Math. Proc. Camb. Phil. Soc. (1987), **102**, 385
Printed in Great Britain

The Artin–Rees equations

By A. CARUTH

Department of Mathematics and Statistics, Portsmouth Polytechnic, Portsmouth PO1 2EG

(*Received* 9 *February* 1987)

Let R denote a commutative Noetherian ring with an identity element and N a finitely generated R-module. When K is a submodule of N and A an ideal of R the Artin–Rees lemma states that there is an integer $q \geqslant 0$ such that $A^nN \cap K = A^{n-q}(A^qN \cap K)$ for all $n \geqslant q$ (Rees[**4**]; Northcott[**3**], theorem 20, p. 210; Atiyah and Macdonald [**1**], proposition 10·9, p. 107; Nagata[**2**], theorem (3·7), p. 9). The above equation belongs to the family of module equations involving A and K which is considered below. We characterize, in terms of A and K, the set of submodules X of N for which there is an integer $q = q(X) \geqslant 0$ satisfying the equation

$$A^nN \cap X = A^{n-q}(A^qN \cap K) \quad \text{for all } n \geqslant q. \tag{1}$$

Equation (1), which we call the Artin–Rees equation related to A and K, gets its maximal force when X is largest and we determine the best possible solution in this sense. Notice that for any submodule X satisfying (1), $X \subseteq K :_N A^n$ for all $n \geqslant q(X)$. Since N is a Noetherian R-module ([**3**], proposition 1 (corollary), p. 177), there is an integer $t \geqslant 1$ such that $K :_N A^t = K :_N A^{t+n}$ for all $n \geqslant 0$. We define $M = K :_N A^t$ and prove, in Theorem 1, that $X = Q$ satisfies equation (1), for a suitable integer $q(Q) \geqslant 0$, if and only if $K \subseteq Q :_N A^v \subseteq M$ for some integer $v \geqslant 0$. In topological terms, the A-adic topology of K coincides with the topology induced by the A-adic topology of N on the subspace Q if the inequality $K \subseteq Q :_N A^v \subseteq M$ is satisfied. It follows that the solution set of equation (1) includes every submodule of N of the form $A^{n-r}K$ when $n \geqslant r = q(K)$ as well as every submodule lying between K and M. Hence, $X = M$ is the strongest solution, in the sense that M is the largest such submodule contained in $A^{n-s}(A^sN \cap K) :_N A^n$ for all $n \geqslant s = q(M)$. Recall that M is strictly larger than K if and only if A is contained in at least one prime ideal of R belonging to K ([**3**], theorem 14 (corollary 1), p. 193). Thus, equation (1) has a unique solution (necessarily $X = K$) if and only if A is not contained in any prime ideal of R belonging to any solution.

If L is a submodule of N and B an ideal of R, $L :_N B$ will be written as $L : B$. We need the following result which uses only the simplest properties of Noetherian modules.

Lemma 1. *Let L denote a submodule of N and k a positive integer such that $S = L : A^k = L : A^{k+u}$ for all $u \geqslant 0$. There exists an integer $m \geqslant 0$ such that $A^nN \cap L = A^nN \cap S$ for all $n \geqslant m$.*

Proof. Let $\{a_1, a_2, \ldots, a_r\}$ denote a minimal generating set for the ideal A. Since the submodules of N satisfy the ascending chain condition, integers $a(j) \geqslant 1$ can be chosen such that, for $j = 0, 1, \ldots, r-1$,

$$(L + a_1^{a(1)}N + \ldots + a_j^{a(j)}N) : a_{j+1}^{a(j+1)} = (L + a_1^{a(1)}N + \ldots + a_j^{a(j)}N) : a_{j+1}^{a(j+1)+s}$$

https://doi.org/10.1017/S0305004100067438 Published online by Cambridge University Press

for all integers $s \geqslant 0$. Let $p = k + \max\{a(1), \ldots, a(r)\}$ and when $1 \leqslant j \leqslant r$, let $b_j = a_j^p$, $L_j = L + b_1 N + \ldots + b_j N$ and $C_j = (b_1 N + \ldots + b_j N) \cap (L : b_1 R + \ldots + b_j R)$. We show first, using induction on j, that $C_j = (b_1 N + \ldots + b_j N) \cap L$ when $1 \leqslant j \leqslant r$.

When $j = 1$, $C_1 = ((L : b_1 R) : b_1 R) b_1 = (L : b_1 R) b_1 = b_1 N \cap L$, as required. Now assume that $C_{j-1} = (b_1 N + \ldots + b_{j-1} N) \cap L$ for all $j \leqslant r$ and consider

$$\begin{aligned} C_j = L_j \cap C_j &= L_j \cap (b_1 N + \ldots + b_j N) \cap (L : b_1 R + \ldots + b_{j-1} R) \cap (L : b_j R) \\ &\subseteq L_j \cap (b_1 N + \ldots + b_j N) \cap (L : b_1 R + \ldots + b_{j-1} R) \cap (L_{j-1} : b_j R). \end{aligned}$$

But

$$L_j \cap (L_{j-1} : b_j R) = L_{j-1}$$

(see [3], lemma 10, p. 111). Therefore,

$$\begin{aligned} C_j &\subseteq L_{j-1} \cap (L : b_1 R + \ldots + b_{j-1} R) \cap (b_1 N + \ldots + b_j N) \\ &= (L + C_{j-1}) \cap (b_1 N + \ldots + b_j N) = L \cap (b_1 N + \ldots + b_j N) \end{aligned}$$

using the modular law and the inductive hypothesis.

Hence, $C_j = (b_1 N + \ldots + b_j N) \cap L$ when $1 \leqslant j \leqslant r$. We now put $m = rp$ and deduce that $A^n N \cap S \subseteq C_r \subseteq L$ for all $n \geqslant m$. Accordingly, $A^n N \cap S \subseteq A^n N \cap L$ and the proof is complete, since the opposite inclusion is trivial.

COROLLARY. *If A is contained in a prime ideal P belonging to $\bigcap_{i=1}^{\infty} A^i N$, there is an integer $m \geqslant 1$ such that P is a prime ideal belonging to $A^n N$ for all $n \geqslant m$.*

Proof. Let $L = \bigcap_{i=1}^{\infty} A^i N$ in Lemma 1 and $d \in N$ such that $P = L : d$ ([3], theorem 19, p. 110). Then, by hypothesis, $Ad \in L$. Now, $(A^n N : d) \cap (S : d) \subseteq L : d$ for all $n \geqslant m$. But $S : d = R$, therefore $A^n N : d \subseteq L : d \subseteq A^n N : d$ when $n \geqslant m$. Accordingly, $P = A^n N : d$ for all $n \geqslant m$ implying that P is a prime ideal belonging to $A^n N$ for all $n \geqslant m$.

We are now in a position to characterize the solutions of the Artin–Rees equation related to A and K.

THEOREM 1. *Let Q denote a submodule of N. There exists an integer $q = q(Q) \geqslant 0$ such that $X = Q$ is a solution of the equation*

$$A^n N \cap X = A^{n-q}(A^q N \cap K) \quad \textit{for all } n \geqslant q$$

if and only if $K \subseteq Q : A^v \subseteq M$ for some integer $v \geqslant 0$.

Proof. Suppose that $K \subseteq Q : A^v \subseteq M$ for some integer $v \geqslant 0$. Then, from Lemma 1 with $L = K$, there is an integer $m_1 \geqslant 0$ such that

$$A^n N \cap K = A^n N \cap (Q : A^v) = A^n N \cap M \quad \text{for all } n \geqslant m_1. \tag{2}$$

Again, using Lemma 1 with $L = Q$, there is an integer $m_2 \geqslant 0$ such that

$$A^n N \cap Q = A^n N \cap (Q : A^v) = A^n N \cap (Q : A^s) \quad \text{for all } n \geqslant m_2, \tag{3}$$

where s is a positive integer having the property $Q : A^s = Q : A^{s+u}$ for all $u \geqslant 0$. Now put $q = q(Q) = \max\{m_1, m_2, q(K)\}$, where $q(K)$ denotes the Artin–Rees integer corresponding to the known solution $X = K$ in the stated equation. Then, from equations (2) and (3),

$$A^n N \cap Q = A^{n-q}(A^q N \cap K) \quad \text{for all } n \geqslant q.$$

https://doi.org/10.1017/S0305004100067438 Published online by Cambridge University Press

On the other hand, if there is an integer $q \geqslant 0$ such that $X = Q$ satisfies the above equation for all $n \geqslant q$, then $Q \subseteq M$ and $A^n K \subseteq Q$. Accordingly, $K \subseteq Q : A^n \subseteq M$ when $n \geqslant q$.

From Theorem 1 it follows immediately that the solution set of equation (1) is closed under intersection. For if Q_1, Q_2 are solutions of (1) there exist integers $c \geqslant 0$, $d \geqslant 0$ such that $K \subseteq Q_1 : A^c \subseteq M$ and $K \subseteq Q_2 : A^d \subseteq M$ and then $K \subseteq (Q_1 \cap Q_2) : A^e \subseteq M$, where $e = \max\{c, d\}$. Also, from the definition of M, if $K \subseteq Q \subseteq M$, then $K \subseteq Q : A^r \subseteq M$ for any integer $r \geqslant 0$, from which it follows that every submodule between K and M is a solution of the Artin–Rees equation related to A and K. Another subset of solutions is obtained by putting $X = K$ so that $K \subseteq A^{n-r}K : A^n \subseteq M$ when $n \geqslant r = q(K)$. Thus, every submodule $A^{n-r}K$ $(n \geqslant r)$ satisfies (1). When equation (1) takes its largest value $X = M$ and $n \geqslant s = q(M)$,

$$M = A^{n-s}(A^s N \cap K) : A^n = A^{n-s}K : A^n. \tag{4}$$

The well-known solution $X = K$ of the Artin–Rees equation related to A and K was originally formulated in terms of ideals of R (see [4]). The following statement can be inferred from Theorem 1 (with $X = M$) and equation (4).

COROLLARY. *Let a, b denote ideals of R. There exists an integer $k \geqslant 0$ such that, when $n \geqslant k$,*

(1) $a^n \cap (b : a^n) = a^{n-k}(a^k \cap b)$;

(2) $b : a^n = a^{n-k}(a^k \cap b) : a^n = a^{n-k}b : a^n = b : a^{n+1}$.

REFERENCES

[1] M. F. ATIYAH and I. G. MACDONALD. *Introduction to commutative algebra* (Addison-Wesley, 1969).

[2] M. NAGATA. *Local rings* (Interscience Publishers, 1962).

[3] D. G. NORTHCOTT. *Lessons on rings, modules and multiplicities* (Cambridge University Press, 1968).

[4] D. REES. Two classical theorems of ideal theory. *Proc. Cambridge Philos. Soc.* **52** (1956), 155–157.

https://doi.org/10.1017/S0305004100067438 Published online by Cambridge University Press