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Spinors in Hilbert space

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1. *Introduction.* In 1913, É. Cartan discovered that the special orthogonal group $SO(k)$ has a 'two-valued' representation (i.e. a projective representation) on a complex vector space S of dimension 2^n, where $k = 2n$ or $2n+1$. The projective representation in question lifts to a true representation of the double cover Spin (k) of $SO(k)$. We restrict attention to the case $k = 2n$. Under the action of Spin $(2n)$, S breaks up into 2 irreducible subspaces:

$$S = S^+ + S^-.$$

The vectors in S are called *spinors* (relative to $SO(2n)$), those in S^+ or S^- are called *half-spinors* (4).

The embedding $SU(n) \longrightarrow SO(2n)$ lifts to Spin $(2n)$. The restriction to $SU(n)$ of the spin representation Δ_{2n} of Spin $(2n)$ is the exterior product representation Λ of $SU(n)$ on the exterior vector space $\Lambda(\mathbb{C}^n)$. Indeed, $\Lambda(\mathbb{C}^n)$ is a model for the space S of spinors. Further, the half-spin representations Δ_{2n}^+ and Δ_{2n}^- restrict to the even and odd parts of Λ, respectively. This connexion between spinors and complex structure is described in the article on Clifford Modules by Atiyah, Bott and Shapiro (1), and more recently by N. Hitchin in his article on Harmonic Spinors (6).

Let H be a real separable Hilbert space. Let $O(H)_1$ = all orthogonal operators on H of the form $1 + A$, where $\mathrm{Tr}\,|A| < \infty$. Thus each operator in $O(H)_1$ differs from the identity by a trace-class perturbation. Let $SO(H)_1$ be the connected component of the identity in $O(H)_1$. $SO(H)_1$ is a Banach-Lie group with fundamental group $\mathbb{Z}_2$.

In 1965, Shale & Stinespring (9) discovered the infinite-dimensional spinors relative to $SO(H)_1$. In 1972, de la Harpe (5) explicitly constructed the universal covering group Spin $(H)_\infty$ of $SO(H)_1$.

In this article, we construct the spin representation Δ of Spin $(H)_\infty$. To do this, we insert a complex structure J in H, thereby determining a complex Hilbert space $H_{\mathbb{C}}$. We consider $\Lambda(H_{\mathbb{C}}) = \bigoplus_{n \geqslant 0} \Lambda^n(H_{\mathbb{C}})$ as the Hilbert space direct sum of the complete exterior powers $\Lambda^n(H_{\mathbb{C}}) = H_{\mathbb{C}} \wedge \ldots \wedge H_{\mathbb{C}}$ (n factors). Then $\Lambda(H_{\mathbb{C}})$ is a model for the space of spinors relative to $SO(H)_1$.

We construct the half-spin representations Δ^+ and Δ^-. Δ^+ and Δ^- are faithful, norm continuous, irreducible representations of Spin $(H)_\infty$ on Λ^0 and Λ^1, the even and odd parts of $\Lambda(H_{\mathbb{C}})$ (Proposition 1).

We show that the connexion between spinors and complex structure persists in the infinite-dimensional case. The embedding $U(H_{\mathbb{C}})_1 \longrightarrow SO(H)_1$ of the unitary subgroup of $SO(H)_1$ lifts to the *complex* spin group Spin^c $(H)_\infty$; and the restriction of the *complex*

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spin representation Δ^c to the unitary subgroup is the exterior product representation Λ (Proposition 2).

The spin representations Δ_n of Cartan fit together nicely. The direct limit representation $\Delta_\infty = \varinjlim \Delta_n$ of $\mathrm{Spin}(\infty) = \varinjlim \mathrm{Spin}(n)$ is well defined. We prove that

$$\Delta|\mathrm{Spin}(\infty) = \Delta_\infty.$$

Incidentally, we construct a unitary group $\varinjlim U(2^n) = U(2^\infty)$, and a unitary representation k of $U(2^\infty)$ on the space $\Lambda(H_{\mathbb{C}})$ of spinors (Proposition 3).

The Λ-functor is, in the context of quantum field theory, the antisymmetric second-quantization functor (11), and $\Lambda(H_{\mathbb{C}})$ is the antisymmetric Fock space over $H_{\mathbb{C}}$. In § 5, we explain the intimate connexion between our spin representation Δ and the second quantization prescription of Irving Segal.

A rather different approach to spinors in Hilbert space may be found in the remarkable book by Dirac (12).

We begin by reviewing the connexion between spinors and complex structure in the finite-dimensional case. In § 2 we follow (1) pretty closely.

2. *Finite-dimensional spinors.* Give $\mathbb{C}^k$ the standard inner product. Then the complex Grassmann algebra

$$\Lambda(\mathbb{C}^k) = \sum_{j=0}^{k} \Lambda^j(\mathbb{C}^k)$$

inherits a natural inner product. In terms of an orthonormal basis $f_1, \ldots, f_k$ of $\mathbb{C}^k$ the elements

$$f_{i_1} \wedge f_{i_2} \wedge \ldots \wedge f_{i_r} \quad (i_1 < i_2 < \ldots < i_r)$$

form an orthonormal basis of $\Lambda(\mathbb{C}^k)$. For each $v \in \mathbb{C}^k$ let d_v denote the (vector space) endomorphism of $\Lambda(\mathbb{C}^k)$ given by the exterior product

$$d_v(w) = v \wedge w,$$

and let δ_v denote its adjoint with respect to the inner product. We now define a pairing

$$\mathbb{C}^k \otimes_{\mathbb{R}} \Lambda(\mathbb{C}^k) \longrightarrow \Lambda(\mathbb{C}^k) \tag{1}$$

by

$$v \otimes w \longrightarrow d_v(w) + \delta_v(w).$$

One verifies that

$$(d_v - \delta_v)^2 = \|v\|^2$$

so that (1) makes $\Lambda(\mathbb{C}^k)$ into a complex module for the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{2k})$ (identifying $\mathbb{C}^k$ with $\mathbb{R}^{2k}$ as usual). Here, $\mathrm{Cl}(\mathbb{R}^{2k})$ is the Clifford algebra over $\mathbb{R}^{2k}$ relative to the quadratic form $Q(v) = \|v\|^2$. Moreover $\Lambda(\mathbb{C}^k)$ has a natural $\mathbb{Z}_2$-grading

$$\Lambda^0 = \Sigma\Lambda^{2r},$$

$$\Lambda^1 = \Sigma\Lambda^{2r+1},$$

compatible with (1). A dimension count then shows that $\Lambda(\mathbb{C}^k)$ must be one of the two irreducible $\mathbb{Z}_2$-graded modules for $\mathrm{Cl}(\mathbb{R}^{2k})$.

The natural homomorphism

$$j\colon SU(k) \longrightarrow SO(2k)$$

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lifts to Spin $(2k)$. Explicitly the lifted map

$$l\colon SU(k) \longrightarrow \mathrm{Spin}\,(2k)$$

is given as follows. Let $T \in SU(k)$ be expressed, relative to an orthonormal base $f_1, \ldots, f_k$ of $\mathbb{C}^k$, by the diagonal matrix

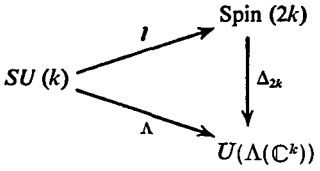
$$\begin{bmatrix} \exp(it_1) & & & \\ & \exp(it_2) & & \\ & & \ddots & \\ & & & \exp(it_k) \end{bmatrix}$$

Let $e_1, \ldots, e_{2k}$ be the corresponding base of $\mathbb{R}^{2k}$, so that

$$e_{2j-1} = f_j, \quad e_{2j} = if_j.$$

Then
$$l(T) = \prod_{j=1}^{k} (\cos t_j/2 - \sin t_j/2 \,.\, e_{2j-1}e_{2j}) \exp(i\Sigma t_j/2).$$

Moreover, the following triangle is commutative:



(2)

Here, $\Lambda(T)$ is defined by the equation

$$\Lambda(T) f_{i_1} \wedge \ldots \wedge f_{i_r} = (Tf_{i_1}) \wedge \ldots \wedge (Tf_{i_r}),$$

$U(\Lambda(\mathbb{C}^k))$ is the unitary group on $\Lambda(C^k)$ and

$$\Delta_{2k} = \sigma | \mathrm{Cl}(\mathbb{R}^{2k}),$$

where σ is the representation of $\mathrm{Cl}\,(\mathbb{R}^{2k})$ determined by (1). We have, of course, Spin $(2k) \subset \mathrm{Cl}\,(\mathbb{R}^{2k})$.

In order to verify (2), we may proceed as follows.

$$(\sigma \circ l)(T) = \exp(i\Sigma t_j/2) \prod_{j=1}^{k} \{\cos t_j/2 - \sin t_j/2 \,.\, \sigma(e_{2j-1})\,\sigma(e_{2j})\}.$$

But
$$\begin{aligned} \sigma(e_{2j-1})\,\sigma(e_{2j}) &= \sigma(f_j)\,\sigma(if_j) \\ &= (d_{f_j} + \delta_{f_j})\, i(d_{f_j} - \delta_{f_j}) \\ &= i(\delta_{f_j} d_{f_j} - d_{f_j}\delta_{f_j}). \end{aligned}$$

Note that σ is $\mathbb{R}$-linear but not $\mathbb{C}$-linear, that $d_{iv} = id_v$, and that

$$\delta_{iv} = (d_{iv})^* = -id_v^* = -i\delta_v.$$

Let $S = \{i_1, \ldots, i_r\} \subset \{1, 2, \ldots, k\}$, and let $f_S = f_{i_1} \wedge \ldots \wedge f_{i_r}$. Then

$$\begin{aligned} (\delta_f\, d_{f_j} - d_{f_j}\delta_{f_j}) f_S &= -f_S \quad (j \in S) \\ &= +f_S \quad (j \notin S) \end{aligned}$$

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by an elementary computation. Therefore

$$\begin{aligned}(\sigma \circ l)(T) f_S &= \exp(\Sigma i t_j/2) \prod_{j \in S} \exp(i t_j/2) \prod_{j \notin S} \exp(-i t_j/2) . f_S \\ &= \prod_{j \in S} (\exp(i t_j)) . f_S \\ &= \Lambda(T) f_S\end{aligned}$$

as required.

A typical element in the standard maximal torus of Spin $(2k)$ is

$$A = \prod_{j=1}^{k} (\cos t_j/2 - \sin t_j/2 . e_{2j-1} e_{2j})$$

hence

$$\Delta_{2k}(A) f_S = \prod_{j \in S} \exp(i t_j/2) \prod_{j \notin S} \exp(-i t_j/2) . f_S$$

so that

$$\begin{aligned}\text{character}(A) &= \operatorname{Tr}(\Delta_{2k}(A)) \\ &= \sum_{\pm \cdots \pm} \exp\{i(\pm t_1 \pm t_2 \pm \ldots \pm t_k)/2\}\end{aligned}$$

the classical formula for the character of the spin representation Δ_{2k} of the spin group Spin $(2k)$ [(3), p. 57].

3. *Infinite-dimensional spinors.* Let H be a separable real Hilbert space, with inner product $(.,.)$. Let J be a complex structure in H, i.e. an orthogonal operator on H such that $J^2 = -1$. Then H may be made into a complex Hilbert space by setting $if = Jf$, $\langle f, g \rangle = (f, g) + i(Jf, g)$. The complex Hilbert space in question is denoted $H_{\mathbb{C}}$. Note, incidentally, that $\langle f, f \rangle = (f, f)$. Let $\Lambda(H_{\mathbb{C}})$ be the complex separable Hilbert space defined in § 1. In terms of an orthonormal basis $f_1, f_2, \ldots$ of $H_{\mathbb{C}}$ the elements

$$f_{i_1} \wedge f_{i_2} \wedge \ldots \wedge f_{i_r} \quad (i_1 < i_2 < \ldots < i_r)$$

form an orthonormal basis of $\Lambda(H_{\mathbb{C}})$. For each $v \in H_{\mathbb{C}}$ let C_v denote the (vector space) endomorphism of $\Lambda(H_{\mathbb{C}})$ given by the exterior product:

$$C_v(w) = v \wedge w$$

and let $A_v = C_v^*$. Define

$$F(v) = C_v + A_v. \tag{3}$$

Let $\mathrm{Cl}(H)$ denote the Clifford algebra over H relative to the quadratic form

$$Q(h) = \|h\|^2, \quad h \in H.$$

$\mathrm{Cl}(H)$ is the quotient of the tensor algebra over H by the two-sided ideal generated by the elements $h \otimes h - Q(h) . 1$. Since

$$\begin{aligned}F(v)^2 &= (C_v + A_v)^2 \\ &= C_v A_v + A_v C_v \\ &= \|v\|^2 1\end{aligned}$$

it follows that F lifts to a representation of $\mathrm{Cl}(H)$ on $\Lambda(H_{\mathbb{C}})$. F is faithful, since $\mathrm{Cl}(H)$ is simple. $\mathrm{Cl}(H)$ acquires a norm, denoted $\|.\|_\infty$, the pull-back by F of the operator

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norm on $\Lambda(H_{\mathbb{C}})$. $\mathrm{Cl}(H)_\infty$ is the completion of $\mathrm{Cl}(H)$ in the $\|\,.\,\|_\infty$ norm, and F extends uniquely to a representation F of $\mathrm{Cl}(H)_\infty$ by bounded linear operators on $\Lambda(H_{\mathbb{C}})$:

$$F\colon \mathrm{Cl}(H)_\infty \to \mathscr{L}(\Lambda(H_{\mathbb{C}}), \Lambda(H_{\mathbb{C}}))$$

F is irreducible (see, for example, (7)).

$\Lambda(H_{\mathbb{C}})$ has a natural $\mathbb{Z}_2$-grading

$$\Lambda^0 = \bigoplus_{r=0}^{\infty} \Lambda^{2r}$$

$$\Lambda^1 = \bigoplus_{r=0}^{\infty} \Lambda^{2r+1}. \tag{4}$$

$\mathrm{Cl}(H)_\infty$ has a natural $\mathbb{Z}_2$-grading, acquired as follows. The map $\Omega\colon H \to H$, $h \to -h$ satisfies

$$\Omega(h)^2 = \|h\|^2,$$

hence lifts to an automorphism of $\mathrm{Cl}(H)$, then extends by continuity to an automorphism Ω of $\mathrm{Cl}(H)_\infty$. Let $\mathscr{C} = \mathrm{Cl}(H)_\infty$. Define

$$\mathscr{C}^0 = \{u \in \mathscr{C}\colon \Omega u = u\}$$
$$\mathscr{C}^1 = \{u \in \mathscr{C}\colon \Omega u = -u\}. \tag{5}$$

The $\mathbb{Z}_2$-grading of $\mathscr{C}$ is compatible with the $\mathbb{Z}_2$-grading of $\Lambda(H_{\mathbb{C}})$ in the sense that

$$F(\mathscr{C}^i)\,\Lambda^j \subset \Lambda^{i+j} \tag{6}$$

for $i, j \in \mathbb{Z}_2$.

To prove (6), which is crucial, consider $u = h_1 \dots h_n$, $h_i \in H$, n odd. Then $u \in \mathscr{C}^1$, by (5). But, by (3),

$$F(u) = (C_{h_1} + A_{h_1}) \dots (C_{h_n} + A_{h_n}).$$

Let $v = f_1 \wedge \dots \wedge f_m$, $f_i \in H_{\mathbb{C}}$, m odd. Then $v \in \Lambda^1$ by (4). But $F(u)(v) \in \Lambda^0$, by an elementary computation. By linearity and continuity,

$$F(\mathscr{C}^1)\,\Lambda^1 \subset \Lambda^0.$$

By similar consideration of the three other cases, (6) is proved.

Let

$$\mathrm{Spin}(H)_\infty = \{u \in \mathscr{C}\colon uHu^* \subset H,\ uu^* = 1 = u^*u,\ \Omega u = u\}.$$

The topology on $\mathrm{Spin}(H)_\infty$ is inherited from $\mathscr{C}$. $O(H)_1$ denotes the Banach–Lie group of all orthogonal operators on H of the form $1 + A$, where A is trace-class. The topology on $O(H)_1$ is derived from the metric $d(1+A, 1+B) = \mathrm{Tr}\,|A - B|$ where

$$|A - B| = ((A^* - B^*)(A - B))^{\frac{1}{2}}.$$

$SO(H)_1$ is the identity component of $O(H)_1$.

$\mathrm{Spin}(H)_\infty$ is a Banach–Lie group and is the universal covering group of $SO(H)_1$. This result is due to Shale, Stinespring and de la Harpe [(9), p. 321] and [(5), p. 258]. The covering map is defined as follows:

$$\rho\colon \mathrm{Spin}(H)_\infty \longrightarrow SO(H)_1$$
$$\rho(u)(h) = uhu^*.$$

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As in § 2, let $F|\mathrm{Spin}\,(H)_\infty = \Delta$

Δ is a representation of $\mathrm{Spin}\,(H)_\infty$ on $\Lambda(H_\mathbb{C})$. Λ^0 and Λ^1 are invariant subspaces, by (6), since $\mathrm{Spin}\,(H)_\infty \subset \mathscr{C}^0$. Let the subrepresentations of Δ determined by Λ^0, Λ^1 be denoted Δ^+, Δ^-.

PROPOSITION 1. *Δ^+ and Δ^- are norm continuous, faithful, irreducible representations of* $\mathrm{Spin}\,(H)_\infty$.

Proof. Norm continuity of Δ follows from norm continuity of F. Let $f_1, f_2, \ldots$ be an orthonormal basis in $H_\mathbb{C}$. Then

$$F(f_{i_r} \ldots f_{i_1}).(f_{i_1} \wedge \ldots \wedge f_{i_r}) = \Omega_0$$

by (3). Here, $\Omega_0 = 1 \in \mathbb{C} \subset \Lambda(H_\mathbb{C})$. The explicit formula for A_f, given in, for example, (7), is helpful here. But

$$F(f_{j_1} \ldots f_{j_s})\,\Omega_0 = f_{j_1} \wedge \ldots \wedge f_{j_s}.$$

If $s+r$ is even, then $f_{j_1} \ldots f_{j_s} . f_{i_r} \ldots f_{i_1} \in \mathrm{Spin}\,(H)_\infty$. Let r, s both be even. Let $u = f_{j_1} \ldots f_{j_s} . f_{i_r} \ldots f_{i_1}$. Then $\Delta(u)$ sends $f_{i_1} \wedge \ldots \wedge f_{i_r} \in \Lambda^0$ to $f_{j_1} \wedge \ldots \wedge f_{j_s} \in \Lambda^0$. But $\Omega u = u$, hence $\mathrm{Spin}\,(H)_\infty$ acts irreducibly on Λ^0. Let r, s be odd. Then $\Omega u = u$ still, hence $\mathrm{Spin}\,(H)_\infty$ acts irreducibly on Λ^1. When $H = \mathbb{R}^{2n}$, Δ^+ and Δ^- are denoted Δ^+_{2n} and Δ^-_{2n} to conform with [(3), p. 57].

If $U: H_\mathbb{C} \longrightarrow H_\mathbb{C}$ is unitary, then the assignment $f_1 \wedge \ldots \wedge f_n \longmapsto (Uf_1) \wedge \ldots \wedge (Uf_n)$ has a unique unitary extension $\Lambda(U): \Lambda(H_\mathbb{C}) \longrightarrow \Lambda(H_\mathbb{C})$. For more on the Λ-functor, consult (11).

LEMMA. *If U is orthogonal on H and $UJ = JU$, so that U is unitary on $H_\mathbb{C}$, then*

$$\Lambda(U).F(x).\Lambda(U^*) = F(Ux) \tag{7}$$

for all $x \in H$.

Proof. This fundamental equation is well known [(9), p. 316]. We prove it in the special case when U has a discrete spectrum. Let $p_1, p_2, \ldots$ be an orthonormal basis in $H_\mathbb{C}$, and let $Up_k = \omega_k p_k$ with $\omega_k \in \mathbb{C}$, $|\omega_k| = 1$. Then

$$\begin{aligned}\Lambda(U)\,F(p_1)\,\Lambda(U^*)\,p_1 \wedge \ldots \wedge p_n &= \Lambda(U)\,F(p_1)\,\omega_1^* \ldots \omega_n^* p_1 \wedge \ldots \wedge p_n \\ &= \Lambda(U)\,\omega_1^* \ldots \omega_n^* p_2 \wedge \ldots \wedge p_n \\ &= \omega_1^* p_2 \wedge \ldots \wedge p_n\end{aligned}$$

and

$$\begin{aligned}F(Up_1)\,p_1 \wedge \ldots \wedge p_n &= F(\omega_1 p_1)\,p_1 \wedge \ldots \wedge p_n \\ &= A_{\omega_1 p_1} p_1 \wedge \ldots \wedge p_n \\ &= \omega_1^* p_2 \wedge \ldots \wedge p_n.\end{aligned}$$

Similar elementary computations establish (7) whenever $x = p_1, p_2, \ldots$. (7) then follows by linearity and continuity.

Next, let

$$U(H_\mathbb{C})_1 = \{U \in SO(H)_1 : UJ = JU\}.$$

Then $U(H_\mathbb{C})_1$ is a subgroup of $SO(H)_1$. $U(H_\mathbb{C})_1$ inherits its topology from $SO(H)_1$. Extend F and Ω to $\mathscr{C} \otimes_\mathbb{R} \mathbb{C}$ by the equations

$$F(u \otimes \alpha) = F(u).\alpha, \quad \Omega(U \otimes \alpha) = \Omega(u) \otimes \alpha.$$

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Let $$\mathrm{Spin}^c(H)_\infty = \{v \in \mathscr{C} \otimes_{\mathbb{R}} \mathbb{C} : vHv^* \subset H,\ vv^* = 1 = v^*v,\ \Omega v = v\}$$

$$\Delta^c = F|\mathrm{Spin}^c(H)_\infty.$$

If $U \in U(H_{\mathbb{C}})_1$ then (7) holds and there exists $w \in \mathrm{Spin}^c(H)_\infty$ such that $wxw^* = Ux$. Thus

$$\Lambda(U)F(A)\Lambda(U^*) = F(Ux) = F(wxw^*) = F(w)F(x)F(w^*),$$

hence

$$\Lambda(U)F(A)\Lambda(U^*) = F(w)F(A)F(w^*)$$

for all $A \in \mathscr{C} \otimes_{\mathbb{R}} \mathbb{C}$. Hence $F(w)^{-1}\Lambda(U)$ commutes with $F(A)$ for all $A \in \mathscr{C} \otimes_{\mathbb{R}} \mathbb{C}$. Therefore $F(w)^{-1}\Lambda(U) = \lambda 1$, with $\lambda \in U(1)$, since F is an irreducible representation of the C^*-algebra $\mathscr{C} \otimes_{\mathbb{R}} \mathbb{C}$. Hence $\Lambda(U) = F(\lambda w)$. Set $v = \lambda w$. Then

$$\Lambda(U) = F(v) = \Delta^c(v).$$

Set $L(U) = v$. L is well defined since F is faithful. Then $F(L(U)) = \Lambda(U)$. Hence

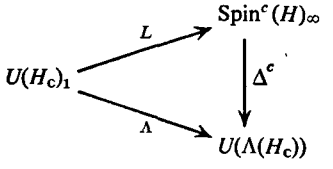
$$L = F^{-1} \circ \Lambda.$$

But $\Lambda(UV) = \Lambda(U)\Lambda(V)$, hence L is a homomorphism: $U(H_{\mathbb{C}})_1 \longrightarrow \mathrm{Spin}^c(H)_\infty$. This establishes the following

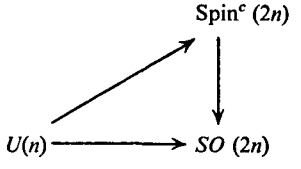
PROPOSITION 2. *The embedding* $U(H_{\mathbb{C}})_1 \longrightarrow SO(H)_1$ *lifts to a homomorphism*

$$L\colon U(H_{\mathbb{C}})_1 \longrightarrow \mathrm{Spin}^c(H)_\infty.$$

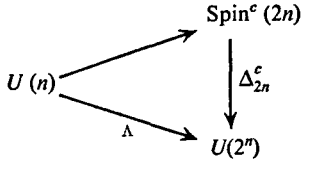
Moreover, the following triangle is commutative:



COROLLARY. *Let* $H = \mathbb{R}^{2n}$. *Then the embedding* $U(n) \longrightarrow SO(2n)$ *lifts to* $\mathrm{Spin}^c(2n)$:



and we have a commutative triangle:



Both assertions in the corollary are well known [(3), pp. 58, 154].

In the finite-dimensional case, there is a unique spin representation up to unitary equivalence. That is, different choices of J, the complex structure, lead to unitarily

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equivalent representations. This fact seems unlikely to be true in the infinite-dimensional case. We are grateful to the referee for emphasizing this point.

4. *Relations among the spin representations.* Let $p_1, q_1, p_2, q_2, \ldots$ be an orthonormal basis of H. Let H_{2n} be the subspace spanned by $p_1, q_1, \ldots, p_n, q_n$, and let H^{2n} be the subspace spanned by $p_{n+1}, q_{n+1}, \ldots$. Let J be the complex structure in H defined by

$$Jp_k = q_k, \quad Jq_k = -p_k.$$

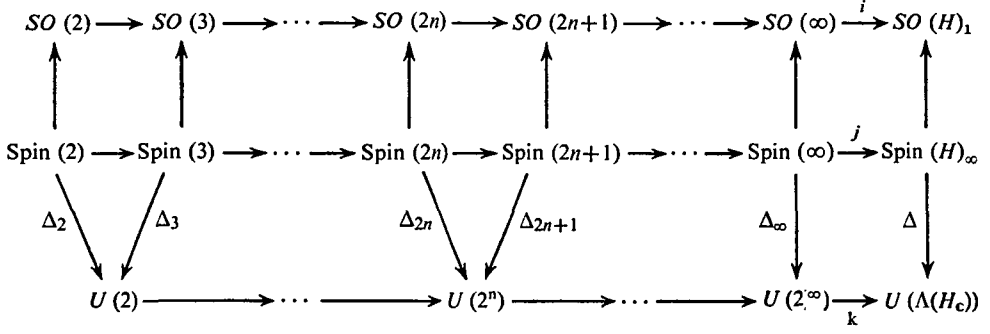
Then J induces complex structures in H_{2n} and H^{2n}; the complex Hilbert spaces so obtained are denoted K_n and K^n. Then $K^n = \mathbb{C}^n$ and $K_n \oplus K^n = H_{\mathbb{C}}$. Further, the basis $p_1, q_1, p_2, q_2, \ldots$ determines monomorphisms

$$SO(2) \longrightarrow \ldots \longrightarrow SO(n) \longrightarrow SO(n+1) \longrightarrow \ldots \longrightarrow SO(\infty),$$

$$\text{Spin}\,(2) \longrightarrow \ldots \longrightarrow \text{Spin}\,(n) \longrightarrow \text{Spin}\,(n+1) \longrightarrow \ldots \longrightarrow \text{Spin}\,(\infty),$$

where $SO(\infty) = \varinjlim SO(n)$, $\text{Spin}\,(\infty) = \varinjlim \text{Spin}\,(n)$ in the direct limit topology.

PROPOSITION 3. *The following diagram is commutative:*



Proof. The bottom row calls for a word of explanation. As in § 2, the spin representation Δ_{2n} is a unitary representation on $\Lambda(\mathbb{C}^n)$. We have identified $U(2^n)$ with the unitary group on $\Lambda(\mathbb{C}^n)$. Further, the monomorphism

$$U(2^n) \longrightarrow U(2^{n+1})$$

is given by sending V to $V \oplus V$. The direct limit

$$U(2^\infty) = \varinjlim U(2^n)$$

is then well defined. The map

$$U(2^n) \longrightarrow U(\Lambda(H_{\mathbb{C}}))$$

is obtained as follows. We have $K_n \oplus K^n = H_{\mathbb{C}}$. Set

$$S \subset \{1, 2, \ldots, n\}, \quad T \subset \{n+1, n+2, \ldots\},$$

T finite, and set

$$p_S = p_{i_1} \wedge \ldots \wedge p_{i_k},$$

when $S = \{i_1, \ldots, i_k\}$. Let $V \in U(\Lambda(K_n))$. Define $\tilde{V}$ by the equation

$$\tilde{V}(p_S \wedge p_T) = (Vp_S) \wedge p_T.$$

Since $\{p_S \wedge p_T\}$ is an orthonormal basis in $\Lambda(H_{\mathbb{C}})$, this defines $\tilde{V} \in U(\Lambda(H_{\mathbb{C}}))$.

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An elementary computation with characters [(3), p. 57] shows that

$$\Delta_{2n+1}|\mathrm{Spin}\,(2n) = \Delta_{2n},$$
$$\Delta_{2n+2}|\mathrm{Spin}\,(2n) = \Delta_{2n} \oplus \Delta_{2n}.$$

Hence the direct limit $\Delta_\infty = \varinjlim \Delta_n$ is well defined. Here, Δ_n is the classical spin representation of Cartan. It remains to show that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Spin}\,(2n) & \longrightarrow & \mathrm{Spin}\,(H)_\infty \\ \downarrow{\scriptstyle \Delta_{2n}} & & \downarrow{\scriptstyle \Delta} \\ U(\Lambda(K_n)) & \longrightarrow & U(\Lambda(H_\mathbf{C})) \end{array}$$

This will be so if $\tilde{d}_x = C_x$ for all $x \in K_n$. But this is true because

$$\begin{aligned} \tilde{d}_x(p_S \wedge p_T) &= (d_x p_S) \wedge p_T \\ &= (x \wedge p_S) \wedge p_T \\ &= x \wedge (p_S \wedge p_T) \\ &= C_x(p_S \wedge p_T). \end{aligned}$$

This establishes Proposition 3.

It is known that i and j are homotopy equivalences [(7), p. 258]. The following observation is due to de la Harpe. The fundamental group of $U(2^\infty)$ may be computed as follows:

$$\begin{aligned} \pi_1 \varinjlim (U(2^n)) &= \varinjlim \pi_1(U(2^n)) \\ &= \varinjlim (\ldots \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \ldots) \\ &= \mathbb{Z}[\tfrac{1}{2}] \end{aligned}$$

the group of dyadic rationals. $\mathbb{Z}[\frac{1}{2}]$ contains all rationals of the form $m/2^n$ with m and n integers. But $U(\Lambda(H_\mathbf{C}))$ is contractible (Kuiper's theorem), therefore k is certainly not a homotopy equivalence.

We take this opportunity to thank de la Harpe for this remark, and for several interesting letters.

5. *Quantization of Fermi systems.* We have

$$\begin{aligned} F(Jx) &= C_{Jx} + A_{Jx} \\ &= iC_x - iA_x \\ &= i(C_x - A_x) \end{aligned}$$

since $H_\mathbf{C}$ and $\Lambda(H_\mathbf{C})$ are constructed from the complex structure J. Also

$$F(x) = C_x + A_x.$$

There is no element $y \in \mathscr{C}$ such that $F(y) = C_x$. But there is an element $y \in \mathscr{C} \otimes_\mathbb{R} \mathbb{C}$ which can be found as follows. First, of course, we extend F to $\mathscr{C} \otimes_\mathbf{R} \mathbb{C}$ by the equation

$$F(u \otimes \alpha) = F(u).\alpha.$$

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Then

$$\begin{aligned} F(x+iJx) &= F(x)+iF(J_x) \\ &= C_x+A_x-C_x+A_x \\ &= 2A_x, \\ F(x-iJx) &= C_x+A_x+C_x-A_x \\ &= 2C_x. \end{aligned}$$

Hence we recover the quantization prescription of I. E. Segal [(2), p. 207]: represent

$\frac{1}{2}(x-iJx)$ by the creator C_x on $\Lambda(H_{\mathbb{C}})$

$\frac{1}{2}(x+iJx)$ by the annihilator A_x on $\Lambda(H_{\mathbb{C}})$.

It is easy to verify that $\frac{1}{2}(x-iJx)$, $\frac{1}{2}(x+iJx)$ obey the CAR (canonical anti-commutation relations).

The subtlety of the above construction arises from the following facts: F is $\mathbb{R}$-linear over $\mathscr{C}$, $\mathbb{C}$-linear over $\mathscr{C} \otimes_{\mathbb{R}} \mathbb{C}$, but *not* $\mathbb{C}$-linear over $\mathscr{C}$ relative to J.

A good example of a complex structure J is the one associated with the Dirac Hamiltonian [(2), p. 207].

Contact with step 1 of Mackey's quantization prescription [(8), p. 32] can be effected as follows.

Consider a Hamiltonian dynamical system with n degrees of freedom. Let P be a point in phase space representing a point of stable equilibrium. Near P, the dynamics can be approximated by a *linear system* (approximate the manifold near P by the tangent space at P). In a suitable co-ordinate system (normal co-ordinates) the tangent space can be identified with $\mathbb{R}^{2n}$. $\mathbb{R}^{2n}$ breaks up as a sum of orthogonal planes

$$\mathbb{R}^{2n} = \mathbb{R}^2 \oplus \ldots \oplus \mathbb{R}^2$$

and the motion breaks down into circular motions in each $\mathbb{R}^2$ (harmonic oscillators) with frequencies $\omega_1, \ldots, \omega_n$. It is now obvious how to give each $\mathbb{R}^2$, hence $\mathbb{R}^{2n}$, a complex structure such that the motion is described by a *unitary matrix* in diagonal form

$$\begin{bmatrix} \exp(i\omega_1 t) & & \\ & \ddots & \\ & & \exp(i\omega_n t) \end{bmatrix}.$$

The complex structure required is given by

$$Jp_j = q_j, \quad Jq_j = -p_j,$$

where $\{p_j, q_j\}$ span the jth plane $\mathbb{R}^2$. Thus we are back in the set-up in § 2.

6. *Miscellaneous remarks.* (1) One reason for the appearance of $U(2^\infty)$ is that the complex Clifford algebras have period 2, in the sense of [(3), p. 151]:

$$\mathrm{Cl}(H_{2n+2}) \otimes_{\mathbb{R}} \mathbb{C} = (\mathrm{Cl}(H_{2n}) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}(2),$$

where $\mathbb{C}(2) = 2 \times 2$ matrix algebra over $\mathbb{C}$.

(2) $\mathscr{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a uniformly hyperfinite C^*-algebra of type $\{2^n\}$. $\mathscr{C} \otimes_{\mathbb{R}} \mathbb{C}$ is the CAR algebra, F its Fock representation.

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(3) We have used $\Lambda(H_{\mathbb{C}})$ as a model of the space of spinors. Two other models (i.e. unitarily equivalent descriptions) are

(i) Hilbert space of spin-waves described in, e.g. (10): the two spaces of half-spinors are realized as the two spaces of N spin-waves, with N even, odd, respectively.

(ii) Hilbert space of holomorphic spinors (9): the two spaces of half-spinors are realized as the two spaces spanned by $z_1 \dots z_N$ with N even, odd, respectively, and $z_j = (p_j + iq_j)/\sqrt{2}$.

(4) Let $\mathrm{Pin}\,(H)_\infty = \{u \in \mathscr{C}\colon uHu^* \subset H,\ uu^* = 1 = u^*u\}$. Then $\mathrm{Pin}\,(H)_\infty$ acts irreducibly on $\Lambda(H_{\mathbb{C}})$. Restriction to the subgroup $\mathrm{Spin}\,(\mathrm{H})_\infty$ yields the decomposition $\Delta^+ \oplus \Delta^-$. Our notation Spin, Spin^c, Pin is of course compatible with that in (1).

(5) k is a unitary representation of $U(2^\infty)$ on $U(\Lambda(H_{\mathbb{C}}))$. Now

$$H_{\mathbb{C}} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \dots$$

hence

$$\begin{aligned} \Lambda(H_{\mathbb{C}}) &\cong \Lambda(\mathbb{C}) \otimes \Lambda(\mathbb{C}) \otimes \Lambda(\mathbb{C}) \otimes \dots \\ &= \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \end{aligned}$$

the ill-named incomplete tensor product with reference vector

$$u \otimes u \otimes u \otimes \dots,$$

where $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. With this identification of $\Lambda(H_{\mathbb{C}})$, k becomes the 'obvious' representation of

$$U(2) \otimes U(2) \otimes U(2) \otimes \dots$$

on

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots .$$

REFERENCES

(1) Atiyah, M. F., Bott, R. and Shapiro, A. Clifford modules. *Topology* **3**, suppl. 1 (1964), 3–38. Reprinted in (3).

(2) Bongaarts, P. J. M. Linear fields according to I. E. Segal. *Mathematics of contemporary physics*, ed. R. F. Streater (London, Academic Press, 1972).

(3) Bott, R. *Lectures on $K(X)$* (New York, Benjamin, 1969).

(4) Chevalley, C. *The algebraic theory of spinors* (New York, Columbia University Press, 1954).

(5) De la Harpe, P. The Clifford algebra and the spinor group of a Hilbert space. *Compositio Math.* **25** (1972), 245–261.

(6) Hitchin, N. Harmonic spinors. *Advances in Math.* **14** (1974), 1–55.

(7) Hugenholtz, N. M. and Kadison, R. V. Automorphisms and quasi-free states of the CAR algebra. *Comm. Math. Phys.* **43** (1975), 181–197.

(8) Mackey, G. *Mathematical foundations of quantum mechanics* (New York, Benjamin, 1963).

(9) Shale, D. and Stinespring, W. F. Spinor representations of infinite orthogonal groups. *J. Math. Mech.* **14** (1965), 315–322.

(10) Streater, R. F. The Heisenberg ferromagnet as a quantum field theory. *Comm. Math. Phys.* **6** (1967), 233–247.

(11) Schrader, R. and Uhlenbrock, D. A. Markov structures on Clifford algebras. *J. Functional Analysis* **18** (1975), 369–413.

(12) Dirac, P. A. M. *Spinors in Hilbert space* (Centre for Theoretical Studies, University of Miami, Coral Gables, Florida 33124, 1970).

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