Some polygon area theorems

JOHN R. SILVESTER

1 *Introduction*

Figure 1 shows an arbitrary triangle $Z_1Z_2Z_3$, the base triangle, with similar isosceles triangles $Z_1Z_2V_1$, $Z_2Z_3V_2$ and $Z_3Z_1V_3$ drawn outwards on the respective sides Z_1Z_2, Z_2Z_3 and Z_3Z_1, all with base angles (the angles at Z_1, Z_2 and Z_3) equal to α. We call $\Delta V_1V_2V_3$ the *outer α-triangle* of $\Delta Z_1Z_2Z_3$. Note that, if $\Delta Z_1Z_2Z_3$ is positively oriented (that is, if the labels go around the triangle in anticlockwise order), then so is $\Delta V_1V_2V_3$, but the triangles $Z_1Z_2V_1$, $Z_2Z_3V_2$ and $Z_3Z_1V_3$ are negatively oriented (that is, the labels go around the triangle in clockwise order).

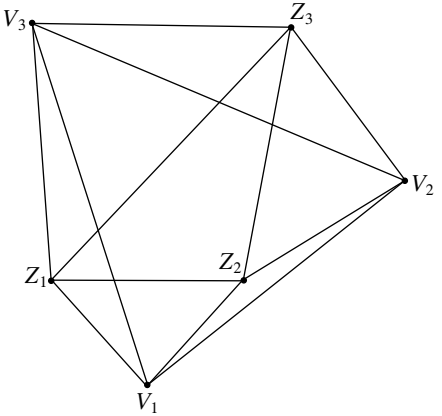


FIGURE 1

Next, Figure 2 shows similar isosceles triangles $Z_1Z_2W_1$, $Z_2Z_3W_2$ and $Z_3Z_1W_3$ drawn inwards on the respective sides Z_1Z_2, Z_2Z_3 and Z_3Z_1, also all with base angles α. We call $\Delta W_1W_2W_3$ the *inner α-triangle* of $\Delta Z_1Z_2Z_3$. This time, if $\Delta Z_1Z_2Z_3$ is positively oriented, then so are the triangles $Z_1Z_2W_1$, $Z_2Z_3W_2$ and $Z_3Z_1W_3$, but $\Delta W_1W_2W_3$ may be (i) positively or (ii)

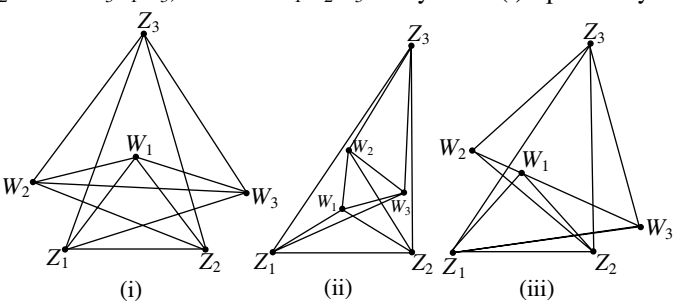


FIGURE 2

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negatively oriented, or (iii) may even degenerate into three collinear points, as shown in Figure 2.

More generally, given an n-gon $Z_1Z_2\ldots Z_n$, if similar negatively oriented isosceles triangles $Z_iZ_{i+1}V_i$ and similar positively oriented isosceles triangles $Z_iZ_{i+1}W_i$, $i = 1, \ldots, n$ (subscripts modulo n) are drawn, all $2n$ triangles having the same base angles (the angles at the Z_i) equal to α, then we call $V_1V_2\ldots V_n$ and $W_1W_2\ldots W_n$ the *outer* and *inner* α-polygons of $Z_1Z_2\ldots Z_n$, respectively. See Figure 3, where we have drawn the case $n = 4$, and see also sections 3 and 4, below. But note that the terms inner and outer only have their intuitive meanings if $Z_1Z_2\ldots Z_n$ is a simple, positively oriented polygon. If it is a crossed polygon then it doesn't have an inside or an outside; see Figure 4, where we have drawn a crossed 9-gon with the negatively oriented triangles $Z_iZ_{i+1}V_i$, $i = 1, \ldots, 9$.

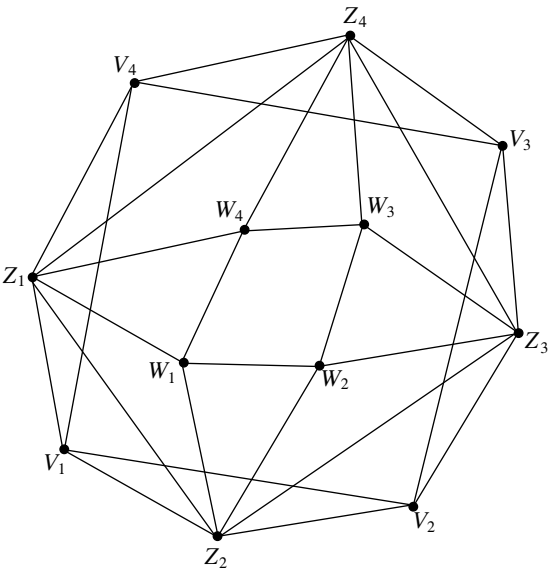


FIGURE 3

In the case $n = 3$ and $\alpha = \frac{\pi}{6}$, the V_i and W_j are the centres of equilateral triangles drawn on the sides of $\Delta Z_1Z_2Z_3$, and Napoleon's theorem says that the triangles $V_1V_2V_3$ and $W_1W_2W_3$ are themselves equilateral; they are respectively the outer and inner Napoleon triangles of $\Delta Z_1Z_2Z_3$. See [1, pp. 63-65], where it is also shown that the outer and inner Napoleon triangles differ in area by the area of the base triangle. It is this last result that we shall now generalise, (i) to arbitrary α, (ii) to quadrilaterals, and then (iii) to n-gons with $n \geqslant 5$, but in this case only for $\alpha = \frac{\pi}{4}$.

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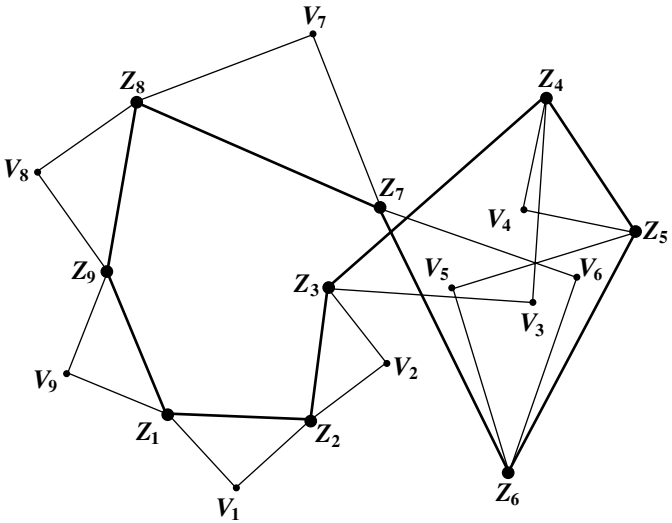


FIGURE 4

2 *Triangles*

We work in the complex plane, and adopt the convention that complex numbers a, b.… correspond to points labelled A, B, ….

Suppose ΔABC is positively oriented and isosceles. Specifically, let the angles at A and B both be equal to α. Then $c - \frac{1}{2}(a+b) = \frac{1}{2}(b-a)i\tan\alpha$, or $c = \bar{u}a + ub$, where $u = \frac{1}{2}(1 + i\tan\alpha)$.

Now the *signed area* of $\Delta Z_1Z_2Z_3$, which we write as $[Z_1Z_2Z_3]$, is given by $\frac{i}{4}f(z_1, z_2, z_3)$, where

$$f(z_1, z_2, z_3) = \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix}.$$

See [2, proposition 5]. Here $[Z_1Z_2Z_3]$ is positive or negative according as $\Delta Z_1Z_2Z_3$ is positively or negatively oriented, respectively. Then, for $\Delta V_1V_2V_3$,

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$$f(v_1, v_2, v_3) = \begin{vmatrix} uz_1 + \bar{u}z_2 & \overline{uz_1} + u\overline{z_2} & 1 \\ uz_2 + \bar{u}z_3 & \overline{uz_2} + u\overline{z_3} & 1 \\ uz_3 + \bar{u}z_1 & \overline{uz_3} + u\overline{z_1} & 1 \end{vmatrix}$$

$$= u\bar{u}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} + u^2\begin{vmatrix} z_1 & \overline{z_2} & 1 \\ z_2 & \overline{z_3} & 1 \\ z_3 & \overline{z_1} & 1 \end{vmatrix} + \bar{u}^2\begin{vmatrix} z_2 & \overline{z_1} & 1 \\ z_3 & \overline{z_2} & 1 \\ z_1 & \overline{z_3} & 1 \end{vmatrix} + u\bar{u}\begin{vmatrix} z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \\ z_1 & \overline{z_1} & 1 \end{vmatrix}$$

$$= 2u\bar{u}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} + u^2\begin{vmatrix} z_1 & \overline{z_2} & 1 \\ z_2 & \overline{z_3} & 1 \\ z_3 & \overline{z_1} & 1 \end{vmatrix} + \bar{u}^2\begin{vmatrix} z_2 & \overline{z_1} & 1 \\ z_3 & \overline{z_2} & 1 \\ z_1 & \overline{z_3} & 1 \end{vmatrix}.$$

Now w_i is obtained from v_1 merely by changing u to $\bar{u}$, and so, for $\Delta W_1W_2W_3$,

$$f(w_1, w_2, w_3) = 2u\bar{u}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} + \bar{u}^2\begin{vmatrix} z_1 & \overline{z_2} & 1 \\ z_2 & \overline{z_3} & 1 \\ z_3 & \overline{z_1} & 1 \end{vmatrix} + u^2\begin{vmatrix} z_2 & \overline{z_1} & 1 \\ z_3 & \overline{z_2} & 1 \\ z_1 & \overline{z_3} & 1 \end{vmatrix}.$$

Then

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} + \begin{vmatrix} z_1 & \overline{z_2} & 1 \\ z_2 & \overline{z_3} & 1 \\ z_3 & \overline{z_1} & 1 \end{vmatrix} + \begin{vmatrix} z_2 & \overline{z_1} & 1 \\ z_3 & \overline{z_2} & 1 \\ z_1 & \overline{z_3} & 1 \end{vmatrix}$$

$$= \begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} + \begin{vmatrix} z_1 & \overline{z_2} & 1 \\ z_2 & \overline{z_3} & 1 \\ z_3 & \overline{z_1} & 1 \end{vmatrix} + \begin{vmatrix} z_1 & \overline{z_3} & 1 \\ z_2 & \overline{z_1} & 1 \\ z_3 & \overline{z_2} & 1 \end{vmatrix}$$

$$= \begin{vmatrix} z_1 & \overline{z_1} + \overline{z_2} + \overline{z_3} & 1 \\ z_2 & \overline{z_1} + \overline{z_2} + \overline{z_3} & 1 \\ z_3 & \overline{z_1} + \overline{z_2} + \overline{z_3} & 1 \end{vmatrix}$$

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$$= \left(\overline{z_1} + \overline{z_2} + \overline{z_3}\right)\begin{vmatrix} z_1 & 1 & 1 \\ z_2 & 1 & 1 \\ z_3 & 1 & 1 \end{vmatrix}$$

$$= 0,$$

whence

$$f(v_1, v_2, v_3) + f(w_1 + w_2 + w_3) = (4u\bar{u} - u^2 - \bar{u}^2) f(z_1, z_2, z_3).$$

Multiplying through by $\frac{1}{4}i$, and substituting $u = \frac{1}{2}(1 + i\tan\alpha)$, we obtain:
Theorem 1: $[V_1V_2V_3] + [W_1W_2W_3] = \lambda[Z_1Z_2Z_3]$, where $\lambda = \frac{1}{2}(1 + 3\tan^2\alpha)$.

If $\alpha = \frac{\pi}{6}$, then $\lambda = 1$, and this, after taking account of the signs, is [1, p. 64, theorem 3.38]. Other notable cases are $\alpha = \frac{\pi}{4}$, when $\lambda = 2$ and $\alpha = \frac{\pi}{3}$, when $\lambda = 5$.

As noted above, if $[Z_1Z_2Z_3] > 0$, then also $[V_1V_2V_3] > 0$. So now we investigate the sign of $[W_1W_2W_3]$. Choose axes so that $z_1 = -1$ and $z_2 = 1$, and put $z_3 = z$, where, since $\Delta Z_1Z_2Z_3$ is positively oriented, the imaginary part of z must be positive. Then

$$f(w_1, w_2, w_3) = \begin{vmatrix} -\bar{u} + u & -u + \bar{u} & 1 \\ \bar{u} + uz & u + \bar{u}\bar{z} & 1 \\ \bar{u}z - u & u\bar{z} - \bar{u} & 1 \end{vmatrix}$$

$$= u^2\begin{vmatrix} 1 & -1 & 1 \\ z & 1 & 1 \\ -1 & \bar{z} & 1 \end{vmatrix} + \bar{u}^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & \bar{z} & 1 \\ z & -1 & 1 \end{vmatrix} + u\bar{u}\begin{vmatrix} 1 & 1 & 1 \\ z & \bar{z} & 1 \\ -1 & -1 & 1 \end{vmatrix} + u\bar{u}\begin{vmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ z & \bar{z} & 1 \end{vmatrix}$$

$$= u^2(z\bar{z} + z - \bar{z} + 3) + \bar{u}^2(-z\bar{z} + z - \bar{z} - 3) + 4u\bar{u}(-z + \bar{z})$$

$$= (u^2 - \bar{u}^2)(z\bar{z} + 3) + (u^2 - 4u\bar{u} + \bar{u}^2)(z - \bar{z}).$$

Put $t = \tan\alpha$, so that $u = \frac{1}{2}(1 + it)$. Then $u^2 - \bar{u}^2 = it$, and

$$u^2 - 4u\bar{u} + \bar{u}^2 = -\tfrac{1}{2}(3t^2 + 1).$$

So

$$f(w_1, w_2, w_3) = it\left(z\bar{z} + i\left(\frac{3t^2 + 1}{2t}\right)(z - \bar{z}) + 3\right)$$

$$= it\left(\left(z - i\left(\frac{3t^2 + 1}{2t}\right)\right)\left(\bar{z} + i\left(\frac{3t^2 + 1}{2t}\right)\right) - \left(\frac{3t^2 - 1}{2t}\right)^2\right).$$

Multiplying by $\frac{1}{4}i$, we obtain:
Theorem 2:

$$[W_1W_2W_3] = \frac{t}{4}\left(\left(\frac{3t^2 - 1}{2t}\right)^2 - \left|z - i\left(\frac{3t^2 + 1}{2t}\right)\right|^2\right).$$

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Now, fixing α, we may regard $[W_1W_2W_3]$ as a function of z. As such, the *critical circle* for Z_3 is the circle, centre $i\left(\frac{3t^2+1}{2t}\right)$, radius $\left|\frac{3t^2-1}{2t}\right|$. Then:

Corollary 1: The level curves of $[W_1W_2W_3]$ are circles concentric with the critical circle.

Corollary 2: $[W_1W_2W_3]$ is (i) positive, (ii) negative or (iii) zero when z is respectively (i) inside, (ii) outside or (iii) on the critical circle.

For example, when $\alpha = \frac{\pi}{4}$ then $t = 1$, so the critical circle is $|z - 2i| = 1$. See Figure 5.

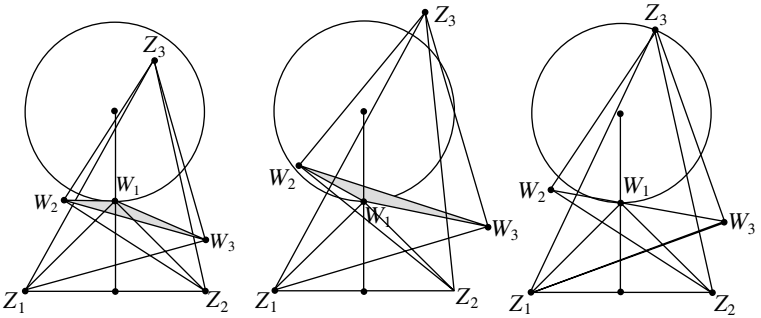


FIGURE 5

Note that if Z_3 is inside/outside/on its critical circle, then also Z_1 and Z_2 must be respectively inside/outside/on *their* critical circles. See Figure 6, where we have drawn the case $t = 0.8$, with all three Z_i *on* their critical circles.

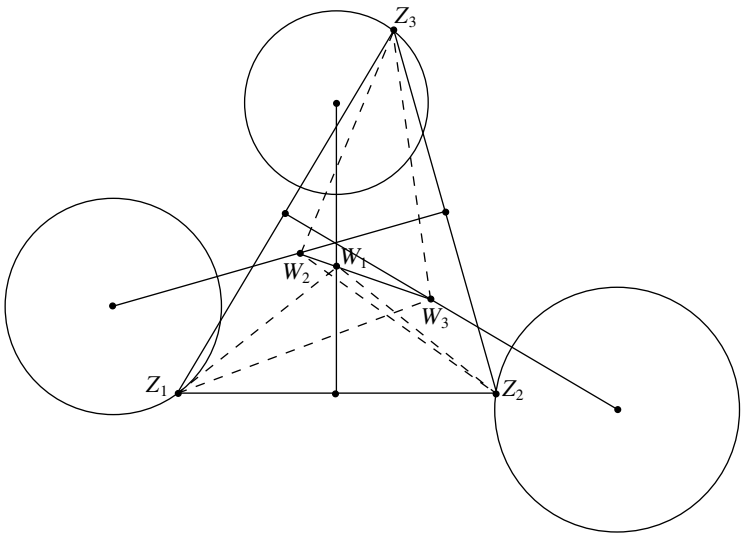


FIGURE 6

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The case $\alpha = \frac{\pi}{6}$ is special, in that the radius of the critical circle is zero, so it is reduced to the point $i\sqrt{3}$, and if z is at this critical point then $\Delta Z_1Z_2Z_3$ is equilateral, and, uniquely, $W_1 = W_2 = W_3$; but otherwise $[W_1W_2W_3] < 0$, so

$$|[V_1V_2V_3]| - |[W_1W_2W_3]| = |[Z_1Z_2Z_3]|,$$

which is [1, p. 64, theorem 3.38].

The next two results will be needed in Section 5. They are special for the case $\alpha = \frac{\pi}{4}$:

Theorem 3: When $\alpha = \frac{\pi}{4}$, we have $[W_1V_2V_3] + [V_1W_2W_3] = 0$. (See Figure 7.)

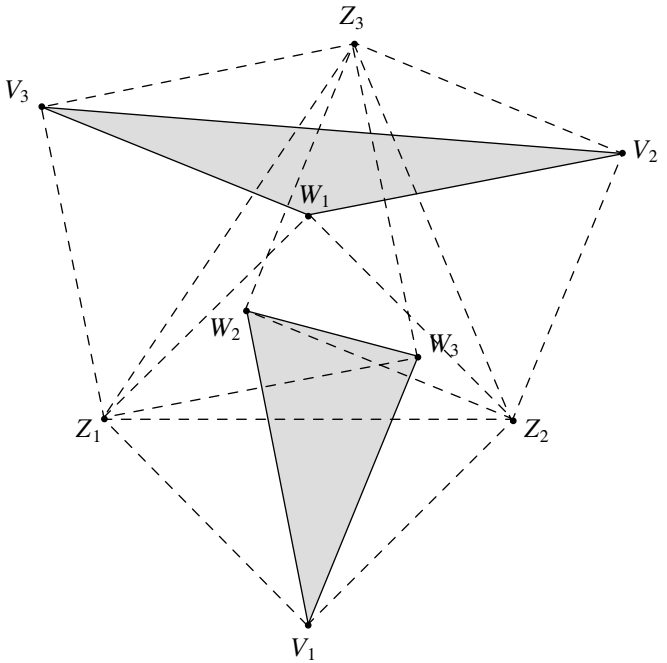


FIGURE 7

Proof: We must show that $f(w_1, v_2, v_3) + f(v_1, w_2, w_3) = 0$, i.e.,

$$\begin{vmatrix} \bar{u}z_1 + uz_2 & u\bar{z}_1 + \bar{u}\bar{z}_2 & 1 \\ uz_2 + \bar{u}z_3 & \bar{u}\bar{z}_2 + u\bar{z}_3 & 1 \\ uz_3 + \bar{u}z_1 & \bar{u}\bar{z}_3 + u\bar{z}_1 & 1 \end{vmatrix} + \begin{vmatrix} uz_1 + \bar{u}z_2 & \bar{u}\bar{z}_1 + u\bar{z}_2 & 1 \\ \bar{u}z_2 + uz_3 & u\bar{z}_2 + \bar{u}\bar{z}_3 & 1 \\ \bar{u}z_3 + uz_1 & u\bar{z}_3 + \bar{u}\bar{z}_1 & 1 \end{vmatrix} = 0,$$

where $u = \frac{1}{2}(1 + i)$. Now, taking the u-terms from the first column and the $\bar{u}$-terms from the second column of the first determinant above, we get

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$u\bar{u}\begin{vmatrix} z_2 & \bar{z}_2 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$, which is zero; and similar calculations show that all the $u\bar{u}$-terms vanish. Then taking the u-terms from the first and second columns of the first determinant gives $u^2\begin{vmatrix} z_2 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_3 & 1 \\ z_3 & \bar{z}_1 & 1 \end{vmatrix}$, and this cancels with the $\bar{u}^2$-terms from the second determinant, since $u^2 + \bar{u}^2 = 0$. Similarly, the $\bar{u}^2$-terms from the first determinant cancel with the $\bar{u}^2$-terms from the second, and the proof is complete.

The reader might like to give a second proof of this result by using rotations about Z_1 and Z_2 to show that $|V_1W_2| = |W_1V_3|$, $|V_1W_3| = |W_1V_2|$, and $\angle V_2W_1V_3 + \angle W_3V_1W_2 = \pi$, so that the result follows by use of the formula $\frac{1}{2}ab\sin C$. (In fact even more is true, namely that $V_1W_3Z_3W_2$ and $V_2Z_3V_3W_1$ are congruent parallelograms, related by a quarter-turn about the midpoint of Z_1Z_3.)

So now suppose we have an extra point Z_0, and isosceles triangles $Z_0Z_1V_0$ (negatively oriented) and $Z_1Z_0W_0$ (positively oriented), both with baseangles $\frac{\pi}{4}$. See Figure 8.

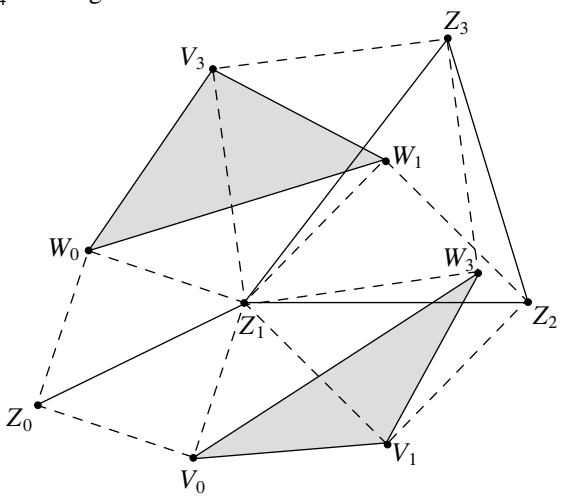


FIGURE 8

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Theorem 4: When $\alpha = \frac{\pi}{4}$, we have $[V_0V_1W_3] + [W_0W_1V_3] = [Z_1Z_2Z_3]$.

Proof: We must show that $f(v_0, v_1, w_3) + f(w_0, w_1, v_3) = f(z_1, z_2, z_3)$. The LHS of this is

$$\begin{vmatrix} uz_0 + \bar{u}z_1 & \bar{u}\bar{z}_0 + u\bar{z}_1 & 1 \\ uz_1 + \bar{u}z_2 & \bar{u}\bar{z}_1 + u\bar{z}_2 & 1 \\ uz_1 + \bar{u}z_3 & \bar{u}\bar{z}_1 + u\bar{z}_3 & 1 \end{vmatrix} + \begin{vmatrix} uz_1 + \bar{u}z_0 & \bar{u}\bar{z}_1 + u\bar{z}_0 & 1 \\ uz_2 + \bar{u}z_1 & \bar{u}\bar{z}_2 + u\bar{z}_1 & 1 \\ uz_3 + \bar{u}z_1 & \bar{u}\bar{z}_3 + u\bar{z}_1 & 1 \end{vmatrix},$$

where $u = \frac{1}{2}(1 + i)$. Now taking the u-terms from the first and second columns of the first determinant gives $u^2 \begin{vmatrix} z_0 & \bar{z}_1 & 1 \\ z_1 & \bar{z}_2 & 1 \\ z_1 & \bar{z}_3 & 1 \end{vmatrix}$, and this cancels with the $\bar{u}^2$-terms from the second determinant, since $u^2 + \bar{u}^2 = 0$. Similarly, the $\bar{u}^2$-terms from the first determinant cancel with the u^2-terms from the second. Then taking the u-terms from the first column and the $\bar{u}$-terms from the second column of the first determinant gives $u\bar{u} \begin{vmatrix} z_0 & \bar{z}_0 & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_1 & \bar{z}_1 & 1 \end{vmatrix}$, which is zero, and similarly for the $u\bar{u}$-terms from the $\bar{u}$-terms of the first column and the u-terms of the second column of the second determinant. Finally, the remaining $u\bar{u}$-terms from the first and second determinants each give $u\bar{u} = \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$; but $u\bar{u} = \frac{1}{2}$, so the proof is complete.

3 *Quadrilaterals*

We are now going to extend the above results to quadrilaterals. First, the area $[Z_1Z_2Z_3Z_4]$ of either a convex or a re-entrant quadrilateral $Z_1Z_2Z_3Z_4$ is given by $[Z_1Z_2Z_3] + [Z_1Z_3Z_4]$, and we take this as the definition for a crossed quadrilateral also. Then

$$f(z_1, z_2, z_3) + f(z_1, z_3, z_4) = \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} + \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_3 & \bar{z}_3 & 1 \\ z_4 & \bar{z}_4 & 1 \end{vmatrix}$$

$$= z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_4 + z_4\bar{z}_1 - z_2\bar{z}_1 - z_3\bar{z}_2 - z_4\bar{z}_3 - z_1\bar{z}_4.$$

By cyclic permutation of subscripts, this is the same as $f(z_2, z_3, z_4) + f(z_2, z_4, z_1)$. Now draw isosceles triangles $Z_iZ_{i+1}V_i$ and $Z_iZ_{i+1}W_i$, $i = 1, \ldots, 4$ (subscripts modulo 4), the former negatively and the latter positively oriented, and all with base angle α, on the sides of $Z_1Z_2Z_3Z_4$, See Figure 3 again. So $v_i = uz_i + \bar{u}z_{i+1}$ and $w_i = \bar{u}z_i + uz_{i+1}$, where, as before, $u = \frac{1}{2}(1 + ia)$ and $a = \tan\alpha$.

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In order to work out $[V_1V_2V_3V_4] + [W_1W_2W_3W_4]$, we must evaluate

$$f(v_1, v_2, v_3) + f(v_1, v_3, v_4) + f(w_1, w_2, w_3) + f(w_1, w_3, w_4), \qquad (1)$$

which is quite horrendous; so we attack it piecemeal. So,

$$f(v_1, v_2, v_3) = \begin{vmatrix} uz_1 + \bar{u}z_2 & u\bar{z}_1 + \bar{u}\bar{z}_2 & 1 \\ uz_2 + \bar{u}z_3 & u\bar{z}_2 + \bar{u}\bar{z}_3 & 1 \\ uz_3 + \bar{u}z_4 & u\bar{z}_3 + \bar{u}\bar{z}_4 & 1 \end{vmatrix},$$

which has terms in u^2, $\bar{u}^2$ and $u\bar{u}$. The terms in u^2 are

$$u^2 \begin{vmatrix} z_1 & \bar{z}_2 & 1 \\ z_2 & \bar{z}_3 & 1 \\ z_3 & \bar{z}_4 & 1 \end{vmatrix} = u^2(z_1\bar{z}_3 + z_2\bar{z}_4 + z_3\bar{z}_2 - z_1\bar{z}_4 - z_2\bar{z}_2 - z_3\bar{z}_3).$$

Similar calculations for $f(v_1, v_3, v_4)$, $f(w_1, w_2, w_3)$ and $f(w_1, w_3, w_4)$ yield, respectively,

$$u^2(z_1\bar{z}_4 + z_3\bar{z}_1 + z_4\bar{z}_2 - z_1\bar{z}_1 - z_3\bar{z}_2 - z_4\bar{z}_4),$$

$$u^2(z_2\bar{z}_2 + z_3\bar{z}_3 + z_4\bar{z}_1 - z_2\bar{z}_3 - z_3\bar{z}_1 - z_4\bar{z}_2),$$

$$u^2(z_2\bar{z}_3 + z_4\bar{z}_4 + z_1\bar{z}_1 - z_2\bar{z}_4 - z_4\bar{z}_1 - z_1\bar{z}_3)$$

Adding these last four expressions gives zero, so there are *no* terms in u^2 in (1), and a similar calculation shows there are no terms in $\bar{u}^2$ either. So now we need to calculate the terms in $u\bar{u}$. For $f(v_1, v_2, v_3)$ we have

$$u\bar{u}\left(\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} + \begin{vmatrix} z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \\ z_4 & \bar{z}_4 & 1 \end{vmatrix}\right)$$

$$= u\bar{u}(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1 + z_2\bar{z}_3 + z_3\bar{z}_4 + z_4\bar{z}_2 - z_1\bar{z}_3 - z_2\bar{z}_1 - z_3\bar{z}_2 - z_2\bar{z}_4 - z_3\bar{z}_2 - z_4\bar{z}_3),$$

and the calculation for $f(w_1, w_2, w_3)$ yields the same result. A similar calculation for $f(v_1, v_3, v_4)$ gives

$$u\bar{u}(z_1\bar{z}_3 + z_3\bar{z}_4 + z_4\bar{z}_1 + z_2\bar{z}_4 + z_4\bar{z}_1 + z_1\bar{z}_2 - z_1\bar{z}_4 - z_3\bar{z}_1 - z_4\bar{z}_3 - z_2\bar{z}_1 - z_4\bar{z}_2 - z_1\bar{z}_4),$$

and the calculation for $f(w_1, w_3, w_4)$ yields the same result. Adding up, we obtain

$$f(v_1, v_2, v_3) + f(v_1, v_3, v_4) + f(w_1, w_2, w_3) + f(w_1, w_3, w_4)$$

$$= 4u\bar{u}(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_4 + z_4\bar{z}_1 - z_2\bar{z}_1 - z_3\bar{z}_2 - z_4\bar{z}_3 - z_1\bar{z}_4).$$

Multiplying through by $\frac{1}{4}i$ and substituting $u = \frac{1}{2}(1 + i\tan\alpha)$, we obtain: *Theorem* 7:

$$[V_1V_2V_3V_4] + [W_1W_2W_3W_4] = \mu[Z_1Z_2Z_3Z_4].$$

where $\mu = 1 + \tan^2\alpha$.

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(Compare Theorem 1, above.) Once again, notable cases are $\alpha = \frac{\pi}{6}$ when $\mu = \frac{4}{3}$; $\alpha = \frac{\pi}{4}$ when $\mu = 2$; and $\alpha = \frac{\pi}{3}$ when $\mu = 4$.

We now investigate the signs of these areas. Certainly, if $Z_1Z_2Z_3Z_4$ is convex and positively oriented, then $V_1V_2V_3V_4$ is also positively oriented. See Figure 3 once more. But the same does not necessarily hold for $W_1W_2W_3W_4$. Choose axes so that Z_1 and Z_3 are on the real axis and Z_2 is on the imaginary axis. So let $z_1 = a$, $z_3 = b$, $z_2 = ic$ and $z_4 = z$, where $a; b; c \in \mathbb{R}$. Then $w_1 = iuc + \bar{u}a$, $w_2 = ub + i\bar{u}c$, $w_3 = uz + \bar{u}b$, and $w_4 = ua + \bar{u}z$. So

$$\begin{aligned} & f(w_1, w_2, w_3) + f(w_1, w_3, w_4) \\ &= \begin{vmatrix} iuc + \bar{u}a & -i\bar{u}c + ua & 1 \\ ub + i\bar{u}c & \bar{u}b - iuc & 1 \\ uz + \bar{u}b & \bar{u}\bar{z} + ub & 1 \end{vmatrix} + \begin{vmatrix} iuc + \bar{u}a & -i\bar{u}c + ua & 1 \\ uz + \bar{u}b & \bar{u}\bar{z} + ub & 1 \\ ua + \bar{u}z & \bar{u}a + u\bar{z} & 1 \end{vmatrix} \\ &= (u^2 - \bar{u}^2)z\bar{z} - d(z - \bar{z}) + e \end{aligned}$$

where $d = 2u\bar{u}(b - a) - ic(u^2 - \bar{u}^2)$ and

$$e = ((b - a)^2 + c^2)(u^2 - \bar{u}^2) + 4ic(b - a)u\bar{u}.$$

Now $u = \frac{1}{2}(1 + it)$, where $t = \tan\alpha$, so $u^2 - \bar{u}^2 = it$. So

$$\begin{aligned} f(w_1, w_2, w_3) + f(w_1, w_3, w_4) &= itz\bar{z} - d(z - \bar{z}) + e \\ &= it(z\bar{z} + it^{-1}d(z - \bar{z}) - it^{-1}e) \\ &= it((z - it^{-1}d)(\bar{z} + it^{-1}d) - it^{-2}d^2 - it^{-1}ed). \end{aligned}$$

Then, after some work, $it^{-1}d = i\left(\frac{t^2+1}{2t}\right)(b - a) + ic$ and

$$t^{-2}d^2 + it^{-1}e = \left(\frac{t^2 - 1}{2t}\right)^2 (b - a)^2.$$

Putting $k = b - a = |Z_1Z_3|$ and multiplying by $\frac{1}{4}i$ we obtain:

Theorem 6:

$$[W_1W_2W_3W_4] = \frac{t}{4}\left(\left(\frac{k(t^2 - 1)}{2t}\right)^2 - \left|z - \frac{ik(t^2 + 1)}{2t} - ic\right|^2\right).$$

Now, fixing α, this time we regard $[W_1W_2W_3W_4]$ as a function of z. As such, the *critical circle* for $[W_1W_2W_3W_4]$ is the circle, centre $\frac{ik(t^2+1)}{2t} + ic$, radius $\left|\frac{k(t^2-1)}{2t}\right|$. Then:

Corollary 3: The level curves of $[W_1W_2W_3W_4]$ are circles concentric with the critical circle.

Corollary 4: $[W_1W_2W_3W_4]$ is (i) positive, (ii) negative, or (iii) zero when z is respectively (i) inside, (ii) outside, or (iii) on the critical circle. See Figure 9. This time the special case is when $\alpha = \frac{\pi}{4}$, so that $t = 1$ and the critical circle has zero radius, and is the point $ik + ic$. So if $Z = Z_4$ is at this point, then the diagonals Z_1Z_3 and Z_2Z_4 are equal and orthogonal, and $[W_1W_2W_3W_4] = 0$.

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Further, in this case,

$$w_1 = w_3 = \tfrac{1}{2}(1-i)(a-c), \text{ and } w_2 = w_4 = \tfrac{1}{2}(1+i)(b+c).$$

See Figure 10. And, if Z is not at the critical point, then $[W_1W_2W_3W_4] < 0$, and so when $\alpha = \frac{\pi}{4}$ we have

$$|[V_1V_2V_3V_4]| - |[W_1W_2W_3W_4]| = 2|[Z_1Z_2Z_3Z_4]|.$$

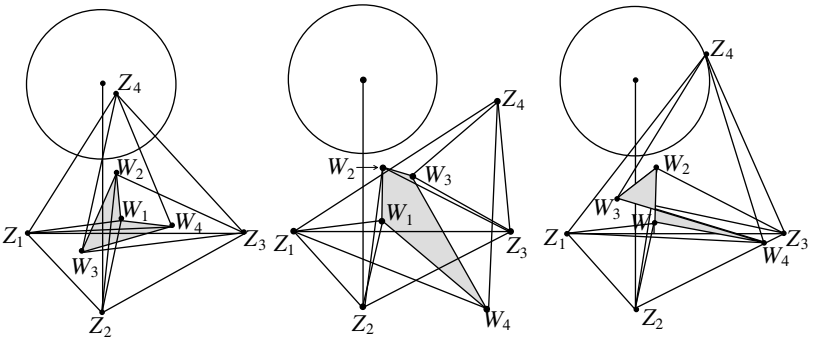


FIGURE 9

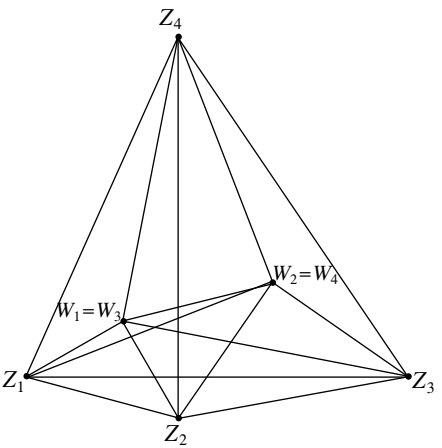


FIGURE 10

4 *n-gons*

In this section we are going to extend Theorems 1 and 5 to n-gons, but this seems to be possible only in the case $\alpha = \frac{\pi}{4}$. So, given an n-gon $Z_1Z_2\ldots Z_n$, we erect right-angled isosceles triangles $Z_1Z_2V_1, \ldots, Z_nZ_1V_n$ (negatively oriented) and $Z_1Z_2W_1, \ldots, Z_nZ_1W_n$ (positively oriented) on its sides, so that $Z_1V_1Z_2W_1, \ldots, Z_nV_nZ_1W_n$ are squares. See Figure 11, where we have drawn an example with $n = 8$. Then we shall prove the following.

Theorem 7: $[V_1V_2\ldots V_N] + [W_1W_2\ldots W_n] = 2[Z_1Z_2\ldots Z_n]$.

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The general proof is below, but there is an easy proof in a special case. Suppose $Z_1Z_2\ldots Z_n$ is a simple polygon, i.e. it has no self-intersections; and suppose, further, that its internal angles are all between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, so that $V_1V_2\ldots V_n$ and $W_1W_2\ldots W_n$ are also simple polygons, with $W_1W_2\ldots W_n$ inside $Z_1Z_2\ldots Z_n$ and $Z_1Z_2\ldots Z_n$ inside $V_1V_2\ldots V_n$. See Figure 11 again. Here the squares $Z_1V_1Z_2W_1$, $Z_2V_2Z_3W_2$, … have their areas bisected by the segments Z_1Z_2, Z_2Z_3, …, and the shaded triangles meeting at Z_1 have equal area, likewise those meeting at Z_2, and so on. This is because their angles, where they meet at Z_i, are supplementary, so have the same sine, and their sides that meet at Z_I are equal in pairs, being sides of the same squares. So their areas are equal, by use of the formula $\frac{1}{2}ab\sin C$. Thus the area of the region inside $V_1V_2\ldots V_n$ but outside $Z_1Z_2\ldots Z_n$ is equal to the area inside $Z_1Z_2\ldots Z_n$ but outside $W_1W_2\ldots W_n$, whence the result. But in the case of self-intersecting polygons, it is more complicated, so we resort to algebra.

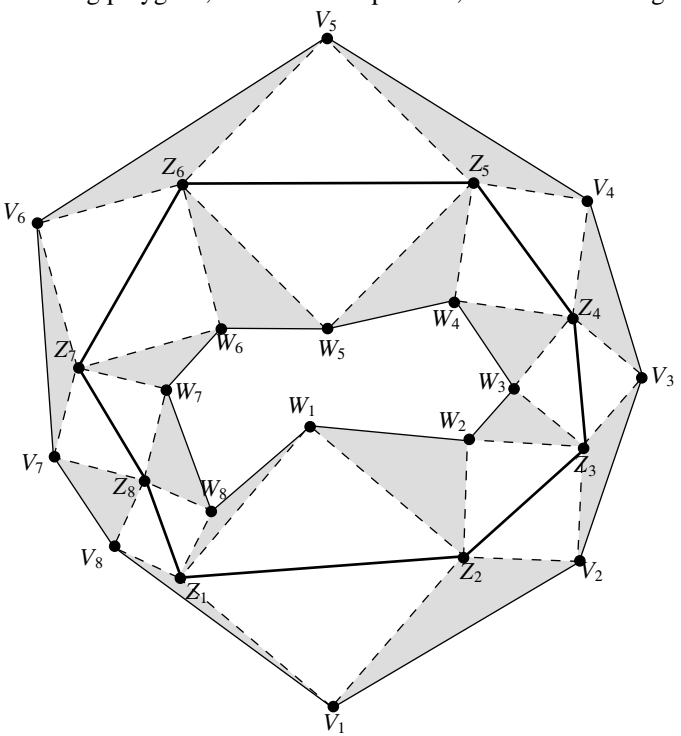


FIGURE 11

First, we need an expression for the area $[Z_1Z_2\ldots Z_n]$ of the n-gon $Z_1Z_2\ldots Z_n$. Now,

$$\begin{aligned} f(z_1, z_2, z_3) &= (z_1\bar{z}_2 - \bar{z}_1z_2) + (z_2\bar{z}_3 - \bar{z}_2z_3) + (z_3\bar{z}_1 - \bar{z}_3z_1) \\ &= f(0, z_1, z_2) + f(0, z_2, z_3) + f(0, z_3, z_1). \end{aligned}$$

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Consequently

$$\begin{aligned} & f(z_1, z_2, z_3) + f(z_1, z_3, z_4) \\ &= \big(f(0, z_1, z_2) + f(0, z_2, z_3) + f(0, z_3, z_1)\big) + \big(f(0, z_1, z_3) + f(0, z_3, z_4) + f(0, z_4, z_1)\big) \\ &= f(0, z_1, z_2) + f(0, z_2, z_3) + f(0, z_3, z_4) + f(0, z_4, z_1), \end{aligned}$$

and, more generally,

$$\begin{aligned} & f(z_1, z_2, z_3) + f(z_1, z_3, z_4) + \ldots + f(z_1, z_{n-1}, z_n) \\ &= f(0, z_1, z_2) + f(0, z_2, z_3) + \ldots + f(0, z_{n-1}, z_n) + f(0, z_n, z_1), \end{aligned}$$

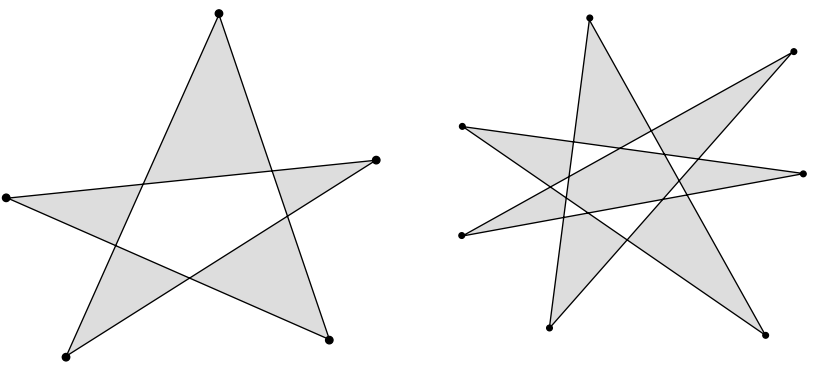
which is invariant under the cyclic permutation $(1\ 2\ \ldots\ n)$ of subscripts. So if we define

$$[Z_1Z_2\ldots Z_n] = [Z_1Z_2Z_3] + [Z_1Z_3Z_4] + \ldots + [Z_1Z_{n-1}Z_n],$$

then this too is invariant under the same permutation of subscripts. Note also that, if O is the origin, then

$$[Z_1Z_2\ldots Z_n] = [OZ_1Z_2] + [OZ_2Z_3] + \ldots + [OZ_{n-1}Z_n] + [OZ_nZ_1].$$

This definition of area, while obviously correct for convex polygons, can produce unexpected results for self-intersecting polygons. If a star pentagon is drawn in *Geometer's Sketchpad*, for example, and the program is asked to construct the pentagon interior, it shades the five small triangles but not the central pentagonal area, as shown in Figure 12(i). But if asked to measure the area, it produces the sum of the areas of the five small triangles plus (surprisingly) *twice* the unshaded pentagonal area. However, as we shall leave the reader to verify, this is the correct answer according the above formulæ!—apart maybe from sign, as *Sketchpad* areas are always positive. The reader might also like to try *predicting* the result for the star heptagon, Figure 12(ii), before working it out.



(i) (ii)

FIGURE 12

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Proof of Theorem 7: The cases $n = 3$ and 4 follow from Theorems 1 and 5. So now we assume the result is true for some $n \geqslant 4$ and use induction. Introduce an extra point Z_{n+1} and extra right-angled isosceles triangles $Z_nZ_{n+1}V'_n$, $Z_{n+1}Z_1V'_{n+1}$, $Z_nZ_{n+1}W'_n$ and $Z_{n+1}Z_1W'_{n+1}$. See Figure 13. We must show that

$$[V_1V_2\ldots V_{n-1}V'_nV'_{n+1}] + [W_1W_2\ldots W_{n-1}W'_nW'_{n+1}] = 2[Z_1Z_2\ldots Z_{n+1}].$$

Now

$$[V_1V_2\ldots V_{n-1}V'_nV'_{n+1}] = [V_1V_2\ldots V_n] + [V_1V_nV_{n-1}V'_nV'_{n+1}],$$

and

$$[W_1W_2\ldots W_{n-1}W'_nW'_{n+1}] = [W_1W_2\ldots W_n] + [W_1W_nW_{n-1}W'_nW'_{n+1}],$$

and of course

$$[Z_1Z_2\ldots Z_{n+1}] = [Z_1Z_2\ldots Z_n] + [Z_1Z_nZ_{n+1}].$$

We know

$$[V_1V_2\ldots V_n] + [W_1W_2\ldots W_n] = 2[Z_1Z_2\ldots Z_n]$$

by the inductive hypothesis, so we need to show that

$$[V_1V_nV_{n-1}V'_nV'_{n+1}] + [W_1W_nW_{n-1}W'_nW'_{n+1}] = 2[Z_1Z_nZ_{n+1}]. \quad (2)$$

Now, comparing Figure 13 with Figure 8, we see that if we apply Theorem 4 to $\Delta Z_1Z_nZ_{n+1}$ with extra point Z_2, we get

$$[W_1W_nW'_{n+1}] + [V_1V_nV'_{n+1}] = [Z_1Z_nZ_{n+1}],$$

and if we apply it to $\Delta Z_nZ_1Z_{n+1}$ with extra point Z_{n-1}, we get

$$[W_{n-1}W_nW'_n] + [V_{n-1}V_nV'_n] = [Z_nZ_1Z_{n+1}].$$

These last two equations can be rewritten as

$$[V_nV'_{n+1}V_1] + [W_nW'_{n+1}W_1] = [Z_1Z_nZ_{n+1}], \quad (3)$$

and

$$[V_nV_{n-1}V'_n] + [W_nW_{n-1}W'_n] = [Z_1Z_nZ_{n+1}]. \quad (4)$$

Then, comparing Figure 13 with Figure 7, we see that if we apply Theorem 3 to $\Delta Z_1Z_nZ_{n+1}$, we get

$$[V_nV'_nV'_{n+1}] + [W_nW'_nW'_{n+1}] = 0. \quad (5)$$

But

$$[V_nV_{n-1}V'_n] + [V_nV'_nV'_{n+1}] + [V_nV'_{n+1}V_1] = [V_nV_{n-1}V'_nV'_{n+1}V_1]$$
$$= [V_1V_nV_{n-1}V'_nV'_{n+1}],$$

and

$$[W_nW_{n-1}W'_n] + [W_nW'_nW'_{n+1}] + [W_nW'_{n+1}W_1] = [W_nW_{n-1}W'_nW'_{n+1}W_1]$$
$$= [W_1W_nW_{n-1}W'_nW'_{n+1}],$$

so adding equations (3), (4) and (5) yields (2), and the proof is complete.

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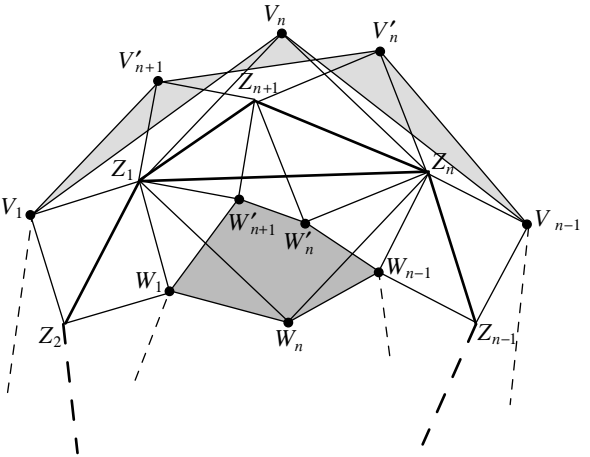


FIGURE 13

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JOHN R. SILVESTER
Department of Mathematics,
King's College,
Strand, London WC2R 2LS
e-mail: *john.silvester@cantab.net*

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