(\psi))$ are not isomorphic (Theorem 4.1) so that σ_ϕ and σ_ψ are not flow equivalent by Theorem 3.4 and the tiling spaces $\mathcal{T}_\phi$ and $\mathcal{T}_\psi$ are not homeomorphic (Remark 3.5). For this example, the one-sided augmented dimension groups are just the dimension group of the original transition matrix and hence are incapable of distinguishing the flow equivalence classes of σ_ϕ and σ_ψ.

Example 4.4. Now consider the example:

$$\begin{aligned} \tau(a) &= dbdba^5d^6b^6c^3abd, \\ \tau(b) &= d^{11}b^{10}c^4a^{10}bdb^3d^4b^4ccaadd, \\ \tau(c) &= dbdba^4cd^6b^6caacbd, \\ \tau(d) &= d^{11}b^{10}c^4a^{10}d^5bd^{31}b^{34}c^{12}a^{24}d^5b^4ccaadd. \end{aligned}$$

Carlsen and Eilers [**CE4**] show that the Matsumoto K_0-groups associated with τ and its reverse, $\bar{\tau}$, are not order isomorphic. Hence σ_τ and $\sigma_{\bar{\tau}} = \sigma_\tau^{-1}$ are not flow equivalent and consequently there is no orientation-reversing automorphism of the tiling space $\mathcal{T}_\tau$ (both the forward and backward groups of §3 would show the same thing). The two-sided groups, $(DG(\tau), DG^+(\tau))$ and $(DG(\tilde{\tau}), DG^+(\tilde{\tau}))$, associated with τ and $\bar{\tau}$ are, however, order isomorphic (see Remark 3.5).

5. *The relationship between $K_0(X_\phi)$ and $DG(\widetilde{A}_F(\phi))$*

In this section we compare the one-sided group $DG(\widetilde{A}_F(\phi))$ (defined in §3) with the Matsumoto K_0-group associated with the shift space X_ϕ. We begin with a definition of $K_0(X_\phi)$ (following [**CE1**]).

Suppose that ϕ is a prepared substitution. We denote by X_ϕ, X_ϕ^+, X_ϕ^- the spaces of allowed bi-infinite, one-sided right-infinite and one-sided left-infinite words, respectively, with shifts

$$\begin{aligned} \sigma(\ldots x_{-1}.x_0x_1\ldots) &= \ldots x_{-1}x_0.x_1\ldots, \\ \sigma^+(x_0x_1x_2\ldots) &= x_1x_2\ldots, \end{aligned}$$

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and

$$\sigma^{-}(\ldots x_{-3}x_{-2}x_{-1}) = \ldots x_{-3}x_{-2}.$$

For $u \in \mathcal{A}^\star = \mathcal{A}^\star(\phi)$, let $|u|$ denote the number of letters in u, and, for each $\underline{x} \in X_\phi^+$, $n \in \mathbb{N}$, let $p_n(\underline{x}) = \{u : u \in \mathcal{A}^\star, |u| = n, u\underline{x} \in X_\phi^+\}$. Then $\underline{x}, \underline{y} \in X_\phi^+$ are *n-past-equivalent*, $\underline{x} \underset{n}{\sim} \underline{y}$, provided $p_n(\underline{x}) = p_n(\underline{y})$. Let $e^n(\underline{x}) = \{\underline{y} \mid \underline{y} \underset{n}{\sim} \underline{x}\}$ denote the n-past-equivalence class of $\underline{x}$. We occasionally write $p_n(e)$ for $p_n(\underline{x})$ in the case $e = e^n(\underline{x})$.

Let $E^n := \{e^n(\underline{x}) \mid \underline{x} \in \mathcal{X}_\phi^+\}$ be the collection of all n-past-equivalence classes and let $F_n \simeq \mathbb{Z}^{k_n}$, $k_n = \#E^n$, be the free abelian group with generators E^n. Let $i_n : F_n \to F_{n+1}$ be the homomorphism defined on generators by

$$i_n(e^n(\underline{x})) = \sum_{\substack{e \in E^{n+1} \\ e \subset e^n(\underline{x})}} e.$$

For each $a \in \mathcal{A}$, define the homomorphism $l_n^a : F_n \to F_{n+1}$ by

$$l_n^a(e^n(\underline{x})) = \sum_{\substack{e \in E^{n+1} \\ \emptyset \neq ae \subset e^n(\underline{x})}} e.$$

Let $l_n = \sum_{a \in \mathcal{A}} l_n^a$. (In the definition of l_n^a, $ae := \{a\underline{y} \mid \underline{y} \in e, a\underline{y} \in X_\phi^+\}$.) Then $i_{n+1} \circ l_n = l_{n+1} \circ i_n$ so that the l_n induce a homomorphism

$$\hat{l} : \mathbb{Z}_\phi \to \mathbb{Z}_\phi, \quad \mathbb{Z}_\phi := \varinjlim i_n.$$

The Matsumoto K_0-group is defined as follows.

Definition 5.1. $K_0(X_\phi) := \mathbb{Z}_\phi/(\mathrm{Id} - \hat{l})\mathbb{Z}_\phi$.

This article was motivated by our desire to understand the topological content of the above definition and, in particular, to understand why $K_0(X_\phi)$ is an invariant of the topological type of the tiling space $\mathcal{T}_\phi$. We will gain this understanding below by showing that $K_0(X_\phi)$ is isomorphic with a certain natural quotient of a trivial extension of the first Čech cohomology of a one-sided augmentation $\widetilde{\mathcal{T}}_\phi^F$ of the tiling space $\mathcal{T}_\phi$.

An element $\underline{x} \in X_\phi^+$ is *left-special* if there are $a, b \in \mathcal{A}$, $a \neq b$, with $a\underline{x}, b\underline{x} \in X_\phi^+$. The words $\underline{x} = x_0x_1 \ldots$ and $\underline{y} = y_0y_1 \ldots$ are *right-tail-equivalent* if there is a k so that $x_n = y_{n+k}$ for all sufficiently large n.

LEMMA 5.2. *Given a prepared substitution ϕ there is a prepared substitution τ with $\mathcal{T}_\phi$ and $\mathcal{T}_\tau$ orientation-preserving homeomorphic and so that each right-tail-equivalence class in X_τ^+ contains at most one left-special element.*

Proof. It is proved in [**BD**, Theorem 3.16, with 'suffix' replacing 'prefix'], that if ϕ is primitive, aperiodic and proper, then there is a primitive, aperiodic and proper substitution τ that is shift equivalent to ϕ and so that for no $i \neq j \in \mathcal{A}(\tau)$ is $\tau(i)$ a suffix of $\tau(j)$. For such τ, the inflation and substitution homeomorphism $\widetilde{\tau}$ on $\mathcal{T}_\tau$ is (orientation-preserving) conjugate with Φ on $\mathcal{T}_\phi$ so that $\mathcal{T}_\tau$ is orientation-preserving homeomorphic with $\mathcal{T}_\phi$ and

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the asymptotic tilings are fixed by $\widetilde{\tau}$ if the asymptotic tilings are fixed by Φ. That is, τ is prepared if ϕ is.

Now suppose that for this τ there are elements $\underline{x} = x_0x_1x_2\ldots$ and $\underline{y} = x_nx_{n+1}\ldots \in X_\tau^+$ that are both left-special, with $n \geq 1$. There are then tilings $T, T', T'' \in \mathcal{T}_\tau$, fixed by inflation and substitution and forward asymptotic with right tails following the pattern of words $a\underline{x}$, $b\underline{x}$, and $c\underline{y}$ with $a = b$ and $c \neq x_{n-1}$. It follows from [**BD**, Proposition 3.6] that there are distinct letters i, j, k; words (perhaps empty) p_i, p_j, p_k; and a non-empty word s so that $\tau(i) = p_iis$, $\tau(j) = p_jjs$, $\tau(k) = p_kks$ and so that the right-infinite words $p_is\tau(s)\tau^2(s)\ldots$, $p_js\tau(s)\tau^2(s)\ldots$, $p_ks\tau(s)\tau^2(s)\ldots$, describe the right tails of T, T', T''. Let $\underline{z} := s\tau(s)\tau^2(s)\ldots$. We have that one of $a\underline{x}$ and $i\underline{z}$, and one of $b\underline{x}$ and $j\underline{z}$, is a shift of the other (where $i \neq j$, $a \neq b$). There are several cases, all of which conclude with $i = a$, $j = b$, and $\underline{x} = \underline{z}$ or some non-trivial shift of $\underline{x}$ equals $\underline{x}$. If the latter happens, $\underline{x}$ is shift-periodic, contrary to the aperiodicity of τ. Thus $\underline{x} = \underline{z}$ and one of $k\underline{z} = k\underline{x} = kx_0x_1\ldots$ and $c\underline{y} = cx_nx_{n+1}\ldots$, with $c \neq x_{n-1}$, is a shift of the other. But this again would imply that $\underline{x}$ is a non-trivial shift of itself, contradicting aperiodicity of τ. Thus any right-tail-equivalence class of X_τ^+ contains at most one left-special element. □

For ϕ and τ as in Lemma 5.2, $DG_F(\phi) \simeq DG_F(\tau)$ (Theorem 3.6) and $K_0(X_\phi) \simeq K_0(X_\tau)$ (see [**M**]). Thus, for the purpose of comparing $DG_F(\phi)$ with $K_0(X_\phi)$ we may make the following assumption.

Standing assumption 5.3. If $\underline{x} \in X_\phi^+$ is left-special then $\#\{a \mid a\underline{x} \in X_\phi^+\} = \#\{\underline{y} \in X_\phi^+ \mid (\sigma^+)^n\underline{y} = \underline{x}\}$ for all $n \geq 1$.

Another way of saying the above assumption is: if $\underline{x} \in X_\phi^+$ is left-special and $u \in \mathcal{A}^\star$ is such that $u\underline{x} \in X_\phi^+$, then $u\underline{x}$ is not left-special.

For each $\underline{x} \in X_\phi^+$, fix, once and for all, $a = a(\underline{x}) \in \mathcal{A}$ so that $a\underline{x} \in X_\phi^+$. We define three sequences of topological graphs. For $n = 2, 3, \ldots$, let $\widetilde{G}_n$ be the graph with edge set E^n and vertex set $\widetilde{V}_n := E^{n-1}$; e_t, $0 \leq t \leq 1$, will denote a parametrization of the edge $e = e^n(\underline{x}) \in E^n$ with initial vertex $e_0 := e^{n-1}(a\underline{x}) \in E^{n-1}$, $a = a(\underline{x})$, and with terminal vertex $e_1 := e^{n-1}(\underline{x}) \in E^{n-1}$. Let G_n be the subgraph of $\widetilde{G}_n$ consisting of those edges $e \in E^n$ with $\#p_n(e) = 1$ and with vertices

$$\mathcal{V}_n := \{e \in E^{n-1} = \widetilde{\mathcal{V}}_n \mid \#p_{n-1}(e) = 1\}.$$

Finally, for $n \geq 2$, let $\bar{G}_n$ be the graph $\widetilde{G}_n$ with additional edges as follows. By (5.3) the collection of left-special words, $LS = \{\underline{x}(1), \ldots, \underline{x}(m_F(\phi))\}$, is in one-to-one correspondence with the collection of equivalence classes of forward asymptotic composants of $\mathcal{T}_\phi$ (hence the $m_F(\phi)$). For each $i \in \{1, \ldots, m_F(\phi)\}$, let $\{b_{i,j} \mid j = 1, \ldots, k_i\}$ be the collection of distinct letters such that $b_{i,j}\underline{x}(i) \in X_\phi^+$ with $b_{i,1} = a(\underline{x}(i))$. For each $i \in \{1, \ldots, m_F(\phi)\}$ and $j \in \{2, \ldots, k_i\}$, let $e^{n,i,j}$ be an edge in $\bar{G}_n$ with initial vertex $e_0^{n,i,j} := e^{n-1}(b_{i,j}\underline{x}(i)) \in E^{n-1} = \widetilde{V}_n$ and terminal vertex $e_1^{n,i,j}$ denoted by $s^{n,i,j}$. Let $\bar{S}_n = \{s^{n,i,j} \mid i = 1, \ldots, m_F(\phi);\ j = 2, \ldots, k_i\}$. Thus $\bar{G}_n$ is the graph $\widetilde{G}_n$ with

$$\sum_{i=1}^{m_F(\phi)} (k_i - 1)$$

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'stickers' (the $e^{n,i,j}$) attached, whose terminal ends make up the collection $\bar{S}_n$ of 'special vertices'. The vertex set of $\bar{G}_n$ is $\bar{V}_n \equiv \widetilde{V}_n \cup \bar{S}_n$.

For $n \geq 2$, let $\bar{f}_n : \bar{G}_{n+1} \to \bar{G}_n$ be the continuous map defined by

$$\bar{f}_n(e_t^{n+1}(\underline{x})) = e_t^n(\underline{x}) \quad \text{and} \quad \bar{f}_n(e_t^{n+1,i,j}) = e_t^{n,i,j}.$$

Then $\bar{f}_n$ is a map of the triple,

$$\bar{f}_n : (\bar{G}_{n+1}, \bar{V}_{n+1}, \bar{S}_{n+1}) \to (\bar{G}_n, \bar{V}_n, \bar{S}_n),$$

and restricting $\bar{f}_n$ gives the maps of pairs

$$\tilde{f}_n : (\widetilde{G}_{n+1}, \widetilde{V}_{n+1}) \to (\widetilde{G}_n, \widetilde{V}_n) \quad \text{and} \quad f_n : (G_{n+1}, V_{n+1}) \to (G_n, V_n).$$

Identifying $H^1(\widetilde{G}_n, \widetilde{V}_n)$ with the dual of $H_1(\widetilde{G}_n, \widetilde{V}_n)$ and choosing the dual basis $\{e^\star \mid e \text{ is a positively oriented edge of } \widetilde{G}_n\}$, we see that

$$\tilde{f}_n^\star : H^1(\widetilde{G}_n, \widetilde{V}_n) \to H^1(\widetilde{G}_{n+1}, \widetilde{V}_{n+1})$$

is identified with $i_n : F_n \to F_{n+1}$. Let $\pi_n : H^1(\bar{G}_n, \bar{V}_n) \to H^1(\widetilde{G}_n, \widetilde{V}_n)$ be the homomorphism determined by

$$\pi_n(e^\star) = e^\star|_{H_1(\widetilde{G}_n, \widetilde{V}_n)}$$

for e an edge of $\widetilde{G}_n \subset \bar{G}_n$ and

$$\pi_n((e^{n,i,j})^\star) = (e^n(\underline{x}(i)))^\star$$

on the extra edges. Then

$$\pi_{n+1} \circ \bar{f}_n^\star = \tilde{f}_n^\star \circ \pi_n$$

and there is an induced homomorphism

$$\hat{\pi} : \varinjlim \bar{f}_n^\star =: H^1(\bar{G}, \bar{V}) \to \varinjlim \tilde{f}_n^\star =: H^1(\tilde{G}, \tilde{V}) \simeq \mathbb{Z}_\phi.$$

The short exact sequence

$$0 \longrightarrow H^0(\bar{V}_n, \bar{S}_n) \xrightarrow{\delta_n^\star} H^1(\bar{G}_n, \bar{V}_n) \xrightarrow{q_n^\star} H^1(\bar{G}_n, \bar{S}_n) \longrightarrow 0$$

gives rise to the short exact sequence of direct limits

$$0 \longrightarrow H^0(\bar{V}, \bar{S}) \xrightarrow{\hat{\delta}} H^1(\bar{G}, \bar{V}) \xrightarrow{\hat{q}} H^1(\bar{G}, \bar{S}) \longrightarrow 0. \tag{5.1}$$

(Here $H^0(\bar{V}, \bar{S})$ denotes $\varinjlim \bar{f}_n^\star : H^0(\bar{V}_n, \bar{S}_n) \to H^0(\bar{V}_{n+1}, \bar{S}_{n+1})$, etc.)

LEMMA 5.4. *We have* $H^1(\widetilde{G}, \widetilde{V})/\hat{\pi} \circ \hat{\delta}(H^0(\bar{V}, \bar{S})) \simeq K_0(X_\phi)$.

Proof of Lemma 5.4. Let $e \in \bar{V}_n \setminus \bar{S}_n$ and let $e^\star$ be its dual in $H^0(\bar{V}_n, \bar{S}_n)$. Then $\delta_n^\star(e^\star)$ is the sum of the duals of the edges in $\bar{G}_n$ that terminate in e minus the sum of the duals

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of the edges in $\bar{G}_n$ that initiate with e. If e is *not* of the form $e_0^{n-1,i,j} = e^{n-1}(b_{i,j}\underline{x}(i))$, $i \in \{1, \ldots, m_F(\phi)\}$, $j \in \{2, \ldots, k_i\}$, then

$$\pi_n(\delta_n^\star(e^\star)) = \pi_n\left(f_{n-1}^\star(e^\star) - \sum_{a\in\mathcal{A}} \sum_{\substack{e'\in E^n \\ \emptyset\neq ae'\subset e}} (e')^\star\right)$$
$$= f_{n-1}^\star(e^\star) - \sum_{a\in\mathcal{A}} \sum_{\substack{e'\in E^n \\ \emptyset\neq ae'\subset e}} (e')^\star \in H^1(\widetilde{G}_n, \widetilde{V}_n),$$

and this element of $H^1(\widetilde{G}_n, \widetilde{V}_n)$ coincides with $i_{n-1}(e) - l_{n-1}(e) \in F_n$ under the identification of $H^1(\widetilde{G}_n, \widetilde{V}_n)$ with F_n. If $e = e^{n-1}(b_{i,j}\underline{x}(i))$ for some $i \in \{1, \ldots, m_F(\phi)\}$, $j \in \{2, \ldots, k_i\}$, then

$$\pi_n(\delta_n^\star(e^\star)) = \pi_n\left(f_{n-1}^\star(e^\star) - \left(\sum_{a\in\mathcal{A}} \sum_{\substack{e'\in E^n \\ e'\neq e^n(\underline{x}(i)) \\ \emptyset\neq ae'\subset e}} (e')^\star\right) - (e^{n,i,j})^\star\right)$$
$$= f_{n-1}^\star(e^\star) - \sum_{a\in\mathcal{A}} \sum_{\substack{e'\in E^n \\ \emptyset\neq ae'\subset e}} (e')^\star,$$

since $\pi_n((e^{n,i,j})^\star) = (e^n(\underline{x}(i)))^\star$. So in this case also $\pi_n(\delta_n^\star(e^\star))$ coincides with $i_{n-1}(e) - l_{n-1}(e)$ in F_n. Thus, for $e \in \bar{V}_n \setminus \bar{S}_n = E^{n-1}$, $\hat{\pi} \circ \hat{\delta}([(e^\star, n-1)])$ in $H^1(\widetilde{G}, \widetilde{V})$ corresponds with

$$[(i_{n-1}(e) - l_{n-1}(e), n)] = [(i_{n-1}(e), n)] - [(l_{n-1}(e), n)] = [(e, n-1)] - \hat{l}([(e, n-1)])$$
$$= (\mathrm{Id} - \hat{l})([(e, n-1)])$$

in $\mathbb{Z}_\phi$. Thus

$$H^1(\widetilde{G}, \widetilde{V})/\hat{\pi} \circ \hat{\delta}(H^0(\bar{V}, \bar{S})) \simeq \mathbb{Z}_\phi/(\mathrm{Id} - \hat{l})\mathbb{Z}_\phi \simeq K_0(X_\phi). \qquad \square$$

As $\hat{\pi}$ is surjective, the exact sequence (5.1) together with Lemma 5.4 yields the following corollary.

COROLLARY 5.5. *We have* $K_0(X_\phi) \simeq H^1(\bar{G}, \bar{S})/\hat{q}(ker(\hat{\pi}))$.

Now $\bar{f}_n : \bar{S}_{n+1} \to \bar{S}_n$ is a bijection from the finite set of special vertices (exactly $\sum_{i=1}^{m_F(\phi)}(k_i - 1)$ in number) at level $n+1$ to those at level n. Thus, denoting by $\bar{S}_\phi$ the finite subset $\bar{S}_\phi := \varprojlim(\bar{f}_n : \bar{S}_{n+1} \to \bar{S}_n)$ of $\varprojlim(\bar{f}_n : \bar{G}_{n+1} \to \bar{G}_n) =: \bar{G}_\phi$ we see that the direct limit

$$H^1(\bar{G}, \bar{S}) := \varinjlim(\bar{f}_n^\star : H^1(\bar{G}_n, \bar{S}_n) \to H^1(\bar{G}_{n+1}, \bar{S}_{n+1}))$$

is isomorphic with $\check{H}^1(\bar{G}_\phi, \bar{S}_\phi)$ by continuity of the Čech theory. With the analogous definitions of inverse limit spaces $\bar{V}_\phi$, $\widetilde{G}_\phi$, and $\widetilde{V}_\phi$, so that $\hat{q} : \check{H}^1(\bar{G}_\phi, \bar{S}_\phi) \to \check{H}^1(\widetilde{G}_\phi, \widetilde{V}_\phi)$ and $\hat{\pi} : \check{H}^1(\bar{G}_n, \bar{V}_n) \to \check{H}^1(\widetilde{G}_\phi, \widetilde{V}_\phi)$, Corollary 5.5 becomes

$$K_0(X_\phi) \simeq \check{H}^1(\bar{G}_\phi, \bar{S}_\phi)/\hat{q}(\ker(\hat{\pi})). \tag{5.2}$$

We will now examine the space $\bar{G}_\phi$. Let G_ϕ denote the inverse limit

$$G_\phi := \varprojlim(f_n : G_{n+1} \to G_n) \subset \widetilde{G}_\phi \subset \bar{G}_\phi.$$

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Let

$$SX_\phi^- := (X_\phi^- \times [0, 1])/(\underline{x}, 1) \sim (\sigma^-(\underline{x}), 0)$$

be the suspension of the shift on the space of left-infinite allowable words X_ϕ^-.

PROPOSITION 5.6. *We have that G_ϕ is homeomorphic with SX_ϕ^-.*

Proof of Proposition 5.6. There is a one-to-one correspondence between the edges e of G_n and the allowable words of length n: $e \leftrightarrow w_1 \dots w_n$ if and only if $p_n(e) = \{w_1 \dots w_n\}$; it is convenient to denote the parametrization of e by $[w_1 \dots w_n]_t$, $0 \le t \le 1$. For $n = 2, 3, \dots$ let $\Psi_n : SX_\phi^- \to G_n$ be the continuous surjection defined by $\Psi_n([(\dots x_{-n} \dots x_{-1}, t)]) = [x_{-n} \dots x_{-1}]_{1-t}$. Then $f_n \circ \Psi_{n+1} = \Psi_n$ and there is an induced surjection $\hat{\Psi} : SX_\phi^- \to G_\phi$. If

$$\hat{\Psi}([(\dots x_{-n} \dots x_{-1}, t)]) = \hat{\Psi}([(\dots y_{-n} \dots y_{-1}, s)]),$$

then $[x_{-n} \dots x_{-1}]_{1-t} = [y_{-n} \dots y_{-1}]_{1-s}$ for all $n \ge 2$. Either $1 - t$ and $1 - s$ are both strictly between 0 and 1 or $t, s \in \{0, 1\}$. In the first case, $s = t$ and $\dots x_{-n} \dots x_{-1} = \dots y_{-n} \dots y_{-1}$. If $t = 0$ and $s = 1$ then

$$[x_{-n} \dots x_{-1}]_1 = [x_{-n+1} \dots x_{-1}] = [y_{-n} \dots y_{-1}]_0 = [y_{-n} \dots y_{-2}]$$

for all $n \ge 2$. Then $\sigma^-(\dots y_{-n} \dots y_{-1}) = \dots x_{-n} \dots x_{-1}$ so that

$$[(\dots y_{-n} \dots y_{-1}, 1)] = [(\sigma^-(\dots y_{-n} \dots y_{-1}), 0)] = [(\dots x_{-n} \dots x_{-1}, 0)].$$

Similar arguments apply in the other cases so that $\hat{\Psi}$ is injective. As SX_ϕ^- is compact and G_ϕ is Hausdorff, $\hat{\Psi}$ is a homeomorphism. □

There is a positive semi-flow defined on SX_ϕ^- by

$$\alpha_s([(\underline{x}, t)]) = [((\sigma^-)^k(\underline{x}), s + t - k)]$$

where $k = \lfloor s + t \rfloor$ is the greatest integer less than or equal to $s + t$. By the inverse limit of the semi-flow α we mean

$$\varprojlim \alpha := \{\gamma \mid [0, \infty) \to SX_\phi^- \mid \gamma \text{ continuous}, \alpha_{s_1}(\gamma(s_2)) = \gamma(s_2 - s_1) \text{ for } s_2 \ge s_1\}$$

with the compact-open topology. Let SX_ϕ denote the suspension of the full shift on X_ϕ

$$SX_\phi := (X_\phi \times [0, 1])/(\underline{x}, 1) \sim (\alpha(\underline{x}), 0).$$

SX_ϕ and SX_ϕ^- are quite similar spaces. SX_ϕ^- is just SX_ϕ with finitely many arc components 'zippered' (halfway) together. (The zippered arc components correspond to the backward asymptotic composants in $\mathcal{T}_\phi$.) To recover SX_ϕ from SX_ϕ^- we may take the inverse limit of α.

PROPOSITION 5.7. *We have that SX_ϕ is homeomorphic with $\varprojlim \alpha$.*

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Proof of Proposition 5.7. Given $\underline{x} = \ldots x_{-2}x_{-1}.x_0 \ldots \in X_\phi$, let $\underline{x}^- = \ldots x_{-2}x_{-1} \in X_\phi^-$. Let $\Psi : SX_\phi \to SX_\phi^-$ be the continuous surjection given by

$$\Psi([(\underline{x}, t)]) = [(\underline{x}^-, 1 - t)]$$

and let

$$\hat{\alpha}_s([(\underline{x}, t)]) = [(\sigma^k(\underline{x}), t - s - k)],$$

where $k = \lfloor t - s \rfloor$. Then $\Psi \circ \hat{\alpha}_s = \alpha_s \circ \Psi$ for all $s \geq 0$ and we have a well-defined continuous surjection $\hat{\Psi} : SX_\phi \to \varprojlim \alpha$; $\hat{\Psi}([(\underline{x}, t)]) = \gamma$, where $\gamma : [0, \infty) \to SX_\phi^-$ is given by $\gamma(s) = \Psi(\hat{\alpha}_{-s}([(\underline{x}, t)]))$. To see that $\hat{\Psi}$ is injective, suppose that $\Psi(\hat{\alpha}_{-s}([(\underline{x}, t_1)])) = \Psi(\hat{\alpha}_{-s}([(\underline{y}, t_2)]))$ for all $s \geq 0$. Then $[((\sigma^{k_1}(\underline{x}))^-, 1 - (t_1 + s - k_1))] = [((\sigma^{k_2}(\underline{y}))^-, 1 - (t_2 + s - k_2))]$, $k_1 = \lfloor t_1 + s \rfloor$, $k_2 = \lfloor t_2 + s \rfloor$, for all $s \geq 0$. Then either $t_1 = t_2$ or $t_1, t_2 \in \{0, 1\}$. If $t_1 = t_2$ we have $k_1 = k_2 = k$ so that $\ldots x_{-n} \ldots x_{-1}x_0 \ldots x_{k-1}$ for all k. That is, $[(\underline{x}, t_1)] = [(\underline{y}, t_2)]$. The other cases are similar. Since $\varprojlim \alpha$ is Hausdorff and SX_ϕ is compact, $\hat{\Phi}$ is a homeomorphism. □

COROLLARY 5.8. *We have* $\check{H}^1(G_\phi) \simeq \check{H}^1(\mathcal{T}_\phi)$.

Proof of Corollary 5.8. We have remarked that $\mathcal{T}_\phi$ is homeomorphic with SX_ϕ. Thus, by Proposition 5.6, it suffices to prove that $\check{H}^1(SX_\phi) \simeq \check{H}^1(SX_\phi^-)$. To obtain this, note that $(z_0, z_1, \ldots) \mapsto \gamma : [0, \infty) \to SX_\phi^-$, $\gamma(s) := \alpha_{n-s}(z_n)$, $n \geq s$, defines a homeomorphism from $\varprojlim \alpha_1$, α_1 the time-one map of the semi-flow α, onto $\varprojlim \alpha$. Therefore, $\check{H}^1(\varprojlim \alpha_1) \simeq \check{H}^1(SX_\phi)$ by Proposition 5.7. By continuity,

$$\check{H}^1(\varprojlim \alpha_1) \simeq \varinjlim (\alpha_1^\star : \check{H}^1(SX_\phi^-) \to \check{H}^1(SX_\phi^-)).$$

But α_s, $0 \leq s \leq 1$ provides a homotopy between $\alpha_0 = \mathrm{id}$ and α_1, so that $\alpha_1^\star$ is the identity and $\varinjlim \alpha_1^\star \simeq \check{H}^1(SX_\phi^-)$. □

Remark 5.9. The principle applied in the proof of Corollary 5.8, that zippering asymptotic rays (provided they are positive flow lines of a globally defined semi-flow) does not change cohomology, will be employed without proof several times in what follows.

At this point we see that the space G_ϕ is homeomorphic with a quotient of $\mathcal{T}_\phi$ in which backward asymptotic composants have been zippered together (but just along their backward 'halves') and that G_ϕ and $\mathcal{T}_\phi$ have the same cohomology. We now consider $\bar{G}_\phi \supset G_\phi$. We shall interpret the cohomology of the quotient $\bar{G}_\phi / \bar{S}_\phi$ in terms of that of a one-sided augmented tiling space $\widetilde{\mathcal{T}}_\phi^F$ and this will yield the final description of

$$K_0(X_\phi) \simeq \check{H}^1(\bar{G}_\phi, \bar{S}_\phi)/\hat{q}(\ker(\hat{\pi}))$$

in terms of a quotient of a trivial extension of $DG_F(\phi) \simeq \check{H}^1(\widetilde{\mathcal{T}}_\phi^F)$.

By virtue of Standing assumption 5.3, for each $\underline{x}(i) \in LS$ there is a path of n edges in $\widetilde{G}_n$ that starts at the vertex $e^{n-1}(b_{i,1}\underline{x}(i))$ in G_n and ends at the vertex $e^{n-1}((\sigma^+)^{n-1}(\underline{x}(i)))$ in G_n:

$$\widetilde{L}_n(\underline{x}(i)) := e^n(\underline{x}(i)) \cup e^n(\sigma^+(\underline{x}(i))) \cup \cdots \cup e^n((\sigma^+)^{n-1}(\underline{x}(i))).$$

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Except for endpoints, $\widetilde{L}_n(\underline{x}(i))$ lies in $\widetilde{G}_n \setminus G_n$; furthermore,

$$\widetilde{G}_n = G_n \cup \left(\bigcup_{i=1}^{m_F(\phi)} \widetilde{L}_n(\underline{x}(i)) \right).$$

For $k = 0, \ldots, n-1$ let

$$\widetilde{L}_n^k(\underline{x}(i)) = \bigcup_{j=0}^{k} e^n((\sigma^+)^j(\underline{x}(i)))$$

denote the section of $k+1$ edges of $\widetilde{L}_n(\underline{x}(i))$ and let

$$\operatorname{int}(\widetilde{L}_n(\underline{x}(i))) = \widetilde{L}_n(\underline{x}(i)) \setminus \{e^{n-1}(b_{i,1}\underline{x}(i)),\, e^{n-1}(\sigma^+)^{n-1}(\underline{x}(i))\}$$

denote $\widetilde{L}_n(\underline{x}(i))$ less endpoints.

LEMMA 5.10. *There is an $N \in \mathbb{N}$ so that, if*

$$\operatorname{int}(\widetilde{L}_N(\underline{x}(i))) \cap \operatorname{int}(\widetilde{L}_N(\underline{x}(j))) \neq \emptyset$$

for some $i, j \in \{1, \ldots, m_F(\phi)\}$, then there is a k, $0 \leq k \leq N-1$, so that $\widetilde{L}_n^k(\underline{x}(i)) = \widetilde{L}_n^k(\underline{x}(j))$ and

$$(\widetilde{L}_n(\underline{x}(i)) \setminus \widetilde{L}_n^k(\underline{x}(i))) \cap (\widetilde{L}_n(\underline{x}(j)) \setminus \widetilde{L}_n^k(\underline{x}(j))) = \emptyset$$

for all $n \geq N$.

Proof of Lemma 5.10. Suppose that $\widetilde{L}_n(\underline{x}(i))$ and $\widetilde{L}_n(\underline{x}(j))$ share an interior vertex:

$$e^{n-1}((\sigma^+)^s(\underline{x}(i))) = e^{n-1}((\sigma^+)^t(\underline{x}(j))).$$

This means that

$$p_{n-1}((\sigma^+)^s(\underline{x}(i))) = p_{n-1}((\sigma^+)^t(\underline{x}(j))) \quad \text{for some } 0 \leq s, t < n-1. \tag{5.3}$$

The set $p_{n-1}((\sigma^+)^s(\underline{x}(i)))$ is of the form

$$\{w_1 b_{i,1} x_0 \ldots x_{s-1},\, w_2 b_{i,2} x_0 \ldots x_{s-1}, \ldots,\, w_{k_i} b_{i,k_i} x_0 \ldots x_{s-1}\}$$

for some words $w_l \in \mathcal{A}^\star$ with $|w_l| = n-s-2$. Since $b_{i,l} \neq b_{i,r}$ for $l \neq r$, for (5.3) to hold it must be the case that $s = t$. From this we see that $\widetilde{L}_n(\underline{x}(i))$ does not self-intersect; that is, $\operatorname{int}(\widetilde{L}_n(\underline{x}(i)))$ is homeomorphic with an open arc for each $i \in \{1, \ldots, m_F(\phi)\}$, $n \geq 2$.

Let N_0 be large enough so that, if $i \neq j$ and $\underline{x}(i) = x_0 x_1 x_2 \ldots$, $\underline{x}(j) = y_0 y_1 y_2 \ldots$, then $x_t \neq y_t$ for some $t \in \{0, \ldots, N_0 - 1\}$. Then

$$e^{n-1}((\sigma^+)^s(\underline{x}(i))) = e^{n-1}((\sigma^+)^t(\underline{x}(j))), \quad i \neq j,$$

implies that $s = t < N_0$. It is straightforward to check that, if

$$p_{(n+1)-1}((\sigma^+)^s(\underline{x}(i))) = p_{(n+1)-1}((\sigma^+)^s(\underline{x}(j))),$$

then

$$p_{n-1}((\sigma^+)^s(\underline{x}(i))) = p_{n-1}((\sigma^+)^s(\underline{x}(j)))$$

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and

$$p_{n-1}((\sigma^+)^{s-1}(\underline{x}(i))) = p_{n-1}((\sigma^+)^{s-1}(\underline{x}(j)))$$

for $0 < s < n-1$. It follows that, if

$$N_s = \{n \mid e^{n-1}((\sigma^+)^s(\underline{x}(i))) = e^{n-1}((\sigma^+)^s(\underline{x}(j)))\} \quad \text{then } N_1 \geq N_2 \geq \cdots \geq N_{N_0-1}.$$

Let $k = \sup\{t \mid 0 < t < N_0,\ N_t = \infty\}$ and let $N = \sup\{N_t \mid k < t < N_0 \text{ or } t = 0\}$. Then k, N are as desired. □

For each $i \in \{1, \ldots, m_F(\phi)\}$ let $p_n(\underline{x}(i)) = \{w^1 b_{i,1}, w^2 b_{i,2}, \ldots, w^{k_i} b_{i,k_i}\}$ with $w^i \in \mathcal{A}^\star$, $|w^i| = n-1$, $j = 1, \ldots, k_i$. For each $j \in \{1, \ldots, k_i\}$ and $s \in \{0, \ldots, n-1\}$ let $q(n, i, j, s)$ denote the prefix of length n of $(\sigma^+)^s(w^j b_{i,j}\underline{x}(i))$ and let $e^{n,i,j,s}$ be the edge of G_n with $p_n(e^{n,i,j,s}) = \{q(n, i, j, s)\}$. Then

$$L_n^j(\underline{x}(i)) := \bigcup_{s=0}^{n-1} e^{n,i,j,s}$$

is a path in G_n that starts at the vertex $e^{n-1}(b_{i,j}\underline{x}(i))$, which, for $j = 1$, is where $\widetilde{L}_n(\underline{x}(i))$ starts and, for $j \geq 2$, is where the extra edge $e^{n,i,j}$ in $\bar{G}_n$ is attached. Furthermore, each $L_n^j(\underline{x}(i))$ ends at $e^{n-1}((\sigma^+)^{n-1}(\underline{x}(i)))$, where $\widetilde{L}_n(\underline{x}(i))$ ends.

For each $n \geq 2$, $i \in \{1, \ldots, m_F(\phi)\}$, $j \in \{1, \ldots, k_i\}$, f_n maps

$$\bigcup_{s=0}^{n-1} e^{n+1,i,j,s} \subset L_{n+1}^j(\underline{x}(i))$$

one-to-one and onto $L_n^j(\underline{x}(i))$ and

$$f_n(e_t^{n+1,i,j,n}) = \tilde{f}_n(e_t^{n+1}((\sigma^+)^n(\underline{x}(i)))) = e_t^n((\sigma^+)^n(\underline{x}(i)))$$

for $0 \leq t \leq 1$. From this we see that

$$L^j(i) := \{\underline{z} = (z_2, z_3, \ldots) \in \varprojlim f_n = G_\phi \mid z_n \in L_n^j(\underline{x}(i)) \text{ for all sufficiently large } n\},$$
$$j = 1, \ldots, k_i,$$

are forward asymptotic in G_ϕ. (Under the composition of maps $\mathcal{T}_\phi \to SX_\phi \to SX^+ \to G_\phi$ the $L^j(i)$ are images of forward asymptotic rays with right infinite word $\underline{x}(i)$.) Also, $\hat{f}_n$ maps

$$\bigcup_{s=0}^{n-1} e^{n+1}((\sigma^+)^s(\underline{x}(i))) \subset \widetilde{L}_{n+1}(\underline{x}(i))$$

onto $\widetilde{L}_n(\underline{x}(i))$ and, since $\widetilde{L}_n(\underline{x}(i))$ is non-self-intersecting by Lemma 5.10,

$$\widetilde{L}(i) := \{\underline{z} = (z_2, z_3, \ldots) \in \varprojlim \tilde{f}_n = \widetilde{G}_\phi \mid z_n \in \widetilde{L}_n(\underline{x}(i)) \text{ for all sufficiently large } n\}$$

is a ray in $\widetilde{G}_\phi$ that meets G_ϕ only at

$$\underline{v}(i) := (e^1(b_{i,1}\underline{x}(i)), e^2(b_{i,1}\underline{x}(i)), \ldots) \in L^1(i)$$

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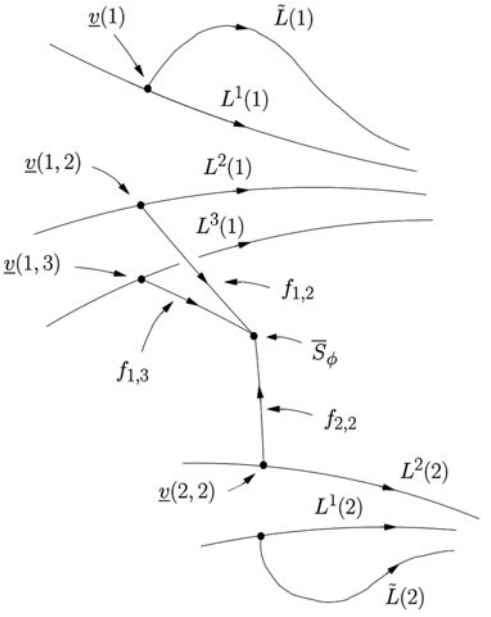


FIGURE 4. The quotient space $\bar{G}_\phi/\bar{S}_\phi$.

and is forward asymptotic to the $L^j(i)$, $j = 1, \ldots, k_i$. Furthermore,

$$(\widetilde{L}(i) \cap \widetilde{L}(j)) \setminus G_\phi = \emptyset$$

for $i \neq j$ by Lemma 5.10.

We now see that the quotient space $\bar{G}_\phi/\bar{S}_\phi$ (Figure 4) consists of the space G_ϕ with: $m_F(\phi)$ rays $\widetilde{L}(i)$, $i = 1, \ldots, m_F(\phi)$; and $\sum_{i=1}^{m_F(\phi)}(k_i - 1)$ edges $f_{i,j}$, $i = 1, \ldots, m_F(\phi)$, $j = 2, \ldots, k_i$. (The edge $f_{i,j}$ is the inverse limit $e^{2,i,j} \leftarrow e^{3,i,j} \leftarrow \ldots$.) Each ray $\widetilde{L}(i)$ is asymptotic to a different equivalence class of forward asymptotic rays in G_ϕ (the $L^j(i)$) and is attached to one of those rays, $L^1(i)$, at the point $\underline{v}(i)$. The edges $f_{i,j}$ attach to G_ϕ at points $\underline{v}(i, j) = (e^1(b_{i,j}\underline{x}), e^2(b_{i,j}\underline{x}), \ldots)$. The terminal ends of the $f_{i,j}$ are all identified to the single point $\bar{S}_\phi$ in $\bar{G}_\phi/\bar{S}_\phi$. It is possible that $\widetilde{L}(i)$ meets $\widetilde{L}(j)$ for $i \neq j$, but if this happens the intersection consists of an initial compact arc (by Lemma 5.10). For the purpose of considering cohomology, such intersections can be ignored.

To simplify the picture, we zipper up the forward asymptotic composants without affecting the cohomology (see Remark 5.9). The *asymptotic loops* $\widetilde{L}(i) \cup L^1(i)$ become actual loops, which we denote by f_i. Call this space $Z\bar{G}_\phi/\bar{S}_\phi$ (Figure 5).

The forward asymptotic tiling space $\widetilde{\mathcal{T}}_\phi^F$ consists of the tiling space $\mathcal{T}_\phi$ together with $m_F(\phi)$ rays, $R(i)$, $i = 1, \ldots, m_F(\phi)$. Each $R(i)$ is forward asymptotic to the ith equivalence class of forward asymptotic composants in $\mathcal{T}_\phi$. The rays initiate from a common vertex s, are otherwise pairwise disjoint, and

$$\left(\bigcup_{i=1}^{m_F(\phi)} R(i)\right) \cap \mathcal{T}_\phi = \emptyset.$$

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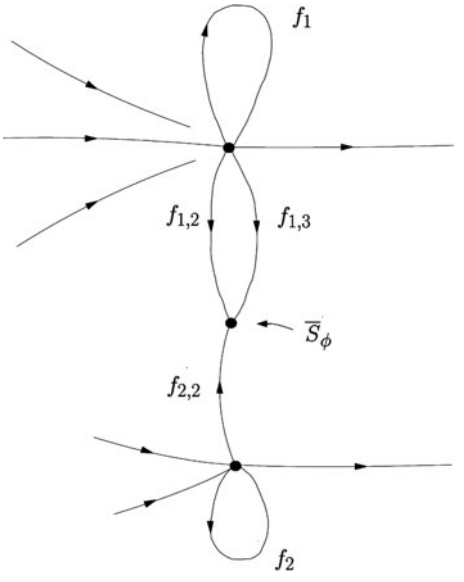


FIGURE 5. The space $Z\bar{G}_\phi/\bar{S}_\phi$.

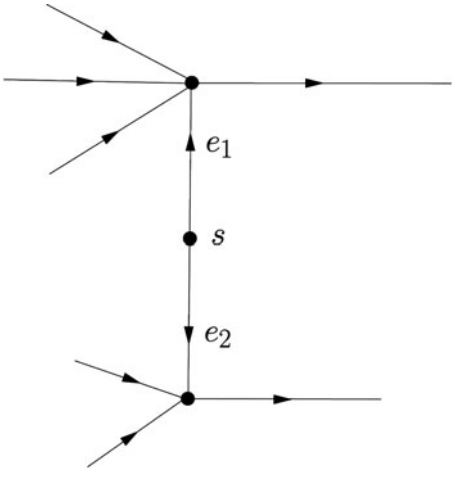


FIGURE 6. The space $Z\widetilde{\mathcal{T}}_\phi^F$.

(A precise definition in terms of inverse limits can be formulated as for the two-sided augmentation in §3.) If we zipper up the forward asymptotic composants as well as most of the $R(i)$ (cohomology is not affected by Remark 5.9), we obtain a space $Z\widetilde{\mathcal{T}}_\phi^F$ caricatured in Figure 6. The edges e_i are the remnants of the rays $R(i)$.

As in Remark 5.9 and §3, $\check{H}^1(Z\widetilde{\mathcal{T}}_\phi^F) \simeq \check{H}^1(\widetilde{\mathcal{T}}_\phi^F) \simeq DG(\widetilde{A}_\phi^F)$. Denoting by ZG_ϕ the zippered image of G_ϕ in $Z\bar{G}_\phi$, we have $\check{H}^1(ZG_\phi) \simeq \check{H}^1(G_\phi) \simeq \check{H}^1(\mathcal{T}_\phi)$ (Corollary 5.8 and Remark 5.9). Thus

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$$\check{H}^1\left(ZG_\phi \cup \left(\bigcup_{i=1}^{m_F(\phi)} f_{i,2}\right)\right) \simeq \check{H}^1(\mathcal{T}_\phi^F).$$

Returning to K_0, we recall that $K_0(X_\phi) \simeq \check{H}^1(\bar{G}_\phi, \bar{S}_\phi)/\hat{q}(\ker \hat{\pi})$ and this latter is isomorphic with $\check{H}^1(Z\bar{G}_\phi/\bar{S}_\phi)/K$, with K the subgroup of $\check{H}^1(Z\bar{G}_\phi/\bar{S}_\phi)$ generated by

$$\{[f_i^*] - [f_{i,j}^*] \mid i = 1, \ldots, m_F(\phi),\ j = 2, \ldots, k_i\}$$

(here and in what follows, [ ] denotes equivalence class mod coboundaries). Collapsing the edges f_i in $Z\bar{G}_\phi/\bar{S}_\phi$, each to its vertex, and collapsing the edges $f_{i,j}$ to $f_{i,2}$, $j = 3, \ldots, k_i$, yields the space

$$ZG_\phi \cup \left(\bigcup_{i=1}^{m_F(\phi)} f_{i,2}\right)$$

as a retract of $Z\bar{G}_\phi/\bar{S}_\phi$. We then have a split short exact sequence

$$0 \to \check{H}^1\left(Z\bar{G}_\phi/\bar{S}_\phi,\ ZG_\phi \cup \left(\bigcup_{i=1}^{m_F(\phi)} f_{i,2}\right)\right)$$
$$\xrightarrow{i^*} \check{H}^1(Z\bar{G}_\phi/\bar{S}_\phi) \underset{r^*}{\overset{j^*}{\rightleftarrows}} \check{H}^1\left(ZG_\phi \cup \left(\bigcup_{i=1}^{m_F(\phi)} f_{i,2}\right)\right) \to 0,$$

where r is the retraction described above. The group

$$\check{H}^1\left(Z\bar{G}_\phi/\bar{S}_\phi,\ ZG_\phi \cup \left(\bigcup_{i=1}^{m_F(\phi)} f_{i,2}\right)\right)$$

is the free abelian group on generators

$$\{[f_i^*], [f_{i,j}^*] \mid i = 1, \ldots, m_F(\phi),\ \ j = 3, \ldots, k_i\}.$$

We have an isomorphism

$$\psi : \check{H}^1(Z\bar{G}_\phi/\bar{S}_\phi) \to \check{H}^1\left(ZG_\phi \cup \left(\bigcup_{i=1}^{m_F(\phi)} f_{i,2}\right)\right)$$
$$\oplus \check{H}^1\left(Z\bar{G}_\phi/\bar{S}_\phi,\ ZG_\phi \cup \left(\bigcup_{i=1}^{m_F(\phi)} f_{i,2}\right)\right)$$

by

$$\psi(c) = (j^*(c),\ (i^*)^{-1}(c - r^*(j^*(c)))).$$

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Calculation shows that $\psi(K)$ is generated by

$$\left\{\left(-[f_{i,2}^*],\, [f_i^*] + \sum_{j=3}^{m_F(\phi)} [f_{i,j}^*]\right),\, (0,\, [f_i^*] - [f_{i,j}^*]) \;\middle|\; i = 1, \ldots, m_F(\phi),\, j = 3, \ldots, k_i\right\}.$$

Expressing $\check{H}^1(Z\bar{G}_\phi/\bar{S}_\phi,\ ZG_\phi \cup (\bigcup_{i=1}^{m_F(\phi)} f_{i,2}))$ as

$$\left(\bigoplus_{i=1,\ldots,m_F(\phi)} \langle [f_i^*] \rangle\right) \oplus \left(\bigoplus_{\substack{i=1,\ldots,m_F(\phi)\\ j=3,\ldots,k_i}} \langle [f_i^*] - [f_{i,j}^*] \rangle\right),$$

we see that 'modding' out

$$\check{H}^1\left(Z\bar{G}_\phi \cup \left(\bigcup_{i=1}^{m_F(\phi)} f_{i,2}\right)\right) \oplus \left(\bigoplus_{i=1,\ldots,m_F(\phi)} \langle [f_i^*] \rangle\right) \oplus \left(\bigoplus_{\substack{i=1,\ldots,m_F(\phi)\\ j=3,\ldots,k_i}} \langle [f_i^*] - [f_{i,j}^*] \rangle\right)$$

by $\psi(K)$ annihilates the third summand and introduces the relation $[f_{i,2}^*] = (k_i - 1)[f_i^*]$ between elements $[f_{i,2}^*]$ of $\check{H}^1(ZG_\phi \cup (\bigcup_{i=1}^{m_F(\phi)} f_{i,2}))$ and generators $[f_i^*]$ of

$$\left(\bigoplus_{i=1,\ldots,m_F(\phi)} \langle [f_i^*] \rangle\right),$$

$i = 1, \ldots, m_F(\phi)$. Since $\check{H}^1(ZG_\phi \cup (\bigcup_{i=1}^{m_F(\phi)} f_{i,2}))$ is isomorphic with $\check{H}^1(\widetilde{\mathcal{T}}_\phi^F)$ by an isomorphism that takes $[f_{i,2}^*]$ to $[r_i^*]$, r_i^* a cocycle along the ray $R(i)$ of $\widetilde{\mathcal{T}}_\phi^F$, $i = 1, \ldots, m_F(\phi)$, we have the following theorem.

THEOREM 5.11. *If ϕ is a prepared substitution satisfying Standing assumption 5.3 then*

$$K_0(X_\phi) \simeq (\check{H}^1(\widetilde{\mathcal{T}}_\phi^F) \oplus \mathbb{Z}^{m_F(\phi)})/G,$$

with G the subgroup generated by

$$\{[r_i^*] \oplus (k_i - 1)e_i \mid e_i \textit{ the } i\textit{th unit vector in } \mathbb{Z}^{m_F(\phi)},\ i = 1, \ldots, m_F(\phi)\}.$$

As $\check{H}^1(\widetilde{\mathcal{T}}_\phi^F) \simeq DG_F(\phi)$, Theorem 5.11 can be used to express $K_0(X_\phi)$ as a stationary matrix direct limit. For one such presentation, see [**CE1**, Theorem 5.8].

A flow equivalence between shifts X_ϕ and X_ψ induces an orientation-preserving homeomorphism of $\mathcal{T}_\phi$ with $\mathcal{T}_\psi$, which in turn induces a homeomorphism of $\widetilde{\mathcal{T}}_\phi^F$ with $\widetilde{\mathcal{T}}_\psi^F$ (as in Theorem 3.1 for the two-sided case). The latter takes the rays $\{R_\phi(i)\}$ to the rays $\{R_\psi(j)\}$, and since the homeomorphism of $\mathcal{T}_\phi$ with $\mathcal{T}_\psi$ takes equivalence classes of forward asymptotic composants to equivalence classes of forward asymptotic composants, the numbers $k_i(\phi)$ and $k_i(\psi)$, properly indexed, are equal. The induced isomorphism of $\check{H}^1(\widetilde{\mathcal{T}}_\phi^F)$ with $\check{H}^1(\widetilde{\mathcal{T}}_\psi^F)$ extends to an isomorphism from $\check{H}^1(\widetilde{\mathcal{T}}_\phi^F) \oplus \mathbb{Z}^{m_F(\phi)}$ to $\check{H}^1(\widetilde{\mathcal{T}}_\phi^F) \oplus \mathbb{Z}^{m_F(\psi)=m_F(\phi)}$ that takes $G(\phi)$ to $G(\psi)$, and we recover from Theorem 5.11 the result of Matsumoto [**M**] that $K_0(X_\phi)$ is a flow equivalence invariant.

A final note: from Theorem 5.11 we see that if $k_i = 2$ for $i = 1, \ldots, m_F(\phi)$ then $K_0(X_\phi) \simeq \check{H}^1(\widetilde{\mathcal{T}}_\phi^F)$.

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