2D and 3D low frequency aerodynamics

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ABSTRACT

Unsteady aerodynamic loads on aircraft configurations are used for aeroelastic or flight dynamic analyses. The sources for deriving these loads include strip theory aerodynamics and three-dimensional panel methods. In some applications the behaviour of the unsteady air loads as the frequency approaches zero is important, and it is well known that the behaviour of strip theory aerodynamics employing the exact circulation function differs qualitatively from that of the three-dimensional panel methods such as the subsonic doublet lattice method (DLM). Theoretical results from an earlier study of the low frequency behaviour of the DLM are used here to show the relationship between the DLM and strip theory and the relationship is verified by a numerical example.

NOMENCLATURE

b reference length, usually mean semi-chord
$[B]$ modal damping matrix in flutter equation
$\{C_p\}$ pressure coefficient distribution, $\{C_p^R\}+i\{C_p^I\}$
$\{C_p^R\}$ real part of the pressure coefficient distribution
$\{C_p^I\}$ imaginary part of the pressure coefficient distribution
$[D]$ downwash matrix, $[D^R]+i[D^I]$
$[D^R]$ real part of the downwash matrix
$[D^I]$ imaginary part of the downwash matrix
D_{1rs} planar part of the incremental downwash factor
e aerodynamic strip semi-width
k reduced frequency, $\omega b/U$
k_1 $\omega r_1/U$
K_1 factor in numerator of planar parts of kernel, also Bessel function in Equation (8)
K_{10} steady part of the planar kernel function
$[K]$ modal stiffness matrix in flutter equation
$[M]$ modal mass matrix in flutter equation
$[Q]$ matrix of generalised unsteady aerodynamic forces, $[Q^R]+i[Q^I]$
$[Q^R]$ real part of Q
$[Q^I]$ imaginary part of Q
r_1 radial distance from sending point to receiving point
T_1 direction cosine function
U air speed
$\{w\}$ downwash distribution, $\{w^R\} + i\omega_r\{w^I\}$
$\{w^R\}$ real part of the downwash distribution
$\{w^I\}$ factor in the imaginary part of the downwash distribution
x streamwise distance from doublet line midpoint to receiving point
Δx_s box chord length
λ_s doublet line sweep angle
ρ density
ω angular frequency
ω_r wave number, ω/U

1.0 INTRODUCTION

Unsteady aerodynamic analysis is often used to generate generalised unsteady aerodynamic forces (GAFs) as input to the flutter equation. In the *p-k* formulation of Rodden, Harder and Bellinger[1] the imaginary parts of the generalised forces appear as the calculated

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values divided by the reduced frequency. In the g-method of Chen[2] the derivative of the generalised forces are taken with respect to reduced frequency. In cases involving static aeroelastic divergence, the limits of the imaginary parts of the generalised forces divided by reduced frequency are required in the case of the *p-k* formulation, and the derivatives of the generalised forces with respect to reduced frequency at zero reduced frequency are required in the g-method – essentially the same thing. The behaviour of the quotient or the derivative as the reduced frequency approaches zero depends on the aerodynamic method used. In the case of the subsonic DLM the limit exists and the exponential approximation used in the kernel function evaluation introduces some subtle changes in how the limit is approached[3]. In the case of strip theory using the exact Theodorsen circulation function, the limit does not exist, but when using the W.P. Jones approximation to the circulation function the limit is finite.

The expressions derived in Ref. 3 for the limiting behaviour of the DLM at low frequency are used to show the link between the low frequency behaviour of strip theory using the exact circulation function and that of the DLM.

2.0 FLUTTER EQUATION

The general form of the flutter equation is;

$$\left[[M]p^2+[B]p+[K]-\tfrac{1}{2}\rho V^2[Q(pb/U)]\right]\{q\}=0 \qquad \ldots (1)$$

The problem with solving this form of the flutter equation is that the aerodynamic coefficients are generally only available for harmonic motion. Rodden, Harder and Bellinger[1] divided the aerodynamic matrix into an aerodynamic stiffness matrix and an aerodynamic damping matrix, so that the approximation reads;

$$Q(pb/U)\approx \mathrm{Re}(Q(ik))+\frac{pb}{kU}\mathrm{Im}(Q(ik)) \qquad \ldots (2)$$

This approximation leads to the equation of motion.

$$\left[[M]p^2+\left([B]-\tfrac{1}{2}\rho bU\,\mathrm{Im}(Q(ik))/k\right)p+[K]-\tfrac{1}{2}\rho U^2\,\mathrm{Re}(Q(ik))\right]\{q\}=0 \qquad \ldots (3)$$

This equation is always a real general eigenvalue problem, with the roots being either real or complex conjugate pairs. The aerodynamic approximation used in the g-method of Chen[2] is a first-order expansion from the imaginary axis, *viz*;

$$Q(p)\approx Q(ik)+\left(\frac{pb}{U}-ik\right)\frac{\mathrm{d}(Q(ik))}{\mathrm{d}(ik)} \qquad \ldots (4)$$

leading to the equation of motion;

$$\left[[M]p^2+\left([B]+\tfrac{1}{2}\rho Ub\frac{\mathrm{d}(Q(ik))}{\mathrm{d}k}\right)p+[K]-\tfrac{1}{2}\rho U^2\left[Q(ik)-k\frac{\mathrm{d}(Q(ik))}{\mathrm{d}k}\right]\right]\{q\}=0 \qquad \ldots (5)$$

For real roots, i.e. $k = 0$, both Equations (3) and (5) reduce to;

$$\left[[M]p^2+\left([B]-\tfrac{1}{2}\rho Ub\frac{\mathrm{d}(\mathrm{Im}(Q(ik)))}{\mathrm{d}k}\bigg|_{k=0}\right)p+[K]-\tfrac{1}{2}\rho U^2Q(0)\right]\{q\}=0 \qquad \ldots (6)$$

The term $\mathrm{d}(\mathrm{Im}(Q(ik)))/\mathrm{d}k\big|_{k=0}$ is the subject of this study.

3.0 STRIP THEORY AERODYNAMICS

Von Kármán and Sears[4] expressed the lift and moment about the midchord of an aerofoil executing pitching and heaving motions in terms of Bessel functions. Rodden and Stahl[5] rewrote the expressions for lift and quarter-chord moment given by Spielberg[6] in the following convenient form;

$$L(\omega)=\pi\rho b^2\left[\omega^2h+\left(\frac{\omega^2}{2}-\frac{i\omega U}{b}\right)b\alpha\right]+2\pi\rho Ub\left[\frac{C(k)}{i\omega}\right]\left[\omega^2h+\left(\omega^2-\frac{i\omega U}{b}\right)b\alpha\right]$$

$$M(\omega)=\pi\rho b^3\left[\frac{\omega^2}{2}h+\left(\frac{3\omega^2}{8}-\frac{i\omega U}{b}\right)b\alpha\right] \qquad \ldots (7)$$

where h and α are the heave and pitch amplitudes, respectively, and $C(k)$ is the Theodorsen circulation function. The cambering deformation and loads were omitted here for clarity. W.P. Jones[7] suggested a two-term exponential approximation to the circulation function:

$$C(k)=\frac{K_1(ik)}{K_1(ik)+K_0(ik)}\approx 1-\sum\alpha_n/(1-i\beta_nk) \qquad \ldots (8)$$

with $\alpha_1 = 0{\cdot}165$, $\alpha_2 = 0{\cdot}335$, $\beta_1 = 0{\cdot}041$ and $\beta_2 = 0{\cdot}320$. The derivative of the lift force with respect to reduced frequency, at zero reduced frequency, will only be finite if the derivative of the circulation function is finite. The derivative of Jones's approximation is finite at $k = 0$, but the derivative of the exact circulation function is infinite at $k = 0$.

4.0 DOUBLET LATTICE METHOD

In Ref. 3 the limiting value of Q^l/k as k approaches zero, as calculated by the DLM, was determined. The essential part of the derivation is repeated here for completeness. The limiting value of the quotient is simply the derivative of Q^l with respect to k at $k = 0$. However, unless the geometry is normalised by the reference length, the reduced frequency does not have significance at all levels of the DLM and it is more convenient to take the derivative with respect to the wave number. Only the planar case is considered, however, the non-planar case can be treated in a similar way.

$$\lim_{k=0}\frac{Q^l}{k}=\left(\frac{\mathrm{d}Q^l}{dk}\right)_{k=0}=\frac{1}{b}\left(\frac{\mathrm{d}Q^l}{\mathrm{d}\omega_r}\right)_{\omega_r=0} \qquad \ldots (9)$$

The generalised forces are determined as area integrals of the pressure distribution, which in turn is solved from the downwash equation. The downwash equation reads;

$$\{w\}=[D]\{C_p\} \qquad \ldots (10)$$

Differentiating this expression with respect to wave number and making the appropriate substitutions for zero frequency leads to the following expressions for the steady pressure distribution and the derivative of the imaginary part of the pressure distribution with respect to wave number;

$$\{C_p^R\}=\left[D^R\right]^{-1}\{w^R\} \qquad \ldots (11)$$

$$\frac{\mathrm{d}\{C_p^I\}}{\mathrm{d}\omega_r}=\left[D^R\right]^{-1}\left(\{w^I\}-\frac{\mathrm{d}[D^I]}{\mathrm{d}\omega_r}\{C_p^R\}\right) \qquad \ldots (12)$$

The limiting value of Q^l/k at $k = 0$ is found by integrating the derivative of the imaginary part of the pressure distribution over the surface area of the configuration. It remains to determine the derivative of the imaginary part of the downwash matrix. The planar part of the incremental downwash factor is given in Ref. 8 (after substituting ω_r for ω/U) as;

$$D_{1rs}=\frac{\Delta x_s}{8\pi}\int_e\frac{\{K_1\exp[-i\omega_r(\bar{x}-\bar{\eta}\,\mathrm{Tan}\lambda_s)]-K_{10}\}T_1}{r_1^2}\mathrm{d}\eta \qquad \ldots (13)$$

The derivative of the downwash factor with respect to wave number at zero frequency is;

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$$\frac{dD_{1rs}}{d\omega_r} = \frac{\Delta x_s}{8\pi}\int_{-e}^{e} \frac{\left\{\frac{dK_1}{d\omega_r} - i(x-\eta \mathrm{Tan}\lambda_s)K_{10}\right\}T_1}{r_1^2}d\eta \quad \ldots (14)$$

The integral for the derivative of the downwash factor is evaluated in the same way as the integral for the incremental downwash. The kernel numerator is given in Ref. 4 as;

$$K_1 = -I_1 - \left(\exp(-ik_1u_1)\right)Mr_1 / R\left(1+u_1^2\right)^{1/2} \quad \ldots (15)$$

where;

$$R = \left[(\bar{x}-\bar{\eta}\mathrm{Tan}\lambda_s)^2 + \beta^2 r_1^2\right]^{1/2} \quad \ldots (16)$$

$$u_1 = \left(MR - \bar{x} + \bar{\eta}\mathrm{Tan}\lambda_s\right)/\beta^2 r_1 \quad \ldots (17)$$

$$I_1 = \int_{u_1}^{\infty}\left(1+u^2\right)^{-3/2}\exp(-ik_1u)du \quad \ldots (18)$$

Noting the relationship between k_1 and ω_r, the derivative of the kernel numerator with respect to ω_r at zero frequency is:

$$\frac{dK_1}{d\omega_r} = r_1\left(-\frac{dI_1}{dk_1} + iu_1Mr_1 / R\left(1+u_1^2\right)^{1/2}\right) \quad \ldots (19)$$

The derivative of I_1 with respect to k_1 at zero frequency is:

$$\frac{dI_1}{dk_1} = \int_{u_1}^{\infty} -iu\left(1+u^2\right)^{-3/2}du = -i\left(1+u_1^2\right)^{-1/2} \quad \ldots (20)$$

5.0 APPLICATION

The imaginary parts of some generalised forces for the classic BAH wing example[9], calculated using strip theory with the exact circulation function, with the Jones approximation to the circulation function, and using the DLM, is plotted against reduced frequency in Fig. 1. The difference in the behaviours of the methods at zero reduced frequency is readily apparent.

The relationship between the three-dimensional DLM and strip theory is investigated by spanwise integration of the three-dimensional DLM influence coefficients for incompressible flow. Integrating Equations(13) and (14) with M = 0 and λ_s = 0, and ignoring the subtraction of the steady part of the kernel numerators, yields

$$D_{rs} = -\frac{\Delta x_s}{4\pi x}\left\{\frac{x}{e} + \sqrt{1+(x/e)^2}\right\} \quad \ldots (21)$$

$$\frac{dD_{rs}}{d\omega_r} = -i\frac{\Delta x_s}{4\pi}\left\{\frac{1}{2}\ln\frac{\sqrt{1+(x/e)^2}-1}{\sqrt{1+(x/e)^2}+1} + \frac{x}{e} + \sqrt{1+(x/e)^2}\right\} \quad \ldots (22)$$

The expression for the real part of the downwash factor is essentially the vortex formula and converges rapidly to the two-dimensional limit with increasing box aspect ratio, *e/x*. For high aspect ratios the behaviour is given by;

$$\lim_{e/x\to\infty} D_{rs} = -\frac{\Delta x_s}{4\pi x} \quad \ldots (23)$$

$$\lim_{e/x\to\infty} \frac{dD_{rs}}{d\omega_r} = i\frac{\Delta x_s}{4\pi}\left\{\ln\left(2(e/x)\right) - 1\right\} \quad \ldots (24)$$

One would therefore expect the three-dimensional DLM results for the limiting value of Q^I/k to approach infinity logarithmically with increasing aspect ratio. Although this agrees with the exact two-dimensional result, the two-dimensional analysis does not give any indication of the dependence on aspect ratio. The author is indebted to one of the referees for pointing out that the application of small perturbation analysis to the oblique sinusoidal gust problem[10] leads to logarithmic terms in spanwise wave number in the expressions for the wing lift coefficient. Since the spanwise wave number is analogous to the inverse of aspect ratio, this gives more credibility to the result derived above.

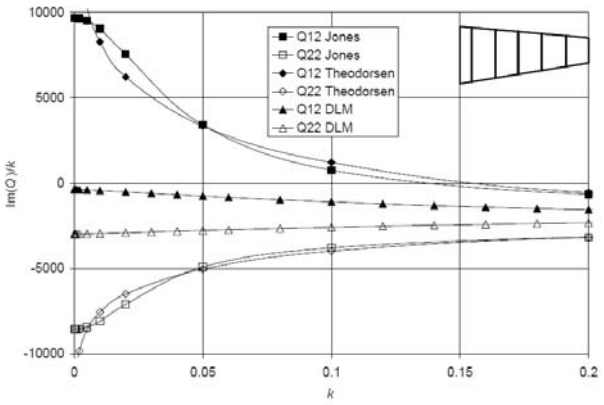


Figure 1. Low frequency behaviour of three different aerodynamic methods, BAH wing generalised forces.

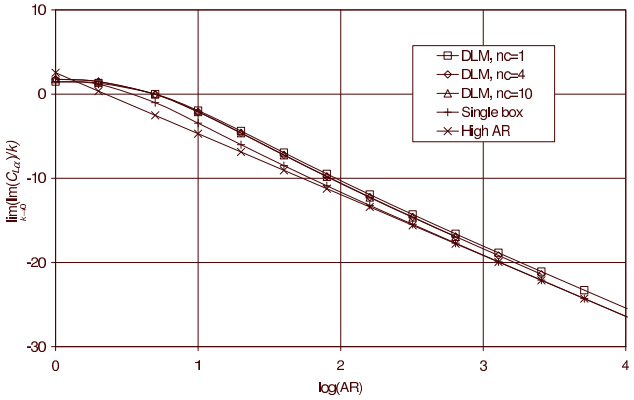


Figure 2. Comparison of three-dimensional DLM and single box results for a rectangular wing of varying aspect ratio.

The limiting value of $\mathrm{Im}(C_{l\alpha})/k$ at k = 0 for a rectangular wing of aspect ratios ranging from 1 to 10,240, represented by a single aerodynamic box, was calculated using Equations (21) and (22) as well as Equations (23) and (24). These results are compared in Fig. 2 to three-dimensional DLM results for 1, 4 and 10 chordwise boxes. The box aspect ratios were kept constant within each series by varying the number of spanwise strips. The comparison confirms the logarithmic relationship between the limiting value of $\mathrm{Im}(C_{l\alpha})$/k at k = 0 and the wing aspect ratio. This relationship suggests that Q^I/k for wings of practical interest will not exhibit singular behaviour at k = 0 and that the limit of AR = 10 suggested by Miles[11] may be disregarded.

6.0 CONCLUSION

The relationship between the analytical behaviour of the generalised forces of three-dimensional wings and the non-analytical behaviour of two-dimensional aerodynamics at k = 0 was investigated and revealed a logarithmic dependence on aspect ratio. This result adequately explains the apparent discrepancy between the DLM and strip theory, and implies that wings of practical interest will not display non-analytical behaviour.

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