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# Tiling spaces are Cantor set fiber bundles

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*Abstract*. We prove that fairly general spaces of tilings of $\mathbb{R}^d$ are fiber bundles over the torus $T^d$, with totally disconnected fiber. This was conjectured (in a weaker form) in the second author's recent work, and proved in certain cases. In fact, we show that each such space is homeomorphic to the $d$-fold suspension of a $\mathbb{Z}^d$ subshift (or equivalently, a tiling space whose tiles are marked unit $d$-cubes). The only restrictions on our tiling spaces are that (1) the tiles are assumed to be polygons (polyhedra if $d > 2$) that meet full-edge to full-edge (or full-face to full-face), (2) only a finite number of tile types are allowed, and (3) each tile type appears in only a finite number of orientations. The proof is constructive and we illustrate it by constructing a 'square' version of the Penrose tiling system.

## 1. *Introduction and results*

Anderson and Putnam [**AP**] have found a relation between tiling theory (specifically, substitution tiling theory) and a theory of attractors studied in the dynamics of diffeomorphisms of manifolds with hyperbolic structure. They show that *substitution* tiling spaces are a special case of expanding attractors, a concept introduced in [**W2**] to study the dynamics of diffeomorphisms. It is well known that an expanding attractor is locally the topological product of a Cantor set and a disk of the appropriate dimension. But what is it globally? Can the local neighborhoods be stitched together to form a bundle over a manifold whose fiber is a Cantor set? In 1973, Williams [**W2**] conjectured that they could. This conjecture has long been known to be true in dimension 1 ([**BJV, W1**], etc). However, it fails in higher dimensions; Farrell and Jones [**FJ**] constructed a counterexample in dimension 2.

In this paper, however, we show (Theorem 1) that the conjecture is true for tiling spaces in all dimensions, even without the substitution assumption. Tiling spaces in general are *flat*, suggesting that the appropriate base spaces should be tori. (In fact, the torus is the only candidate for any expanding attractor since we know that the area must grow as a polynomial of the 'radius'.) In addition, we show (Theorem 2) that tiling spaces are suspensions of $\mathbb{Z}^d$ subshifts, which are manifestly bundles over tori.

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The eventual goal is to classify tiling spaces up to homeomorphism, and the existence of a bundle structure is an important first step. Although Cantor set fiber bundles are harder to deal with than, say, vector bundles, topological invariants have been constructed for bundles over $S^1$ ([**BF, BJV, F, S**]). In higher dimensions, the bundles are generically less 'flabby' and thus, hopefully, allow stronger invariants than those of Parry–Sullivan [**PS**] and Bowen–Franks [**BF**] type. In addition, Theorem 2 suggests that much of the theory of subshifts [**B, LM, Wa**] can be brought into play.

The proofs given below are easy and are too abstract and general to be of direct use in classification. Note that given a bundle structure $p : X \to T^d$ and an $n$-to-1 self covering $f : T^d \to T^d$, the composition $f \circ p : X \to T^d$ is another bundle structure. The latter is 'coarser' in that each of its fibers contains $n$ fibers of the former; is there a finest possible bundle structure? The current construction does not address this problem. However, see [**W3**] for the case of the Penrose tiling. The 40 tiles of the Penrose tiling cover the torus 10 times in [**W3**], but 232 times in the current version.

The basic ingredients of this paper are tiling systems $P$ of $\mathbb{R}^d$ and the corresponding tiling spaces $X(P)$. These spaces are assumed to satisfy the following hypotheses.

(1) The tiles are (triangulated) polyhedra that meet full-face to full-face.
(2) Only a finite number of tile types (proto-tiles) appear. In this counting, tiles that are translations of one another are considered to be the same type, but tiles that are rotations of one another are considered to be different.
(3) The space $X(P)$ is a closed, non-empty and translation-invariant subset of the space of all tilings that can be formed from the tiles in $P$.

We will henceforth refer to both the tiling system and the associated topological space by the same letter $P$.

Our first result is the following.

THEOREM 1. *A tiling space that satisfies the above hypotheses is a fiber bundle over the torus, with totally disconnected fiber.*

Note that we do not assume that the tilings are quasi-periodic, or generated by a substitution, or even that they are non-periodic. The only difference between these cases is the nature of the fiber. The fiber for a substitution tiling, or a quasi-periodic tiling, will be a Cantor set, while the fiber for a ($d$-fold) periodic tiling system will be a finite collection of points.

The requirement of polygonal tiles is mostly for convenience. A tiling, such as the Penrose chickens, whose edges follow standard shapes, can be deformed to a tiling system with polygonal tiles and therefore is a fiber bundles over a torus. The requirement that tiles appear in only a finite number of orientations is more serious. The techniques of this paper do not apply to pinwheel-like tiling spaces [**ORS, R**].

Our second result is the following.

THEOREM 2. *A tiling space $P$ that satisfies the above hypotheses is homeomorphic to a tiling space $S$ whose tiles are marked $d$-cubes or equivalently to the $d$-fold suspension of a $\mathbb{Z}^d$ subshift. The space $S$ is defined by local matching rules if and only if $P$ is.*

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Note that this theorem proves the existence of a homeomorphism, not a topological conjugacy. The homeomorphism typically does *not* commute with translations, much less with rotations. For more information on $\mathbb{Z}^d$ subshifts, see [**Wa**].

The proofs proceed as follows. We call a tiling space rational (integral) if each edge of each tile is given by a vector with rational (integral) coordinates. In §2 we show that every tiling space $P$ can be deformed to a rational tiling space $R$. This deformation is a homeomorphism of tiling spaces, but not a topological conjugacy. We then show that every rational tiling is a fiber bundle over the torus. This proves Theorem 1.

In §3 we prove Theorem 2. We rescale the rational tiling space $R$ into an integral tiling space and replace the straight edges with zig–zags consisting of unit segments in the several coordinate directions. The faces then become unions of unit squares, the three-cells become unions of unit cubes, and so on. This gives a 'zig–zag' system $Z$. The tiles of $Z$ may take on odd shapes and may even be disconnected, but are unions of $d$-cubes. The space $Z$ is homeomorphic (topologically conjugate, in fact) to the rescaled $R$. Finally, we consider each constituent $d$-cube of a tile $z$ in $Z$ to be a tile in a tiling space $S$, with the matching rule that wherever one such constituent appears, the other constituents of $t$ also appear nearby. $S$ is a suspension of a subshift, but is also topologically conjugate to $Z$ and therefore homeomorphic to $P$.

## 2. *Tiling spaces as fiber bundles*

LEMMA 3. *A tiling space $P$ satisfying the above hypotheses is homeomorphic to a rational tiling space $R$. Furthermore, $R$ has finite type if and only if $P$ does.*

*Proof.* For greater clarity, we go through the proof in dimension 2 and illustrate how each step applies to the Penrose system. We then indicate how the proof applies, with small changes, in any dimension.

Let the tiles of a tiling space $P$ be represented by polygonal disks $C_i, i = 1, \ldots, c$, in the plane; let each edge of each tile be given a fixed orientation. If tiles $C_i$ and $C_j$ can meet along a common edge in a tiling, then these edges must be parallel, of the same length and have the same orientation. That is, the displacement vectors of these edges must be equal. Thus, altogether we have a finite set of these vectors, say $v_1, \ldots, v_n$, which we will call *edge vectors*.

In the Penrose 'B-tile' system, there are 40 triangular tiles, namely those shown in Figure 1 and their rotations by multiples of $2\pi/10$. We let $t$ denote the rotation by $2\pi/10$, so $t^5 A$ means tile $A$ rotated by $\pi$. Although $A$ is congruent to $B$, they are considered separate tiles. Similarly, $C$ and $D$ are considered distinct. Because of the identifications, there are 40 edge vectors, not 120. Five of the edge vectors are

$$\begin{aligned} a &= (2(\tau - 1), 2\sqrt{\tau + 2}) \\ ta &= (-2(\tau - 1), 2\sqrt{\tau + 2}) \\ t^2 a &= (-2\tau, 2\sqrt{3 - \tau}) \\ t^3 a &= (-4, 0) \\ t^4 a &= (-2\tau, -2\sqrt{3 - \tau}), \end{aligned} \tag{1}$$

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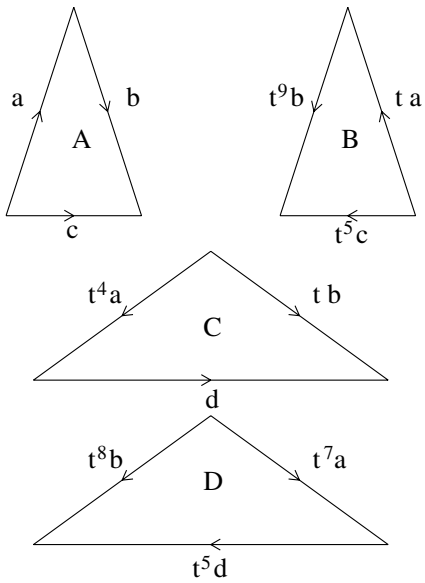



FIGURE 1. The tiles of the Penrose system.

where $\tau = (1+\sqrt{5})/2$. The other vectors are given by

$$\begin{aligned} t^{5+n}a &= -t^n a & n &= 0,\dots,4 \\ t^n b &= t^{n-4}a & n &= 0,\dots,9 \\ t^n c &= (\tau-1)t^{n-2}a & n &= 0,\dots,9 \\ t^n d &= \tau t^{n-2}a & n &= 0,\dots,9. \end{aligned} \tag{2}$$

We wish to construct new tiles $C'_k$, $k = 1,\dots,c$, so that $n$ (same number!) edge vectors, $v'_1,\dots,v'_n$ of the new tiles all have rational coordinates. Furthermore, if an edge $I_i$ of $C_k$ has edge vector $v_i$, the corresponding edge $I'_i$ of $C'_k$ has edge vector $v'_i$. To see that this can be done, note that there is a single constraint for each tile, i.e. the sum of its edge vectors must be zero. Thus we have a finite system of homogeneous equations:

$$v_{i_1} + \cdots + v_{i_k} = 0. \tag{3}$$

As the coefficients in this system are rational (in fact, integers) it follows by elementary linear algebra, that the solutions with rational entries are dense in the space of all solutions, which is non-empty, as the original vectors are solutions. Thus there are rational vectors $v'_1,\dots,v'_n$ approximating the original vectors, as close as we wish, which satisfy the constraints.

The crux here is that we have finitely many loops among the edge vectors so that the constraints consist of finitely many equations with integral coefficients. In higher dimensions the choice of these loops is a little different. In fact, since our tiles are triangulated, the boundary of each two-dimensional simplex is a loop, and there are only

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TABLE 1. Integer set of solutions.

| $n$ | $v'(t^n a)$ | $v'(t^n b)$ | $v'(t^n c)$ | $v'(t^n d)$ |
|---|---|---|---|---|
| 0 | $(1, 4)$ | $(1, -4)$ | $(2, 0)$ | $(6, 0)$ |
| 1 | $(-1, 4)$ | $(3, -2)$ | $(2, 2)$ | $(5, 4)$ |
| 2 | $(-3, 2)$ | $(4, 0)$ | $(1, 2)$ | $(2, 6)$ |
| 3 | $(-4, 0)$ | $(3, 2)$ | $(-1, 2)$ | $(-2, 6)$ |
| 4 | $(-3, -2)$ | $(1, 4)$ | $(-2, 2)$ | $(-5, 4)$ |
| 5 | $(-1, -4)$ | $(-1, 4)$ | $(-2, 0)$ | $(-6, 0)$ |
| 6 | $(1, -4)$ | $(-3, 2)$ | $(-2, -2)$ | $(-5, -4)$ |
| 7 | $(3, -2)$ | $(-4, 0)$ | $(-1, -2)$ | $(-2, -6)$ |
| 8 | $(4, 0)$ | $(-3, -2)$ | $(1, -2)$ | $(2, -6)$ |
| 9 | $(3, 2)$ | $(-1, -4)$ | $(2, -2)$ | $(5, -4)$ |

finitely many of these, which certainly suffice. Thus in any dimension, the constraint can be taken to consist of a finite system of equations of the form (3).

For the Penrose tiling, our system of equations is

$$\begin{aligned} v'(t^n a) + v'(t^n b) - v'(t^n c) &= 0, \quad n = 0, \ldots, 9 \\ v'(t^{n+6} a) + v'(t^{4+n} b) - v'(t^n c) &= 0, \quad n = 0, \ldots, 9 \\ -v'(t^{n+4} a) + v'(t^{n+1} b) - v'(t^n d) &= 0, \quad n = 0, \ldots, 9 \\ -v'(t^{n+2} a) + v'(t^{n+3} b) - v'(t^n d) &= 0, \quad n = 0, \ldots, 9 \end{aligned} \tag{4}$$

Table 1 is an integer set of solutions. Note that by picking integer solutions, we have broken the ten-fold rotational symmetry of the Penrose system. This is to be expected, as one cannot represent $\mathbb{Z}_{10}$ in $GL(2, \mathbb{Q})$. We have, however, preserved the two-fold rotational symmetry, the reflection symmetry about the $x$-axis and the fact that $v'(t^n a) = v'(t^{n+4} b)$.

Returning to the general case, there is a linear isomorphism from each edge $I_i$ to the corresponding $I_i'$. These can be extended to homeomorphisms from each $C_k$ to $C_k'$. (If $C_k$ is a triangle there is a linear extension. In general, we can always find a continuous extension.) We now use these homeomorphisms to convert an arbitrary tiling by the tiles $\{C_k\}$ into a tiling by the tiles $\{C_k'\}$. As we shall see, this procedure is continuous and has a continuous inverse and so defines a homeomorphism between the tiling space $P$ and a rational tiling space $R$.

For each tiling $t$ in the tiling space $P$, we construct a corresponding tiling $t' \in R$, beginning at the origin. The origin in $t$ sits at a point in a closed tile $C_k$; we let the origin in $t'$ sit at the corresponding point in $C_k'$. If the origin lies on an edge $I_i$ that is shared by $C_j$ and $C_k$, then we may start with either $C_j$ or $C_k$. There is no ambiguity in the location of the origin, since the maps $C_j \to C_j'$ and $C_k \to C_k'$ are both extensions of the linear map from $I_i$ to $I_i'$.

This procedure determines the position of a single tile $C_k'$ of $t'$ that contains the origin. We then grow outwards, so that the tiling $t'$ is combinatorially identical to $t$, only with each tile of type $C_j$ replaced with a tile of type $C_j'$ and each edge of type $I_j$ replaced by $I_j'$.

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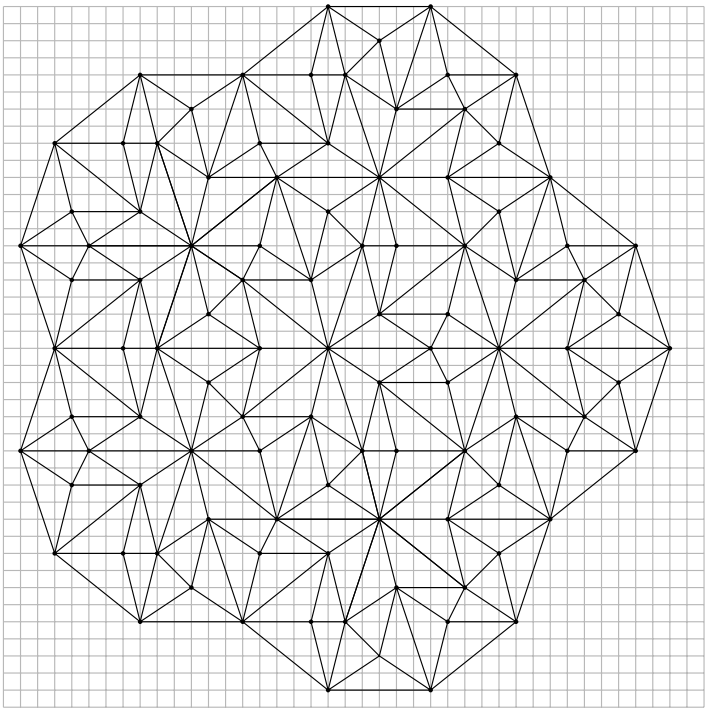


FIGURE 2. A patch of the rational Penrose tiling (180 tiles).

This is shown in Figure 2, where a patch of the original Penrose tiling is replaced by a patch of rational Penrose tiling.

To see that this construction does result in a tiling, we must show that the vertices of $t'$ are well defined. Let $x$ be a vertex of $t$ and consider two paths from a vertex $y$ of the central seed tile to $x$. The algebraic difference of these two paths, namely zero, is the boundary of a sum of tiles in $t$. By equations (4), the algebraic difference of the corresponding sum of vectors $v'$ is also zero. This means that either path can be used to determine the position of $x'$, the vertex in $t'$ that corresponds to $x$. Once the vertices are defined, the edges and faces follow.

This transformation is continuous. If two tilings $t$ and $\tilde{t}$ agree on a large neighborhood of the origin, then $t'$ and $\tilde{t}'$ agree on a large neighborhood of the origin. If $t$ and $\tilde{t}$ differ by a small translation, then $t'$ and $\tilde{t}'$ differ by a small translation, as determined by the homeomorphism between the center tile of $t$ and that of $t'$. (As noted above, there is no ambiguity, and no discontinuity, if the origin in $t$ sits on the boundary of a tile.) Similarly, the reverse transformation, from tilings in $R$ to tilings in $P$, is also continuous. Thus $P$ and $R$ are homeomorphic tiling spaces.

Finally, since each tiling $t$ in $P$ is combinatorially equivalent to a tiling $t'$ in $R$, any local atlas for the $P$ system can be naturally transformed into a local atlas for the $R$ system, and *vice-versa*.

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In higher dimensions, the argument is essentially as in two dimensions. As mentioned above, since we are taking our tiles in higher dimensions to be *triangulated*, we can proceed with a finite system of homogeneous equations—all of the form $a + b + c = 0$. As before, we can find a rational solution arbitrarily close to the original vectors. To construct homeomorphisms between tiles $C_k$ and $C'_k$, one must start with homeomorphisms (e.g. linear maps) between edges $I_i$ and $I'_i$, extend these to homeomorphisms of the two-skeleton, then of the three-skeleton and so on. There are no topological obstructions. □

To complete the proof of Theorem 1, we must only prove

LEMMA 4. *Every rational tiling space is a fiber bundle over the torus.*

*Proof.* Let $R$ be a rational tiling and let $D$ be the least common multiple of all the denominators of all the coordinates of displacement vectors $v_i$ for the tiles in $R$. Rescale $R$ by $D$, so that all displacement vectors are integers. Then all the vertices in any fixed tiling have the same coordinates (mod $\mathbb{Z}^d$). These coordinates give a natural projection from the space of tilings to the $d$-torus $\mathbb{R}^d/\mathbb{Z}^d$. □

## 3. *Square tiling spaces*

We have shown that our general tiling space $P$ is homeomorphic to a rational tiling space $R$ that is of finite type if $P$ is (and is not if $P$ is not). By rescaling, we can assume that $R$ is in fact integral. Topological conjugacies preserve finite type [**RS**]. To complete the proof of Theorem 2, it suffices to prove the following lemma.

LEMMA 5. *Every rational tiling space $R$ is, after rescaling, topologically conjugate to a square-type tiling space $S$.*

*Proof.* As before, we work first in two dimensions and then sketch what modifications need to be made in higher dimensions. Also as before, we illustrate each step of our construction with the Penrose system.

We first rescale $R$ so that $R$ becomes an integral tiling. Furthermore, we assume that each tile contains a circle of radius greater than $\sqrt{2}/2$; this can always be achieved by further scaling. Next we replace each of our straight edges $I'_i$ with zig–zags $J_i$, that is with sequences of unit displacements in the coordinate directions. We do this in such a way that the maximum distance of a point in $J_i$ from the original edge $I'_i$ is minimized. In particular, one can always choose $J_i$ such that this distance is no greater than $\sqrt{2}/2$. There is sometimes more than one way to minimize this distance. For example, one could replace a diagonal edge from $(0, 0)$ to $(1, 1)$ with a zig–zag from $(0, 0)$ to $(1, 0)$ to $(1, 1)$, or with a zig–zag from $(0, 0)$ to $(1, 0)$ to $(1, 1)$. In such a case, one must make a choice and apply it consistently. A possible set of zig–zags for the rational Penrose system is given in Figure 3. Under this replacement, the 180-tile patch of Figure 2 turns into the patch of Figure 4.

This defines a space $Z$ of tilings whose edges are zig–zags. To each tiling $t'$ in $R$ we generate a tiling $z$ in $Z$ by replacing each straight edge in $t'$ with its corresponding zig–zag. If a tile type $C'_k$ in the $R$ system is bounded by several straight edges $I'_i$, then the tile type $D_k$ in the $Z$ system is defined to be the region bounded by the corresponding zig–zags $J_i$. The condition that $C'_k$ contains a circle of radius greater than $\sqrt{2}/2$ ensures that $D_k$ is

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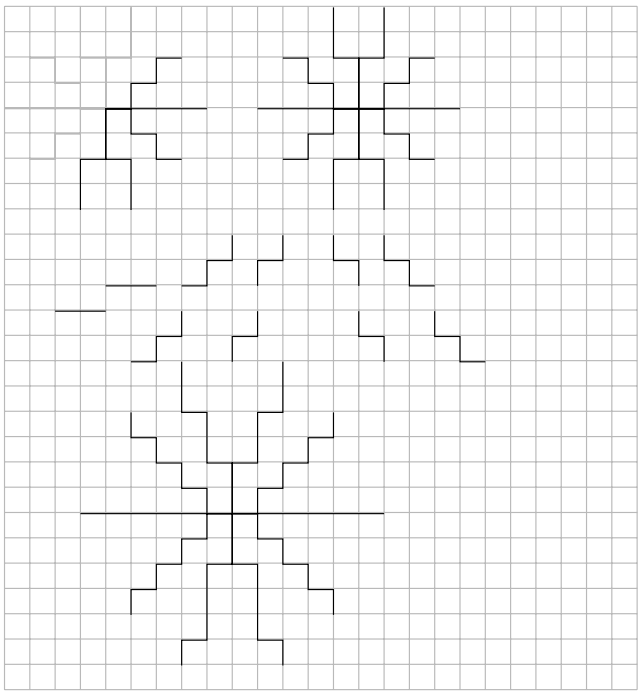


FIGURE 3. The 40 zig–zag edges. The 'a' edges are in the upper left, the 'b' edges in the upper right, the 'c' edges in the middle and the 'd' edges at the bottom.

non-empty. (It may, however, be disconnected.) It may happen that geometrically non-congruent tile types $C'_k$ generate congruent tile types $D_k$; however, as marked tiles, these $D_k$ should be considered distinct.

The operation of replacing straight edges with zig–zags is reversible and does not require a choice of origin. It therefore commutes with translation and defines a topological conjugacy between $R$ and $Z$.

In the tiling system $Z$, the basic tiles are irregularly shaped regions $D_k$ bounded by zig–zags and we have already seen that each $D_k$ is non-empty. Suppose that the tile $D_k$ has area $n$. Then $D_k$ can be decomposed as the union of $n$ unit squares $D_k^1, \ldots, D_k^n$. In the tiling system $S$, the basic tiles are the squares $D_k^i$ and we apply a matching rule that says that wherever one of the $D_k^i$ squares is found, the other $n-1$ squares that make up $D_k$ are also found nearby, arranged to form the larger region $D_k$. A tiling in $S$ can therefore be amalgamated into a tiling by tiles $D_k$. We allow in $S$ those tilings, and only those tilings, that amalgamate into tilings in $Z$. In this way, the tiling system $S$ is naturally conjugate to $Z$.

The proof in higher dimensions is almost identical. In dimension 3, one must pick zig–zags $J_i$ to replace the straight edges $I'_i$. If several edges $I'_i$ bound a two-face of a tile $C'_k$, we must find a union of unit squares (oriented in the coordinate directions), bounded by the appropriate zig–zags $J_i$, that approximates this face. The tile $D_k$ is then the solid region bounded by these zig–zag faces. In dimension $d > 3$, we work recursively, replacing edges

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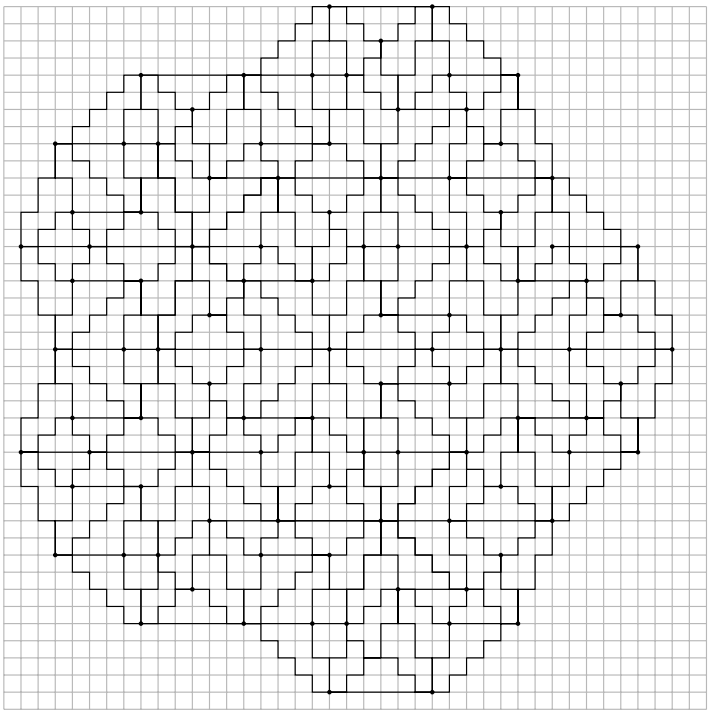


FIGURE 4. A patch of the square Penrose tiling (180 tiles).

$I_i'$ with zig–zags $J_i$, then replacing two-cells with unions of squares, three-cells with unions of cubes, and so on up through dimension $d-1$. The tiles $D_k$ are the $d$-cells bounded by the $d-1$ cells constructed in this manner. We can compute a universal bound for each dimension, so that the $(d-1)$-dimensional zig–zags are within that universal bound of the original faces of the $C_k'$. As long as the $C_k'$ contain a sphere of radius greater than that bound, the resulting $D_k$ will be non-empty. □

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