# BLOWING UP CHERN CLASSES

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*Introduction.* The behaviour of the Chern classes or of the canonical classes of an algebraic variety under a dilatation has been studied by several authors (Todd (8)–(11), Segre (5), van de Ven (12)). This problem is of interest since a dilatation is the simplest form of birational transformation which does not preserve the underlying topological structure of the algebraic variety. A relation between the Chern classes of the variety obtained by dilatation of a subvariety and the Chern classes of the original variety has been conjectured by the authors cited above but a complete proof of this relation is not in the literature.

In the present paper the formula conjectured by Todd and Segre is proved for algebraic varieties over an algebraically closed field of characteristic zero with the Chern/canonical classes defined as Chow classes of rational equivalence (2). The formula is true modulo torsion for varieties over an algebraically closed field of arbitrary characteristic. The proof leans heavily on the recent theory of Grothendieck ((1), (2)) on the Riemann–Roch Theorem and the writer understands from conversation with Grothendieck that he has obtained results similar to those contained in this paper. Theorem 1 has been anticipated by Washnitzer (13) who has proved the 'dual' exact sequence. Finally, several details of the proof are due to Dr M. F. Atiyah, to whom the writer wishes to express his thanks for his interest and the encouragement which he has given in the preparation of this paper. He is likewise indebted to Dr J. A. Todd. The writer was supported during this work by a scholarship from the Carnegie Trust for the Universities of Scotland.

*The Grothendieck ring.* (For the details of Grothendieck's theory we refer to (1), (2).) Let $X$ be a non-singular irreducible projective algebraic variety over an algebraically closed field of arbitrary characteristic. Let $'\mathscr{C}$ be the set of vector bundles on $X$ and $\mathscr{C}$ be the set of algebraic coherent sheaves on $X$. It is well known (7) that the vector bundles may be identified with the locally free sheaves on $X$, where the sheaf corresponding to a given vector bundle is the sheaf of germs of regular sections of the bundle. We shall make this identification in what follows.

From either $'\mathscr{C}$ or $\mathscr{C}$ we may derive a group in the following manner. Let $\mathscr{W}$ be either $'\mathscr{C}$ or $\mathscr{C}$ and let $F(X)$ be the free abelian group having as base the elements of $\mathscr{W}$. An element of $F(X)$ is then a formal linear combination

$$x = \Sigma n_i \mathscr{F}_i, \quad n_i \in \mathbf{Z}, \quad \mathscr{F}_i \in \mathscr{W}.$$

Let $S$ denote the exact sequence of elements of $\mathscr{W}$

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0,$$

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and let us associate with this exact sequence the element

$$Q(S) = \mathscr{F} - \mathscr{F}' - \mathscr{F}'' \in \mathscr{W}.$$

We form the quotient group of $F(X)$ by the subgroup generated by the $Q(S)$, where $S$ runs through all exact sequences of three terms. This group we denote by $'K(X)$ or by $K(X)$, according as $\mathscr{W} = {}'\mathscr{C}$ or $\mathscr{C}$. It is then an important result of the theory that the canonical homomorphism $'K(X) \to K(X)$ induced by the inclusion ${}'\mathscr{C} \subset \mathscr{C}$ is an isomorphism.

Because of this isomorphism operations defined in $'K(X)$ can be extended to $K(X)$. In $'K(X)$ we have a multiplication induced by the tensor product of vector bundles. The induced product $\mathscr{F}.\mathscr{G}$, $\mathscr{F}$, $\mathscr{G} \in K(X)$, is commutative and associative and may be directly defined by the formula

$$\mathscr{F}.\mathscr{G} = \sum_p (-1)^p \operatorname{Tor}_p^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}).$$

Thus $K(X)$ has ring structure. We call $K(X)$ the Grothendieck ring of the variety $X$. (Note that if $\mathscr{F} \in \mathscr{C}$ then by abuse of notation we still denote by $\mathscr{F}$ its image in $K(X)$.)

*The map $f^!$.* Let $f: Y \to X$ be a regular map, $Y$, $X$ non-singular irreducible projective algebraic varieties. Then we denote by $f^!$ the induced map of bundles on $X$ into bundles on $Y$. Since $'K(X)$ is isomorphic with $K(X)$ this extends to

$$f^! : K(X) \to K(Y).$$

The image $f^!(\mathscr{F})$, $\mathscr{F} \in K(X)$, is directly defined by the formula

$$f^!(\mathscr{F}) = \sum_p (-1)^p \operatorname{Tor}_p^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{F});$$

$f^!$ is a ring homomorphism and $(fg)^! = g^! f^!$ for the composition of maps

$$g: Z \to Y, \quad f: Y \to X.$$

*The map $f_!$.* Let $Y$ be a non-singular subvariety of $X$, and let $j$ be the injection $j: Y \to X$. Then an algebraic coherent sheaf $\mathscr{F}$ on $Y$ can be extended by $O$ over $X$ to give an algebraic coherent sheaf over $X$, which we denote by $j_!(\mathscr{F})$. In this way we obtain an additive homomorphism $j_! : K(Y) \to K(X)$. More generally $f_!$ can be defined for any proper regular map $f$. However, we shall consider here only the case of an injection.

$j_!$ is not compatible with multiplication. Instead we have for $x \in K(X)$, $y, y' \in K(Y)$ the formulae

$$\begin{aligned} j_!(y.j^!(x)) &= j_!(y).x, \\ j_!(y).j_!(y') &= j_!(y.y').j_!(\mathscr{O}_{YX}). \end{aligned} \tag{1}$$

*Chern classes.* The fact that $'K(X)$ is isomorphic to $K(X)$ also enables us to extend the definition of the Chern classes from vector bundles to algebraic coherent sheaves. This is done in a straightforward manner in the case where the ground field is the field of complex numbers. Grothendieck has extended this definition to the case of an arbitrary ground field, replacing $H^*(X)$ by the graded ring $A(X)$ of classes of cycles on $X$ under Chow rational equivalence ((2), (6)). We can only summarize here the results which we shall require.

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Denoting by $A(X)$ either $H^*(X)$ or the ring $A(X)$ just mentioned, we can define for an element $x \in K(X)$ its Chern class $c(x) \in A(X)$ and its Chern character $\mathrm{ch}(x) \in A(X) \otimes \mathbf{Q}$, $\mathbf{Q}$ being the field of rationals. If we write formally $c(x) = \Pi(1+a_i)$, then $\mathrm{ch}(x) = \mathrm{rg}(x) + \Sigma(e^{a_i} - 1)$ where rg: $K(X) \to \mathbf{Z}$ is the unique homomorphism which maps a vector bundle on to its dimension. Now $c$ satisfies the usual axioms for the Chern classes; in particular $c(x+x') = c(x).c(x')$. Also ch: $K(X) \to A(X) \otimes \mathbf{Q}$ is a ring homomorphism. We define the Todd class $T(x) \in A(X) \otimes \mathbf{Q}$ of an element $x \in K(X)$ by the formula $T(x) = \Pi a_i/(1-e^{-a_i})$; then $T(x+x') = T(x).T(x')$.

*The Grothendieck Riemann–Roch theorem.* If $f$: $Y \to X$ is a regular map we have an induced ring homomorphism $f^*$: $A(X) \to A(Y)$, and if $f$ is also proper we have an induced group homomorphism $f_*$: $A(Y) \to A(X)$. For $j$ an injection and $a \in A(X)$, $b, b' \in A(Y)$, we have formulae analogous to (1)

$$j_*(b.j^*(a)) = j_*(b).a,$$
$$j_*(b).j_*(b') = j_*(b.b').j_*(1_Y). \tag{2}$$

Also $cf^! = f^*c$ and $\mathrm{ch} f^! = f^* \mathrm{ch}$. However, $\mathrm{ch} f_!$ is not equal to $f_* \mathrm{ch}$. Instead we have the formula

$$f_*(\mathrm{ch}(y).T(Y)) = \mathrm{ch}(f_!(y)).T(X),$$

where $y \in K(Y)$ and the two sides are considered as elements of $A(X) \otimes \mathbf{Q}$. This formula constitutes the Grothendieck Riemann–Roch theorem. For the proof of this theorem and its relationship to other 'Riemann–Roch' theorems see (1).

When $j$: $Y' \to X'$ is an injection and $Y'$ is a subvariety of $X'$ of codimension one, the Riemann–Roch theorem takes the form

$$\mathrm{ch} j_!(y) = j_*(\mathrm{ch}(y).T(\mathscr{L})^{-1}), \tag{3}$$

where $y \in K(Y')$ and $\mathscr{L}$ is the normal line bundle of $Y'$ in $X'$. Let $j_*(1_{Y'}) = u$, $v = j^*u$. then $c(\mathscr{L}) = 1+v$ ((1), Prop. 13) and we may write (3) in the form

$$\mathrm{ch} j_!(y) = j_*[(1/v)(1-e^{-v}).\mathrm{ch}(y)]. \tag{3'}$$

Modulo torsion, (3') is equivalent to a formula for Chern *classes*

$$c(j_!(y)) = 1 + j_*(1/v)[c(y)/(\sum_i (1-v)^i c_{q-i}(y)) - 1], \tag{3''}$$

where $q = \mathrm{rg}(y)$, which has been proved directly by Grothendieck (unpublished) only for the case where the ground field has characteristic zero. The right-hand side of (3'') is unambiguously defined; for the expression, $B$ say, in square brackets is equal to $v.P(v, c_i)$, where $P(v, c_i)$ is a well-defined formal power series in $v$ and the $c_i$, and we set $(1/v)B = P(v, c_i)$. We shall use the symbol $(1/v)$ only in such a sense. Note that if $c(y)$ is expressed as a formal product $\Pi(1+b_i)$ then

$$\sum_i (1-v)^i c_{q-i}(y) = \Pi(1+b_i-v).$$

*Dilatations.* Let $X$ be a non-singular irreducible projective algebraic variety of dimension $n$ and $Y$ a non-singular irreducible subvariety of $X$ of dimension $m$. Then

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by dilatation (4) of $X$ along $Y$ we obtain new varieties $X'$ and $Y'$ of dimensions $n$ and $n-1$, respectively. $Y'$ is a subvariety of $X'$ of codimension one. We have the commutative diagram of varieties and maps:

$$\begin{array}{ccc} X' & \longleftarrow & Y' \\ {\scriptstyle f}\downarrow & {\scriptstyle j} & \downarrow{\scriptstyle g} \\ X & \underset{i}{\longleftarrow} & Y \end{array}$$

where $f$, $g$ are projections, $i$, $j$ injections. $f$ is biregular except on $Y'$, and $g = f \mid Y'$. $Y'$ has the structure of a projective fibre bundle over $Y$ with fibre projective space of $n-m-1$ dimensions.

Associated with this configuration we have the following fibre bundles and their images in the appropriate Grothendieck rings:

$\mathscr{T}_X, \mathscr{T}_Y, \mathscr{T}_{X'}, \mathscr{T}_{Y'}$ the tangent bundles of $X$, $Y$, $X'$, $Y'$, respectively;
$\mathscr{E}$ the normal bundle of $Y$ in $X$;
$\mathscr{L}$ the normal line bundle of $Y'$ in $X'$;
$\mathscr{G}$ the tangent bundle along the fibres of $Y'$.

From the definitions of $\mathscr{E}$, $\mathscr{L}$ and $\mathscr{G}$, it follows directly that

$$i^!\mathscr{T}_X = \mathscr{T}_Y + \mathscr{E}, \tag{4}$$

$$j^!\mathscr{T}_{X'} = \mathscr{T}_{Y'} + \mathscr{L} = g^!\mathscr{T}_Y + \mathscr{G} + \mathscr{L}. \tag{5}$$

*Local equations.* It can be shown ((3), (4)) that if $p'$ and $p = f(p')$ are a pair of corresponding points in the dilatation and if $p$ lies on the subvariety $Y$, then it is possible to choose local uniformizing parameters $(x_1, \ldots, x_n)$ on $X$ at $p$ and $(x'_1, \ldots, x'_n)$ on $X'$ at $p'$ such that

$$x_i \circ f = x'_i \qquad (i = 1, \ldots, m),$$

$$x_j \circ f = x'_j x'_n \quad (j = m+1, \ldots, n-1),$$

$$x_n \circ f = x'_n,$$

where the local equations of $Y$ at $p$ are $x_j = 0$ $(j = m+1, \ldots, n)$ and the local equation of $Y'$ at $p'$ is $x'_n = 0$. The map $f$ induces a map of the tangents on $X'$ into the tangents on $X$ and therefore induces maps $\phi$, say, of $\mathscr{T}_{X'}$ into $f^!\mathscr{T}_X$ and $\psi$, say, of $j^!\mathscr{T}_{X'}$ into $j^!f^!\mathscr{T}_X$. It will be convenient to regard these maps both as maps of vector bundles mapping fibres into fibres and as maps of locally free sheaves mapping stalks into stalks. Locally $\phi$ is an isomorphism except on $Y'$. At $p'$ on $Y'$ $\phi$ is given locally by the equations

$$\phi\left(\frac{\partial}{\partial x'_i}\right) = \frac{\partial}{\partial x_i} \quad (i = 1, \ldots, m), \tag{6a}$$

$$\phi\left(\frac{\partial}{\partial x'_j}\right) = x'_n \frac{\partial}{\partial x_j} \quad (j = m+1, \ldots, n-1), \tag{6b}$$

$$\phi\left(\frac{\partial}{\partial x'_n}\right) = \frac{\partial}{\partial x_n} + \sum_j x'_j \frac{\partial}{\partial x_j}, \tag{6c}$$

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where $\partial/\partial x'_k$ and $\partial/\partial x_k$ $(k = 1, \ldots, n)$ are local bases either for the bundles or for the sheaves $\mathscr{T}_{X'}$ and $f^!\mathscr{T}_X$, according as we interpret the symbols as vectors or as local sections. The local equations of $\psi: j^!\mathscr{T}_{X'} \to j^!f^!\mathscr{T}_X$ will be

$$\psi\left(\frac{\partial}{\partial x'_i}\right) = \frac{\partial}{\partial x_i}, \tag{6a'}$$

$$\psi\left(\frac{\partial}{\partial x'_j}\right) = 0, \tag{6b'}$$

$$\psi\left(\frac{\partial}{\partial x'_n}\right) = \frac{\partial}{\partial x_n} + \sum_j x'_j \frac{\partial}{\partial x_j}, \tag{6c'}$$

$x'_n$ being equal to zero on the subvariety $Y'$.

First, we regard $\phi$ and $\psi$ as sheaf maps. Let us denote by $\mathscr{I}(Y')$ the sheaf of local ideals of $Y'$ defined by the inclusion $j: Y' \to X'$. Then we have the exact sequence

$$0 \to \mathscr{I}(Y') \to \mathscr{O}_{X'} \to j_!\mathscr{O}_{Y'} \to 0$$

of sheaves on $X'$. Now by (6) we have

(*a*) $\phi$ is a monomorphism, since $\phi$ has determinant $(x'_n)^{n-1}$ at $p'$ and $x'_n$ is a non-zero element of $\mathscr{O}_{p',X'}$;

(*b*) the quotient sheaf $\operatorname{coker}\phi = f^!\mathscr{T}_X/\phi(\mathscr{T}_{X'})$ is annihilated by $\mathscr{I}(Y')$; for, since $\mathscr{I}(Y')_{p'}$ is a free $\mathscr{O}_{p',X'}$-module with generator $x'_n$, the equations for $\phi$ show that

$$\mathscr{I}(Y')_{p'} \otimes_{\mathscr{O}_{p',X'}} (f^!\mathscr{T}_X)_{p'} \subset \phi(\mathscr{T}_{X'})_{p'}.$$

From (*b*) we have

(*c*) $\operatorname{coker}\phi = j_!(\mathscr{O}_{Y'} \otimes \operatorname{coker}\phi)$; that is, $\operatorname{coker}\phi$ is the extension by zero of the sheaf of $\mathscr{O}_{Y'}$-modules on $Y'$, $\mathscr{O}_{Y'} \otimes \operatorname{coker}\phi$. Since $\mathscr{T}_{X'}$ and $f^!\mathscr{T}_X$ are locally free

$$j^!\mathscr{T}_{X'} = \mathscr{O}_{Y'} \otimes \mathscr{T}_{X'} \quad \text{and} \quad j^!f^!\mathscr{T}_X = \mathscr{O}_{Y'} \otimes f^!\mathscr{T}_X.$$

Also $\mathscr{O}_{Y'} \otimes$ is a right exact functor. Therefore, by operating with this functor on the exact sequence

$$0 \to \mathscr{T}_{X'} \to f^!\mathscr{T}_X \to \operatorname{coker}\phi \to 0$$

we obtain $\mathscr{O}_{Y'} \otimes \operatorname{coker}\phi = \operatorname{coker}\psi$. This, together with (*c*), gives

(*d*) $\operatorname{coker}\phi = j_!\operatorname{coker}\psi$.

Let us now regard $\psi$ as a map of bundles. Then

(*e*) equation (6*c'*) is just the local equation of the natural embedding $\mathscr{L} \to g^!\mathscr{E}$;

(*f*) $\operatorname{coker}\psi$ is a bundle and is the cokernel, $\mathscr{F}$ say, of this embedding.

This follows directly from equations (6') together with (*e*). By (*f*) and the correspondence between vector bundles and locally free sheaves we have finally

(*g*) the sheaf $\operatorname{coker}\psi$ is locally free. The associated vector bundle is the bundle $\mathscr{F}$ defined in (*f*).

We summarize all these results in

THEOREM 1. (cf. Washnitzer (13)).

(i) *$\phi: \mathscr{T}_{X'} \to f^!\mathscr{T}_X$ regarded as a sheaf homomorphism is a monomorphism.*

(ii) *The quotient sheaf $f^!\mathscr{T}_X/\phi(\mathscr{T}_{X'})$, though not itself locally free, is the extension by zero of a locally free sheaf of $\mathscr{O}_{Y'}$-modules $\mathscr{F}$ on $Y'$.*

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(iii) *$\mathscr{F}$ regarded as a vector bundle on $Y'$ is the cokernel of the natural embedding of $\mathscr{L}$ in $g^!\mathscr{E}$.*

(iv) *In $K(X')$ we have the identity*

$$f^!\mathscr{T}_X - \mathscr{T}_{X'} = j_!(g^!\mathscr{E} - \mathscr{L}). \tag{7}$$

(iv) is an immediate consequence of (i), (ii) and (iii), which we have just proved.

By Theorem 1,

$$0 \to \mathscr{T}_{X'} \xrightarrow{\phi} f^!\mathscr{T}_X \to j_!\mathscr{F} \to 0$$

is an exact sequence of sheaves. If we denote the dual tangent bundles by $\mathscr{T}_X{}^*$, $\mathscr{T}_{X'}{}^*$, then by a similar argument there is a locally free sheaf of $\mathscr{O}_{Y'}$-modules $\mathscr{G}$ such that the sequence

$$0 \to f^!\mathscr{T}_X{}^* \xrightarrow{\phi^*} \mathscr{T}_{X'}{}^* \to j_!\mathscr{G}^* \to 0$$

is exact. The relation between $\mathscr{F}$ and $\mathscr{G}$ becomes clear if we tensor each sequence by $\mathscr{O}_{Y'}$, for we obtain in this way two four-term exact sequences of locally free sheaves on $Y'$ which are dual to each other as exact sequences of vector bundles, namely

$$0 \to \mathrm{Tor}_1^{\mathscr{O}_{X'}}(\mathscr{F}, \mathscr{O}_{Y'}) \to j^!\mathscr{T}_{X'} \xrightarrow{\psi} j^!f^!\mathscr{T}_X \to \mathscr{F} \to 0$$

and

$$0 \to \mathrm{Tor}_1^{\mathscr{O}_{X'}}(\mathscr{G}^*, \mathscr{O}_{Y'}) \to j^!f^!\mathscr{T}_X{}^* \xrightarrow{\psi^*} j^!\mathscr{T}_{X'}{}^* \to \mathscr{G}^* \to 0.$$

By use of the exact sequence

$$0 \to \mathscr{I}(Y') \to \mathscr{O}_{X'} \to j_!\mathscr{O}_{Y'} \to 0,$$

$\mathrm{Tor}_1^{\mathscr{O}_{X'}}(\mathscr{F}, \mathscr{O}_{Y'}) = \mathscr{G}$ is easily seen to be isomorphic to $\mathscr{F} \otimes \mathscr{L}^*$, the restriction of $\mathscr{I}(Y')$ to $Y'$ being equal to $\mathscr{L}^*$. So $\mathscr{F} = \mathscr{G} \otimes \mathscr{L}$. Also $\mathscr{G}$, being the kernel of the map $\psi\colon j^!\mathscr{T}_{X'} \to j^!f^!\mathscr{T}_X$, is just the tangent bundle along the fibres of $Y'$.

*The Todd–Segre formula.* Retaining the notations introduced and writing $\mathrm{ch}\, X$ for $\mathrm{ch}\, \mathscr{T}_X$, etc., we have

THEOREM 2. *The behaviour of the Chern character of an irreducible non-singular projective algebraic variety under dilatation is expressed by the formula*

$$f^*\,\mathrm{ch}\, X - \mathrm{ch}\, X' = j_*\left[(g^*\,\mathrm{ch}\, \mathscr{E} - e^v)\left(\frac{1-e^{-v}}{v}\right)\right]. \tag{8}$$

*This formula is equivalent modulo torsion to the formula expressing the behaviour of the Chern classes under dilatation conjectured by Todd and Segre, which we write* (cf. van de Ven (12)) *in the form*

$$c(X') = f^*c(X) + j_*\{g^*c(Y)\,(1/v)\,[(1+v)\sum_i (1-v)^i g^*c_{n-m-i}(\mathscr{E}) - g^*c(\mathscr{E})]\}. \tag{8'}$$

(8) *holds for varieties defined over an algebraically closed field of arbitrary characteristic.* (8′) *is true if the field has characteristic zero.*

*Proof.* (*a*) By Theorem 1 we have the identity in $K(X')$

$$f^!\mathscr{T}_X - \mathscr{T}_{X'} = j_!(g^!\mathscr{E} - \mathscr{L}). \tag{7}$$

Since ch is a ring homomorphism,

$$\mathrm{ch}\, f^!\mathscr{T}_X - \mathrm{ch}\, \mathscr{T}_{X'} = \mathrm{ch}\, j_!(g^!\mathscr{E} - \mathscr{L}).$$

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Using the Grothendieck Riemann–Roch theorem (3′) to evaluate the right-hand side we obtain equation (8).

(*b*) Putting $\mathscr{L}-g^!\mathscr{E}$ for $y$ in (3″) we have, again by (7),

$$\begin{aligned} c(\mathscr{T}_{X'}-f^!\mathscr{T}_X) &= c(X')/f^*c(X) \\ &= 1+j_*\frac{1}{v}\left[\frac{c(\mathscr{L})\sum_i(1-v)^i g^*c_{n-m-i}(\mathscr{E})}{g^*c(\mathscr{E})\sum_i(1-v)^i c_{1-i}(\mathscr{L})}-1\right] \\ &= 1+j_*(1/v)\,[(1+v)\sum_i(1-v)^i g^*c_{n-m-i}(\mathscr{E})/g^*c(\mathscr{E})-1]. \end{aligned} \tag{9}$$

Now by (4) $$i^!\mathscr{T}_X = \mathscr{T}_Y+\mathscr{E}.$$

Therefore $$j^!f^!\mathscr{T}_X = g^!i^!\mathscr{T}_X = g^!\mathscr{T}_Y+g^!\mathscr{E},$$

and so $$j^*f^*c(X) = g^*c(Y)\,g^*c(\mathscr{E});$$

that is $$1/g^*c(\mathscr{E}) = g^*c(Y)j^*[1/f^*c(X)].$$

Using this relation in (9) together with (2) we obtain finally formula (8′).

The above argument constitutes a proof of the Todd–Segre formula for an algebrai variety under dilatation. The same argument will clearly apply to the case of a complex manifold under dilatation so soon as the Riemann–Roch theorem is proved in this case.

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