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ARTICLE

# Transversal Hamilton cycles in digraph collections

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**Abstract**

Given a collection $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$ of digraphs on the common vertex set $V$, an $m$-edge digraph $H$ with vertices in $V$ is *transversal* in $\mathcal{D}$ if there exists a bijection $\varphi : E(H) \to [m]$ such that $e \in E(D_{\varphi(e)})$ for all $e \in E(H)$. Ghouila-Houri proved that any $n$-vertex digraph with minimum semi-degree at least $\frac{n}{2}$ contains a directed Hamilton cycle. In this paper, we provide a transversal generalisation of Ghouila-Houri's theorem, thereby solving a problem proposed by Chakraborti, Kim, Lee, and Seo. Our proof utilises the absorption method for transversals, the regularity method for digraph collections, as well as the transversal blow-up lemma and the related machinery. As an application, when $n$ is sufficiently large, our result implies the transversal version of Dirac's theorem, which was proved by Joos and Kim.

**Keywords:** Transversal; Hamilton cycle; Digraph collection; Semi-degree

**2020 MSC Codes:** 05C35

## 1. Introduction

A transversal $X$ over a collection $\mathcal{F} = \{F_1, \ldots, F_m\}$ of objects represents an object that intersects every $F_i$. Here, the object can be any mathematical object, such as sets, spaces, set systems, matroids, and so forth. Several classical theorems, which guarantee the existence of certain objects, have been restated in the context of transversals, illustrating the feasibility or infeasibility of obtaining such an object as a transversal over a certain collection. To name a few, transversals are extensively studied for Carathéodory's theorem [4], Helly's theorem [22], Erdős-Ko-Rado theorem [2], Rota's basis conjecture [20, 24], and others.

Following this trend, there has been ample research studying transversal generalisations of classical results in graph theory. These generalisations are not only interesting in their own right but also often provide a strengthening of the original results, reflecting the robustness of graph properties to some extent. Transversals over a collection of graphs have been implicitly used in the literature under the name of rainbow colouring and explicitly defined in [21] as follows.

**Definition 1.1.** *For a given collection $\mathcal{G} = \{G_1, \ldots, G_m\}$ of graphs/hypergraphs/digraphs with the same vertex set $V$, an $m$-edge graph/hypergraph/digraph $H$ with vertices in $V$ is transversal in $\mathcal{G}$ if there exists a bijection $\varphi : E(H) \to [m]$ such that $e \in E(G_{\varphi(e)})$ for all $e \in E(H)$.*

By interpreting each $G_i$ as the set of edges coloured with colour $i$, the function $\varphi$ is often referred to as a colouring. We say that the colouring $\varphi$ is *rainbow* if it is injective, and $H$ is a *rainbow subgraph* if there exists a rainbow colouring on it. Many classical results in extremal graph theory

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have been extended to such transversal settings, exhibiting interesting phenomena. The central question in studying transversal generalisations can be stated as follows.

**Problem A** ([21]). *Let $H$ be a graph/hypergraph/digraph with $m$ edges, $\mathcal{G} = \{G_1, G_2, \ldots, G_m\}$ be a collection of graphs/hypergraphs/digraphs on a common vertex set of size $n$. Which property $\mathcal{P}$ imposed on $\mathcal{G}$ will guarantee the existence of a transversal copy of $H$?*

Since every object in the family $\mathcal{G}$ can be the same, a necessary condition for a positive answer to the above question is that every $n$-vertex graph/hypergraph/digraph satisfying the property $\mathcal{P}$ contains a copy of $H$. However, this alone is not always sufficient. For example, Aharoni, DeVos, de la Maza, Montejano, and Šámal [1] proved that if $\mathcal{G} = \{G_1, G_2, G_3\}$ is a collection of graphs on a common vertex set of size $n$ and $|E(G_i)| > (\frac{26-2\sqrt{7}}{81})n^2$ for all $i \in [3]$, then $\mathcal{G}$ contains a rainbow triangle. Moreover, the constant $\frac{26-2\sqrt{7}}{81}$ is optimal, while Mantel's theorem states that any $n$-vertex graph with more than $\lfloor \frac{n^2}{4} \rfloor$ edges must contain a triangle.

In 2020, Aharoni [1] conjectured that Dirac's theorem can be extended to a transversal version. Cheng, Wang and Zhao [12] solved this conjecture asymptotically, and it was completely confirmed by Joos and Kim [21].

**Theorem 1.2** ([21]). *Suppose $\mathcal{G} = \{G_1, \ldots, G_n\}$ is a collection of graphs with the same vertex set of size $n$. If $\delta(G_i) \geq \frac{n}{2}$ for all $i \in [n]$, then $\mathcal{G}$ contains a transversal Hamilton cycle.*

There are numerous established results on transversal problems in graph collections. For more results on transversals, we refer the reader to the survey [26].

A natural next step is to study the rainbow analogues in digraph collections. Cheng, Han, Wang and Wang [8] studied the transversal version of Hajnal-Szemerédi theorem in digraph collections, which asymptotically generalised the corresponding results by Czygrinow-DeBiasio-Kierstead-Molla [15] and Treglown [30]. It is well-known that every tournament contains a Hamilton path, and every strongly connected tournament contains a Hamilton cycle. Chakraborti, Kim, Lee and Seo [6] proved the existence of a transversal directed Hamilton path and cycle in a family of tournaments. They [7] also proved that in a digraph collection consisting of tournaments, there exist transversal Hamilton cycles of all possible orientations possibly except the consistently oriented one. These results generalised the classical theorem of Thomason [28]. Babiński, Grzesik and Prorok [3] studied the transversal Mantel's theorem in digraph and oriented graph collections. Gerbner, Grzesik, Palmer and Prorok [18] determined the minimum number of edges in each colour which guarantees the existence of any rainbow directed star.

### *1.1 Main result*

In this paper, we study the transversal problem in digraph collections. Let $D$ be a digraph with no loops and at most one edge in each direction between every pair of vertices. Denote the set of vertices of $D$ by $V(D)$. Define $\delta^+(D)$ and $\delta^-(D)$ as the minimum out-degree and in-degree of $D$, respectively, and let the *minimum semi-degree* be $\delta^0(D) = \min \delta^+(D), \delta^-(D)$. For a collection $\mathcal{D} = \{D_1, \ldots, D_m\}$ of digraphs (not necessarily distinct) on a common vertex set $V$, define $\delta^0(\mathcal{D}) = \min\{\delta^0(D_i) : i \in [m]\}$.

In 1960, Ghouila-Houri [19] generalised Dirac's theorem to digraphs.

**Theorem 1.3** ([19]). *For each integer $n \geq 3$, every $n$-vertex digraph $D$ with $\delta^0(D) \geq \frac{n}{2}$ contains a directed Hamilton cycle.*

Chakraborti, Kim, Lee, and Seo [6] suggested that it would be interesting to consider a transversal version of Ghouila-Houri's theorem. Our main result solves this.

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**Theorem 1.4.** *Let $n$ be a sufficiently large integer. Suppose $\mathcal{D} = \{D_1, \dots, D_n\}$ is a collection of digraphs on a common vertex set of size $n$. If $\delta^0(\mathcal{D}) \geq \frac{n}{2}$, then $\mathcal{D}$ contains a transversal directed Hamilton cycle.*

In an undirected graph, one may replace each undirected edge by two arcs with opposite directions. Thus, for sufficiently large $n$, Theorem 1.2 is a direct consequence of Theorem 1.4. Furthermore, by setting $D_1 = D_2 = \cdots = D_n$ in Theorem 1.4, we immediately obtain Theorem 1.3 for all sufficiently large $n$.

### 1.2 Notation and organisation

**Notation**. Throughout the paper, we use standard graph theory notation and terminology. Let $D$ be a digraph with vertex set $V(D)$ and edge set $E(D)$, and sometimes we write $e(D) := |E(D)|$. For $x, y \in V(D)$, denote by $xy$ the directed edge from $x$ to $y$ and we always write $xy \in D$. Denote by $P_n$ and $C_n$ the directed path and directed cycle on $n$ vertices, respectively.

We use $H \subseteq D$ to denote that $H$ is a sub-digraph of $D$. For $S \subseteq V(D)$, we write $D[S]$ for the sub-digraph of $D$ induced by $S$ and $D - S$ for $D[V(D) \setminus S]$. For two not necessarily disjoint vertex subsets $A, B \subseteq V(D)$, let $D[A, B]$ be the sub-digraph of $D$ with vertex set $A \cup B$ and edge set $\{xy \in E(D) : x \in A, y \in B\}$ and let $e_D(A, B) := |E(D[A, B])|$. Denote $D^{\pm}[A, B] = D[A, B] \cup D[B, A]$. When $A = B$, we often abbreviate $D[A, B]$ and $e_D(A, B)$ as $D[A]$ and $e_D(A)$ respectively.

For a digraph $D$ and $v \in V(D)$, we denote the sets of out-neighbours and in-neighbours of $v$ by $N_D^+(v)$ and $N_D^-(v)$, respectively, with the corresponding out- and in-degrees given by $d_D^+(v) := |N_D^+(v)|$ and $d_D^-(v) := |N_D^-(v)|$. For $S \subseteq V(D)$, we write $d_D^+(v, S) := |N_D^+(v) \cap S|$ and $d_D^-(v, S) := |N_D^-(v) \cap S|$. We often omit subscripts when they are clear from the context.

Let $\mathcal{D} = \{D_i : i \in [m]\}$ be a collection of digraphs on a common vertex set $V$ and $H$ be a rainbow digraph inside $\mathcal{D}$. For a vertex $v \in V$ and each $i \in [m]$, let $N_i^+(v) := N_{D_i}^+(v)$, $N_i^-(v) := N_{D_i}^-(v)$, $d_i^+(v) := |N_i^+(v)|$ and $d_i^-(v) := |N_i^-(v)|$. For any two vertex subsets $X, Y \subseteq V$, define $\mathcal{D}[X] := \{D_i[X] : i \in [m]\}$, $\mathcal{D}[X, Y] := \{D_i[X, Y] : i \in [m]\}$ and $\mathcal{D}^{\pm}[X, Y] := \{D_i^{\pm}[X, Y] : i \in [m]\}$. If $E(D) = \emptyset$, then we simply write $D = \emptyset$. Similarly, if there is no edge in $D_i$ for all $i \in [m]$, then we write $\mathcal{D} = \emptyset$. Denote by $\mathrm{col}(H)$ the set of colours appearing in $H$. Two rainbow graphs $H_1$ and $H_2$ are said to be *disjoint* if $V(H_1) \cap V(H_2) = \emptyset$ and $\mathrm{col}(H_1) \cap \mathrm{col}(H_2) = \emptyset$.

The notation above extends to undirected graphs in the obvious ways. For $n \in \mathbb{N}$, we denote the set $\{1, 2, \dots, n\}$ by $[n]$. For any two constants $\alpha, \beta \in (0, 1)$, we write $\alpha \ll \beta$ if there exists a function $f = f(\beta)$ such that the subsequent arguments hold for all $0 < \alpha \leq f$.

**Organisation.** We conclude this section with a sketch of the proof of Theorem 1.4. In Section 2, we introduce two key techniques that we will need later, i.e., the regularity lemma in digraph collections and a stability result for transversal perfect matchings in bipartite graph collections. For readability, we postpone their proofs to the appendix. The proof of Theorem 1.4 will be divided into "stable case" and "extremal case". Section 3 introduces notions of extremality and stability for digraphs and digraph collections; we prove several results about absorption that will be used in the proof of the stable case. In Section 4, we show that there is always a transversal directed Hamilton cycle in the stable case. Sections 5–7 deal with the extremal case. We finish our paper with some concluding remarks in the final section.

### 1.3 Discussion of the strategy

Joos and Kim [21] proved the transversal version of Dirac's theorem by introducing the auxiliary digraph technique, along with some rotations. However, this technique is not feasible in digraph collections because we cannot extend an existing directed cycle into a longer cycle by using the

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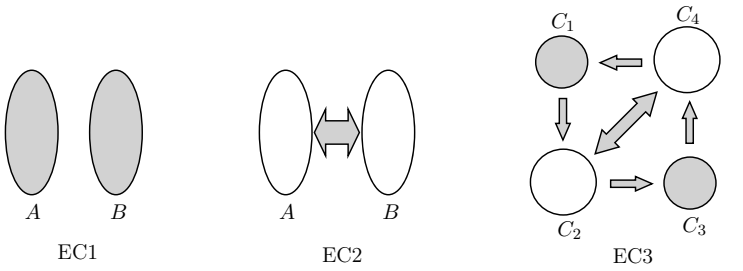



**Figure 1.** Extremal digraphs EC1, EC2, and EC3. The grey shaded elliptical indicates that the digraph induced by this vertex set is complete, the grey shaded arrow between two vertex sets indicates that the induced digraph by them is complete in this direction.

same rotation. In [9], the first author and Staden further developed the absorption method for graph collections, combined with the application of the transversal blow-up lemma, they finally proved a stability version of Joos and Kim's result [21]. The absorption technique in [9] originates from [12] and [8], which adapted the absorption technique first introduced in [25]. This approach has become a powerful tool for solving various transversal graph embedding problems.

In this paper, we adapt the method developed in [9] and generalise their stability from graph collections to digraph collections. By analysing the extremal cases, we proved a tight bound for the transversal Ghouila-Houri's theorem. We believe that our method could be used to demonstrate stabilities or tight bounds for other transversal graph embedding problems (see the survey [26]).

### *1.4 Sketch of the proof of Theorem 1.4*

Our proof utilizes the regularity blow-up method in digraph collections, and is divided into two parts: stable case and extremal case.

**Stable case.** We first characterise the extremality of a single digraph, which forms the basis of our extremal case distinction. Let $0 < \frac{1}{n} \ll \epsilon \ll 1$ and $D$ be an $n$-vertex digraph with minimum semi-degree at least $(\frac{1}{2} - \epsilon)n$. Suppose $D$ has no directed Hamilton cycle. Then $D$ contains a sub-digraph that is close to one of the digraphs in Figure 1. By Lemma 3.2, such a sub-digraph induces a characteristic partition of $D$; we say that $D$ is $(\epsilon, \mathrm{EC}k)$-extremal for some $k \in [3]$. Furthermore, this $k$ is unique.

Assume that $\mathcal{D} = \{D_1, \ldots, D_n\}$ is a collection of digraphs with the same vertex set $V$ of size $n$. Suppose that $\delta^0(\mathcal{D}) \geq (\frac{1}{2} - \epsilon)n$ and $\mathcal{D}$ contains no transversal directed Hamilton cycles. We suspect that either almost all digraphs in $\mathcal{D}$ are $(\epsilon, \mathrm{EC}k)$-extremal for some $k \in [2]$ (allowing $k = 1$ for some digraphs and $k = 2$ for others), or almost all digraphs in $\mathcal{D}$ are $(\epsilon, \mathrm{EC3})$-extremal; moreover those digraphs have similar vertex partitions. Based on this observation, we define two kinds of stability for a digraph collection with minimum degree at least $(\frac{1}{2} - \epsilon)n$. We say that

- $\mathcal{D}$ is strongly stable if $\mathcal{D}$ contains many digraphs each of which is not $(\epsilon, \mathrm{EC}k)$-extremal for any $k \in [3]$,
- $\mathcal{D}$ is weakly stable if for almost all colours $i \in [n]$, each $D_i$ is $(\epsilon, \mathrm{EC}k)$-extremal for some $k \in [3]$, but their vertex partitions are not similar,
- $\mathcal{D}$ is stable if it is either strongly stable or weakly stable.

In the stable case, we are to show that if $\mathcal{D}$ is stable, then it contains a transversal directed Hamilton cycle.

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**Step 1. Build an absorbing cycle when $\mathcal{D}$ is stable. (Section 3)**

We build a "directed absorbing rainbow cycle" $C$ for $\mathcal{D}$ with the property that $C$ is very small and there is a set $A$ consisting of some colour-vertex-vertex triples $(c, u, v)$ such that whenever $A_0 \subseteq A$ is sufficiently small compared to $|C|$, $C$ can absorb all of its elements. Additionally, we construct an additional auxiliary set $\mathcal{Q}$ consisting of colour-colour-vertex-vertex tuples $(c, c', v, v')$ to absorb those colours and vertices which $C$ cannot. Delete all colours and vertices in $C$ and $\mathcal{Q}$ from $\mathcal{D}$.

**Step 2. Use the regularity-blow-up method for transversals in digraph collections to cover almost all vertices and colours with long rainbow directed paths.**

We first prove a regularity lemma for digraph collections (Section 2.1), and then apply it to $\mathcal{D}$, yielding a reduced digraph collection $\mathcal{R}$ that inherits the minimum degree condition and the stability property of $\mathcal{D}$. By considering the characteristic bipartite graph of each digraph, we get a bipartite graph collection $\mathcal{B}_{\mathcal{R}}$ for the reduced digraph collection $\mathcal{R}$. Notice that $\mathcal{R}$ has a disjoint rainbow directed cycle partition if and only if $\mathcal{B}_{\mathcal{R}}$ has a transversal perfect matching. We establish a stability result for transversal perfect matching in bipartite graph collections (Section 2.2), ensuring that the disjoint rainbow directed cycle partition of $\mathcal{R}$ exists. Using the blow-up method, we obtain a set of almost spanning disjoint rainbow directed paths within each directed cycle of $\mathcal{R}$, which covers almost all vertices outside the absorbing cycle $C$ and the auxiliary set $\mathcal{Q}$ (Section 4).

**Step 3. Connect the obtained directed paths and cover the remaining vertices via the absorbing cycle.**

In this step, we apply the absorbing property of $C$ as well as the auxiliary set $\mathcal{Q}$ to connect all the directed rainbow paths and to cover the remaining vertices, which ultimately forms a transversal directed Hamilton cycle.

**Extremal case (Sections 5–7).** Assume that $\delta^0(\mathcal{D}) \geq \frac{n}{2}$ and $\mathcal{D}$ contains no transversal Hamilton cycles. Let $0 < \frac{1}{n} \ll \epsilon \ll \delta \ll \eta \ll 1$. Denote $\mathcal{C}_k := \{i \in [n] : D_i \text{ is } (\epsilon, \mathrm{EC}k)\text{-extremal}\}$ for $k \in [3]$ and $\mathcal{C}_{\mathrm{bad}} := [n] \setminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3)$. Notice that $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for $1 \leq i < j \leq 3$. By "stable case", we know that almost all digraphs in $\mathcal{D}$ are extremal and $|\mathcal{C}_1| + |\mathcal{C}_2| \leq 2\sqrt{\delta}n$ if $|\mathcal{C}_3| \geq 2\sqrt{\delta}n$. In view of Lemma 3.2, we can fix a characteristic partition $(A_i, B_i, L_i)$ for each graph $D_i$ with $i \in \mathcal{C}_1 \cup \mathcal{C}_2$, and a characteristic partition $(C_i^1, C_i^2, C_i^3, C_i^4, L_i)$ for each graph $D_i$ with $i \in \mathcal{C}_3$. By swapping labels, we know that $|A_1 \triangle A_i|, |B_1 \triangle B_i| < \delta n$ for every $i \in (\mathcal{C}_1 \cup \mathcal{C}_2) \setminus \{1\}$ and $|C_1^k \triangle C_i^k| < 2\delta n$ for every $i \in \mathcal{C}_3 \setminus \{1\}$. We proceed with our proof by considering the sizes of $|\mathcal{C}_1|$, $|\mathcal{C}_2|$ and $|\mathcal{C}_3|$. We will take the case $|\mathcal{C}_1| < \eta n$, $|\mathcal{C}_2| \geq \eta n$ and $|\mathcal{C}_3| \leq 2\sqrt{\delta}n$ as an example to illustrate the general idea of our proof.

Firstly, expand $A_1 \cup B_1$ into an equitable partition $A \cup B$ of $V$. To apply the transversal blow-up lemma (Lemma 6.2) for finding long rainbow directed paths in $\mathcal{D}$, we need to ensure most vertices in the partition $A \cup B$ satisfy uniform neighbourhood properties (for example, a vertex $v \in A$ should satisfy $v \in A_i$ for the majority of colours $i \in \mathcal{C}_2$). We identify three vertex types in $A$ (and symmetrically in $B$) that may disrupt the desired uniformity:

Type 1. $v \in A$ belongs to $L_i$ for many $i \in \mathcal{C}_2$;

Type 2. $v \in A$ belongs to $A_i$ for many $i \in \mathcal{C}_2$, and $v \in B_j$ for many other $j \in \mathcal{C}_2$;

Type 3. $v \in A$ belongs to nearly all $B_i$ for $i \in \mathcal{C}_2$.

We handle these cases as follows: For Type 3 vertices in $A$, we move them to $B$ (and perform the analogous operation for Type 3 vertices in $B$). We define $V_{\mathrm{bad}}$ as the union of vertices of Type 1 and Type 2. After this reallocation, every vertex in $A \setminus V_{\mathrm{bad}}$ (resp. $B \setminus V_{\mathrm{bad}}$) lies in $A_i$ (resp. $B_i$) for the majority of colours $i \in \mathcal{C}_2$. However, the partition sizes $|A|$ and $|B|$ may become unbalanced, with their difference bounded by $9\delta n$. To balance them, we require the following four steps.

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**Step 1. Balance the number of vertices in $A$ and $B$.**
If there exists a set of disjoint rainbow paths in $\mathcal{D}[B]$ such that after deleting their vertices, the number of vertices in $A$ and $B$ is balanced, then we are done. Otherwise, we will show that $\mathcal{D}$ must contain a transversal directed Hamilton cycle (see Lemma 6.5).

**Step 2. Cover vertices in $V_{\text{bad}}$.**
Using colours in $\mathcal{C}_2$, we are to find a sequence of disjoint rainbow directed paths $P_3$ such that their centres are all vertices in $V_{\text{bad}}$ and endpoints are unused vertices in $(A \cup B) \setminus V_{\text{bad}}$.

**Step 3. Deal with colours in $\mathcal{C}_{\text{bad}} \cup \mathcal{C}_3$.**
Choose a maximal rainbow matching $M$ using colours in $\mathcal{C}_{\text{bad}}$ and avoiding vertices used in Steps 1-2. It is routine to verify that for each $j \in (\mathcal{C}_{\text{bad}} \cup \mathcal{C}_3) \setminus \text{col}(M)$, the graph $D_j$ is EC1-extremal.

**Step 4. Deal with colours in $(\mathcal{C}_1 \cup \mathcal{C}_{\text{bad}} \cup \mathcal{C}_3) \setminus \text{col}(M)$.**
We greedily select two rainbow directed paths using colours from $(\mathcal{C}_1 \cup \mathcal{C}_{\text{bad}} \cup \mathcal{C}_3) \setminus \text{col}(M)$ while avoiding the vertices used in the previous three steps.

Based on the minimum degree condition and the characteristic partition of extremal digraphs, one may use colours in $\mathcal{C}_2$ to connect all rainbow paths obtained in the above four steps into a single short rainbow directed path, say $P$, such that $V_{\text{bad}} \subseteq V(P)$, $\mathcal{C}_1 \cup \mathcal{C}_3 \cup \mathcal{C}_{\text{bad}} \subseteq \text{col}(P)$ and $|A \setminus V(P)|$ and $|B \setminus V(P)|$ are nearly equal. By applying the transversal blow-up lemma and some structural analysis, we obtain that $\mathcal{D}$ contains a transversal directed Hamilton cycle.

## 2. Preliminaries

### *2.1 Regularity for digraph collections*

We use the following version of the regularity lemma for digraph collections, which is obtained by applying the degree version of the weak regularity lemma to the 4-graph of $\mathcal{D}$. The proofs of Lemmas 2.2 and 2.4 are postponed to the appendix. We begin by defining regularity for digraph collections.

**Definition 2.1.** *Suppose that $\mathcal{D} = \{D_c : c \in \mathcal{C}\}$ is a collection of bipartite digraphs on a common vertex partition $V_1 \cup V_2$. We say that $\mathcal{D}$ is $(\epsilon, d)$-regular if whenever $V_i' \subseteq V_i$ with $|V_i'| \geq \epsilon |V_i|$ for $i \in [2]$ and $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C}'| \geq \epsilon |\mathcal{C}|$ we have*

$$\left| \frac{\sum_{c \in \mathcal{C}'} e_{D_c}(V_1', V_2')}{|\mathcal{C}'||V_1'||V_2'|} - \frac{\sum_{c \in \mathcal{C}} e_{D_c}(V_1, V_2)}{|\mathcal{C}||V_1||V_2|} \right| < \epsilon$$

*and $\sum_{c \in \mathcal{C}} e_{D_c}(V_1, V_2) \geq d|\mathcal{C}||V_1||V_2|$.*

**Lemma 2.2** (Regularity lemma for digraph collections). *For every integer $L_0 \geq 1$ and every $\epsilon, \delta > 0$, there is an integer $n_0 := n_0(\epsilon, \delta, L_0)$ such that for every $d \in [0, 1)$ and every digraph collection $\mathcal{D} = \{D_i : i \in \mathcal{C}\}$ on vertex set $V$ of size $n \geq n_0$ with $\delta n \leq |\mathcal{C}| \leq \frac{n}{\delta}$, there exists a partition of $V$ into $V_0, V_1, \ldots, V_L$, of $\mathcal{C}$ into $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_M$ and a spanning sub-digraph $D_c'$ of $D_c$ for each $c \in \mathcal{C}$ satisfying the following properties:*

(i) *$L_0 < L, M < n_0$ and $|V_0| + |\mathcal{C}_0| \leq \epsilon n$,*
(ii) *$|V_1| = \cdots = |V_L| = |\mathcal{C}_1| = \cdots = |\mathcal{C}_M| =: m$,*
(iii) *$\sum_{c \in \mathcal{C}} d^+_{D_c'}(v) \geq \sum_{c \in \mathcal{C}} d^+_{D_c}(v) - (\frac{24d}{\delta^3} + \epsilon)n^2$ and $\sum_{c \in \mathcal{C}} d^-_{D_c'}(v) \geq \sum_{c \in \mathcal{C}} d^-_{D_c}(v) - (\frac{24d}{\delta^3} + \epsilon)n^2$ for all $v \in V$, and $e(D_c') \geq e(D_c) - (\frac{24d}{\delta^3} + \epsilon)n^2$ for all $c \in \mathcal{C}$,*
(iv) *if for each $c \in \mathcal{C}$, the digraph $D_c'$ has an edge with both end vertices in a single cluster $V_i$ for some $i \in [L]$, then $c \in \mathcal{C}_0$,*

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(v) *for all triples* $\{(h,i),j\} \in \binom{[L]}{2} \times [M]$, *we have either* $D'_c[V_h, V_i] = \emptyset$ *for all* $c \in \mathcal{C}_j$, *or* $\mathcal{D}'_{hi,j} := \{D'_c[V_h, V_i] : c \in \mathcal{C}_j\}$ *is* $(\epsilon, d)$*-regular.*

The sets $V_i$ are called *vertex clusters* and the sets $\mathcal{C}_i$ are called *colour clusters*, while $V_0$ and $\mathcal{C}_0$ are the exceptional vertex and colour sets respectively. Now, we define the reduced digraph collection.

**Definition 2.3** (Reduced digraph collection). *Given a digraph collection* $\mathcal{D} = \{D_i : i \in \mathcal{C}\}$ *on* $V$ *and parameters* $\epsilon > 0$, $d \in [0,1)$ *and* $L_0 \geq 1$, *the reduced digraph collection* $\mathcal{R} := \mathcal{R}(\epsilon, d, L_0)$ *of* $\mathcal{D}$ *is defined as follows. Apply Lemma 2.2 to* $\mathcal{D}$ *with parameters* $\epsilon, \delta, d, L_0$ *to obtain* $\mathcal{D}'$, *a partition* $V_0, V_1, \ldots, V_L$ *of* $V$, *and a partition* $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_M$ *of* $\mathcal{C}$, *where* $V_0, \mathcal{C}_0$ *are the exceptional sets and* $V_1, \ldots, V_L$ *are the vertex clusters and* $\mathcal{C}_1, \ldots, \mathcal{C}_M$ *are the colour clusters. Then* $\mathcal{R} := \{R_1, \ldots, R_M\}$ *is a digraph collection of* $M$ *digraphs each on the same vertex set* $[L]$, *where, for* $\{(h,i),j\} \in \binom{[L]}{2} \times [M]$, *we have* $hi \in R_j$ *if and only if* $\mathcal{D}'_{hi,j}$ *is* $(\epsilon, d)$*-regular.*

The next lemma states that clusters inherit a minimum semi-degree bound in the reduced digraph collection.

**Lemma 2.4** (Degree inheritance). *Suppose* $L_0 \geq 1$ *and* $0 < \frac{1}{n} \ll \epsilon \leq d \ll \delta, \gamma, p \leq 1$. *Let* $\mathcal{D} = \{D_i : i \in \mathcal{C}\}$ *be a digraph collection on vertex set* $V$ *of size* $n$ *with* $\delta^0(\mathcal{D}) \geq (p+\gamma)n$ *and* $\delta n \leq |\mathcal{C}| \leq \frac{n}{\delta}$. *Let* $\mathcal{R} := \mathcal{R}(\epsilon, d, L_0)$ *be the reduced digraph collection of* $\mathcal{D}$ *consisting of* $M$ *digraphs on a common vertex set* $[L]$. *Then*

(i) *for every* $i \in [L]$, *there are at least* $(1 - d^{\frac{1}{4}})M$ *colours* $j \in [M]$ *for which* $d^+_{R_j}(i) \geq (p + \frac{\gamma}{2})L$ *and* $d^-_{R_j}(i) \geq (p + \frac{\gamma}{2})L$,

(ii) *for every* $j \in [M]$, *there are at least* $(1 - d^{\frac{1}{4}})L$ *vertices* $i \in [L]$ *for which* $d^+_{R_j}(i) \geq (p + \frac{\gamma}{2})L$ *and* $d^-_{R_j}(i) \geq (p + \frac{\gamma}{2})L$.

**Proof.** (i) Choose a vertex $v \in V \setminus V_0$. By Lemma 2.2 (i) and (iii), we have

$$\sum_{c \in \mathcal{C}} d^+_{D'_c - V_0}(v) \geq \sum_{c \in \mathcal{C}} d^+_{D_c}(v) - \left(\frac{24d}{\delta^3} + \epsilon\right) n^2 - \epsilon n|\mathcal{C}| \geq \sum_{c \in \mathcal{C}} d^+_{D_c}(v) - \frac{26dn^2}{\delta^3}.$$

Let $\mathcal{D}_v := \left\{c \in \mathcal{C} \setminus \mathcal{C}_0 : d^+_{D'_c - V_0}(v) \geq d^+_{D_c}(v) - \sqrt{d}n\right\}$. Then

$$\sum_{c \in \mathcal{C} \setminus \mathcal{C}_0} d^+_{D'_c - V_0}(v) \leq \sum_{c \in \mathcal{C} \setminus \mathcal{C}_0} d^+_{D_c}(v) - \sqrt{d}n|\mathcal{C} \setminus (\mathcal{C}_0 \cup \mathcal{D}_v)|.$$

Therefore, $|\mathcal{C} \setminus (\mathcal{C}_0 \cup \mathcal{D}_v)| \leq \frac{d^{\frac{1}{3}}n}{2}$ since $d \ll \delta$. It follows that $|\mathcal{D}_v| \geq |\mathcal{C}| - \frac{d^{\frac{1}{3}}n}{2} - \epsilon n \geq |\mathcal{C}| - d^{\frac{1}{3}}n$. Recall that $mM \leq |\mathcal{C}| \leq mM + \epsilon n$ and $mL \leq n \leq mL + \epsilon n$. Together with $\delta n \leq |\mathcal{C}| \leq \frac{n}{\delta}$, one has $\delta L \leq 2(1-\epsilon)M$. Hence the number of colour clusters $\mathcal{C}_j$ containing at least one colour of $\mathcal{D}_v$ is at least

$$\frac{|\mathcal{D}_v|}{m} \geq M - \frac{d^{\frac{1}{3}}n}{m} \geq M - \frac{d^{\frac{1}{3}}L}{1-\epsilon} \geq M - \frac{2d^{\frac{1}{3}}M}{\delta} \geq \left(1 - \frac{d^{\frac{1}{4}}}{2}\right)M.$$

Now let $i \in [L]$ and $v \in V_i$. For each cluster $\mathcal{C}_j$ as above, choose an arbitrary colour $c_j \in \mathcal{C}_j \cap \mathcal{D}_v$. Thus, the number of vertex clusters $V_h$ containing some vertices in $N^+_{D'_{c_j}}(v)$ is at least

$$\frac{d^+_{D_{c_j}}(v) - \sqrt{d}n}{m} \geq \frac{(p + \gamma - \sqrt{d})n}{m} \geq \left(p + \frac{\gamma}{2}\right)L.$$

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Lemma 2.2 (v) implies that $i$ is adjacent to each such $V_h$ in $R_j$. Therefore, for every $i \in [L]$, $d^+_{R_j}(i) \geq (p + \frac{\gamma}{2})L$ for at least $(1 - \frac{d^{\frac{1}{4}}}{2})M$ colours $j$. The proofs of other cases are similar and we omit the proof. □

### *2.2 Transversal perfect matching in bipartite graph collections*

In the study of digraphs, a natural approach is to consider their characteristic bipartite graphs. This subsection introduces the corresponding characteristic bipartite graph collection associated with a digraph system, and studies the existence of transversal perfect matchings in this bipartite system.

Let $D$ be a digraph with vertex set $V$. We define $B_D$ to be the characteristic bipartite graph of $D$ with bipartition $V_1 \cup V_2$ where $V_1 = V_2 = V$; for each $u \in V_1$ and $v \in V_2$, $uv \in E(B_D)$ if and only if $uv \in E(D)$. Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a collection of digraphs on a common vertex set $V$. Define the characteristic bipartite graph collection of $\mathcal{D}$ to be $\mathcal{B}_\mathcal{D} = \{B_{D_1}, \ldots, B_{D_n}\}$. A *matching* in a (di)graph $G$ is a collection of vertex-disjoint edges $M \subseteq E(G)$. We say that $M$ is a *perfect matching* if $V(M) = V(G)$.

It is straightforward to check that $\mathcal{B}_\mathcal{D}$ contains a transversal perfect matching if and only if the vertex set of $\mathcal{D}$ can be covered by a set of disjoint rainbow cycles inside $\mathcal{D}$. Therefore, in the remainder of this section, we are to find a transversal perfect matching in a bipartite graph collection.

Joos and Kim [21] established a minimum degree condition that guarantees the existence of transversal perfect matchings in a graph collection. Bradshaw [5] proved the analogue result in bipartite graph collections, which can be stated as follows.

**Theorem 2.5** ([5]). *Let $\mathcal{G} = \{G_1, \ldots, G_n\}$ be a collection of bipartite graphs on a common vertex bipartition $V_1 \cup V_2$ with $|V_1| = |V_2| = n$. If for each $i \in [n]$ we have the following hold:*

(i) $d_{G_i}(v) > \frac{n}{2}$ *for each* $v \in V_1$,

(ii) $d_{G_i}(v) \geq \frac{n}{2}$ *for each* $v \in V_2$,

*then $\mathcal{G}$ contains a transversal perfect matching.*

In this subsection, we establish a stability result for transversal perfect matching in bipartite graph collections, that is, the minimum degree required to guarantee a transversal perfect matching can be below that of Theorem 2.5, as long as the bipartite graph collection $\{G_1, \ldots, G_n\}$ is far in edit distance from several extremal cases. For this, we first define the extremality for a single bipartite graph.

**Definition 2.6** (nice, extremal). *Let $\epsilon > 0$ and $G$ be a balanced bipartite graph on the vertex set $V_1 \cup V_2$ of size $2n$. We say that*

- *$G$ is $\epsilon$-nice if for any two sets $A \subseteq V_1$ and $B \subseteq V_2$ of size at least $(\frac{1}{2} - \epsilon)n$ we have $e_G(A, B) \geq \epsilon n^2$;*
- *$G$ is $\epsilon$-extremal if it is not $\epsilon^5$-nice.*

The following lemma is the basis for our proof.

**Lemma 2.7.** *Suppose that $0 < \frac{1}{n} \ll d \ll \epsilon \leq 1$. Let $G$ be a balanced bipartite graph on a vertex set $V = V_1 \cup V_2$ of size $2n$ with $d_G(x) \geq (\frac{1}{2} - \epsilon^5)n$ for all but at most $dn$ vertices $x \in V$ which is $\epsilon$-extremal. Then there is a characteristic partition $(A_1, B_1, A_2, B_2, C_1, C_2)$ of $G$ such that the following hold:*

(i) $A_i, B_i, C_i \subseteq V_i$, $|A_i| = |B_i| = (\frac{1}{2} - \epsilon)n$ *and* $|C_i| = 2\epsilon n$ *for each* $i \in [2]$,

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(ii) $d_G(v, X_i) \geq (\frac{1}{2} - 2\epsilon)n$ *for* $X \in \{A, B\}$ *and* $v \in X_{3-i}$ *with* $i \in [2]$,

(iii) *either* $e_G(A_1, B_2) \leq \epsilon n^2$ *or* $e_G(A_2, B_1) \leq \epsilon n^2$.

**Definition 2.8** (crossing, cross graph). *Let* $0 < \frac{1}{n}, \epsilon, \delta < 1$ *where* $n \in \mathbb{N}$ *and let* $\mathcal{G} = \{G_1, \ldots, G_n\}$ *be a balanced bipartite graph collection on a common vertex set* $V$ *of size* $2n$. *Given* $i, j \in [n]$ *such that* $G_i$ *and* $G_j$ *are both* $\epsilon$*-extremal, assume that the characteristic partition corresponding to* $G_\ell$ *is* $(A_1^\ell, B_1^\ell, C_1^\ell, A_2^\ell, B_2^\ell, C_2^\ell)$ *for* $\ell \in \{i, j\}$. *We say that they are* $\delta$*-crossing if* $|A_1^i \triangle A_1^j| \geq \delta n$ *and* $|A_1^i \triangle B_1^j| \geq \delta n$.

*We define the cross graph* $C_{\mathcal{G}}^{\epsilon,\delta}$ *to be the graph with vertex set* $[n]$ *where* $i$ *is adjacent to* $j$ *if and only if* $G_i$ *and* $G_j$ *are both* $\epsilon$*-extremal and* $\delta$*-crossing.*

**Definition 2.9** (strongly stable, weakly stable[1]). *Let* $0 < \gamma, \alpha, \epsilon, \delta < 1$. *Suppose that* $\mathcal{G} = \{G_1, \ldots, G_n\}$ *is a collection of bipartite graphs on a common vertex partition* $V_1 \cup V_2$ *with* $|V_1| = |V_2| = n$. *We say that*

- $\mathcal{G}$ *is* $(\gamma, \alpha)$*-strongly stable if* $G_i$ *is* $\alpha$*-nice for at least* $\gamma n$ *colours* $i \in [n]$;
- $\mathcal{G}$ *is* $(\epsilon, \delta)$*-weakly stable if* $e(C_{\mathcal{G}}^{\epsilon,\delta}) \geq \delta n^2$;
- $\mathcal{G}$ *is* $(\gamma, \alpha, \epsilon, \delta)$*-stable if it is either* $(\gamma, \alpha)$*-strongly stable or* $(\epsilon, \delta)$*-weakly stable.*

The following is a stability result for transversal perfect matching in bipartite graph collections.

**Theorem 2.10.** *Let* $0 < \frac{1}{n} \ll d \ll \mu \ll \alpha \ll \gamma, \epsilon \ll \delta \ll 1$. *Suppose that* $\mathcal{G} = \{G_1, \ldots, G_n\}$ *is a collection of bipartite graphs on a common vertex partition* $V = V_1 \cup V_2$ *with* $|V_1| = |V_2| = n$. *If the following hold:*

- $\mathcal{G}$ *is* $(\gamma, \alpha, \epsilon, \delta)$*-stable,*
- *for every* $i \in [n]$, $d_{G_i}(x) \geq \left(\frac{1}{2} - \mu\right) n$ *for all but at most* $dn$ *vertices* $x \in V_1 \cup V_2$,
- *for every* $x \in V_1 \cup V_2$, $d_{G_i}(x) \geq \left(\frac{1}{2} - \mu\right) n$ *for all but at most* $dn$ *colours* $i \in [n]$,

*then* $\mathcal{G}$ *contains a transversal perfect matching.*

For convenience, we postpone the proofs of Lemma 2.7 and Theorem 2.10 to the appendix.

### 2.3 Probabilistic tool

We use the following version of Chernoff's bound.

**Lemma 2.11.** *Let* $X$ *be a random variable with binomial or hypergeometric distribution, and let* $0 < \epsilon < \frac{3}{2}$. *Then*

$$\mathbb{P}[|X - \mathbb{E}(X)| \geq \epsilon \mathbb{E}(X)] \leq 2e^{-\frac{\epsilon^2}{3}\mathbb{E}(X)}.$$

## 3. Absorbing

In this section, we first define the extremality for a single digraph and distinguish the characteristic partition of such digraphs. We then introduce our absorbing structure and prove that every digraph collection contains a directed absorbing cycle whenever either a large proportion of its

[1] To avoid introducing too many terms, we have similarly defined the concepts of "nice, extremal, crossing, cross graph, strongly stable, weakly stable, stable" in the context of digraph collections. We should note that, unless explicitly stated otherwise, those concepts in bipartite graph collections are only discussed in Section 2.2 and the appendix.

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members are non-extremal, or almost all of them are extremal while their characteristic partitions differ significantly.

**Definition 3.1** (nice, extremal). *Let $\epsilon > 0$ and $D$ be a digraph on vertex set $V$ of size $n$. We say that*

- *$D$ is $\epsilon$-nice if for any two sets $A, B \subseteq V$ of size at least $(\frac{1}{2} - \epsilon)n$, we have $e_D(A, B) \geq \epsilon n^2$;*
- *$D$ is $\epsilon$-extremal if it is not $\epsilon$-nice.*

Cheng, Wang and Yan [13] characterised the structure of $\epsilon$-extremal digraphs, which forms the basis of our extremal case distinction.

**Lemma 3.2** ([13]). *Suppose that $0 < \frac{1}{n} \ll d \ll \mu \ll \epsilon \leq 1$. Let $D$ be an $\epsilon$-extremal digraph on vertex set $V$ of size $n$ with $\min\{d^+(x), d^-(x)\} \geq (\frac{1}{2} - \mu)n$ for all but at most $dn$ vertices $x \in V$. Then exactly one of the following holds:*

(i) *there is a partition $A \cup B \cup L$ of $V$ such that $|A| = |B| = (\frac{1}{2} - \epsilon)n$, $d^+(a, A), d^-(a, A) \geq (\frac{1}{2} - 2\epsilon)n$ for all $a \in A$, $d^+(b, B), d^-(b, B) \geq (\frac{1}{2} - 2\epsilon)n$ for all $b \in B$ and either $e(A, B) \leq \epsilon n^2$ or $e(B, A) \leq \epsilon n^2$; here we say $D$ is $(\epsilon, EC1)$-extremal,*

(ii) *there is a partition $A \cup B \cup L$ of $V$ such that $|A| = |B| = (\frac{1}{2} - \epsilon)n$, $d^+(a, B), d^-(a, B) \geq (\frac{1}{2} - 2\epsilon)n$ for all $a \in A$, $d^+(b, A), d^-(b, A) \geq (\frac{1}{2} - 2\epsilon)n$ for all $b \in B$ and either $e(A) \leq \epsilon n^2$ or $e(B) \leq \epsilon n^2$; here we say $D$ is $(\epsilon, EC2)$-extremal,*

(iii) *there is a constant $\zeta$ with $\epsilon^{1/3} \leq \zeta \leq \frac{1}{2} - \epsilon - \epsilon^{1/3}$ and a partition $C_1 \cup C_2 \cup C_3 \cup C_4 \cup L$ of $V$ such that $|C_1| = |C_3| = \zeta n$, $|C_2| = |C_4| = (\frac{1}{2} - \zeta - \epsilon)n$, for all $a \in C_i$ we have $d^+(a, C_i) \geq (\zeta - \epsilon)n$ if $i \in \{1, 3\}$, $d^+(a, C_{6-i}) \geq (\frac{1}{2} - \zeta - 2\epsilon)n$ if $i \in \{2, 4\}$, and $d^+(a, C_{i+1}) \geq (|C_{i+1}| - \epsilon)n$ for all $i \in [4]$ (here we identify $C_5$ with $C_1$), and either $e(C_1, C_3) \leq \epsilon n^2$ or $e(C_3, C_1) \leq \epsilon n^2$, either $e(C_2) \leq \epsilon n^2$ or $e(C_4) \leq \epsilon n^2$; here we say $D$ is $(\epsilon, EC3)$-extremal, (see Figure 1 for the structures of EC1, EC2 and EC3).*

We call the partition obtained in Lemma 3.2 a *characteristic partition* of $D$. Assume that $0 < \frac{1}{n} \ll \mu \ll \epsilon \leq 1$ and $\mathcal{D} = \{D_1, \ldots, D_n\}$ is a collection of digraphs on a common vertex set $V$ of size $n$ with $\delta^0(\mathcal{D}) \geq (\frac{1}{2} - \mu)n$. In view of Lemma 3.2, whenever $D_i$ is $\epsilon$-extremal, one may fix a characteristic partition $(A_i, B_i, L_i)$ of $D_i$ if it is $(\epsilon, \mathrm{EC}k)$-extremal for some $k \in [2]$, a characteristic partition $(C_i^1, C_i^2, C_i^3, C_i^4, L_i)$ of $D_i$ if it is $(\epsilon, \mathrm{EC3})$-extremal. For convenience, define $W_i^j := C_i^j \cup C_i^{j+1}$ for $j \in [4]$ (here we identify $C_i^5$ with $C_i^1$). We say that a vertex $v \in V$ is *$D_i$-good* if either $D_i$ is not $\epsilon$-extremal or $D_i$ is $\epsilon$-extremal and $v \in V \setminus L_i$. For every pair of distinct vertices $x, y \in V$, we define $L(xy) := \{i \in [n] : xy \in E(D_i)\}$ to be the set of colours appearing on $xy$.

**Definition 3.3** (directed absorbing path and directed absorbing cycle). *Given any two not necessarily distinct vertices $u, v \in V$ and a rainbow directed path $P = v_1v_2v_3v_4$ with $u, v \notin V(P)$, we call $P$ a Type-*I *directed $c$-absorbing path of $(v, u)$ if $c \in L(v_2v)$ and $\mathrm{col}(v_2v_3) \in L(uv_3)$ (see Figure 2), a Type-*II *directed $c$-absorbing path of $(v, u)$ if $c \in L(vv_3)$ and $\mathrm{col}(v_2v_3) \in L(v_2u)$ (see Figure 3).*

*Given $\delta, \delta', \gamma, \gamma' > 0$ and $\mathrm{K} \in \{\mathrm{I}, \mathrm{II}\}$, a rainbow directed cycle $C = v_1v_2 \ldots v_tv_1$ is a Type-*K *directed absorbing cycle with parameters $(\delta, \delta', \gamma, \gamma')$ if $t \leq \gamma n$ and there exists a colour set $\mathcal{C}$ of size at least $\delta n$ such that*

- *given any colour $c \in \mathcal{C}$ and any $D_c$-good vertex $v$, for all but at most $\delta' n$ vertices $u \in V$, there are at least $\gamma' n$ disjoint Type-K directed $c$-absorbing paths of $(v, u)$ inside $C$,*
- *given any colour $c \in \mathcal{C}$, for all but at most $\delta' n$ $D_c$-good vertices $v$, there are at least $\gamma' n$ disjoint Type-K directed $c$-absorbing paths of $(v, v)$ inside $C$.*

If there is no ambiguity, we typically refer to the Type-I and Type-II directed $c$-absorbing path (cycle) simply as a directed $c$-absorbing path (cycle).

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**Figure 2.** A type-I directed $c$-absorbing path of $(v, v)$ and a type-I directed $c$-absorbing path of $(v, u)$ with $v \neq u$.

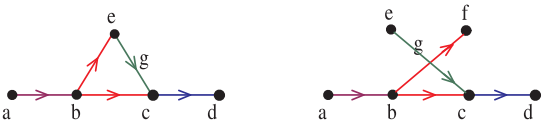



**Figure 3.** A type-II directed $c$-absorbing path of $(v, v)$ and a type-II directed $c$-absorbing path of $(v, u)$ with $v \neq u$.

Denote by $V^k$ the set of all $k$-tuples of distinct elements of $V$. A pair $H = (V, E)$ where $V$ is a set and $E \subseteq V^k$ is called a *directed $k$-graph*, while if we allow $E$ to be a multiset, it is a *directed multi-$k$-graph*. We use the same notation for directed $k$-graphs as for graphs. The following lemma was obtained by Cheng and Staden [9], which is the key tool that we will use to build an absorbing structure.

**Lemma 3.4** ([9]). *Suppose that $k, C, n \in \mathbb{N}$, $0 < \frac{1}{n} \ll \gamma \ll \epsilon \ll \frac{1}{k}, \frac{1}{C} \leq 1$, $m \in [n^C]$ and $t := \gamma n$. Let $\mathbf{H} = \{H_1, \ldots, H_t\}$ be a collection of directed $k$-graphs and let $\mathbf{Z} = \{Z_1, \ldots, Z_m\}$ be a collection of directed multi-$k$-graphs all defined on a common vertex set $V$ of size $n$. Suppose that $|E(H_i)| \geq \epsilon n^k$ for all $i \in [t]$, and for each $j \in [m]$ we have $|E(Z_j) \cap E(H_i)| \geq \epsilon n^k$ for at least $\epsilon t$ indices $i \in [t]$. Then there is a rainbow matching $M$ in $\mathbf{H}$ of size at least $(1 - \frac{\epsilon^2}{4})t$ and $|E(Z_j) \cap E(M)| \geq \frac{\epsilon^2 t}{4}$ for each $j \in [m]$.*

We first consider the case where a digraph collection contains many non-extremal digraphs; such a collection is said to be strongly stable.

**Definition 3.5** (strongly stable). *Let $0 < \gamma, \alpha < 1$. Suppose that $\mathcal{D} = \{D_1, \ldots, D_n\}$ is a collection of digraphs on a common vertex set $V$ of size $n$. We say that $\mathcal{D}$ is $(\gamma, \alpha)$-strongly stable if $D_i$ is $\alpha$-nice for at least $\gamma n$ colours $i \in [n]$.*

By applying Lemma 3.4, we establish the following lemma, which yields a directed absorbing cycle when $\mathcal{D}$ is strongly stable.

**Lemma 3.6.** *Let $0 < \frac{1}{n} \ll \lambda, \mu \ll \gamma, \alpha \ll 1$. Assume that $\mathcal{D} = \{D_1, \ldots, D_n\}$ is a collection of digraphs on a common vertex set $V$ of size $n$ with $\delta^0(\mathcal{D}) \geq \frac{1-\mu}{2} n$. If $\mathcal{D}$ is $(\gamma, \alpha)$-strongly stable, then there exists a Type- I directed absorbing cycle with parameters $(1, 0, \lambda, \lambda^2)$.*

**Proof.** Since $\mathcal{D}$ is $(\gamma, \alpha)$-strongly stable, one may assume that $D_1, \ldots, D_{\gamma n}$ are $\alpha$-nice and $6|\lambda n$. We divide the colour set $[\frac{\lambda n}{2}]$ into consecutive sets $\{1, 2, 3\}, \{4, 5, 6\}, \ldots, \{\frac{\lambda n}{2} - 2, \frac{\lambda n}{2} - 1, \frac{\lambda n}{2}\}$. For each $i \in I := [\frac{\lambda n}{6}]$, define $\mathcal{F}_i$ to be a directed 4-graph with vertex $V$ and edge set

$$\left\{(v_1, v_2, v_3, v_4) \in V^4 : 3(i-1) + \ell \in L(v_\ell v_{\ell+1}) \text{ for each } \ell \in [3]\right\}.$$

The minimum semi-degree condition implies that $e(\mathcal{F}_i) \geq n\left(\frac{1}{2} - \mu\right)^3 n^3 \geq \frac{n^4}{9}$.

Given a colour $c \in [n]$ and two not necessarily distinct vertices $v, v' \in V$ (which are not required to be $D_c$-good), let $Z_i(c, vv')$ be the collection of $(v_1, v_2, v_3, v_4)$ for which $v_1 v_2 v_3 v_4$ is a Type-I directed $c$-absorbing path of $(v, v')$. Since $D_{3i+2}$ is $\alpha$-nice, there are at least $\alpha n^2$ ways to choose $v_2, v_3$ with $v_2 \in N_c^-(v)$, $v_3 \in N_{3i+2}^+(v')$ and $v_2 v_3 \in E(D_{3i+2})$. When $v_2, v_3$ are fixed, there are at least $\left(\frac{1}{2} - \mu\right)^2 n^2$ ways to choose $v_1$ and $v_4$. Thus $|Z_i(c, vv')| \geq \alpha\left(\frac{1}{2} - \mu\right)^2 n^4 \geq \frac{\alpha n^4}{5}$.

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Applying Lemma 3.4 with

$$t := |I|, \epsilon := \frac{\alpha}{5}, \mathbf{H} := \{\mathcal{F}_i : i \in I\}, \text{ and}$$

$$\mathbf{Z} := \left\{ Z(c, vv') := \bigcup_{i \in I} Z_i(c, vv') : c \in [n], v, v' \in V \right\},$$

we obtain a rainbow matching $M$ inside $\mathbf{H}$ of size at least $(1 - \frac{\alpha^2}{100})\frac{\lambda n}{6}$ such that $|E(Z(c, vv')) \cap E(M)| \geq \frac{\alpha^2 \lambda n}{600}$ for all $c \in [n]$ and $v, v' \in V$. That is, there is a set $I' \subseteq I$ with $|I'| \geq (1 - \frac{\alpha^2}{100})\frac{\lambda n}{6}$ such that there is a rainbow directed path $P^i = v_1^i v_2^i v_3^i v_4^i$ with $(v_1^i, v_2^i, v_3^i, v_4^i) \in \mathcal{F}_i$ for each $i \in I'$, and for every $c \in [n]$ and $v, v' \in V$, there are at least $\frac{\alpha^2 \lambda n}{600}$ directed $c$-absorbing paths $P^i$ of $(v, v')$.

By relabelling indices, one may assume $I' =: [s]$ with $s = (1 - \frac{\alpha^2}{100})\frac{\lambda n}{6}$. We are to connect the directed paths $P^1, \ldots, P^s$ one by one into a rainbow directed cycle $C$. Firstly, we connect $P^1$ and $P^2$ into a single rainbow directed path $P^1 xy P^2$. For this, choose distinct unused colours $c_1, c_2 \in [\lambda n + 1, n]$ and $c_3 \in [\frac{\lambda n}{2} + 1, \lambda n]$. Let $U := V \setminus V(\bigcup_{i \in [s]} P^i)$. Clearly, $|U| \geq n - \lambda n$. It follows from $\delta^0(\mathcal{D}) \geq \frac{1-\mu}{2} n$ that

$$d^+_{c_1}(v_4^1, U),\ d^-_{c_2}(v_1^2, U) \geq \left(\frac{1-\mu}{2} - \lambda\right) n \geq \left(\frac{1}{2} - \alpha\right) n. \tag{1}$$

Notice that $D_{c_3}$ is $\alpha$-nice. Therefore, $|E(D_{c_3}[N^+_{c_1}(v_4^1, U), N^-_{c_2}(v_1^2, U)])| \geq \alpha n^2$. Hence, there is an edge $xy \in E(D_{c_3})$ with $x \in N^+_{c_1}(v_4^1, U)$ and $y \in N^-_{c_2}(v_1^2, U)$ such that $P^1 xy P^2$ is a rainbow directed path with colour set $\mathrm{col}(P^1 \cup P^2) \cup \{c_1, c_2, c_3\}$.

We apply the above process for each pair $(P^i, P^{i+1})$ for each $i \in [s]$ with $P^{s+1} := P^1$. In each process, we need two unused colours from $[\lambda n + 1, n]$ and one unused colour from $[\frac{\lambda n}{2} + 1, \lambda n]$. This is feasible since $2s \leq \frac{\lambda n}{3}$. The total number of used vertices at each step is at most $6s \leq \lambda n$, so we have the same degree bounds as (1). It is straightforward to verify that the resulting directed cycle $C$ is a Type-I directed absorbing cycle with parameters $(1, 0, \frac{6s}{n}, \frac{\alpha^2 \lambda}{600})$ and hence with parameters $(1, 0, \lambda, \lambda^2)$ since $\lambda \ll \alpha$. □

The extremality of a digraph collection also depends crucially on the relationships between different digraphs. This leads us to introduce the following definitions.

**Definition 3.7** (crossing, cross graph, weakly stable). *Let $0 < \frac{1}{n} \ll \epsilon, \delta < 1$ with $n \in \mathbb{N}$. Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a collection of digraphs on a common vertex set $V$ of size $n$. Choose $i, j \in [n]$ such that $D_i$ is $(\epsilon, \mathrm{EC}t)$-extremal and $D_j$ is $(\epsilon, \mathrm{EC}s)$-extremal with $t, s \in [3]$. We say that $D_i$ and $D_j$ are $\delta$-crossing if one of the following holds:*

- **(A1)** *$t, s \in \{1, 2\}$, $|A_i \triangle A_j| \geq \delta n$ and $|A_i \triangle B_j| \geq \delta n$;*
- **(A2)** *$t = s = 3$, either $|W_i^1 \triangle W_j^1| \geq \delta n$ and $|W_i^1 \triangle W_j^3| \geq \delta n$, or $|W_i^2 \triangle W_j^2| \geq \delta n$ and $|W_i^2 \triangle W_j^4| \geq \delta n$;*
- **(A3)** *$t \in \{1, 2\}, s = 3$, either $|A_i \triangle W_j^1| \geq \delta n$ and $|A_i \triangle W_j^3| \geq \delta n$, or $|A_i \triangle W_j^2| \geq \delta n$ and $|A_i \triangle W_j^4| \geq \delta n$.*

*For $k \in [3]$, define the cross graph $C^{\epsilon,\delta}_{\mathcal{D},k}$ to be the graph with vertex set $[n]$ where $i$ and $j$ are adjacent if and only if $D_i, D_j$ are satisfying **(Ak)**. We say that $\mathcal{D}$ is $(\epsilon, \delta)$-weakly stable if $e(C^{\epsilon,\delta}_{\mathcal{D},k}) \geq \delta n^2$ for some $k \in [3]$.*

The subsequent observation is an immediate consequence of the above definition.

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**Observation 3.8.** *Assume that* $0 < \epsilon \le \frac{\delta}{8}$.

- *If* $D_i$ *and* $D_j$ *are satisfying* ***(A1)****, then* $|X_i \cap Y_j| \ge \frac{\delta n}{4}$*, where* $X, Y \in \{A, B\}$.
- *If* $D_i$ *and* $D_j$ *are satisfying* ***(A2)****, then* $|W_i^x \cap W_j^y| \ge \frac{\delta n}{4}$*, where either* $x, y \in \{1, 3\}$ *or* $x, y \in \{2, 4\}$.
- *If* $D_i$ *and* $D_j$ *are satisfying* ***(A3)****, then* $|X_i \cap W_j^y| \ge \frac{\delta n}{4}$*, where* $X \in \{A, B\}$ *and either* $y \in \{1, 3\}$ *or* $y \in \{2, 4\}$.

Our following lemma finds a directed absorbing cycle when $\mathcal{D}$ is weakly stable.

**Lemma 3.9.** *Let* $0 < \frac{1}{n} \ll \lambda, \mu \ll 8\epsilon^{1/2} \le \delta \le \epsilon^{1/3} \ll 1$*. Assume that* $\mathcal{D} = \{D_1, \ldots, D_n\}$ *is a collection of digraphs on a common vertex set* $V$ *of size* $n$ *with* $\delta^0(\mathcal{D}) \ge \frac{1-\mu}{2} n$*. If* $\mathcal{D}$ *is* $(\epsilon, \delta)$*-weakly stable, then there exists a directed absorbing cycle with parameters* $(\frac{\delta}{3}, \sqrt{\epsilon}, \lambda, \lambda^2)$.

**Proof.** Since $\mathcal{D}$ is $(\epsilon, \delta)$-weakly stable, one has $e(C^{\epsilon,\delta}_{\mathcal{D},k}) \ge \delta n^2$ for some $k \in [3]$. In particular, if $k = 2$, then we consider only $|W_i^1 \triangle W_j^1| \ge \delta n$ and $|W_i^1 \triangle W_j^3| \ge \delta n$ for all $ij \in E(C^{\epsilon,\delta}_{\mathcal{D},2})$; if $k = 3$, then we consider only $|A_i \triangle W_j^1| \ge \delta n$ and $|A_i \triangle W_j^3| \ge \delta n$ for all $ij \in E(C^{\epsilon,\delta}_{\mathcal{D},3})$ (since the other case can be discussed similarly).

For each $i \in [\lfloor \frac{n}{3} \rfloor]$, define $\mathcal{F}_i$ to be a directed 4-graph with vertex set $V$ and edge set

$$\left\{(v_1, v_2, v_3, v_4) \in V^4 : 3(i-1) + \ell \in L(v_\ell v_{\ell+1}) \text{ for } \ell \in [3]\right\}.$$

If follows from $\delta^0(\mathcal{D}) \ge \frac{1-\mu}{2} n$ that $e(\mathcal{F}_i) \ge n\left(\left(\frac{1}{2} - \mu\right) n\right)^3 \ge \frac{n^4}{9}$.

In order to apply Lemma 3.4, we first give the following claim, which requires only the minimum semi-degree condition.

**Claim 3.10.** *Assume that* $ij \in E(C^{\epsilon,\delta}_{\mathcal{D},k})$ *for some* $k \in [3]$*. Let* $u$ *be a* $D_j$*-good vertex and* $v$ *be a* $D_i$*-good vertex with* $j + 1 < 3n$*. For* $K \in \{I, II\}$*, let* $Z_j^K(i, vu)$ *be the collection of* $(v_1, v_2, v_3, v_4) \in \mathcal{F}_{\frac{j+1}{3}}$ *where* $v_1v_2v_3v_4$ *is a Type-K directed* $i$*-absorbing path of* $(v, u)$*. Then* $|Z_j^K(i, vu)| \ge 2^{-7}\delta n^4$.

**Proof of Claim** 3.10. Fix any such $i, j, u, v$. We proceed by considering the following cases.

**Case 1.** $k = 1$.

In this case, we prove $|Z_j^{\mathrm{I}}(i, vu)| \ge 2^{-7}\delta n^4$. By Lemma 3.2 (i) and (ii), we have either $d_i^-(v, A_i) \ge (\frac{1}{2} - 2\epsilon)n$ or $d_i^-(v, B_i) \ge (\frac{1}{2} - 2\epsilon)n$. Without loss of generality, we assume that the former case holds since the latter case can be proved similarly. Note that $D_i$ and $D_j$ are $\delta$-crossing. Then $|A_i \cap A_j| \ge \frac{\delta n}{4}$ and $|A_i \cap B_j| \ge \frac{\delta n}{4}$. Therefore,

$$d_i^-(v, A_j) \ge |A_i \cap A_j| - (|A_i| - d_i^-(v, A_i)) \ge \frac{\delta n}{4} - \epsilon n \ge \frac{\delta n}{5} \text{ (since } \epsilon \le \delta^2).$$

Similarly, we have $d_i^-(v, B_j) \ge \frac{\delta n}{5}$.

Since $u$ is $D_j$-good, one has $u \in Y_j$ for some $Y \in \{A, B\}$. Let $Z = Y$ if $D_j$ is $(\epsilon, \mathrm{EC1})$-extremal, and let $Z \in \{A, B\} \setminus \{Y\}$ if $D_j$ is $(\epsilon, \mathrm{EC2})$-extremal. Notice that $|N_j^+(w, Z_j)| \ge (\frac{1}{2} - 2\epsilon)n$ for each vertex $w \in Y_j$. Choose $x \in N_i^-(v, Y_j)$. Then

$$|N_j^+(u, Z_j) \cap N_j^+(x, Z_j)| \ge |N_j^+(u, Z_j)| + |N_j^+(x, Z_j)| - |Z_j| \ge \left(\frac{1}{2} - 3\epsilon\right) n.$$

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Any choice of $y \in N_j^+(u, Z_j) \cap N_j^+(x, Z_j)$, $x' \in N_{j-1}^-(x)$ and $y' \in N_{j+1}^+(y)$ yields a directed $i$-absorbing path $x'xyy'$ of $(v, u)$ in $\mathcal{F}_{\frac{j+1}{3}}$. The number of such paths is therefore at least $\frac{\delta n}{5}(\frac{1}{2} - 3\epsilon)n\left(\frac{1}{2} - \mu\right)^2 n^2 \geq 2^{-7}\delta n^4$.

**Case 2.** $k = 2$.

In this case, we prove $|Z_j^{\mathrm{II}}(i, vu)| \geq 2^{-7}\delta n^4$. By Lemma 3.2 (iii), we have either $d_{D_i}^+(v, W_i^1) \geq (\frac{1}{2} - 3\epsilon)n$ or $d_{D_i}^+(v, W_i^3) \geq (\frac{1}{2} - 3\epsilon)n$. Without loss of generality, we assume that the former case holds since the latter case can be proved similarly. Note that $D_i$ and $D_j$ are $\delta$-crossing. Then $|W_i^1 \cap W_j^1| \geq \frac{\delta n}{4}$ and $|W_i^1 \cap W_j^3| \geq \frac{\delta n}{4}$. Therefore,

$$d_i^+(v, W_j^1) \geq |W_i^1 \cap W_j^1| - \left(|W_i^1| - d_i^+(v, W_i^1)\right) \geq \frac{\delta n}{4} - 2\epsilon n \geq \frac{\delta n}{5} \text{ (since } \epsilon \leq \delta^2\text{)}.$$

Similarly, we have $d_i^+(v, W_j^3) \geq \frac{\delta n}{5}$.

Since $u$ is $D_j$-good, one has $u \in Y_j$ for some $Y \in \{W^1, W^3\}$. Let $Z = W^4$ if $Y = W^1$, and let $Z = W^2$ if $Y = W^3$. Note that $|N_j^-(w, Z_j)| \geq (\frac{1}{2} - 3\epsilon)n$ for each vertex $w \in Y_j$. Choose $x \in N_i^+(v, Y_j)$. Then

$$|N_j^-(u, Z_j) \cap N_j^-(x, Z_j)| \geq |N_j^-(u, Z_j)| + |N_j^-(x, Z_j)| - |Z_j| \geq \left(\frac{1}{2} - 5\epsilon\right) n.$$

Any choice of $y \in N_j^-(u, Z_j) \cap N_j^-(x, Z_j)$, $x' \in N_{j+1}^+(x)$ and $y' \in N_{j-1}^-(y)$ yields a Type-II directed $i$-absorbing path $y'yxx'$ of $(v, u)$ in $\mathcal{F}_{\frac{j+1}{3}}$. Therefore, the number of such paths is at least $\frac{\delta n}{5}$ $(\frac{1}{2} - 5\epsilon)n\left(\frac{1}{2} - \mu\right)^2 n^2 \geq 2^{-7}\delta n^4$.

**Case 3.** $k = 3$.

This case can be proved by a similar discussion, so we omit the proof. □

For each vertex $v$, let

$$\mathcal{C}_v := \{i \in [n] : v \text{ is } D_i\text{-good}\}.$$

Since $e(C_{\mathcal{D},k}^{\epsilon,\delta}) \geq \delta n^2$, there is a subgraph $H$ of $C_{\mathcal{D},k}^{\epsilon,\delta}$ such that $|V(H)| \geq \delta n$ and $\delta(H) \geq \delta n$. For each $i \in V(H)$, define

$$T_i := \left\{u \in V : |N_H(i) \setminus \mathcal{C}_u| \geq \frac{\delta n}{2}\right\}.$$

Notice that $|L_i| = 2\epsilon n$ when $D_i$ is $\epsilon$-extremal. Hence $|T_i|\frac{\delta n}{2} \leq 2\epsilon n^2$, that is, $|T_i| \leq \frac{\sqrt{\epsilon}}{2}n$ since $8\epsilon^{1/2} \leq \delta$.

For each $i \in V(H)$, let $\overline{T_i} := V \setminus T_i$. Then $|\overline{T_i}| \geq (1 - \frac{\sqrt{\epsilon}}{2})n$. For each $u \in \overline{T_i}$ with $i \in V(H)$, one has $|N_H(i) \cap \mathcal{C}_u| \geq \frac{\delta n}{2}$. Now we independently and randomly select vertices from $V(H)$ with probability $\kappa := \frac{\lambda}{14}$ to obtain a colour set $\mathcal{U}$. Using Chernoff's bound, with high probability the following hold:

(∗) $\frac{\kappa|V(H)|}{2} \leq t := |\mathcal{U}| \leq 2\kappa|V(H)|$,
(∗∗) for every $i \in V(H)$ and $u \in \overline{T_i}$, we have $|N_H(i, \mathcal{U}) \cap \mathcal{C}_u| \geq \frac{\delta\kappa n}{4}$.

Fix such a colour set $\mathcal{U}$ and one may assume $\mathcal{U} = \{3j - 1 : j \in [t]\}$. Let $\overline{\mathcal{U}} := V(H) \cap [3t + 1, n]$. It is easy to see that $\mathcal{U} \cap \overline{\mathcal{U}} = \emptyset$, $|\overline{\mathcal{U}}| \geq \frac{\delta n}{2}$ and $\delta(H[\overline{\mathcal{U}}]) \geq \frac{\delta n}{2}$.

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Given $i \in \overline{\mathcal{U}}$ and $u \in \overline{T_i}$, choose $j \in \mathcal{C}_u \cap N_H(i, \mathcal{U})$. Then $\frac{j+1}{3} \in [t]$ and the number of such choices of $j$ is at least $\frac{\delta\kappa n}{4} \geq \frac{\delta t}{8}$ by $(**)$. Let $v$ be a $D_i$-good vertex. It follows from Claim 3.10 that $|Z_j^{\mathrm{K}}(i, vu)| \geq 2^{-7}\delta n^4$ for some $\mathrm{K} \in \{\mathrm{I}, \mathrm{II}\}$.

By $(*)$ and the fact that $\lambda \ll \delta$, one has $\frac{t}{n} \leq 2\kappa = \frac{\lambda}{7} \ll 2^{-7}\delta$. Applying Lemma 3.4 with

$$\gamma := \frac{\lambda}{7}, \epsilon := 2^{-7}\delta, \mathbf{H} := \{\mathcal{F}_j : j \in [t]\} \text{ and}$$

$$\mathbf{Z} := \left\{ Z(i, vu) := \bigcup_{j \in \mathcal{C}_u \cap N_H(i,\mathcal{U})} Z_j^{\mathrm{K}}(i, vu) : i \in \overline{\mathcal{U}},\ u \in \overline{T_i} \text{ and } v \text{ is } D_i\text{-good} \right\},$$

we obtain that there is a rainbow matching $M$ of $\mathbf{H}$ with size at least $(1 - 2^{-16}\delta^2)t$ such that $|E(Z(i, vu)) \cap E(M)| \geq 2^{-16}\delta^2 t$ for all $Z(i, vu) \in \mathbf{Z}$. That is, there is a subset $I \subseteq [t]$ with $|I| \geq (1 - 2^{-16}\delta^2)t$ such that there is a Type-K directed path $P^j = v_1^j v_2^j v_3^j v_4^j \in \mathcal{F}_j$ for each $j \in I$, and for every $i \in \overline{\mathcal{U}}$, $u \in \overline{T_i}$ and a $D_i$-good vertex $v$, there are at least $2^{-16}\delta^2 t$ Type-K directed $i$-absorbing paths $P^j$ of $(v, u)$.

Relabel indices such that $I =: [s]$ with $s \geq (1 - 2^{-16}\delta^2)t$. We are to connect those disjoint rainbow directed paths $P^1, \ldots, P^s$ into a directed absorbing cycle $C$. Notice that $\overline{\mathcal{U}} \cap \mathrm{col}(\bigcup_{i \in I} P^i) = \emptyset$. We first connect $P^1$ and $P^2$ into a single rainbow directed path $P^1 v_4^1 xzyv_1^2 P^2$ or $P^1 v_4^1 xzv_1^2 P^2$. For this, choose distinct colours $c_1, c_2, c_3, c_4 \in \overline{\mathcal{U}}$ such that $c_1c_2 \in E(H)$. Clearly, there are at least $|\overline{\mathcal{U}}| \geq \frac{\delta n}{2}$ choices for $c_1$, and given this at least $\delta(H[\overline{\mathcal{U}}]) \geq \frac{\delta n}{2}$ choices for $c_2$, and at least $|\overline{\mathcal{U}}| - 3 > \frac{\delta n}{3}$ choices for each of $c_3, c_4$. Next, choose an unused $D_{c_1}$-good vertex $x \in N^+_{c_3}(v_4^1)$ and an unused $D_{c_2}$-good vertex $y \in N^-_{c_4}(v_1^2)$. The following claim considers the number of choices for $x, y$.

**Claim 3.11.** *One of the following holds:*

- *there exists a rainbow directed path $P^1 v_4^1 xzyv_1^2 P^2$ with colours $col(P^1 \cup P^2) \cup \{c_1, c_2, c_3, c_4\}$ and $z \in N^+_{c_1}(x) \cap N^-_{c_2}(y)$, each of $x, y, z$ has at least $\frac{\delta n}{10}$ choices;*
- *there exists a rainbow directed path $P^1 v_4^1 xyv_1^2 P^2$ with colours $col(P^1 \cup P^2) \cup \{c_1, c_3, c_4\}$ and $y \in N^+_{c_1}(x) \cap N^-_{c_4}(v_1^2)$, each of $x, y$ has at least $\frac{\delta n}{10}$ choices.*

**Proof of Claim** 3.11. We proceed by considering the following three cases.

**Case 1.** $k = 1$.

In this case, $D_{c_1}$ and $D_{c_2}$ are either $(\epsilon, \mathrm{EC1})$-extremal or $(\epsilon, \mathrm{EC2})$-extremal. Then $x \in A_{c_1} \cup B_{c_1}$ and $y \in A_{c_2} \cup B_{c_2}$. Clearly, each of $x, y$ has at least $\frac{n}{4}$ choices. Recall that $D_{c_1}$ and $D_{c_2}$ are $\delta$-crossing. Then, $|X_{c_1} \cap Y_{c_2}| \geq \frac{\delta n}{4}$ whenever $X, Y \in \{A, B\}$. By Lemma 3.2, there are $Z, W \in \{A, B\}$ such that $d^+_{c_1}(x, Z_{c_1}) \geq |Z_{c_1}| - \epsilon n$ and $d^-_{c_2}(y, W_{c_2}) \geq |W_{c_2}| - \epsilon n$. Choose $z \in N^+_{c_1}(x) \cap N^-_{c_2}(y)$ and the number of choices for $z$ is at least

$$|N^+_{c_1}(x) \cap N^-_{c_2}(y)| - 7e(M) \geq |N^+_{c_1}(x, Z_{c_1}) \cap N^-_{c_2}(y, W_{c_2})| - 14\kappa n \geq |Z_{c_1} \cap W_{c_2}| - 2\epsilon n - 12\kappa n \geq \frac{\delta n}{5},$$

as desired.

**Case 2.** $k = 2$.

In this case, both $D_{c_1}$ and $D_{c_2}$ are $(\epsilon, \mathrm{EC3})$-extremal. Then $|W^i_{c_1} \cap W^j_{c_2}| \geq \frac{\delta n}{4}$ for $i, j \in \{1, 3\}$. Notice that either $|W^1_{c_1} \cap W^2_{c_2}| \geq \frac{n}{5}$ or $|W^3_{c_1} \cap W^2_{c_2}| \geq \frac{n}{5}$ holds. Without loss of generality, assume the latter case hold. Choose $z \in N^+_{c_1}(x) \cap N^-_{c_2}(y)$. Next, we consider the size of $|N^+_{c_1}(x) \cap N^-_{c_2}(y)|$.

If $|W^2_{c_1} \cap N^+_{c_3}(v_4^1)| \geq \delta n$ and $|W^3_{c_2} \cap N^-_{c_4}(v_1^2)| \geq \delta n$, then choose $x \in W^2_{c_1} \cap N^+_{c_3}(v_4^1)$ and $y \in W^3_{c_2} \cap N^-_{c_4}(v_1^2)$. Hence, there are at least $\frac{\delta n}{2}$ choices for each of $x$ and $y$. Furthermore, $|N^+_{c_1}(x) \cap N^-_{c_2}(y)| \geq |W^3_{c_1} \cap W^2_{c_2}| - 4\epsilon n \geq \frac{\delta n}{5}$, as desired.

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If $|W^2_{c_1} \cap N^+_{c_3}(v^1_4)| < \delta n$, then $|W^4_{c_1} \cap N^+_{c_3}(v^1_4)| \geq (\frac{1}{2} - \mu - \delta - 2\epsilon)n$. Choose $x \in W^4_{c_1} \cap N^+_{c_3}(v^1_4)$. Then $d^+_{c_1}(x, W^1_{c_1}) \geq |W^1_{c_1}| - 2\epsilon n$. Notice that either $|W^1_{c_2} \cap N^-_{c_4}(v^2_1)| \geq \delta n$ or $|W^3_{c_2} \cap N^-_{c_4}(v^2_1)| \geq \delta n$. We may assume the former case holds. Choose $y \in W^1_{c_2} \cap N^-_{c_4}(v^2_1)$. Then $d^-_{c_2}(y, W^4_{c_2}) \geq |W^4_{c_2}| - 2\epsilon n$. If $|W^1_{c_1} \cap W^4_{c_2}| \geq \delta n$, then $|N^+_{c_1}(x) \cap N^-_{c_2}(y)| \geq |W^1_{c_1} \cap W^4_{c_2}| - 4\epsilon n \geq \frac{\delta n}{5}$, as desired. If $|W^1_{c_1} \cap W^4_{c_2}| < \delta n$ and $|W^3_{c_2} \cap N^-_{c_4}(v^2_1)| \geq \frac{\delta n}{9}$, then $|W^1_{c_1} \cap W^2_{c_2}| \geq (\frac{1}{2} - \delta - 3\epsilon)n$ and choose $y \in W^3_{c_2} \cap N^-_{c_4}(v^2_1)$. Similarly, $|N^+_{c_1}(x) \cap N^-_{c_2}(y)| \geq |W^1_{c_1} \cap W^2_{c_2}| - 4\epsilon n \geq \frac{\delta n}{5}$, as desired. If $|W^1_{c_1} \cap W^4_{c_2}| < \delta n$ and $|W^3_{c_2} \cap N^-_{c_4}(v^2_1)| < \frac{\delta n}{9}$, then $d^-_{c_4}(v^2_1, W^1_{c_2}) \geq |W^1_{c_2}| - (\mu + 2\epsilon + \frac{\delta}{9})n$. Therefore,

$$|N^+_{c_1}(x) \cap N^-_{c_4}(v^2_1)| \geq |N^+_{c_1}(x, W^1_{c_1}) \cap N^-_{c_4}(v^2_1, W^1_{c_2})| \geq |W^1_{c_1} \cap W^1_{c_2}| - \left(\mu + 4\epsilon + \frac{\delta}{9}\right) n \geq \frac{\delta n}{9}.$$

Now, we connect $P^1$ and $P^2$ by using a rainbow directed path $P^1 v^1_4 x y v^2_1 P^2$ with colours $\mathrm{col}(P^1 \cup P^2) \cup \{c_1, c_3, c_4\}$, as desired.

**Case 3.** $k = 3$.

In this case, $D_{c_1}$ is either $(\epsilon, \mathrm{EC1})$-extremal or $(\epsilon, \mathrm{EC2})$-extremal, and $D_{c_2}$ is $(\epsilon, \mathrm{EC3})$-extremal. Hence $x \in A_{c_1} \cup B_{c_1}$ and $y \in W^1_{c_2} \cup W^3_{c_2}$. Choose $z \in N^+_{c_1}(x) \cap N^-_{c_2}(y)$. Note that $D_{c_1}$ and $D_{c_2}$ are $\delta$-crossing. Then $|X_{c_1} \cap Y_{c_2}| \geq \frac{\delta n}{4}$ whenever $X \in \{A, B\}$ and $Y \in \{W^1, W^3\}$. By Lemma 3.2, there are $Z \in \{A, B\}$ and $W \in \{W^2, W^4\}$ such that $d^+_{c_1}(x, Z_{c_1}) \geq |Z_{c_1}| - \epsilon n$ and $d^-_{c_2}(y, W_{c_2}) \geq |W_{c_2}| - 2\epsilon n$. If $|Z_{c_1} \cap W_{c_2}| \geq \frac{\delta n}{4}$, then $|N^+_{c_1}(x) \cap N^-_{c_2}(y)| \geq |Z_{c_1} \cap W_{c_2}| - 3\epsilon n \geq \frac{\delta n}{5}$, as desired.

Now, we consider $|Z_{c_1} \cap W_{c_2}| < \frac{\delta n}{4}$. Without loss of generality, assume that $Z = A$ and $W = W^2$. It follows that $|A_{c_1} \cap W^4_{c_2}| \geq |W^4_{c_2}| - (\frac{\delta}{4} + 2\epsilon)n$. This implies $|A_{c_1} \cap W^3_{c_2}| \geq |C^4_{c_2}| - (\frac{\delta}{4} + 2\epsilon)n \geq \frac{\delta n}{2}$ since $|C^2_4| \geq \delta n$. If $|W^1_{c_2} \cap N^-_{c_4}(v^1_2)| \geq \frac{\delta n}{9}$, then choose $y \in W^1_{c_2} \cap N^-_{c_4}(v^1_2)$. Hence $|N^+_{c_1}(x) \cap N^-_{c_2}(y)| \geq |A_{c_1} \cap W^4_{c_2}| - 3\epsilon n \geq \frac{\delta n}{5}$. If $|W^1_{c_2} \cap N^-_{c_4}(v^1_2)| < \frac{\delta n}{9}$, then $|W^3_{c_2} \cap N^-_{c_4}(v^1_2)| \geq |W^3_{c_2}| - (\frac{\delta}{9} + 2\epsilon)n$. Thus, $|N^+_{c_1}(x) \cap N^-_{c_4}(v^1_2)| \geq |A_{c_1} \cap W^3_{c_2}| - (3\epsilon + \frac{\delta}{9})n \geq \frac{\delta n}{9}$. Now, we connect $P^1$ and $P^2$ by using a rainbow directed path $P^1 v^1_4 x y v^2_1 P^2$ with colours $\mathrm{col}(P^1 \cup P^2) \cup \{c_1, c_3, c_4\}$, as desired.

This completes the proof of Claim 3.11. □

In view of Claim 3.11, there are at least $\frac{\delta n}{10}$ choices for each of $c_1, c_2, c_3, c_4, x, y, (z)$ given any previous choices. Thus, we obtain the desired rainbow directed path $P^1 v^1_4 x z y v^2_1 P^2$ or $P^1 v^1_4 x y v^2_1 P^2$. Applying the above process for each pair $(P^j, P^{j+1})$ with $j \in [s]$ where $P^{s+1} := P^1$. Every time we use at most four unused colours in $\overline{\mathcal{U}}$ and at most three unused vertices. Therefore, for each choice, at most $4s \leq 4t \leq 8\kappa n \ll \frac{\delta n}{10}$ colours are forbidden and $3s \ll \frac{\delta n}{10}$ vertices are forbidden. Thus we obtain a rainbow directed cycle $C$ of length at most $7s$ which contains every $P^j$ with $j \in [s]$ as a segment.

By the construction of $C$, for every colour $i \in \overline{\mathcal{U}}$, any $D_i$-good vertex $v$ and for every vertex $u \in \overline{T_i}$, there are at least $2^{-16}\delta^2 t \geq 2^{-17}\kappa\delta^3 n$ disjoint directed $i$-absorbing paths of $(v, u)$. Moreover, the number of $D_i$-good vertices inside $\overline{T_i}$ is at least $|\overline{T_i}| - 2\epsilon n \geq (1 - \frac{\sqrt{\epsilon}}{2})n - 2\epsilon n \geq (1 - \sqrt{\epsilon})n$. Thus $C$ is a directed absorbing cycle with parameters $(\frac{\delta}{3}, \sqrt{\epsilon}, \lambda, \lambda^2)$. □

## 4. Stable case

In this section, we employ the regularity-blow-up method for digraph collections $\mathcal{D}$ to prove the existence of a transversal directed Hamilton cycle when $\mathcal{D}$ is either strongly or weakly stable, combining the results from previous sections. Let $0 < \gamma, \alpha, \epsilon, \delta < 1$. We say that $\mathcal{D}$ is $(\gamma, \alpha, \epsilon, \delta)$-*stable* if it is either $(\gamma, \alpha)$-strongly stable or $(\epsilon, \delta)$-weakly stable. We first show that the reduced digraph collection inherits stability.

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**Lemma 4.1.** *Let* $0 < \frac{1}{n} \ll \frac{1}{L_0} \ll \epsilon_0 \ll d \ll \mu, \alpha \ll \gamma, \epsilon \leq \delta^2 \ll 1$. *Assume that* $\mathcal{D} = \{D_1, \ldots, D_n\}$ *is a collection of digraphs on a common vertex set* $V$ *of size* $n$ *and* $\delta^0(\mathcal{D}) \geq \left(\frac{1}{2} - \mu\right) n$. *Let* $\mathcal{R} = \mathcal{R}(\epsilon_0, d, L_0)$ *be the reduced digraph collection of* $\mathcal{D}$. *If* $\mathcal{D}$ *is* $(\gamma, \alpha, \epsilon, \delta)$*-stable, then* $\mathcal{R}$ *is* $(\frac{\gamma}{2}, \alpha^2, \epsilon, \frac{\delta}{2})$*-stable.*

**Proof.** We may assume that $\frac{1}{n} \leq \frac{1}{n_0}$ where $n_0 = n_0(\epsilon, 1, L_0)$ is the constant from Lemma 2.2. Write $[L]$ for the common vertex set of digraphs in $\mathcal{R}$ where $L_0 \leq L \leq n_0$, and $[M]$ for the set of colour clusters. Thus there is a partition $V_0, V_1, \ldots, V_L$ of $V$ and $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_M$ of $[n]$ and a digraph collection $\mathcal{D}'$ satisfying (i)–(v) of Lemma 2.2. Therefore, for each $\{(h, i), j\} \in \binom{[L]}{2} \times [M]$, we have $hi \in R_j$ if and only if $\mathcal{D}'_{hi,j} := \{D'_c[V_h, V_i] : c \in \mathcal{C}_j\}$ is $(\epsilon_0, d)$-regular. Furthermore, $Lm \leq n \leq Mm + \epsilon_0 n$ and $Mm \leq n \leq Lm + \epsilon_0 n$, so $|L - M| \leq \frac{\epsilon_0 n}{m} \leq \frac{\epsilon_0 L}{1-\epsilon_0}$. Thus we may assume that $M = L$ at the expense of assuming the slightly worse bound $|V_0| + |\mathcal{C}_0| \leq 3\epsilon_0 n$. Given $X \subset [L]$, we write $\hat{X} := \bigcup_{j \in X} V_j$. Hence $|\hat{X}| = m|X|$.

**Claim 4.2.** *Let* $d \ll \alpha'$ *and* $i \in [M]$. *Assume that there are sets* $A, B \subset [L]$ *of size at least* $(\frac{1}{2} - \alpha')L$ *such that* $e_{R_i}(A, B) \leq \alpha' L^2$. *Then* $e_{D_c}(\hat{A}, \hat{B}) \leq \sqrt{\alpha'} n^2$ *for all but at most* $2\sqrt{\alpha'} m$ *colours* $c \in \mathcal{C}_i$.

**Proof of Claim** 4.2. Let $t$ be the number of colours $c \in \mathcal{C}_i$ such that $e_{D_c}(\hat{A}, \hat{B}) > \sqrt{\alpha'} n^2$. Since $e(D_c) - e(D'_c) \leq (24d + \epsilon_0)n^2$ for all $c \in [n]$, we have

$$\begin{aligned}\sqrt{\alpha'} n^2 t &\leq \sum_{c \in \mathcal{C}_i} e_{D_c}(\hat{A}, \hat{B}) \leq \sum_{c \in \mathcal{C}_i} \left( e_{D'_c}(\hat{A}, \hat{B}) + (24d + \epsilon_0)n^2 \right) \\ &\leq |\mathcal{C}_i| e_{R_i}(A, B) m^2 + |\mathcal{C}_i|(24d + \epsilon_0)n^2 \\ &\leq \alpha' L^2 m^3 + 25dmn^2 \leq 2\alpha' mn^2.\end{aligned}$$

This implies that $t \leq 2\sqrt{\alpha'} m$. □

We consider the cases where $\mathcal{D}$ is strongly stable and weakly stable, respectively.

**Case 1.** $\mathcal{D}$ is $(\gamma, \alpha)$-strongly stable.

We prove that $\mathcal{R}$ is $(\frac{\gamma}{2}, \alpha^2)$-strongly stable. Suppose that there exists a subset $I \subseteq [L]$ with $|I| \geq (1 - \frac{\gamma}{2})L$ such that $R_i$ is not $\alpha^2$-nice for any $i \in I$. By Definition 3.1, for each $i \in I$, there are two sets $A^i, B^i \subseteq [L]$ of size at least $(\frac{1}{2} - \alpha^2)L$ such that $e_{R_i}(A^i, B^i) < \alpha^2 L^2$. Applying Claim 4.2 with $\alpha' := \alpha^2$, we get that there are at least $(1 - 2\alpha)m$ colours $c \in \mathcal{C}_i$ such that $e_{D_c}(\hat{A^i}, \hat{B^i}) \leq \alpha n^2$. Notice that $|\hat{A^i}|, |\hat{B^i}| \geq (\frac{1}{2} - \alpha^2)Lm > (\frac{1}{2} - \alpha)n$. Then there are at least $(1 - 2\alpha)m$ colours $c \in \mathcal{C}_i$ such that $D_c$ is not $\alpha$-nice. Thus the number of colours $c \in [n]$ for which $D_c$ is not $\alpha$-nice is at least $(1 - 2\alpha)m|I| > (1 - \gamma)n$ since $\alpha \ll \gamma$. This contradicts the fact that $\mathcal{D}$ is $(\gamma, \alpha)$-strongly stable.

**Case 2.** $\mathcal{D}$ is $(\epsilon, \delta)$-weakly stable.

Suppose that $\mathcal{R}$ is not $(\frac{\gamma}{2}, \alpha^2)$-strongly stable. Then there is a subset $I \subseteq [L]$ with $|I| \geq (1 - \frac{\gamma}{2})L$ such that $R_i$ is not $\alpha^2$-nice for any $i \in I$. It suffices to show that $\mathcal{R}$ is $(\epsilon, \frac{\delta}{2})$-weakly stable. Choose $i \in I$, then $R_i$ is $\alpha^2$-extremal. Based on Lemma 2.4 (i), one has $d^+_{R_i}(j), d^-_{R_i}(j) \geq (\frac{1}{2} - 2\mu)L$ for all but at most $d^{\frac{1}{4}} L$ vertices $j \in [L]$. Together with Lemma 3.2, we obtain that

- if $R_i$ is either $(\alpha^2, \mathrm{EC1})$-extremal or $(\alpha^2, \mathrm{EC2})$-extremal, then $R_i$ has a characteristic partition $(A'_i, B'_i, L'_i)$ such that there are $Z, Y \in \{A, B\}$ satisfying $e_{R_i}(Z'_i, Y'_i) \leq \alpha^2 L^2$;
- if $R_i$ is $(\alpha^2, \mathrm{EC3})$-extremal, then $R_i$ has a characteristic partition $(C_1^{i\prime}, C_2^{i\prime}, C_3^{i\prime}, C_4^{i\prime}, L'_i)$ such that $e_{R_i}(Z'_i, Y'_i) \leq \alpha^2 L^2$ with $(Z'_i, Y'_i) \in \{(W_i^{4\prime}, W_i^{3\prime}), (W_i^{2\prime}, W_i^{1\prime})\}$.

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Note that $|\hat{Z}_i'|, |\hat{Y}_i'| = (\frac{1}{2} - \alpha^2)Lm \geq (\frac{1}{2} - 2\alpha)n$. In view of Claim 4.2, we know that there is a subset $\mathcal{B}_i \subseteq \mathcal{C}_i$ with $|\mathcal{C}_i \setminus \mathcal{B}_i| \leq 2\alpha m$ such that for all $c \in \mathcal{B}_i$, we have $e_{D_c}(\hat{Z}_i', \hat{Y}_i') \leq \alpha n^2$. It follows that $D_c$ is $2\alpha$-extremal for each $c \in \mathcal{B}_i$.

If $R_i$ is either $(\alpha^2, \text{EC1})$-extremal or $(\alpha^2, \text{EC2})$-extremal, then each $D_c$ with $c \in \mathcal{B}_i$ admits a characteristic partition $(A_c, B_c, L_c)$, and there are $W \in \{A, B\}$ and $\{Z, Y\} = \{A, B\}$ such that $e_{D_c}(Z_c, W_c) \leq \sqrt{\alpha}n^2$ and $e_{D_c}(Y_c, W_c) \geq (\frac{1}{4} - \sqrt{\alpha})n^2$. Hence, we must have either $|\hat{Z}_i' \triangle Z_c|, |\hat{Y}_i' \triangle Y_c| \leq \epsilon n$ or $|\hat{Z}_i' \triangle Y_c|, |\hat{Y}_i' \triangle Z_c| \leq \epsilon n$. So, either $|\hat{A}_i' \triangle A_c|, |\hat{B}_i' \triangle B_c| \leq \epsilon n$ or $|\hat{A}_i' \triangle B_c|, |\hat{B}_i' \triangle A_c| \leq \epsilon n$. That is, for all $c \in \mathcal{B}_i$, the characteristic partition of $D_c$ is almost the same as the union of clusters corresponding to the characteristic partition of $R_i$.

If $R_i$ is $(\alpha^2, \text{EC3})$-extremal, then each $D_c$ admits a characteristic partition $(C_c^1, C_c^2, C_c^3, C_c^4, L_c)$ for $c \in \mathcal{B}_i$, and it satisfies $e_{D_c}(W_c^2, W_c^1), e_{D_c}(W_c^4, W_c^3) \leq \sqrt{\alpha}n^2$, $e_{D_c}(C_c^1), e_{D_c}(C_c^3) \geq |C_c^1|^2 - \sqrt{\alpha}n^2$ and $e_{D_c}(C_c^2, C_c^4)$, $e_{D_c}(C_c^4, C_c^2) \geq |C_c^2|^2 - \sqrt{\alpha}n^2$. Therefore, either $|\hat{Z}_i' \triangle W_c^2|, |\hat{Y}_i' \triangle W_c^1| \leq \epsilon n$ or $|\hat{Z}_i' \triangle W_c^4|, |\hat{Y}_i' \triangle W_c^3| \leq \epsilon n$. By relabelling, one may assume that $|\hat{W}_i^{4\prime} \triangle W_c^4|, |\hat{W}_i^{3\prime} \triangle W_c^3| \leq \epsilon n$ for each $c \in \mathcal{B}_i$. That is, for all $c \in \mathcal{B}_i$, the characteristic partition of $D_c$ is almost the same as the union of clusters corresponding to the characteristic partition of $R_i$.

Choose $c \in \mathcal{B}_i$, $c' \in \mathcal{B}_j$ for some $i, j \in [L]$. Suppose that $D_c$ and $D_{c'}$ are $\delta$-crossing. We must have $i \neq j$. We consider the following possible cases.

• Suppose that $R_i$ and $R_j$ are either $(\alpha^2, \text{EC1})$-extremal or $(\alpha^2, \text{EC2})$-extremal. If $|\hat{A}_i' \triangle A_c| \leq \epsilon n$ and $|\hat{A}_j' \triangle A_{c'}| \leq \epsilon n$, then

$$|A_i' \triangle A_j'|m = |\hat{A}_i' \triangle \hat{A}_j'| \geq |A_c \triangle A_{c'}| - |\hat{A}_i' \triangle A_c| - |\hat{A}_j' \triangle A_{c'}| \geq \delta n - 2\epsilon n \geq \frac{\delta n}{2} \geq \frac{\delta mL}{2}.$$

The other cases are almost identical. Thus $R_i$ and $R_j$ are $\frac{\delta}{2}$-crossing.

• Suppose that $R_i$ and $R_j$ are $(\alpha^2, \text{EC3})$-extremal. Recall that $|\hat{W}_i^{4\prime} \triangle W_c^4|, |\hat{W}_i^{3\prime} \triangle W_c^3| \leq \epsilon n$. Then $|\hat{W}_i^{1\prime} \triangle W_c^1| \leq 10\epsilon n$ and $|\hat{W}_i^{2\prime} \triangle W_c^2| \leq 10\epsilon n$. Since $D_c$ and $D_{c'}$ are $\delta$-crossing, one may assume, without loss of generality, that $|W_c^1 \triangle W_{c'}^1| \geq \delta n$ and $|W_c^1 \triangle W_{c'}^3| \geq \delta n$. Hence

$$\begin{aligned} |W_i^{1\prime} \triangle W_j^{1\prime}|m = |\hat{W}_i^{1\prime} \triangle \hat{W}_j^{1\prime}| &\geq |W_c^1 \triangle W_{c'}^1| - |\hat{W}_i^{1\prime} \triangle W_c^1| - |\hat{W}_j^{1\prime} \triangle W_{c'}^1| \\ &\geq \delta n - 21\epsilon n \geq \frac{\delta n}{2} \geq \frac{\delta mL}{2}. \end{aligned}$$

and similarly, $|W_i^{1\prime} \triangle W_j^{3\prime}| \geq \frac{\delta L}{2}$. Thus $R_i$ and $R_j$ are $\frac{\delta}{2}$-crossing.

• Suppose that $R_i$ is either $(\alpha^2, \text{EC1})$-extremal or $(\alpha^2, \text{EC2})$-extremal, and $R_j$ is $(\alpha^2, \text{EC3})$-extremal. If $|\hat{A}_i' \triangle A_c| \leq \epsilon n$ and $|\hat{W}_j^{3\prime} \triangle W_c^3| \leq \epsilon n$, then

$$|A_i' \triangle W_j^{3\prime}|m = |\hat{A}_i' \triangle \hat{W}_j^{3\prime}| \geq |A_c \triangle W_{c'}^3| - |\hat{A}_i' \triangle A_c| - |\hat{W}_j^{3\prime} \triangle W_{c'}^3| \geq \delta n - 2\epsilon n \geq \frac{\delta n}{2} \geq \frac{\delta mL}{2}.$$

Similarly, $|B_i' \triangle W_j^{3\prime}| \geq \frac{\delta L}{2}$. The other cases are almost identical. Thus $R_i$ and $R_j$ are $\frac{\delta}{2}$-crossing.

Since $\mathcal{D}$ is $(\epsilon, \delta)$-weakly stable, we have $e(C_{\mathcal{D},k}^{\epsilon,\delta}) \geq \delta n^2$ for some $k \in [3]$. The number of pairs $i, j \in [L]$ such that $R_i$ and $R_j$ are $\frac{\delta}{2}$-crossing is at least $\frac{\delta n^2}{m^2} \geq \frac{\delta L^2}{2}$. Thus $e(C_{\mathcal{R},k}^{\epsilon,\frac{\delta}{2}}) \geq \frac{\delta L^2}{2}$, and hence $\mathcal{R}$ is $(\epsilon, \frac{\delta}{2})$-weakly stable. □

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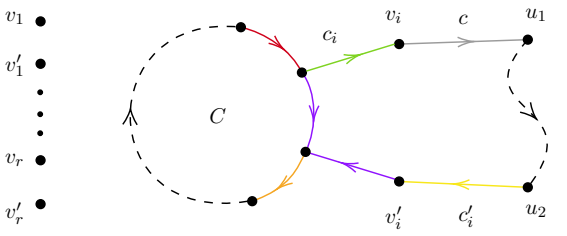



**Figure 4.** Absorbing.

For a digraph collection $\mathcal{D}$ that is either strongly stable or weakly stable, Lemmas 3.6 and 3.9 guarantee the existence of a directed absorbing cycle in $\mathcal{D}$. We shall then employ this absorbing property to prove that $\mathcal{D}$ contains a transversal directed Hamilton cycle.

**Lemma 4.3.** *Let* $0 < \frac{1}{n} \ll \mu \ll \alpha \ll \gamma, 8\epsilon^{1/2} \le \delta \le \epsilon^{1/3} \ll 1$. *Assume that* $\mathcal{D} = \{D_1, \dots, D_n\}$ *is a collection of digraphs on a common vertex set* $V$ *of size* $n$ *with* $\delta^0(\mathcal{D}) \ge \left(\frac{1}{2} - \mu\right) n$. *If* $\mathcal{D}$ *is* $(\gamma, \alpha, \epsilon, \delta)$*-stable, then* $\mathcal{D}$ *contains a transversal directed Hamilton cycle.*

**Proof.** Choose additional parameters $n_0, L_0, \epsilon_0, d, \beta, \lambda$, where $n_0 = n_0(\epsilon, 1, L_0)$ is obtained from Lemma 2.2, such that

$$1 < \frac{1}{n} \ll \frac{1}{n_0} \ll \frac{1}{L_0} \ll \epsilon_0 \ll d \ll \mu \ll \beta \ll \lambda \ll \alpha \ll \gamma, 8\epsilon^{1/2} \le \delta \le \epsilon^{1/3} \ll 1.$$

Since $\mathcal{D}$ is $(\gamma, \alpha, \epsilon, \delta)$-stable, in view of Lemmas 3.6 and 3.9, we know that $\mathcal{D}$ has a directed absorbing cycle $C$ with parameters $(\frac{\delta}{3}, \sqrt{\epsilon}, \lambda, \lambda^2)$. Fix any two colours $c, c' \in [n]$, two vertices $x, y \in V$ and $\mathrm{K} \in \{\mathrm{I}, \mathrm{II}\}$, we say the triple $(c, x, y)$ is absorbable if there are at least $\lambda^2 n$ disjoint Type-K directed $c$-absorbing paths of $(x, y)$ inside $C$; we say the pair $(c, x)$ is absorbable if there are at least $\lambda^2 n$ disjoint Type-K directed $c$-absorbing paths of $(x, x)$ inside $C$; we say $(c, c', x, y)$ is totally absorbable if $(c, x)$, $(c', y)$ and $(c, x, y)$ are all absorbable.

By Definition 3.3, $C$ has length at most $\lambda n$, and there exists a colour set $\mathcal{C} \subseteq [n] \setminus \mathrm{col}(C)$ of size at least $\frac{\delta n}{3}$ such that

(a) given any colour $c \in \mathcal{C}$ and any $D_c$-good vertex $v$, the triple $(c, v, u)$ is absorbable for all but at most $\sqrt{\epsilon} n$ vertices $u$,

(b) given any colour $c \in \mathcal{C}$, for all but at most $\sqrt{\epsilon} n$ $D_c$-good vertices $v$, the pair $(c, v)$ is absorbable.

□

**Claim 4.4.** *There is an integer* $r \in [(2 - 2^{-10})\beta n, 2\beta n]$ *and for each* $i \in [r]$, *disjoint vertex pairs* $(v_i, v_i') \in V^2$ *and disjoint colour pairs* $(c_i, c_i') \in \mathcal{C}^2$ *such that the family* $\mathcal{Q} := \{(c_i, c_i', v_i, v_i') : i \in [r]\}$ *has the following properties:*

(i) $(c_i, c_i', v_i, v_i')$ *is totally absorbable for all* $i \in [r]$,

(ii) *for every pair* $(u_1, u_2) \in V^2$ *and* $c \in [n]$, *there are at least* $2^{-9}\beta n$ *values* $i \in [r]$ *such that* $c \in L(v_i u_1)$ *and* $c_i' \in L(u_2 v_i')$ *(see Figure 4).*

**Proof of Claim** 4.4. For every colour pair $(b_1, b_2) \in \mathcal{C}^2$, every vertex pair $(u_1, u_2) \in V^2$ and colour $c \in \mathcal{C}$, let $S(b_1, b_2, u_1, u_2, c)$ be the collection of pairs $(v_1, v_2) \in V^2$ such that $c \in L(v_1 u_1)$, $b_2 \in L(u_2 v_2)$ and $(b_1, b_2, v_1, v_2)$ is totally absorbable. We will prove that $|S(b_1, b_2, u_1, u_2, c)| \ge 2^{-4} n^2$.

For this, we first count the number of choices for $v_1$. Let $N_1 := \{x \in N_c^-(u_1) : x \text{ is } D_{b_1}\text{-good}\}$. Hence, $|N_1| \ge \left(\frac{1}{2} - \mu\right) n - 2\epsilon n$. By (b), we know that all but at most $\sqrt{\epsilon} n$ vertices $v_1 \in N_1$ satisfying that $(b_1, v_1)$ is absorbable. Hence there are at least $\frac{n}{4}$ choices of $v_1$ such that $v_1 \in N_1$ and $(b_1, v_1)$ is absorbable.

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Now fix $v_1$. Let $N_2 := \{x \in N^+_{b_2}(u_2) : x \text{ is } D_{b_2}\text{-good}\}$. Hence, $|N_2| \geq \left(\frac{1}{2} - \mu\right) n - 2\epsilon n$. By (a) and (b), we know that all but at most $2\sqrt{\epsilon}n + 1$ vertices $v_2 \in N_2$ such that $v_2 \neq v_1$ and $(b_1, b_2, v_1, v_2)$ is totally absorbable. Therefore, $|S(b_1, b_2, u_1, u_2, c)| \geq 2^{-4}n^2$, as desired.

Let $\{c_i, c'_i : i \in [2\beta n]\} \subseteq \mathcal{C}$ be a collection of distinct colours. For each $i \in [2\beta n]$, let $\mathcal{F}_i$ be the directed 2-graph on $V$ with edge set $\{(v_1, v_2) \in V^2 : (c_i, c'_i, v_1, v_2) \text{ is totally absorbable}\}$, and define $\mathbf{H} := \{\mathcal{F}_i : i \in [2\beta n]\}$. Define the collection of directed multi-graphs:

$$\mathbf{Z} := \left\{ S(u_1, u_2, c) := \bigcup_{i \in [2\beta n]} S(c_i, c'_i, u_1, u_2, c) : (u_1, u_2) \in V^2,\ c \in [n] \right\}.$$

For every $S := S(u_1, u_2, c) \in \mathbf{Z}$ and every $i \in [2\beta n]$, we have $|E(S) \cap E(\mathcal{F}_i)| \geq 2^{-4}n^2$. Applying Lemma 3.4 with $t := 2\beta n$ and $\epsilon := 2^{-4}$, we obtain that there is a rainbow matching $M$ of $\mathbf{H}$ with size at least $(2 - 2^{-10})\beta n$ (and at most $2\beta n$) and $|E(S) \cap E(M)| \geq 2^{-9}\beta n$ for all $S \in \mathbf{Z}$, as desired. □

Let

$$V_{\text{abs}} := \bigcup_{i \in [r]} \{v_i, v'_i\},\ U := V \setminus (V(C) \cup V_{\text{abs}}),\ \mathcal{C}_{\text{abs}} := \bigcup_{i \in [r]} \{c_i, c'_i\},$$
$$\mathcal{C}_{\text{rem}} = [n] \setminus (\text{col}(C) \cup \mathcal{C}_{\text{abs}}),\ \mathcal{J} := \{J_i : i \in \mathcal{C}_{\text{rem}}\} \text{ where } J_i = D_i[U].$$

Thus, $\left|[n] \setminus \mathcal{C}_{\text{rem}}\right| = |V \setminus U| \leq \lambda n + 4\beta n < 2\lambda n$ since $\beta \ll \lambda$, and $\delta^0(\mathcal{J}) \geq (\frac{1}{2} - \mu - 2\lambda)n \geq (\frac{1}{2} - 3\lambda)n$.

Since $\lambda \ll \gamma, \alpha, \epsilon, \delta$ and $\mathcal{D}$ is $(\gamma, \alpha, \epsilon, \delta)$-stable, it is easy to see that $\mathcal{J}$ is $(\frac{\gamma}{2}, \frac{\alpha}{2}, 2\epsilon, \frac{\delta}{2})$-stable. Applying Lemma 2.2 to $\mathcal{J}$ with parameters $(\epsilon_0, 1, d, L_0)$, we obtain a reduced digraph collection $\mathcal{R}$. Denote by $[L]$ the common vertex set of the digraphs in $\mathcal{R}$, where $L_0 \leq L \leq n_0$, and by $[M]$ the set of colour clusters. Thus there is a partition $V_0, V_1, \ldots, V_L$ of $U$ and $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_M$ of $\mathcal{C}_{\text{rem}}$ and a digraph collection $\mathcal{J}'$ satisfying (i)-(v) of Lemma 2.2. Therefore, for $\{(h, i), j\} \in \binom{[L]}{2} \times [M]$, we have that $hi \in R_j$ if and only if $\mathcal{J}'_{hi,j} := \{J'_c[V_h, V_i] : c \in \mathcal{C}_j\}$ is $(\epsilon_0, d)$-regular. Furthermore, $Lm \leq |U| \leq Mm + \epsilon_0 n$ and $Mm \leq |\mathcal{C}_{\text{rem}}| \leq Lm + \epsilon_0 n$, so $|L - M| \leq \frac{\epsilon_0 n}{m} \leq \frac{\epsilon_0 L}{1 - \epsilon_0}$. Thus, we may assume that $M = L$ at the expense of assuming the slightly worse bound $|V_0| + |\mathcal{C}_0| \leq 3\epsilon_0 n$. Based on Lemma 2.4, we obtain that, for each vertex $i \in [L]$, there are at least $(1 - d^{1/4})M$ colours $j \in [M]$ such that $d^+_{R_j}(i), d^-_{R_j}(i) \geq (\frac{1}{2} - 4\lambda)L$; for each colour $j \in [M]$, there are at least $(1 - d^{1/4})L$ vertices $i \in [L]$ such that $d^+_{R_j}(i), d^-_{R_j}(i) \geq (\frac{1}{2} - 4\lambda)L$.

It follows from Lemma 4.1 that $\mathcal{R}$ is $(\frac{\gamma}{4}, \frac{\alpha^2}{4}, 2\epsilon, \frac{\delta}{4})$-stable. It is routine to check that the bipartite graph collection corresponding to $\mathcal{R}$, say $\mathcal{B}_{\mathcal{R}}$, is $(\frac{\gamma}{4}, \frac{\alpha^2}{4}, 2\epsilon, \frac{\delta}{4})$-stable. Together with Theorem 2.10, there is a transversal perfect matching inside $\mathcal{B}_{\mathcal{R}}$. Therefore, there exists a set of disjoint rainbow directed cycles inside $\mathcal{R}$, say $C^1, C^2, \ldots, C^t$, such that $|V(C^1)| + \cdots + |V(C^t)| = L$. For each rainbow directed cycle $C^i$ in $\mathcal{R}$, by using colours in $\bigcup_{j \in \text{col}(C^i)} \mathcal{C}_j$, we are to find a rainbow directed path $P^i$ in $\mathcal{J}'$ with length at least $(1 - \sqrt{\epsilon_0})|C^i|m$.

**Claim 4.5.** *Let $C = v_1 v_2 \ldots v_s v_1$ be a rainbow directed cycle in $\mathcal{R}$ with $v_{s+1} := v_1$ and $v_i v_{i+1} \in E(R_i)$ for each $i \in [s]$. Then $\{J'_i[V_1, V_2, \cdots, V_s] : i \in \bigcup_{j \in [s]} \mathcal{C}_j\}$ contains a rainbow directed path with order at least $(1 - \sqrt{\epsilon_0})sm$.*

**Proof of Claim** 4.5. We construct a rainbow directed path greedily. Assume that we already have a rainbow directed path $P = x_1 \ldots x_i$ with $x_j \in V_{j_0}$ and $j_0 \equiv j \pmod{s}$ for each $j \in [i]$. If $i \geq (1 - \sqrt{\epsilon_0})sm$, then we are done. Otherwise, $|V_j \setminus V(P)| \geq \sqrt{\epsilon_0}m - 1$ for each $j \in [s]$. We claim that $\max\{|N^+_{J'_c}(x_i) \cap (V_{i_0+1} \setminus V(P))| : c \in \mathcal{C}_{i_0} \setminus \text{col}(P)\} \geq 2\epsilon_0 m$. If so, then choose $c \in \mathcal{C}_{i_0} \setminus \text{col}(P)$ such

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that $|N^+_{J'_c}(x_i) \cap (V_{i_0+1} \setminus V(P))|$ is maximum and choose $x_{i+1} \in N^+_{J'_c}(x_i) \cap (V_{i_0+1} \setminus V(P))$ such that $\max\{|N^+_{J'_c}(x_{i+1}) \cap (V_{i_0+2} \setminus V(P))| : c \in \mathcal{C}_{i_0+1} \setminus \mathrm{col}(P)\}$ is maximum.

Suppose that $\max\{|N^+_{J'_c}(x_i) \cap (V_{i_0+1} \setminus V(P))| : c \in \mathcal{C}_{i_0} \setminus \mathrm{col}(P)\} < 2\epsilon_0 m$. Clearly, $i \geq 2$. Let $W_{i_0} := N^+_{J'_{c(x_{i-1}x_i)}}(x_{i-1}) \cap (V_{i_0} \setminus V(P))$ and $W_{i_0+1}$ be a subset of $V_{i_0+1} \setminus V(P)$ with $|W_{i_0+1}| \geq \frac{\sqrt{\epsilon_0}}{2} m$. By the choice of each $x_j$ with $j \leq i$, we have $|W_{i_0}| \geq 2\epsilon_0 m$ and each $y \in W_{i_0}$ satisfies $|N^+_{J'_c}(y) \cap (V_{i_0+1} \setminus V(P))| < 2\epsilon_0 m$ for each colour $c \in \mathcal{C}_{i_0} \setminus \mathrm{col}(P)$. Then

$$\frac{\sum_{c \in \mathcal{C}_{i_0} \setminus \mathrm{col}(P)} e_{J'_c}(W_{i_0}, W_{i_0+1})}{|\mathcal{C}_{i_0} \setminus \mathrm{col}(P)||W_{i_0}||W_{i_0+1}|} \leq 2\sqrt{\epsilon_0} \ll d,$$

which contradicts the fact that $\{J'_i[V_{i_0}, V_{i_0+1}] : i \in \mathcal{C}_{i_0}\}$ is $(\epsilon_0, d)$-regular. □

Based on Claim 4.5, we know that there exists a set of disjoint rainbow directed paths $P^i$ in $\mathcal{J}$ with order at least $(1 - \sqrt{\epsilon_0})|C^i|m$ and colours from $\bigcup_{j \in \mathrm{col}(C^i)} \mathcal{C}_j$ for all $i \in [t]$. The number of vertices or colours not in any $P^i$ is at most $\sqrt{\epsilon_0}n + 3\epsilon_0 n \leq 2\sqrt{\epsilon_0}n$. We consider each vertex in $U$ but not in any $P^i$ to be a single path (of length 0).

Without loss of generality, assume that $P^1, P^2, \ldots, P^s$ are all the disjoint rainbow directed paths obtained in the above. Then $s \leq 2\sqrt{\epsilon_0}n + L < 2^{-10}\beta n$. Let $x_i, y_i$ be the start vertex and end vertex of $P^i$ for each $i \in [s]$ (where $x_i = y_i$ if $P^i$ has length 0). Clearly, $U = \bigcup_{i \in [s]} V(P^i)$. It follows that $\left|\mathcal{C}_{\mathrm{rem}} \setminus \mathrm{col}(\bigcup_{i \in [s]} P^i)\right| = s$.

Next, we do the following operation for each $i \in [s]$ in turn. Let $a_i \in \mathcal{C}_{\mathrm{rem}}$ be an arbitrary unused colour. Choose an unused 4-tuple $Q_i := (c_{j_i}, c'_{j_i}, v_{j_i}, v'_{j_i}) \in \mathcal{Q}$ where $j_i \in [r]$, $a_i \in L(v_{j_i}x_i)$ and $c'_{j_i} \in L(y_i v'_{j_i})$. This is feasible since Claim 4.4 implies that there are at least $2^{-9}\beta n$ choices for $Q_i$, of which at most $s \leq 2^{-10}\beta n$ have been used. Now, $(c_{j_i}, c'_{j_i}, v_{j_i}, v'_{j_i})$ is totally absorbable, so there are at least $\lambda^2 n$ disjoint rainbow directed $c_{j_i}$-absorbing path of $(v_{j_i}, v'_{j_i})$ inside $C$. Then choose one of them, say $S_i := x^i_1 x^i_2 x^i_3 x^i_4$, whose vertices have not been previously chosen since $s < 2^{-10}\beta n < 2^{-10}\lambda^2 n$, and whose colours are $b^i_1, b^i_2, b^i_3$ in order.

Notice that there remains $I \subseteq [r]$ such that the $(c_j, c'_j, v_j, v'_j) \in \mathcal{Q}$ with $j \in I$ are precisely the 4-tuples which were not chosen to be some $Q_i$. For each one, there are at least $\lambda^2 n$ disjoint rainbow directed $c_j$-absorbing paths of $(v_j, v_j)$ and disjoint rainbow directed $c'_j$-absorbing paths of $(v'_j, v'_j)$ inside $C$. Since $\beta < \lambda^2$, there are two such rainbow directed paths $T_i$, $T'_i$ for each $(c_j, v_j)$ and $(c'_j, v'_j)$, which are disjoint and whose vertices have not previously been chosen.

At the end of this process, we have a collection $\{S_i : i \in [s]\}$, $\{T_i : i \in I\}$, $\{T'_i : i \in I\}$ of disjoint rainbow directed paths inside $C$ satisfying

- for each $i \in [s]$, $S_i$ is a directed $c_{j_i}$-absorbing path of $(v_{j_i}, v'_{j_i})$,
- for each $i \in I$, $T_i$ is a directed $c_i$-absorbing path of $(v_i, v_i)$ and $T'_i$ is a $c'_i$-directed absorbing path of $(v'_i, v'_i)$.

For each $i \in [s]$, we replace $S_i$ by $x^i_1 x^i_2 v_{j_i} x_i P^i y_i v'_{j_i} x^i_3 x^i_4$ with colours $b^i_1, c_{j_i}, a_i$ followed by the colours inherited from $P^i$, followed by $c'_{j_i}, b^i_2, b^i_3$. That is, we have replaced $S_i$ by a rainbow directed path with the same endpoints, vertices $V(S_i) \cup V(P^i) \cup \{v_{j_i}, v'_{j_i}\}$, and colours $\mathrm{col}(S_i) \cup \mathrm{col}(P^i) \cup \{c_{j_i}, c'_{j_i}, a_i\}$. For each $i \in I$, we replace $T_i := y^i_1 y^i_2 y^i_3 y^i_4$ by $y^i_1 y^i_2 v_i y^i_3 y^i_4$ where colours are inherited except $\mathrm{col}(y^i_2 v_i) = c_i$ and $\mathrm{col}(v_i y^i_3) = \mathrm{col}(y^i_2 y^i_3)$. We do a similar replacement of $T'_i$, using new vertex $v'_i$ and new colour $c'_i$. So we have replaced $T_i$ (resp. $T'_i$) by a rainbow directed path with the same endpoints, vertices $V(T_i) \cup \{v_i\}$ (resp. $V(T'_i) \cup \{v'_i\}$) and colours $\mathrm{col}(T_i) \cup \{c_i\}$ (resp. $\mathrm{col}(T'_i) \cup \{c'_i\}$). Thus we have obtained a rainbow directed cycle using vertices $V(C) \cup \bigcup_{i \in [s]} V(P^i) \cup \{v_i, v'_i : i \in$

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$[r]\} = V$ and colours $\mathrm{col}(C) \cup \bigcup_{i\in[s]} \mathrm{col}(P^i) \cup \{c_i, c_i' : i \in [r]\} = [n]$ where each colour is used at most once (and hence exactly once). That is, $\mathcal{D}$ contains a transversal directed Hamilton cycle, as desired. □

## 5. Extremal case

In this section, we consider the extremal case, in which most digraphs in the collection are $(\epsilon, \mathrm{ECk})$-extremal for some $k \in [3]$. The following result follows directly from Lemma 4.3.

**Lemma 5.1.** *Assume that* $0 < \frac{1}{n} \ll 8\epsilon^{1/2} \le \delta \le \epsilon^{1/3} \ll 1$ *and* $m = (1 - 4\sqrt{\delta})n$. *Let* $\mathcal{D} = \{D_1, \dots, D_n\}$ *be a digraph collection on a common vertex set* $V$ *of size* $n$ *and* $\delta^0(\mathcal{D}) \ge \frac{n}{2}$. *If* $\mathcal{D}$ *contains no transversal directed Hamilton cycles, then by swapping indices if necessary, one of the following holds:*

- **(B1)** *either* $D_i$ *is* $(\epsilon, EC1)$*-extremal or* $(\epsilon, EC2)$*-extremal and it admits an* $\epsilon$*-characteristic partition* $(\tilde{A}_i, \tilde{B}_i, \tilde{L}_i)$ *for each* $i \in [m]$,
- **(B2)** *or* $D_i$ *is* $(\epsilon, EC3)$*-extremal and it admits an* $\epsilon$*-characteristic partition* $(C_i^1, C_i^2, C_i^3, C_i^4, L_i)$ *for each* $i \in [m]$.

*In (B1), for every* $i \in [m]$, $D_i$ *also admits a partition* $(A_i, B_i, L_i)$ *satisfying*

- **(C1)** *if* $i = 1$ *then* $(A_1, B_1, L_1) := (\tilde{A}_1, \tilde{B}_1, \tilde{L}_1)$; *for* $i \ge 2$,
- **(C2)** *if* $D_i$ *is* $(\epsilon, K_{\lceil \frac{n}{2} \rceil} \cup K_{\lfloor \frac{n}{2} \rfloor})$*-extremal, then for* $Y \in \{A, B\}$ *we have* $\tilde{Y}_i \subseteq Y_i$ *and* $d_{D_i}(v, Y_i) \ge (\frac{1}{2} - 3\sqrt{\delta})n$ *for each vertex* $v \in Y_i$,
- **(C3)** *if* $D_i$ *is* $(\epsilon, K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor})$*-extremal, then for* $\{Y, Z\} = \{A, B\}$ *we have* $\tilde{Y}_i \subseteq Y_i$ *and* $d_{D_i}(v, Z_i) \ge (\frac{1}{2} - 3\sqrt{\delta})n$ *for each vertex* $v \in Y_i$,
- **(C4)** *subject to (C2)-(C3), the partition* $(A_i, B_i, L_i)$ *is chosen such that* $|A_i \cup B_i|$ *is maximised for each* $i \in [2, m]$.

*Furthermore, for every* $i \in [m]$ *and* $Y \in \{A, B\}$, *we have*

$$|L_i| \le 2\epsilon n \text{ and } |Y_1 \triangle Y_i| < 2\delta n.$$

**Proof.** Choose constants $\alpha, \gamma, \epsilon, \delta$ such that

$$0 < \frac{1}{n} \ll \alpha \ll \gamma, 8\epsilon^{1/2} \le \delta \le \epsilon^{1/3} \ll 1.$$

Let $\mathcal{D} = \{D_1, \dots, D_n\}$ be a digraph collection on a common vertex set $V$ of size $n$ and $\delta^0(\mathcal{D}) \ge \frac{n}{2}$. Suppose that $\mathcal{D}$ does not contain transversal directed Hamilton cycles. By Lemma 4.3, we know that $\mathcal{D}$ is not $(\gamma, \alpha)$-strongly stable. Hence, there are at least $(1 - \gamma)n$ digraphs in $\mathcal{D}$ such that each of them is $\alpha$-extremal. Without loss of generality, assume that $D_i$ is $\alpha$-extremal for each $i \in [(1 - \gamma)n]$. Note that every $\alpha$-extremal digraph is also $\epsilon$-extremal, since $\alpha < \epsilon$. Denote

$$\tilde{\mathcal{C}}_1 := \{i \in [n] : D_i \text{ is } (\epsilon, \mathrm{EC1})\text{-extremal}\},$$

$$\tilde{\mathcal{C}}_2 := \{i \in [n] : D_i \text{ is } (\epsilon, \mathrm{EC2})\text{-extremal}\},$$

$$\tilde{\mathcal{C}}_3 := \{i \in [n] : D_i \text{ is } (\epsilon, \mathrm{EC3})\text{-extremal}\}.$$

Hence $|\tilde{\mathcal{C}}_1| + |\tilde{\mathcal{C}}_2| + |\tilde{\mathcal{C}}_3| \ge (1 - \gamma)n$. By Lemma 3.2, we know that $D_i$ has a characteristic partition $(\tilde{A}_i, \tilde{B}_i, \tilde{L}_i)$ for each $i \in \tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2$, or a characteristic partition $(C_i^1, C_i^2, C_i^3, C_i^4, L_i)$ for each $i \in \tilde{\mathcal{C}}_3$.

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Under the above characteristic partition, applying Lemma 4.3 again yields that $\mathcal{D}$ is not $(\epsilon, \delta)$-weakly stable. For each $i \in \tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2$, define $I_i^1$ to be the set of $j \in \tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2$ such that $D_i$ and $D_j$ are $\delta$-crossing. For each $i \in \tilde{\mathcal{C}}_3$, define $I_i^2$ to be the set of $j \in \tilde{\mathcal{C}}_3$ such that $D_i$ and $D_j$ are $\delta$-crossing. For each $i \in \tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2$, define $I_i^3$ to be the set of $j \in \tilde{\mathcal{C}}_3$ such that $D_i$ and $D_j$ are $\delta$-crossing. Then we have

$$\sum_{i \in \tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2} |I_i^1| < 2\delta n^2, \sum_{i \in \tilde{\mathcal{C}}_3} |I_i^2| < 2\delta n^2, \text{ and } \sum_{i \in \tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2} |I_i^3| < \delta n^2.$$

Hence there exists a colour set, say $I^k$ (where $I^k \subseteq \tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2$ if $k = 1$ and $I^k \subseteq \tilde{\mathcal{C}}_3$ if $k \in \{2, 3\}$), with size at most $\sqrt{\delta} n$ such that the following hold:

- if $|\tilde{\mathcal{C}}_1| + |\tilde{\mathcal{C}}_2| \geq 2\sqrt{\delta} n$, then by swapping labels one has $|\tilde{A}_1 \triangle \tilde{A}_i| < \delta n$ and $|\tilde{B}_1 \triangle \tilde{B}_i| < \delta n$ for all $i \in (\tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2) \setminus I^1$,
- if $|\tilde{\mathcal{C}}_3| \geq 2\sqrt{\delta} n$, then by swapping labels, for all $k \in [4]$ and all $i \in \tilde{\mathcal{C}}_3 \setminus I^2$ we have $|W_1^k \triangle W_i^k| < \delta n$, in particular $|C_1^k \triangle C_i^k| < 2\delta n$,
- if $|\tilde{\mathcal{C}}_1| + |\tilde{\mathcal{C}}_2| \geq 2\sqrt{\delta} n$ and $|\tilde{\mathcal{C}}_3| \geq 2\sqrt{\delta} n$, then by swapping labels one has $|\tilde{A}_1 \triangle W_i^1| < \delta n$ and either $|\tilde{A}_1 \triangle W_i^2| < \delta n$ or $|\tilde{A}_1 \triangle W_i^4| < \delta n$ for all $i \in \tilde{\mathcal{C}}_3 \setminus I^3$.

Note that if last itemise holds, then for each $i \in \tilde{\mathcal{C}}_3 \setminus I^3$, there is an $i_0 \in \{2, 4\}$ such that

$$2\delta n > |\tilde{A}_1 \triangle W_i^1| + |\tilde{A}_1 \triangle W_i^{i_0}| \geq |W_i^1 \triangle W_i^{i_0}| > 2\delta n,$$

a contradiction. This implies that exactly one of $|\tilde{\mathcal{C}}_1| + |\tilde{\mathcal{C}}_2| \geq 2\sqrt{\delta} n$ and $|\tilde{\mathcal{C}}_3| \geq 2\sqrt{\delta} n$ holds.

Suppose that $|\tilde{\mathcal{C}}_1| + |\tilde{\mathcal{C}}_2| \geq 2\sqrt{\delta} n$. Denote $[m] := (\tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2) \setminus I^1$ and $\mathcal{C}_{\text{bad}} := [m+1, n]$. By adding colours to $\mathcal{C}_{\text{bad}}$ if necessary we may assume $m = (1 - 4\sqrt{\delta})n$. Recall that $\epsilon < \delta$ and $D_i$ has a characteristic partition $(\tilde{A}_i, \tilde{B}_i, \tilde{L}_i)$ for each $i \in [m]$. Hence for each $i \in [2, m]$, there exists a new partition $(A_i, B_i, L_i)$ of $D_i$ such that

- if $D_i$ is $(\epsilon, \text{EC1})$-extremal, then for $Y \in \{A, B\}$ we have $\tilde{Y}_i \subseteq Y_i$ and $d_i^+(v, Y_i) \geq (\frac{1}{2} - 3\sqrt{\delta})n$ for each vertex $v \in Y_i$,
- if $D_i$ is $(\epsilon, \text{EC2})$-extremal, then for $Y \in \{A, B\}$ we have $\tilde{Y}_i \subseteq Y_i$ and $d_i^+(v, Z_i) \geq (\frac{1}{2} - 3\sqrt{\delta})n$ for each vertex $v \in Y_i$,
- subject to the above two conditions, the partition $(A_i, B_i, L_i)$ is chosen such that $|A_i \cup B_i|$ is maximised for each $i \in [2, m]$.

Denote $(A_1, B_1, L_1) := (\tilde{A}_1, \tilde{B}_1, \tilde{L}_1)$. Hence for every $i \in [m]$ and $Y \in \{A, B\}$, we have $|Y_1 \triangle Y_i| \leq |\tilde{Y}_1 \triangle \tilde{Y}_i| + |\tilde{L}_i| \leq \delta n + 2\epsilon n < 2\delta n$, as desired. □

Based on Lemma 5.1, we proceed with the proof by considering the following two theorems.

**Theorem 5.2.** *Assume that* $0 < \frac{1}{n} \ll \epsilon \leq \delta^2 \ll 1$ *and* $m = (1 - 4\sqrt{\delta})n$. *Let* $\mathcal{D} = \{D_1, \ldots, D_n\}$ *be a collection of digraphs on a common vertex set* $V$ *of size* $n$ *and* $\delta^0(\mathcal{D}) \geq \frac{n}{2}$. *If* $D_i$ *is* $(\epsilon, EC1)$*-extremal or* $(\epsilon, EC2)$*-extremal for each* $i \in [m]$, *then* $\mathcal{D}$ *contains a transversal directed Hamilton cycle.*

**Theorem 5.3.** *Assume that* $0 < \frac{1}{n} \ll \epsilon \leq \delta^2 \ll 1$ *and* $m = (1 - 4\sqrt{\delta})n$. *Let* $\mathcal{D} = \{D_1, \ldots, D_n\}$ *be a collection of digraphs on a common vertex set* $V$ *of size* $n$ *and* $\delta^0(\mathcal{D}) \geq \frac{n}{2}$. *If* $D_i$ *is* $(\epsilon, EC3)$*-extremal for each* $i \in [m]$, *then* $\mathcal{D}$ *contains a transversal directed Hamilton cycle.*

In the following two sections, we prove the above two theorems respectively.

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## 6. Proof of Theorem 5.2

In this section we prove Theorem 5.2. Following the outline presented in Section 1.4, we first identify several types of "bad" vertices and cover them by short rainbow directed paths whose endpoints possess additional properties. Next we handle the "bad" colours by choosing matchings. We then connect all these short rainbow directed paths via a connecting lemma. Finally, by the transversal blow-up lemma, we construct a long transversal directed path, which can be closed to form the desired transversal directed cycle.

For the sake of clarity in presenting the following lemmas, we summarise some properties of the digraph collection $\mathcal{D}$.

† Let $0 < \frac{1}{n} \ll \epsilon \leq \delta^2 \ll \eta \ll 1$ and $m = (1 - 4\sqrt{\delta})n$. Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a collection of digraphs on a common vertex set $V$ of size $n$ and $\delta(\mathcal{D}) \geq \frac{n}{2}$. We assume the following conditions hold.

(a) For every $i \in [m]$, $D_i$ is either $(\epsilon, \mathrm{EC1})$-extremal or $(\epsilon,\ \mathrm{EC2})$-extremal and it admits a partition $(A_i, B_i, L_i)$ satisfying **(C1)–(C4)**. We extend $A_1 \cup B_1$ to an equitable partition $A \cup B$ of $V$.

(b) Define

$$\mathcal{C}_1 = \{i \in [m] : D_i \text{ is } (\epsilon, \mathrm{EC1})\text{-extremal}\},$$
$$\mathcal{C}_2 = \{i \in [m] : D_i \text{ is } (\epsilon, \mathrm{EC2})\text{-extremal}\}, \text{ and } \mathcal{C}_{\mathrm{bad}} = \mathcal{C} \setminus (\mathcal{C}_1 \cup \mathcal{C}_2).$$

Let $\hat{\mathcal{C}} := \bigcup_{k\in[2]} \psi(\mathcal{C}_k)$, where $\psi(\mathcal{C}_k) = \mathcal{C}_k$ if $|\mathcal{C}_k| \geq \eta n$ and $\psi(\mathcal{C}_k) = \emptyset$ otherwise.

(c) After moving at most $4\delta n$ vertices from $A$ to $B$ (resp. from $B$ to $A$) and deleting at most $\frac{1}{3}\sqrt{\epsilon}n$ vertices in $A \cup B$, there exists a set $V_{\mathrm{bad}} \subseteq V$ with size at most $(\sqrt{\epsilon} + \frac{1}{2}\sqrt{\delta})n$ such that each vertex in $Y \setminus V_{\mathrm{bad}}$ lies in $Y_i$ for at least $(1 - 13\sqrt{\delta})|\hat{\mathcal{C}}|$ colours $i \in \hat{\mathcal{C}}$, where $Y \in \{A, B\}$.

(d) Define

$$X := \{x \in V : x \notin A_i \cup B_i \text{ for at least } 6\sqrt{\delta}|\hat{\mathcal{C}}| \text{ colors } i \in \hat{\mathcal{C}}\}.$$

Let $X'$ be a subset of $X$ consisting of vertices $x$ such that $d_i^+(x, Y_i) \geq (1 - 3\sqrt{\delta})n$ (resp. $d_i^-(x, Y_i) \geq (1 - 3\sqrt{\delta})n$) for at least $(1 - 3\sqrt{\delta})|\hat{\mathcal{C}}|$ colours $i \in \hat{\mathcal{C}}$ and $d_i^-(x, Z_i) \geq \frac{5}{2}\sqrt{\delta}n$ (resp. $d_i^+(x, Z_i) \geq \frac{5}{2}\sqrt{\delta}n$) for at least $3\sqrt{\delta}|\hat{\mathcal{C}}|$ colours $i \in \hat{\mathcal{C}}$.

The next lemma shows that any vertex contained in $L_i$ for many colours $i \in \mathcal{C}$ can be covered by a rainbow directed path $P_3$ with endpoints not in $V_{\mathrm{bad}}$. To enhance flexibility, we select a rainbow star $S_5$ (instead of a simple directed $P_3$), which allows each endpoint of the resulting path to be replaced by an alternative available vertex when needed.

**Lemma 6.1.** *Suppose that † holds and $\{Y, Z\} = \{A, B\}$. Let $P$ be a rainbow directed path in $\mathcal{D}$ with length at most $\frac{7}{3}\sqrt{\delta}n$. Assume $X \setminus (X' \cup V(P)) = \{x_1, \ldots, x_s\}$ with $s \leq \sqrt{\epsilon}n$. If $x_i \in X \cap Y$, then there exist colours $c_i^1, c_i^2, c_i^3, c_i^4 \in \mathcal{C}_j \setminus (\ col(P) \cup \{c_\ell^1, c_\ell^2, c_\ell^3, c_\ell^4 : \ell \in [i-1]\})$ and vertices $x_i^1, x_i^2, x_i^3, x_i^4 \in W \setminus (V(P) \cup \{x_\ell^1, x_\ell^2, x_\ell^3, x_\ell^4 : \ell \in [i-1]\})$ such that $x_i^k x_i \in E(D_{c_i^k})$ for $k \in [2]$ and $x_i x_i^k \in E(D_{c_i^k})$ for $k \in \{3, 4\}$, where*

- *$j = 2$ and $W = Z$ if $|\mathcal{C}_1| < \eta n$,*
- *$j = 1$ and $W = Y$ if $|\mathcal{C}_2| < \eta n$,*
- *either $j = 1$ and $W = Y$, or $j = 2$ and $W = Z$ otherwise.*

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**Proof.** We only prove the case that $|\mathcal{C}_1| < \eta n$, and the other cases can be proved by similar arguments. Suppose there exists an $i_0 \in [s]$ such that Lemma 6.1 holds for all $i \in [i_0 - 1]$ but does not hold for $i_0$. Assume for convenience that $x_{i_0} \in X \cap A$. Denote $\mathcal{C}_2' := \mathcal{C}_2 \setminus (\mathrm{col}(P) \cup \{c_\ell^1, c_\ell^2, c_\ell^3, c_\ell^4 : \ell \in [i_0 - 1]\})$. Let

$$\mathcal{C}_2^1 := \left\{ j \in \mathcal{C}_2' : d_j^+(x_{i_0}, B) \geq 4(i_0 - 1) + 4 + |V(P)| \right\} \text{ and}$$
$$\mathcal{C}_2^2 := \left\{ j \in \mathcal{C}_2' : d_j^-(x_{i_0}, B) \geq 4(i_0 - 1) + 4 + |V(P)| \right\}.$$

Hence either $|\mathcal{C}_2^1| \leq 3$ or $|\mathcal{C}_2^2| \leq 3$. Without loss of generality, assume that $|\mathcal{C}_2^1| \leq 3$. Thus, for each $j \in \mathcal{C}_2' \setminus \mathcal{C}_2^1$, we have

$$\begin{aligned} d_j^+(x_{i_0}, A_j) &\geq d_j^+(x_{i_0}, A) - |A \setminus A_1| - |A_1 \triangle A_j| - |X_A \setminus X_A^2| - |X'| \\ &\geq \delta^0(D_j) - d_j^+(x_{i_0}, B) - |A \setminus A_1| - |A_1 \triangle A_j| - |X_A \setminus X_A^2| - |X'| \\ &\geq \frac{n}{2} - (4(i_0 - 1) + 4 + |V(P)|) - 2\epsilon n - 2\delta n - 4\delta n - \sqrt{\epsilon} n \\ &\geq \left( \frac{1}{2} - 3\sqrt{\delta} \right) n. \end{aligned}$$

Furthermore, $|\mathcal{C}_2' \setminus \mathcal{C}_2^1| \geq (1 - 3\sqrt{\delta})|\mathcal{C}_2|$.

Notice that $x_{i_0} \notin X'$. Then, for at least $(1 - 3\sqrt{\delta})|\mathcal{C}_2|$ colours $j \in \mathcal{C}_2$, $d_j^-(x_{i_0}, B_j) < \frac{5}{2}\sqrt{\delta} n$, and consequently, $d_j^-(x_{i_0}, A_j) \geq (\frac{1}{2} - 3\sqrt{\delta})n$ for the same set of colours. By **(C4)**, we know that $x_{i_0} \in A_j \cup B_j$ for all but at most $6\sqrt{\delta}|\mathcal{C}_2|$ colours $j \in \mathcal{C}_2$, which leads to $x_{i_0} \notin X$, a contradiction. □

Next, we give an application of the transversal blow-up lemma (see [10]) for embedding transversal undirected Hamilton paths inside very dense bipartite graph collections, which can be proved by minor modifications to the proof of [9, Lemma 6.1], and we omit the proof here. We say that $P$ is an undirected subgraph of a digraph collection $\mathcal{D} = \{D_1, \ldots, D_n\}$ if, for every edge $uv \in E(P) \cap E(D_i)$, the reverse arc $vu \in E(D_i)$.

**Lemma 6.2.** *Suppose † holds and $\{W, Z\} = \{A, B\}$. Let $W^* \subseteq W \setminus V_{bad}$ and $Z^* \subseteq Z \setminus V_{bad}$, where $|W^*|, |Z^*| \geq \eta n$, $W^* \cap Z^* = \emptyset$ and $|W^*| - |Z^*| = t \in \{0, 1\}$. Let $T^* = Z^*$ if $t = 0$ and $T^* = W^*$ if $t = 1$. Let $\mathcal{C}^* \subseteq \mathcal{C}$ satisfy $|\mathcal{C}^*| = |W^*| + |Z^*| - 1$, where $\mathcal{C}^* \subseteq \mathcal{C}_1$ if $W = Z$ and $\mathcal{C}^* \subseteq \mathcal{C}_2$ if $W \neq Z$. Let $W^- \subseteq W^*$ and $T^+ \subseteq T^*$ with $|W^-|, |T^+| \geq \frac{\eta n}{8}$. Then there is a transversal undirected Hamilton path in $\{D_i^{\pm}[W^*, Z^*] : i \in \mathcal{C}^*\}$ starting at $W^-$ and ending at $T^+$.*

The subsequent result can be used to connect two disjoint short rainbow paths into a single short rainbow path.

**Lemma 6.3** (Connecting tool). *Suppose † holds and $\{Y, Z\} = \{A, B\}$. Assume $P = u_1 u_2 \ldots u_s$ and $Q = v_1 v_2 \ldots v_t$ are two disjoint rainbow directed paths inside $\mathcal{D}$ with $u_s, v_1 \notin V_{bad} \cup X'$ and $s + t \leq 5\eta n$.*

(i) *If $u_s \in Y$, $v_1 \in Z$ and $|\mathcal{C}_2 \setminus col(P \cup Q)| \geq 14\sqrt{\delta} n$, then there are three colours $c_1, c_2, c_3 \in \mathcal{C}_2 \setminus col(P \cup Q)$ and two vertices $w_1 \in Z \setminus (V(P \cup Q) \cup V_{bad})$, $w_1' \in Y \setminus (V(P \cup Q) \cup V_{bad})$ such that $u_1 P u_s w_1 w_1' v_1 Q v_t$ is a rainbow directed path with colours $col(P \cup Q) \cup \{c_1, c_2, c_3\}$.*

(ii) *If $u_s, v_1 \in Y$ and $|\mathcal{C}_2 \setminus col(P \cup Q)| \geq 14\sqrt{\delta} n$, then there are two colours $c_1, c_2 \in \mathcal{C}_2 \setminus col(P \cup Q)$ and a vertex $w_1 \in Z \setminus (V(P \cup Q) \cup V_{bad})$ such that $u_1 P u_s w_1 v_1 Q v_t$ is a rainbow directed path with colours $col(P \cup Q) \cup \{c_1, c_2\}$.*

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(iii) *If $u_s, v_1 \in Y$ and $|\mathcal{C}_1 \setminus col(P \cup Q)| \geq 14\sqrt{\delta}n$, then there are two colours $c_1, c_2 \in \mathcal{C}_1 \setminus col(P \cup Q)$ and a vertex $w_1 \in Y \setminus (V(P \cup Q) \cup V_{bad})$ such that $u_1 P u_s w_1 v_1 Q v_t$ is a rainbow directed path with colours $col(P \cup Q) \cup \{c_1, c_2\}$.*

**Proof.** We only give the proof of (i), the other two statements can be proved by similar discussions, whose procedures are omitted.

Note that $u_s, v_1 \notin V_{\text{bad}} \cup X'$. Then $u_s$ (resp. $v_1$) lies in $Y_i$ (resp. $Z_i$) for at least $(1 - 13\sqrt{\delta})|\hat{\mathcal{C}}|$ colours $i \in \hat{\mathcal{C}}$. Since $|\mathcal{C}_2 \setminus \text{col}(P \cup Q)| \geq 14\sqrt{\delta}n$, we obtain that $u_s$ (resp. $v_1$) lies in $Y_i$ (resp. $Z_i$) for at least $\sqrt{\delta}n$ colours $i \in \mathcal{C}_2 \setminus \text{col}(P \cup Q)$. Hence there are two distinct colours $c_1, c_2 \in \mathcal{C}_2 \setminus \text{col}(P \cup Q)$ such that $u_s \in Y_{c_1}$ and $v_1 \in Z_{c_2}$. It is routine to check that

$$\begin{aligned}|N^+_{c_1}(u_s) \cap (Z \setminus (V_{\text{bad}} \cup X'))| \geq& |N^+_{c_1}(u_s) \cap Z_{c_1}| - |Z_1 \triangle Z_{c_1}| - |X \cup X_A \cup X_B| \\ \geq& \left(\frac{1}{2} - 3\sqrt{\delta}\right) n - \left(2\delta + \sqrt{\epsilon} + \frac{1}{2}\sqrt{\delta}\right) n \geq \left(\frac{1}{2} - 4\sqrt{\delta}\right) n.\end{aligned}$$

Hence there exists a vertex $w_1 \in N^+_{c_1}(u_s) \cap Z$ that avoids $V(P \cup Q) \cup V_{\text{bad}} \cup X'$. Similarly, since $w_1 \notin V_{\text{bad}} \cup X'$, there is a colour $c_3 \in \mathcal{C}_2 \setminus (\text{col}(P \cup Q) \cup \{c_1, c_2\})$ such that $w_1 \in B_{c_3}$ and

$$|N^-_{c_2}(v_1) \cap N^+_{c_3}(w_1) \cap (Y \setminus (V_{\text{bad}} \cup X'))| \geq \left(\frac{1}{2} - 10\sqrt{\delta}\right) n.$$

Therefore, there is a vertex $w'_1 \in N^-_{c_2}(v_1) \cap N^+_{c_3}(w_1) \cap A$ that avoids $V(P \cup Q) \cup V_{\text{bad}} \cup X'$. It follows that $u_1 P u_s w_1 w'_1 v_1 Q v_t$ is a rainbow directed path inside $\mathcal{D}$ with colours $\text{col}(P \cup Q) \cup \{c_1, c_2, c_3\}$. □

To prove Theorem 5.2, we need the following lemma. This establishes that every digraph collection $\mathcal{D}$ contains a transversal directed Hamilton cycle if it admits an almost balanced partition, with the small parts inducing an empty graph for almost all colours. We first state a preliminary result from [11].

**Lemma 6.4** ([11]). *Let $\mathcal{C}$ be a set of colours, and $\mathcal{G} = \{G_i[Y, B] : i \in \mathcal{C}\}$ be a collection of bipartite graphs with the common bipartition $Y \cup B$ such that $7|Y| < |B| \leq \frac{3}{5}|\mathcal{C}|$. If $\sum_{i \in \mathcal{C}} |E(D_i[Y, B])| \geq t|B||\mathcal{C}|$ for some integer $t$ with $1 \leq t \leq |Y|$, then $\mathcal{G}$ contains $t$ disjoint rainbow stars on 5 vertices, and each of them has its centre in $Y$ and other vertices in $B$.*

**Lemma 6.5.** *Assume $0 < \frac{1}{n} \ll \delta \ll 1$ and $0 \leq \gamma \leq 3\delta$. Let $\mathcal{C}$ be a set of $n$ colours, and let $\mathcal{D} = \{D_i : i \in \mathcal{C}\}$ be a collection of digraphs with the common vertex set $V$ of size $n$ such that $\delta^0(\mathcal{D}) \geq \lceil \frac{n}{2} \rceil$. Let $A \cup B$ be a partition of $V$ with $|A| = \lceil \frac{n}{2} \rceil + \gamma n$, and let $\mathcal{C}' \cup \mathcal{C}''$ be a partition of $\mathcal{C}$ with $|\mathcal{C}''| \leq \delta n$. Assume that $D_i[B] = \emptyset$ for all $i \in \mathcal{C}'$. Then $\mathcal{D}$ contains a transversal directed Hamilton cycle.*

**Proof.** Define $Y_1 := \{v \in A : d^+_i(v, B) \leq (1 - \delta^{\frac{1}{4}})|B|$ for at least $\delta^{\frac{1}{4}}|\mathcal{C}'|$ colors $i \in \mathcal{C}'\}$, $Y_2 := \{v \in A : d^-_i(v, B) \leq (1 - \delta^{\frac{1}{4}})|B|$ for at least $\delta^{\frac{1}{4}}|\mathcal{C}'|$ colors $i \in \mathcal{C}'\}$. Since $D_i[B] = \emptyset$ for all $i \in \mathcal{C}'$, it is routine to check that

$$\begin{aligned}\left\lceil \frac{n}{2} \right\rceil |B||\mathcal{C}'| \leq& \sum_{i \in \mathcal{C}'} |E(D_i[A, B])| \\ \leq& |Y_1|(1 - \delta^{\frac{1}{4}})|B|\delta^{\frac{1}{4}}|\mathcal{C}'| + |Y_1||B|(1 - \delta^{\frac{1}{4}})|\mathcal{C}'| + (|A| - |Y_1|)|B||\mathcal{C}'| \\ =& (|A| - \delta^{\frac{1}{2}}|Y_1|)|B||\mathcal{C}'|.\end{aligned}$$

It implies that $|Y_1| \leq \frac{\gamma n}{\sqrt{\delta}} \leq 3\sqrt{\delta}n$. Similarly, $|Y_2| \leq \frac{\gamma n}{\sqrt{\delta}} \leq 3\sqrt{\delta}n$. For a vertex $v \in Y_1 \setminus Y_2$ (resp. $v \in Y_2 \setminus Y_1$), we have $d^-_i(v, B) > (1 - \delta^{\frac{1}{4}})|B|$ (resp. $d^+_i(v, B) > (1 - \delta^{\frac{1}{4}})|B|$) for at least $(1 - \delta^{\frac{1}{4}})|\mathcal{C}'|$

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colours $i \in \mathcal{C}'$. Notice that Lemma 6.2 and Lemma 6.3 (i)–(ii) hold by setting $V_{\mathrm{bad}} := Y_1 \cup Y_2$ and $\mathcal{C}_2 = \mathcal{C}'$.

Since $\delta^0(\mathcal{D}) \geq \lceil \frac{n}{2} \rceil$, we have $|E(D_i[A, B])|, |E(D_i[B, A])| \geq |B| \lceil \frac{n}{2} \rceil$ for each $i \in \mathcal{C}'$, i.e., there are at most $|B|(\lceil \frac{n}{2} \rceil + \gamma n) - |B| \lceil \frac{n}{2} \rceil = |B|\gamma n$ non-edges from $A$ to $B$ (resp., from $B$ to $A$) in $D_i$. Hence

$$\sum_{i \in \mathcal{C}'} |E(D_i[Y_1, B])| \geq (|Y_1| - \gamma n)|B||\mathcal{C}'| \text{ and } \sum_{i \in \mathcal{C}'} |E(D_i[B, Y_2])| \geq (|Y_2| - \gamma n)|B||\mathcal{C}'|.$$

According to the sizes of $|Y_1|$ and $|Y_2|$, we proceed by considering the following three cases.

**Case 1.** $|Y_1|, |Y_2| \geq \gamma n$**.**

In view of Lemma 6.4, there exists a subset $Y_1' \subseteq Y_1$ with size $\gamma n$ such that vertices in $Y_1 \setminus Y_1'$ can be covered by disjoint rainbow $P_3$ copies inside $\{D_i[Y_1, B]{:}i \in \mathcal{C}'\}$ with centres in $Y_1 \setminus Y_1'$; and a subset $Y_2' \subseteq Y_2$ with size $\gamma n$ such that vertices in $Y_2 \setminus Y_2'$ can be covered by disjoint rainbow $P_3$ copies inside $\{D_i[B, Y_2]{:}i \in \mathcal{C}'\}$ with centres in $Y_2 \setminus Y_2'$. Recall that each vertex in $Y_1$ (resp. $Y_2$) has at least $\lceil \frac{n}{2} \rceil - (1 - \delta^{\frac{1}{4}})|B| > 4|Y|$ out-neighbours (resp. in-neighbours) in $A$ for at least $\delta^{\frac{1}{4}}|\mathcal{C}'| > \delta^{\frac{1}{4}}(1 - \delta)n > 4|Y|$ digraphs $D_i$ with $i \in \mathcal{C}'$. Therefore, by using colours in $\mathcal{C}'$,

- vertices in $(Y_1 \cup Y_2) \setminus (Y_1' \cup Y_2')$ can be covered by disjoint rainbow directed $P_3$ copies with centres in $(Y_1 \cup Y_2) \setminus (Y_1' \cup Y_2')$ and endpoints in $B$,
- vertices in $Y_1' \setminus Y_2'$ can be covered by disjoint rainbow directed $P_3$ copies starting at $B$ and ending at $A$ with centres in $Y_1' \setminus Y_2'$,
- vertices in $Y_2' \setminus Y_1'$ can be covered by disjoint rainbow directed $P_3$ copies starting at $A$ and ending at $B$ with centres in $Y_2' \setminus Y_1'$,
- vertices in $Y_1' \cap Y_2'$ can be covered by disjoint rainbow directed $P_3$ copies with centres in $Y_1' \cap Y_2'$ and endpoints in $A$.

For each $y \in Y_1 \cup Y_2$, let $P_y := y^1 y y^2$ denote the rainbow directed $P_3$ with centre $y$, where $P_y$ has colours $c_y^1$ and $c_y^2$. Let $\mathbf{P} = \{P_y{:}y \in Y_1 \cup Y_2\}$. By using Lemma 6.3, one may connect all of the above rainbow directed $P_3$ copies into a single rainbow directed path $P^1$ that starts at $A$ and ends at $B$. Then

$$|V(P^1) \cap A| - |V(P^1) \cap B| = |Y_1' \setminus Y_2'| + |Y_2' \setminus Y_1'| + 2|Y_1' \cap Y_2'| = |Y_1'| + |Y_2'| = 2\gamma n.$$

Hence, $|A \setminus V(P^1)| = |B \setminus V(P^1)| + \sigma$, where $\sigma = 0$ if $n$ is odd and $\sigma = 1$ otherwise.

It is routine to check that there exists a rainbow matching inside $\{D_i^{\pm}[A \setminus V(P^1), B \setminus V(P^1)]{:}i \in \mathcal{C}''\}$, say $M$, such that $D_j[A \setminus V(P^1 \cup M), B \setminus V(P^1 \cup M)] \neq \emptyset$ for each $j \in \mathrm{col}(M)$. Furthermore, $D_j$ is $(19\sqrt{\delta}, \mathrm{EC1})$-extremal for all $j \in \mathcal{C}'' \setminus \mathrm{col}(M)$.

**Subcase 1.1.** $n$ and $|\mathcal{C}'' \setminus \mathrm{col}(M)|$ have the same parity.

We greedily choose two disjoint rainbow directed paths $P_A$ and $P_B$ inside $\{D_i[A \setminus V(P^1 \cup M)] \cup D_i[B \setminus V(P^1 \cup M)]{:}i \in \mathcal{C}'' \setminus \mathrm{col}(M)\}$ with lengths $\lceil \frac{|\mathcal{C}'' \setminus \mathrm{col}(M)|}{2} \rceil$ and $\lfloor \frac{|\mathcal{C}'' \setminus \mathrm{col}(M)|}{2} \rfloor$ respectively, such that $V(P_A) \subseteq A$ and $V(P_B) \subseteq B$. Applying Lemma 6.3 (i), we successively connect $P^1$, all rainbow edges in $M$, $P_A$ and $P_B$ into a single rainbow path $P^2$ whose endpoints lie in different parts. Clearly, $|A \setminus V(P^2)| = |B \setminus V(P^2)|$. Together with Lemma 6.2, we know that $\mathcal{D}$ contains a transversal directed Hamilton cycle, as desired.

**Subcase 1.2.** $n$ and $|\mathcal{C}'' \setminus \mathrm{col}(M)|$ have different parity.

Notice that $|B| = \lfloor \frac{n}{2} \rfloor - \gamma n$. Hence for each $D_i$ with $i \in \mathcal{C}$, each vertex in $B$ has at least $\gamma n + 1$ in-neighbours in $A$. Therefore, each vertex in $B$ has at least one in-neighbour outside $Y_1'$.

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We first assume $|\mathcal{C}'' \setminus \mathrm{col}(M)| \geq 1$. Choose an unused vertex $y \in B$ and a colour $c \in \mathcal{C}'' \setminus \mathrm{col}(M)$. Assume that $xy \in E(D_c[A, B])$ with $x \notin Y_2'$. If $x \in A \setminus (Y_1 \cup Y_2)$, then $x$ can be chosen outside $V(P^1 \cup M)$. Using Lemma 6.3 (i)-(ii), we connect $xy$ and $P^1$ into a single rainbow directed path $P$. If $x \in (Y_1 \cup Y_2) \setminus Y_1'$, then we replace the end vertex of $P_x$ by $y$. In either situation the problem reduces to Subcase 1.1, yielding a transversal directed Hamilton cycle in $\mathcal{D}$, as desired.

Now, we consider that $n$ is odd and $|\mathcal{C}'' \setminus \mathrm{col}(M)| = 0$. For each $D_i$ with $i \in \mathcal{C}$, each vertex in $A$ has at least one in-neighbour outside $Y_1'$. Choose an unused vertex $y \in A \setminus (Y_1 \cup Y_2)$ and a colour $c \in \mathcal{C}$. Then there exists a vertex $x \in A \setminus Y_1'$ such that $xy \in E(D_c[A])$. By a similar discussion as above, we obtain a transversal directed Hamilton cycle inside $\mathcal{D}$, as desired.

**Case 2.** At least one of $Y_1$ and $Y_2$ has size less than $\gamma n$.

In this case, we consider only $|Y_1| \geq \gamma n$ and $|Y_2| < \gamma n$; the remaining two cases are analogous. Following the argument of Case 1, we know that all vertices in $Y_1$ can be covered by a set of disjoint rainbow directed copies of $P_3$ with centres in $Y_1$ and endpoints in $A \setminus (Y_1 \cup Y_2)$. Furthermore, each vertex in $Y_2 \setminus Y_1$ can be covered by a rainbow directed $P_3$ with its centre in $Y_2 \setminus Y_1$, starting from $A \setminus (Y_1 \cup Y_2)$ and using colours in $\mathcal{C}'$. (In this process, for the choice of the other endpoint of each rainbow directed $P_3$, we prefer using vertices in $A$ over vertices in $B$.) By using unused colours in $\mathcal{C}$ and unused vertices in $A \setminus Y$, one may extend those rainbow directed $P_3$ as long as possible or choose all other disjoint rainbow directed paths. Let $\mathbf{P} = \{Q_1, Q_2, \ldots, Q_t\}$ be a set consisting of all disjoint rainbow directed paths in the above, where each $Q_i$ has at least one endpoint in $A \setminus Y$ and length $s_i$ $(1 \leq i \leq t)$ inside $A$. We next consider the following two possible cases.

**Subcase 2.1.** $|B| \geq |A| - (s_1 + \cdots + s_t) - \sigma$.

In this subcase, there exists a set $\mathbf{P}' = \{Q_1', Q_2', \ldots, Q_\ell'\}$ such that $|B| = |A| - (s_1' + \cdots + s_\ell') - \sigma$, the endpoints of each $Q_i'$ lie outside $Y$ and $Y_2 \cup Y_1' \subseteq V(Q_1' \cup \ldots \cup Q_\ell')$, where $s_i' = |E(Q_i'[A])|$ for each $i \in [\ell]$. By Lemma 6.3 (i)–(ii), we connect all rainbow directed paths in $\mathbf{P}'$ into a single rainbow directed path $P^1$, whose endpoints are in different parts. Since $|A| - |B| = 2\gamma n + \sigma$, we have $\ell \leq s_1' + \cdots + s_\ell' = 2\gamma n$. Therefore, $|E(P^1)| \leq 8\gamma n$ and $|A \setminus V(P^1)| - \sigma = |B \setminus V(P^1)|$.

Recall that all vertices in $Y_1 \setminus V(\mathbf{P}')$ can also be covered by a set of disjoint rainbow directed copies of $P_3$ with centres in $Y_1 \setminus V(\mathbf{P}')$ and both endpoints in $B$. Applying Lemma 6.3 (i) again to connect those rainbow directed paths with $P^1$, we get a rainbow directed path $P^2$ with length at most $8\gamma n + 12\sqrt{\delta}n$, and with endpoints in different parts.

Choose a maximum rainbow matching inside $\{D_i^{\pm}[A \setminus V(P^2), B \setminus V(P^2)]: i \in \mathcal{C}'' \setminus \mathrm{col}(P^2)\}$, say $M$, such that $D_j^{\pm}[A \setminus V(P^2 \cup M), B \setminus V(P^2 \cup M)] \neq \emptyset$ for each $j \in \mathrm{col}(M)$. Clearly, $D_j$ is $(13\sqrt{\delta}, \mathrm{EC1})$-extremal for all $j \in \mathcal{C}'' \setminus \mathrm{col}(P^1 \cup M)$. In fact, when constructing $P^1 - B - Y$, we may prioritise the use of colours from $\mathcal{C}'' \setminus \mathrm{col}(P^1 \cup M)$ over other colours. Hence either $\mathrm{col}(P^1 - B - Y) \subseteq \mathcal{C}'' \setminus \mathrm{col}(M)$ or $\mathcal{C}'' \setminus \mathrm{col}(M) \subseteq \mathrm{col}(P^1 - B - Y)$.

Based on Lemma 6.3 (i), one may connect $P^1$ and all edges of $M$ into a single rainbow directed path $P^2$, whose endpoints are in different parts and length is at most $8\gamma n + 4\delta n$. Next, we are to choose two disjoint rainbow paths $P_A$ and $P_B$ inside $\{D_i[A \setminus V(P^1 \cup M)] \cup D_i[B \setminus V(P^1 \cup M)]: i \in \mathcal{C}'' \setminus \mathrm{col}(P^1 \cup M)\}$ such that $V(P_A) \subseteq A$ and $V(P_B) \subseteq B$ respectively, whose lengths are determined by the parity of $|\mathcal{C}'' \setminus \mathrm{col}(P^1 \cup M)|$. Clearly, if $n$ and $|\mathcal{C}'' \setminus \mathrm{col}(P^1 \cup M)|$ have the same parity, then by Lemma 6.2, we know that $\mathcal{D}$ contains a transversal directed Hamilton cycle, as desired. Hence $n$ and $|\mathcal{C}'' \setminus \mathrm{col}(P^1 \cup M)|$ have different parity. By an argument analogous to that in the proof of Step 4 in Case 1, we can obtain the desired transversal Hamilton cycle in $\mathcal{D}$.

**Subcase 2.2.** $|B| < |A| - (s_1 + \cdots + s_t) - \sigma$.

In this case, $\sum_{i=1}^{t} s_i < 2\gamma n$. Let $w$ be an arbitrary vertex in $A \setminus V(\mathbf{P})$ and $c_1, c_2$ be two colours in $\mathcal{C} \setminus \mathrm{col}(\mathbf{P})$. Clearly, in $D_{c_1}$ and $D_{c_2}$, $w$ has at least $\lceil \frac{n}{2} \rceil - (\lfloor \frac{n}{2} \rfloor - \gamma n) = \gamma n + \sigma$ out-neighbours (resp. in-neighbours) in $V(\mathbf{P})$ and it cannot be adjacent to the start vertex of $Q_i[A]$ (resp. from the

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end vertex of $Q_i[A]$) for all $i \in [t]$ (by the maximality of **P**). For convenience, assume that $s_0 = 0$ and $V(Q_i) \cap A := \{v_{s_1+\cdots+s_{i-1}+i}, \ldots, v_{s_1+\cdots+s_i+i}\}$ for each $i \in [t]$. Define

$$I_1 := \left\{i : v_i \in N^-_{c_1}(w)\right\} \text{ and } I_2 := \left\{i : v_{i+1} \in N^+_{c_2}(w)\right\}.$$

Hence $I_1, I_2 \subseteq \bigcup_{i\in[t]} \left[\sum_{j\in[i-1]} s_j + i, \sum_{j\in[i]} s_j + i - 1\right]$. Therefore, $I_1 \cap I_2 \neq \emptyset$, which contradicts the maximality of **P**. □

Now, we are ready to give the proof of Theorem 5.2.

**Proof of Theorem** 5.2. Choose constant $\epsilon, \delta, \eta$ such that

$$\frac{1}{n} \ll \epsilon \leq \delta^2 \ll \eta \ll 1,$$

let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a collection of digraphs on a common vertex set $V$ of size $n$ and $\delta^0(\mathcal{D}) \geq \frac{n}{2}$. Suppose that $\mathcal{D}$ does not contain transversal directed Hamilton cycles. By Lemma 5.1, for each $i \in [m]$, $D_i$ has a characteristic partition $(A_i, B_i, L_i)$ satisfying **(C1)**–**(C4)**. Moreover, for every $i \in [m]$ and $Y \in \{A, B\}$, we have

$$|L_i| \leq 2\epsilon n \text{ and } |Y_1 \triangle Y_i| < 2\delta n. \tag{2}$$

Expand $A_1 \cup B_1$ to an equitable partition $A \cup B$ of $V$. For simplicity, we assume $Y \in \{A, B\}$ and $Z \in \{A, B\} \setminus \{Y\}$ in the following. Let $\mathcal{C}_1 \cup \mathcal{C}_2$ be a partition of $[m]$ with

$$\mathcal{C}_1 = \{i \in [m] : D_i \text{ is } (\epsilon, \text{EC1})\text{-extremal}\},$$
$$\mathcal{C}_2 = \{i \in [m] : D_i \text{ is } (\epsilon, \text{EC2})\text{-extremal}\}.$$

Let $\hat{\mathcal{C}} := \bigcup_{k\in[2]} \psi(\mathcal{C}_k)$, where $\psi(\mathcal{C}_k) = \mathcal{C}_k$ if $|\mathcal{C}_k| \geq \eta n$ and $\psi(\mathcal{C}_k) = \emptyset$ otherwise. Clearly, $|\hat{\mathcal{C}}| \geq (1 - 4\sqrt{\delta} - \eta)n$. Define

$$X := \{x \in V : x \notin A_i \cup B_i \text{ for at least } 6\sqrt{\delta}|\hat{\mathcal{C}}| \text{ colors } i \in \hat{\mathcal{C}}\},$$
$$X_Y := \{x \in Y \setminus X : x \notin Y_i \text{ for at least } 10\sqrt{\delta}|\hat{\mathcal{C}}| \text{ colors } i \in \hat{\mathcal{C}}\}.$$

By (2), one has $6\sqrt{\delta}|\hat{\mathcal{C}}||X| \leq 2\epsilon n|\hat{\mathcal{C}}|$ and

$$10\sqrt{\delta}|\hat{\mathcal{C}}||X_Y| \leq \sum_{i\in\hat{\mathcal{C}}} |Y \setminus Y_i| \leq \sum_{i\in\hat{\mathcal{C}}} (|Y \setminus Y_1| + |Y_1 \triangle Y_i|) \leq (2\delta + 2\epsilon)n|\hat{\mathcal{C}}|.$$

It follows that $|X| < \frac{1}{3}\sqrt{\epsilon}n$ and $|X_Y| < \frac{1}{4}\sqrt{\delta}n$. For $k \in [2]$, define

$$X^k_Y := \begin{cases} \{x \in X_Y : x \in Y_i \text{ for at least } 3\sqrt{\delta}n \text{ colors } i \in \mathcal{C}_k\}, & \text{if } |\mathcal{C}_k| \geq \eta n, \\ \emptyset, & \text{otherwise.} \end{cases}$$

For a vertex $x \in X_Y \setminus (\bigcup_{k\in[2]} X^k_Y)$, we know $x \in Y_i$ for at most $6\sqrt{\delta}n$ colours $i \in \hat{\mathcal{C}}$ and $x \in A_i \cup B_i$ for at least $(1 - 6\sqrt{\delta})|\hat{\mathcal{C}}|$ colours $i \in \hat{\mathcal{C}}$. Thus, $x \in Y \setminus Y_i$ for at least $|\hat{\mathcal{C}}| - 6\sqrt{\delta}n$ colours $i \in \hat{\mathcal{C}}$ and $x \in Z_i$ for at least $(1 - 6\sqrt{\delta})|\hat{\mathcal{C}}| - 6\sqrt{\delta}n \geq (1 - 13\sqrt{\delta})|\hat{\mathcal{C}}|$ colours $i \in \hat{\mathcal{C}}$. Hence

$$\left|X_Y \setminus \left(\bigcup_{k\in[2]} X^k_Y\right)\right| \left(|\hat{\mathcal{C}}| - 6\sqrt{\delta}n\right) \leq \sum_{i\in\hat{\mathcal{C}}} |Y \setminus Y_i| \leq (2\delta + 2\epsilon)n|\hat{\mathcal{C}}|.$$

It follows that $|X_Y \setminus (\bigcup_{k\in[2]} X^k_Y)| < 4\delta n$. Then we move vertices in $X_A \setminus (\bigcup_{k\in[2]} X^k_A)$ to $B$ and vertices in $X_B \setminus (\bigcup_{k\in[2]} X^k_B)$ to $A$.

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Let $X'$ be a subset of $X$ consisting of vertices $x$ such that $d_i^+(x, Y_i) \geq (1-3\sqrt{\delta})n$ (resp. $d_i^-(x, Y_i) \geq (1-3\sqrt{\delta})n$) for at least $(1-3\sqrt{\delta})|\hat{\mathcal{C}}|$ colours $i \in \hat{\mathcal{C}}$ and $d_i^-(x, Z_i) \geq \frac{5}{2}\sqrt{\delta}n$ (resp. $d_i^+(x, Z_i) \geq \frac{5}{2}\sqrt{\delta}n$) for at least $3\sqrt{\delta}|\hat{\mathcal{C}}|$ colours $i \in \hat{\mathcal{C}}$. Delete vertices in $X'$ from $A \cup B$. Without loss of generality, assume that $|B| - |A| = r$ with $0 \leq r \leq 8\delta n + \sqrt{\epsilon}n$. Define $V_{\text{bad}} := \bigcup_{k\in[2]} (X_A^k \cup X_B^k) \cup (X \setminus X')$. Obviously, $|V_{\text{bad}}| < (\sqrt{\epsilon} + \frac{1}{2}\sqrt{\delta})n$. Moreover, each vertex in $Y \setminus V_{\text{bad}}$ lies in $Y_i$ for at least $(1-13\sqrt{\delta})|\hat{\mathcal{C}}|$ colours $i \in \hat{\mathcal{C}}$.

Next, based on the sizes of $|\mathcal{C}_1|$ and $|\mathcal{C}_2|$, we distinguish our proof into the following three cases.

**Case 1.** $|\mathcal{C}_1| < \eta n$.

In this case, each vertex in $Y \setminus (X \cup X_Y^2)$ belongs to $Y_i$ for at least $(1-13\sqrt{\delta})|\mathcal{C}_2|$ colours $i \in \mathcal{C}_2$. Recall that $|B| - |A| = r$ with $0 \leq r \leq (8\delta + \sqrt{\epsilon})n$. The proof is divided into four steps.

**Step 1. Balance the number of vertices in $A$ and $B$.**

In this step, we claim that by moving vertices in $B \cap V_{\text{bad}}$ or deleting rainbow directed paths inside $\mathcal{D}[B]$, we can make the number of remaining vertices in $B$ and $A$ differ by $\sigma$, where $\sigma = 1$ if $|A| + |B|$ is odd and $\sigma = 0$ otherwise. Denote $s := |V_{\text{bad}} \cap B|$. If $|B| - |A| - \sigma \leq 2s$, then move $\frac{|B|-|A|-\sigma}{2}$ vertices in $V_{\text{bad}} \cap B$ to $A$, and our desired result holds. Hence, it suffices to consider $|B| - |A| - \sigma > 2s$. Now, we move all vertices in $V_{\text{bad}} \cap B$ to $A$. Hence, $B \cap V_{\text{bad}} = \emptyset$. Assume that $\{Q_1, \ldots, Q_t\}$ is a set of disjoint maximal rainbow directed paths in $\mathcal{D}[B]$.

On the one hand, suppose that $|E(Q_1)| + \cdots + |E(Q_t)| \geq |B| - |A| - \sigma$. Then there must exist a set of disjoint rainbow directed paths, say $\{Q_1', \ldots, Q_{t'}'\}$, such that $|E(Q_1')| + \cdots + |E(Q_{t'}')| = |B| - |A| - \sigma$. That is, $|A| - t' = |B| - (|V(Q_1')| + \cdots + |V(Q_{t'}')|) - \sigma$. Using Lemma 6.3 (ii), we connect $Q_1', \ldots, Q_{t'}'$ into a single rainbow directed path $P^1$ with length at most $27\delta n$, starting at $A$ and ending at $B$. (In fact, by using Lemma 6.3 (ii), we only get a rainbow path with two endpoints in $B$. But we usually want to find a rainbow directed path with endpoints in different parts. Hence one may extend it by using an unused colour of $\mathcal{C}_2$ and an unused vertex in $A$). Therefore, $|A \setminus V(P^1)| = |B \setminus V(P^1)| - \sigma$, as desired.

On the other hand, suppose that $|E(Q_1)| + \cdots + |E(Q_t)| < |B| - |A| - \sigma = r - 2s - \sigma$. In this case, we move all vertices of $Q_1, \ldots, Q_t$ to $A$. Let $\mathcal{C}' := \mathcal{C} \setminus \text{col}(\bigcup_{i\in[t]} Q_i)$. Hence

$$|A| \leq \frac{n - |X'| - r}{2} + s + 2(r - 2s - \sigma) \leq \frac{n}{2} + 18\delta n,$$

$$|B| \geq \frac{n - |X'| + r}{2} - s - 2(r - 2s - \sigma) \geq \frac{n}{2} - 18\delta n,$$

$$|\mathcal{C}'| = |\mathcal{C}| - (|E(Q_1)| + \cdots + |E(Q_t)|) \geq (1 - 9\delta)n.$$

By the maximality of $\{Q_1, \ldots, Q_t\}$, one has $D_i[B] = \emptyset$ for all $i \in \mathcal{C}'$. It follows from $\delta^0(\mathcal{D}) \geq \lceil \frac{n}{2} \rceil$ that $\lceil \frac{n}{2} \rceil \leq |A| + |X'| \leq \frac{n}{2} + 19\delta n$. Now, by Lemma 6.5, we obtain that $\mathcal{D}$ contains a transversal directed Hamilton cycle, a contradiction.

In this step, if we only move vertices in $V_{\text{bad}}$ from $B$ to $A$, then let $P^1$ be a null digraph (i.e., no vertices); otherwise, assume that $P^1$ is a rainbow directed path obtained in the above, whose length is at most $27\delta n$ and endpoints are $u_1 \in A \setminus V_{\text{bad}}$, $v_1 \in B \setminus V_{\text{bad}}$. Hence $|A \setminus V(P^1)| = |B \setminus V(P^1)| - \sigma$.

**Step 2. Construct a sequence of disjoint rainbow directed $P_3$ copies such that all centres of them are exactly all vertices in $V_{\text{bad}} \cup X'$.**

Recall that $|X_A^2 \cup X_B^2| \leq |X_A \cup X_B| < \frac{1}{2}\sqrt{\delta}n$. Let $v \in X_Y^2$. If $v \in Z$, then $v$ is moved from $B$ to $A$ in Step 1, so $v \in X_B$. Therefore, $v \in A_i \cup B_i$ for at least $(1-6\sqrt{\delta})|\mathcal{C}_2|$ colours $i \in \mathcal{C}_2$ and $v \notin B_i$ for at least $10\sqrt{\delta}|\mathcal{C}_2|$ colours $i \in \mathcal{C}_2$. Hence, $v \in A_i$ for at least $4\sqrt{\delta}|\mathcal{C}_2| - 27\delta n > 4|V_{\text{bad}} \cup X'|$

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colours $i \in \mathcal{C}_2 \setminus \mathrm{col}(P^1)$. If $v \in Y$, then $v$ belongs to $Y_i$ for at least $(3\sqrt{\delta} - 27\delta)n > 4|V_{\mathrm{bad}} \cup X'|$ colours $i \in \mathcal{C}_2 \setminus \mathrm{col}(P^1)$. Hence, using colours in $\mathcal{C}_2 \setminus \mathrm{col}(P^1)$, we can greedily choose $|X_A^2 \cup X_B^2|$ disjoint rainbow directed $P_3$ copies, such that their centres correspond to all vertices in $X_A^2 \cup X_B^2$. Moreover, if $v \in (X_A^2 \cup X_B^2) \cap Y$, then the rainbow directed $P_3$ with centre $v$ has endpoints in $Z \setminus (V_{\mathrm{bad}} \cup X' \cup V(P^1))$.

For vertices in $X'$, we know from its definition that by avoiding vertices and colours in $P^1$ and all rainbow directed paths chosen for vertices in $X_A^2 \cup X_B^2$, there exists a set of disjoint rainbow directed $P_3$ copies with centres in $X'$ and endpoints in different parts.

By using Lemma 6.3 (i)–(ii), we connect $P^1$, all rainbow directed $P_3$ copies with centres in $X_A^2 \cup X_B^2 \cup X'$ (as obtained above) into a single rainbow directed path. This results in a rainbow directed path $P^2 = u_1 \dots v_2$ with $v_2 \in B \setminus V_{\mathrm{bad}}$ and length at most

$$|E(P^1)| + 4|X_A^2 \cup X_B^2| + 5|X'| + 6 \le 27\delta n + 2\sqrt{\delta}n + 5\sqrt{\epsilon}n + 6.$$

For vertices in $X \setminus X'$, we consider the following claim. Notice that the position (in $A$ or $B$) of vertices in $X \setminus X'$ does not affect our proof, so if $v \in X$ is a vertex moved from $B$ to $A$, then we only need consider $v \in A$. By Lemma 6.1, for each $x_i \in (X \setminus X') \cap Y$, there exists a rainbow directed path $Q_{x_i} = x_i^1 x_i x_i^2$ with $\mathrm{col}(Q_{x_i}) = \{c_i^1, c_i^2\} \subseteq \mathcal{C}_2$ and $x_i^1, x_i^2 \in Z \setminus V_{\mathrm{bad}}$. Moreover, each colour or endpoint in $Q_{x_i}$ can be replaced by an unused colour or vertex (if needed). Using Lemma 6.3 (i)–(ii), one may connect $P^2$ and all rainbow directed paths in $\{Q_{x_i} : x_i \in X \setminus X'\}$ into a single rainbow directed path $P^3$ with endpoints $u_1 \in A \setminus V_{\mathrm{bad}}$ and $v_3 \in B \setminus V_{\mathrm{bad}}$, whose length is at most $27\delta n + 2\sqrt{\delta}n + 5\sqrt{\epsilon}n + 10$.

**Step 3. Select a rainbow matching with colours in $\mathcal{C}_{\mathrm{bad}} \setminus \mathrm{col}(P^3)$.**

Choose a maximum rainbow matching in $\{D_i^{\pm}[A \setminus V(P^3), B \setminus V(P^3)] : i \in \mathcal{C}_{\mathrm{bad}} \setminus \mathrm{col}(P^3)\}$, say $\tilde{M}$. If there exists an edge with colour $j$ in $\tilde{M}$ such that $D_j^{\pm}[A \setminus V(P^3 \cup \tilde{M}), B \setminus V(P^3 \cup \tilde{M})]$ contains no 2-matching, then delete all such edges from $\tilde{M}$ and denote the resulting rainbow directed matching by $M$. Connecting $P^3$ and all rainbow directed edges in $M$ by Lemma 6.3 (i), we obtain a rainbow directed path $P^4$ with endpoints $u_1 \in A \setminus V_{\mathrm{bad}}$ and $v_4 \in B \setminus V_{\mathrm{bad}}$, whose length is at most $|E(P^3)| + 16\sqrt{\delta}n \le 18\sqrt{\delta}n + 5\sqrt{\epsilon}n + 27\delta n + 10$.

Denote $\mathcal{C}'_{\mathrm{bad}} := \mathcal{C}_{\mathrm{bad}} \setminus \mathrm{col}(P^4)$ and let $j \in \mathcal{C}'_{\mathrm{bad}}$. Hence in $D_j$, all but at most one vertex in $Y \setminus V(P^3 \cup \tilde{M})$ has at least $\frac{n}{2} - |V(P^3 \cup \tilde{M})| - 1 \ge (\frac{1}{2} - 19\sqrt{\delta})n$ out-neighbours in $Y$. Therefore, $D_j$ is $(19\sqrt{\delta}, \mathrm{EC1})$-extremal for all $j \in \mathcal{C}'_{\mathrm{bad}}$.

**Step 4. Construct two rainbow directed paths inside $\mathcal{D}[A \setminus V(P^4)]$ and $\mathcal{D}[B \setminus V(P^4)]$ by using unused colours in $\mathcal{C}_1 \cup \mathcal{C}'_{\mathrm{bad}}$.**

It is straightforward to check that

$$|B \setminus V(P^4)| - |A \setminus V(P^4)| = |B \setminus V(P^1)| - |A \setminus V(P^1)| = \sigma.$$

By the construction of $P^4$, we know that if $w \in V(P^4)$ is a vertex not in $(V(P^1) \cap B) \cup V_{\mathrm{bad}} \cup X$, then it is possible to avoid $w$ when constructing $P^4$; if $i \in \mathrm{col}(P^4)$ is a colour not in $\mathrm{col}(P^1 - A) \cup \mathrm{col}(M)$, then it is possible to avoid $i$ when constructing $P^4$. In other words, we may assume that $w \notin V(P^4)$ unless $w \in (V(P^1) \cap B) \cup V_{\mathrm{bad}} \cup X'$, and $i \notin \mathrm{col}(P^4)$ unless $i \in \mathrm{col}(P^1 - A) \cup \mathrm{col}(M)$.

Furthermore, while selecting rainbow directed paths inside $\mathcal{D}[B]$ (in Step 1), one may give priority to using colours from $\mathcal{C}_1 \cup \mathcal{C}'_{\mathrm{bad}}$ and avoid vertices in $X' \cup V_{\mathrm{bad}} \cup V(M)$. This implies that either $\mathcal{C}_1 \cup \mathcal{C}'_{\mathrm{bad}} \subseteq \mathrm{col}(P^1)$ or $\mathrm{col}(P^1 - A) \subseteq \mathcal{C}_1 \cup \mathcal{C}'_{\mathrm{bad}}$. Denote $\tilde{\mathcal{C}} := (\mathcal{C}_1 \cup \mathcal{C}'_{\mathrm{bad}}) \setminus \mathrm{col}(P^1)$.

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Notice that for each colour $i \in \mathcal{C}_2 \setminus \mathrm{col}(P^4)$, every vertex $v \in A \setminus V(P^4)$ has at least $(\frac{1}{2} - 3\sqrt{\delta})n - |V(P^4)| - |B \triangle B_i|$ out-neighbours (resp. in-neighbours) in $B \setminus V(P^4)$ in $D_i$. Hence,

$$\left|N_i^+(v) \cap N_i^-(v) \cap (B \setminus V(P^4))\right| \geq \left(\frac{1}{2} - 45\sqrt{\delta}\right) n.$$

Therefore, by replacing every pair of arcs with the same endpoints and opposite directions by an undirected edge, the digraph $D_i^{\pm}[A \setminus V(P^4), B \setminus V(P^4)]$ can be viewed as an undirected graph $G_i$ with vertex set $(A \cup B) \setminus V(P^4)$ and minimum degree at least $(\frac{1}{2} - 45\sqrt{\delta})n$.

The next claim employs Lemma 6.2 to construct a transversal directed Hamilton cycle.

**Claim 6.6.** *Let $Q = x_1 \ldots x_s$ be a rainbow directed path inside $\mathcal{D}$ with $s \leq 19\sqrt{\delta}n$, $V_{bad} \subseteq V(Q)$, $(\mathcal{C}_1 \cup \mathcal{C}_{bad}) \setminus \tilde{\mathcal{C}} \subseteq col(Q)$, $x_1 \in A \setminus V_{bad}$ and $x_s \in B \setminus V_{bad}$. If $|\tilde{\mathcal{C}}| - \big||B \setminus V(Q)| - |A \setminus V(Q)|\big|$ is a nonnegative even integer, then $\mathcal{D}$ has a transversal directed Hamilton cycle.*

**Proof of Claim** 6.6. Using colours in $\tilde{\mathcal{C}}$, one may greedily choose two disjoint rainbow directed paths $P^5$ and $P^6$ in $\mathcal{D}[A \setminus V(Q)]$ and $\mathcal{D}[B \setminus V(Q)]$ with lengths

$$\frac{|\tilde{\mathcal{C}}| - (|B \setminus V(Q)| - |A \setminus V(Q)|)}{2} \text{ and } \frac{|\tilde{\mathcal{C}}| + (|B \setminus V(Q)| - |A \setminus V(Q)|)}{2}$$

respectively, whose endpoints are $u_5, v_5 \in A \setminus V_{\mathrm{bad}}$ and $u_6, v_6 \in B \setminus V_{\mathrm{bad}}$. (Note that if a rainbow path has length 0, then it contains one vertex.) Applying Lemma 6.3 (i)-(ii) to connect $Q$, $P^5$ and $P^6$ in turn, we get a rainbow directed path $P$ with endpoints $x_1 \in A \setminus V_{\mathrm{bad}}$ and $v_6 \in B \setminus V_{\mathrm{bad}}$. Clearly, $|V(P)| \leq 19\sqrt{\delta}n + 4\sqrt{\delta}n + \eta n + 4$ and $|A \setminus V(P)| = |B \setminus V(P)|$.

Notice that $x_1, v_6 \notin V_{\mathrm{bad}}$. Hence there exist $i_1, i_2 \in \mathcal{C}_2 \setminus \mathrm{col}(P)$ such that $x_1 \in A_{i_1}$ and $v_6 \in B_{i_2}$. Let $W^* := A \setminus V(P)$, $T^* := B \setminus V(P)$, $\mathcal{C}^* := \mathcal{C} \setminus (\mathrm{col}(P) \cup \{i_1, i_2\})$, $W^- := N_{G_{i_2}}(v_6) \cap W^*$ and $T^+ := N_{G_{i_1}}(x_1) \cap T^*$. It is routine to check that $|W^*| = |T^*|$, $|\mathcal{C}^*| = |W^*| + |T^*| - 1$,

$$\begin{aligned} |W^-| &\geq d_{G_{i_2}}(v_6, A) - |V(P)| \geq d_{G_{i_2}}(v_6, A_{i_2}) - |A_1 \triangle A_{i_2}| - |X_A \setminus X_A^2| - |X'| - |V(P)| \\ &\geq \left(\frac{1}{2} - 45\sqrt{\delta}\right) n - 2\delta n - 4\delta n - \sqrt{\epsilon} n - |V(P)| > \left(\frac{1}{2} - 2\eta\right) n, \end{aligned}$$

and similarly $|T^+| > \left(\frac{1}{2} - 2\eta\right) n$. Applying Lemma 6.2 yields that $\{G_i[W^*, T^*] : i \in \mathcal{C}^*\}$ contains a transversal undirected path $P'$ starting at $v' \in W^-$ and ending at $u' \in T^+$. Thus, $x_1 P v_6 v' P' u' x_1$ is a transversal directed Hamilton cycle inside $\mathcal{D}$. □

If $n$ and $|\tilde{\mathcal{C}}|$ have the same parity, then by Claim 6.6, there exists a transversal directed Hamilton cycle inside $\mathcal{D}$, a contradiction. Hence, it suffices to consider that $n$ and $|\tilde{\mathcal{C}}|$ have different parity. Therefore, either $|\tilde{\mathcal{C}}| - (|B \setminus V(P^4)| - |A \setminus V(P^4)|)$ is a positive odd integer or $|\tilde{\mathcal{C}}| = 0$ and $n$ is odd.

Based on the definition of $X'$, we move vertices in $X'$ to $Y$ if it is covered by a rainbow directed path from $Z$ to $Y$ (in Step 2). The following claim considers the local structure of $D_i$.

**Claim 6.7.**

(i) *If $|\tilde{\mathcal{C}}| = 0$, then $D_i[B \setminus (X' \cup V(P^1))] = \emptyset$ for all $i \in \mathcal{C} \setminus col(P^1 - A)$.*

(ii) *For all $i \in \tilde{\mathcal{C}}$, $D_i[A \setminus (V_{bad} \cup X'), B] = \emptyset$ and $D_i[(B \setminus (V_{bad} \cup X' \cup V(P^1)), A] = \emptyset$.*

**Proof of Claim** 6.7. (i) Suppose that there exists an $i_0 \in \mathcal{C} \setminus \mathrm{col}(P^1 - A)$ such that $D_{i_0}[B \setminus (X' \cup V(P^1))] \neq \emptyset$. Choose $w_1 w_2 \in E(D_{i_0}[B \setminus (X' \cup V(P^1))])$. Recall that each of $w_1$ and $w_2$ can be chosen outside $V(P^4)$ unless it is in $(V(P^1) \cap B) \cup V_{\mathrm{bad}} \cup X'$, and $i_0$ can be chosen outside $\mathrm{col}(P^4)$ unless $i_0 \in \mathrm{col}(M)$. If $i_0 \in \mathrm{col}(M)$, then let $\tilde{P}^4$ be a rainbow directed path obtained by deleting the edge with colour $i_0$ from $P^4$ and connecting its two endpoints by Lemma 6.3 (i). It is routine

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to check that $|B \setminus V(\tilde{P}^4)| - |A \setminus V(\tilde{P}^4)| = |B \setminus V(P^4)| - |A \setminus V(P^4)|$. Let $P^4 := \tilde{P}^4$ if $i_0 \in \mathrm{col}(M)$, which reduces our proof to the case where $i_0 \notin \mathrm{col}(P^4)$. We consider the following several cases.

• Suppose $w_1, w_2 \notin V_{\mathrm{bad}}$. Then $w_1, w_2 \notin V(P^4)$. Connect $w_1$ with the endpoint $v_4$ of $P^4$ by Lemma 6.3 (ii). Denote the resulting rainbow directed path by $Q_1$.

• Suppose $w_1 \notin V_{\mathrm{bad}}$ and $w_2 \in V_{\mathrm{bad}}$. Hence $w_1 \notin V(P^4)$ and there exists a rainbow subpath $u_1P^4w_2^1w_2^2w_2$ of $P^4$ with $w_2^1 \in B \setminus (V_{\mathrm{bad}} \cup X')$, $w_2^2 \in A \setminus (V_{\mathrm{bad}} \cup X')$. Replace $w_2^1w_2^2w_2$ with $w_1w_2$ in $P^4$, and connect $w_1$ with the vertex in $N^-_{P^4}(w_2^1)$ by Lemma 6.3 (i). Let $Q_2$ be the resulting rainbow directed path. The case for $w_1 \in V_{\mathrm{bad}}$ and $w_2 \notin V_{\mathrm{bad}}$ is similar.

• Suppose $w_1, w_2 \in V_{\mathrm{bad}}$. By rearranging, one may assume that $w_1$ and $w_2$ are connected by a rainbow directed path $w_1w_1^1w_2^1w_2^2w_2$ in $P^4$, where $w_1^1, w_2^2 \in A \setminus (V_{\mathrm{bad}} \cup X')$ and $w_2^1 \in B \setminus (V_{\mathrm{bad}} \cup X')$. Replace $w_1w_1^1w_2^1w_2^2w_2$ with $w_1w_2$ in $P^4$; and denote the resulting rainbow directed path by $Q_3$.

In each of the above cases, we get a rainbow directed path $Q_i$ ($i \in [3]$) such that $|\tilde{\mathcal{C}}| - \big||B \setminus V(Q_i)| - |A \setminus V(Q_i)|\big|$ is a nonnegative even integer. It follows from Claim 6.6 that $\mathcal{D}$ contains a transversal directed Hamilton cycle, a contradiction.

(ii) We will only prove that $D_i[A \setminus (V_{\mathrm{bad}} \cup X'), B] = \emptyset$ for all $i \in \tilde{\mathcal{C}}$, the second statement can be proved by a similar discussion. Suppose that $D_{i_1}[A \setminus (V_{\mathrm{bad}} \cup X'), B] \neq \emptyset$ for some $i_1 \in \tilde{\mathcal{C}}$. Choose $z_1z_2 \in E(D_{i_1}[A \setminus (V_{\mathrm{bad}} \cup X'), B])$. Hence $z_1 \notin V(P^4)$. Next, we consider several cases.

• Suppose $z_2 \notin (V(P^1) \cap B) \cup V_{\mathrm{bad}} \cup X'$. Then $z_2 \notin V(P^4)$. Using Lemma 6.3 (i), connect $z_2$ to the endpoint $u_1$ of $P^4$. Denote the resulting rainbow directed path by $Q_1$, and set $\tilde{\mathcal{C}}_1 := \tilde{\mathcal{C}} \setminus \{i_1\}$.

• Suppose $z_2 \in V(P^1) \cap B$. Then $P^4$ can be written as $u_1P^4z_2'z_2P^4v_4$, where $z_2'z_2 \in E(P^1) \cap E(D_{i_2})$ for some $i_2 \in \mathcal{C}$. Clearly, $i_2 \in \mathcal{C}_2$ if $z_2' \in A$, and $i_2 \in (\mathcal{C}_1 \cup \mathcal{C}'_{\mathrm{bad}}) \setminus \tilde{\mathcal{C}}$ if $z_2' \in B$ (since $\mathrm{col}(P^1 - A) \subseteq \mathcal{C}_1 \cup \mathcal{C}'_{\mathrm{bad}}$). Notice that $V(P^1) \cap B \cap V_{\mathrm{bad}} = \emptyset$. Hence $z_2' \notin V_{\mathrm{bad}}$. Apply Lemma 6.3 (i)–(ii) to connect $u_1P^1z_2'$ and $z_1z_2P^4v_4$ into a single rainbow directed path, which we denote by $Q_2$. Let $\tilde{\mathcal{C}}_2 := \tilde{\mathcal{C}} \setminus \{i_1\}$ if $z_2' \in A$, and $\tilde{\mathcal{C}}_2 := (\tilde{\mathcal{C}} \setminus \{i_1\}) \cup \{i_2\}$ if $z_2' \in B$.

• Suppose $z_2 \in V_{\mathrm{bad}} \cup X'$. Then $P^4$ can be written as $u_1P^4z_2^1z_2P^4v_4$, where $z_2^1 \in A \setminus V_{\mathrm{bad}}$. By Lemma 6.3 (ii), we connect $u_1P^4z_2^1$ with $z_1z_2z_2^2P^4v_4$ into a single rainbow directed path, say $Q_3$. Set $\tilde{\mathcal{C}}_3 := \tilde{\mathcal{C}} \setminus \{i_1\}$.

In each of the above cases, we get a rainbow directed path $Q_i$ ($i \in [3]$) such that $|\tilde{\mathcal{C}}| - \big||B \setminus V(Q_i)| - |A \setminus V(Q_i)|\big|$ is a nonnegative even integer. It follows from Claim 6.6 that $\mathcal{D}$ contains a transversal directed Hamilton cycle, a contradiction. □

If $|\tilde{\mathcal{C}}| = 0$, then $n$ is odd. In view of Claim 6.7 (i), we know $D_i[B \setminus (X' \cup V(P^1))] = \emptyset$ for all $i \in \mathcal{C} \setminus \mathrm{col}(P^1 - A)$. Notice that $|B \setminus (X' \cup V(P^1))| \geq \frac{n}{2} - 28\delta n$ and $|\mathrm{col}(P^1 - A)| \leq 27\delta n$. Together with Lemma 6.5 (see the end of this subsection), we know that $\mathcal{D}$ contains a transversal directed Hamilton cycle, a contradiction.

If $|\tilde{\mathcal{C}}| \geq 1$, then based on Claim 6.7 (ii) and the fact that $\delta^0(\mathcal{D}) \geq \frac{n}{2}$, we obtain $|A|, |B| \geq \frac{n+1}{2}$, a contradiction.

**Case 2.** $|\mathcal{C}_2| < \eta n$.

In this case, each vertex in $Y \setminus (X \cup X_Y^1)$ belongs to $Y_i$ for at least $(1 - 13\sqrt{\delta})|\mathcal{C}_1|$ colours $i \in \mathcal{C}_1$. Following the argument from Step 2 of Case 1, we know that there are $|V_{\mathrm{bad}}|$ disjoint rainbow directed $P_3$ copies, each having its centre in $V_{\mathrm{bad}}$ and both endpoints in the same part. We now consider vertices in $X'$. Based on the definition of $X'$, we move a vertex in $X'$ to $Y$ whenever it is covered by a rainbow directed $P_3$ from $Z$ to $Y$ (in Step 2 of Case 1). In order to connect those

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rainbow directed $P_3$ with centres in $X'$ into a single rainbow directed path, we balance the number of rainbow directed paths from $A$ to $B$ and those from $B$ to $A$.

**Claim 6.8.** *There exists a rainbow directed path $P^0$ with length at most $8\sqrt{\epsilon}n$ such that $X' \subseteq V(P^0)$, $col(P^0) \subseteq \mathcal{C}_1$, and its endpoints lie outside $V_{bad} \cup X'$. Furthermore, $P^0$ can be chosen to either start at $A$ and end at $B$, or have both endpoints in the same part.*

**Proof of Claim** 6.8. If $|X'| = 0$, then by the minimum semi-degree condition, our result holds obviously. Assume therefore that $|X'| \geq 1$ and, without loss of generality, that $|A \cap X'| - |B \cap X'| \geq 0$. We first select all disjoint non-trivial maximum rainbow directed paths, say $Q_1, \ldots, Q_t$, inside $\{D_i[A \cap X'] : i \in \mathcal{C}_1\}$. Let $t' := |V(Q_1)| + \cdots + |V(Q_t)|$ and $Q_i := x_i \ldots y_i$ for each $i \in [t]$. Next, we choose a maximum rainbow directed matching, denoted by $\tilde{M}$, inside $\{D_i[A, B] : i \in \mathcal{C}_1 \setminus \mathrm{col}(Q_1 \cup Q_2 \cup \ldots \cup Q_t)\}$ avoiding vertices in $\bigcup_{i\in[t]} V(Q_i - y_i)$. Partition $\tilde{M}$ into $M_1 \cup M_2$ such that $V(M_2) \cap A = V(\tilde{M}) \cap A \cap X'$. Hence, each vertex in $V(M_2) \cap A$ or each $Q_i$ with $y_i \in V(M_2) \cap A$ can be covered by a rainbow directed path with length at most $|X'| + 2$ and both endpoints in $B \setminus (V_{\mathrm{bad}} \cup X')$. Denote $\hat{X} := (A \cap X') \setminus V\big(\bigcup_{i\in[t]} (Q_i - y_i) \cup \tilde{M}\big)$.

Assume that $|\hat{X}| \geq 1$. We are to prove that $|E(M_1)| \geq |\hat{X}| + 1$. Suppose for contradiction that $|E(M_1)| \leq |\hat{X}| \leq \sqrt{\epsilon}n$. Together with the definition of $X'$, one has $B \cap X' \subseteq V(\tilde{M}) \cap B$ and $\tilde{M}$ can be chosen such that each matching edge incident to a vertex in $B \cap X'$ is not incident to vertices in $A \cap X'$.

Choose $u_1 \in \hat{X}$, $u_2 \in B \setminus V(\tilde{M})$ and two colours $c_1, c_2 \in \mathcal{C}_1 \setminus \mathrm{col}(\tilde{M} \cup Q_1 \cup \ldots \cup Q_t)$. We consider $N^+_{c_1}(u_1)$ and $N^-_{c_2}(u_2)$ respectively. It is easy to see that $u_1$ has at most $t' - t$ out-neighbours inside $A \cap X'$ and $N^+_{c_1}(u_1, B) = N^+_{c_1}(u_1, B \cap V(\tilde{M}))$. Hence

$$\left\lceil \frac{n}{2} \right\rceil \leq |N^+_{c_1}(u_1)| \leq |A \setminus X'| + (t' - t) + |N^+_{c_1}(u_1, B \cap V(\tilde{M}))|. \tag{3}$$

Similarly, we have

$$\left\lceil \frac{n}{2} \right\rceil \leq |N^-_{c_2}(u_2)| \leq |N^-_{c_2}(u_2, A \cap V(\tilde{M}))| + |B| - 1. \tag{4}$$

Therefore,

$$|N^+_{c_1}(u_1, B \cap V(\tilde{M}))| + |N^-_{c_2}(u_2, A \cap V(\tilde{M}))| \geq |\hat{X}| + |E(M_2)| + 1.$$

Notice that, if there exists an edge $w_1w_2 \in E(\tilde{M})$ such that $u_1w_1 \in E(D_{c_1})$ and $w_2u_2 \in E(D_{c_2})$, then we will find a large rainbow matching, a contradiction. Hence $|N^+_{c_1}(u_1, B \cap V(\tilde{M}))| + |N^-_{c_2}(u_2, A \cap V(\tilde{M}))| \leq |E(\tilde{M})|$. It follows that $|E(\tilde{M})| \geq |\hat{X}| + |E(M_2)| + 1$, which implies $|E(M_1)| \geq |\hat{X}| + 1$. Thus, together with Lemma 6.3 (iii), the desired rainbow directed path $P^0$ can be obtained.

Now, we consider $|\hat{X}| = 0$ and $|B \cap X'| = 0$. If $|E(M_1)| \geq 1$, then we are done. Otherwise, by the maximality of $\tilde{M}$ and the definition of $X'$, we deduce that $D_i[A \setminus X', B] = \emptyset$. Using a similar argument as (3) and (4), there is a vertex in $A \setminus (V_{\mathrm{bad}} \cup X')$ that is adjacent to some vertex in $\{x_1, \ldots, x_t\} \cup ((A \cap X') \setminus V(Q_1 \cup \ldots \cup Q_t))$. Together with Lemma 6.3 (iii), the desired rainbow directed path can be obtained immediately.

Finally, we consider $|\hat{X}| = 0$ and $|B \cap X'| \geq 1$. In this case, the desired rainbow directed path also follows immediately from part (iii) of Lemma 6.3. (Note that if there is a rainbow directed path with some endpoint in $V_{\mathrm{bad}}$, then it can be extended by at most two vertices so that both of its endpoints remain in the corresponding part and are no longer in $V_{\mathrm{bad}}$.) □

By Lemma 6.1 and a similar discussion as Steps 1–3 in Case 1, the following hold.

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**(D1)** In $\{D_i[A \setminus V(P^0)] \cup D_i[B \setminus V(P^0)] : i \in \mathcal{C}_1 \setminus \mathrm{col}(P^0)\}$, there are $|V_{\mathrm{bad}} \setminus V(P^0)|$ disjoint rainbow directed $P_3$ copies, each with its centre in $V_{\mathrm{bad}} \setminus V(P^0)$ and both endpoints in the same part. Let $\mathbf{P}$ be the set of these rainbow directed $P_3$ copies.

**(D2)** For colours in $\mathcal{C}_{\mathrm{bad}}$, there exists a maximal rainbow matching, say $M$, inside $\{D_i[A \setminus V(\mathbf{P} \cup P^0)] \cup D_i[B \setminus V(\mathbf{P} \cup P^0)] : i \in \mathcal{C}_{\mathrm{bad}}\}$ such that for each edge in $M[Y]$ with colour $j$ we have $D_j[Y \setminus V(\mathbf{P} \cup P^0 \cup M)]$ contains a 2-matching. Denote $\mathcal{C}'_{\mathrm{bad}} := \mathcal{C}_{\mathrm{bad}} \setminus \mathrm{col}(M)$. Then $D_j$ is $(19\sqrt{\delta}, \mathrm{EC2})$-extremal for all $j \in \mathcal{C}'_{\mathrm{bad}}$.

**(D3)** Let $q := |\mathcal{C}_2 \cup \mathcal{C}'_{\mathrm{bad}}|$. In the digraph collection $\{D_i^{\pm}[A, B] : i \in \mathcal{C}_2 \cup \mathcal{C}'_{\mathrm{bad}})\}$, there exists a rainbow directed path $Q$ of length $q-1$ or $q$ that avoids vertices in $\mathbf{P} \cup P^0 \cup M$.

**(D4)** We can assume that a vertex $v \notin V(\mathbf{P} \cup M \cup Q)$ unless $v \in V_{\mathrm{bad}}$, and a colour $c \notin \mathrm{col}(\mathbf{P})$ if $c \in \mathcal{C}_1$.

Next, we give a claim that will be used later to obtain transversal directed Hamilton cycles.

**Claim 6.9.** *Assume that $Q_1 = z_1 \dots z_s$ and $Q_2 = z'_t \dots z'_1$ are two disjoint rainbow directed paths inside $\mathcal{D}$ such that $z_1, z'_1 \in A \setminus V_{bad}$, $z_s, z'_t \in B \setminus V_{bad}$ and $s + t < 4\eta n$. If $V_{bad} \subseteq V(Q_1 \cup Q_2)$ and $\mathcal{C}_2 \cup \mathcal{C}_{bad} \subseteq col(Q_1 \cup Q_2)$, then $\mathcal{D}$ contains a transversal directed Hamilton cycle.*

**Proof of Claim** 6.9. Note that $z_1, z'_1 \in A \setminus V_{\mathrm{bad}}$ and $z_s, z'_t \in B \setminus V_{\mathrm{bad}}$. There exist $i_1, i_2, i_3, i_4 \in \mathcal{C}_1 \setminus \mathrm{col}(Q_1 \cup Q_2)$ such that $z_1 \in A_{i_1}$, $z'_1 \in A_{i_2}$, $z_s \in B_{i_3}$ and $z'_t \in B_{i_4}$. Split $\mathcal{C}_1 \setminus (\mathrm{col}(Q_1 \cup Q_2) \cup \{i_1, i_2, i_3, i_4\})$ into two parts $\mathcal{C}_a \cup \mathcal{C}_b$, where $|\mathcal{C}_a| = |A \setminus V(Q_1 \cup Q_2)| - 1$ and $|\mathcal{C}_b| = |B \setminus V(Q_1 \cup Q_2)| - 1$. Let $I_1 \cup I_2$ and $J_1 \cup J_2$ be equitable partitions of $A \setminus V(Q_1 \cup Q_2)$ and $B \setminus V(Q_1 \cup Q_2)$ with $|I_1| \geq |I_2|$ and $|J_1| \geq |J_2|$, respectively. Define

$$\mathcal{D}_a := \{D_i^{\pm}[I_1, I_2] : i \in \mathcal{C}_a\} \text{ and } \mathcal{D}_b := \{D_i^{\pm}[J_1, J_2] : i \in \mathcal{C}_b\}.$$

Notice that

$$|Y \setminus V(Q_1 \cup Q_2)| \geq \frac{n - |X'| - r}{2} - |V(Q_1 \cup Q_2)| > \frac{n}{2} - 5\eta n,$$

and for $i \in \mathcal{C}_a \cup \{i_1, i_2\}$, $j \in [2]$ and $z \in I_{3-j} \cup \{z_1, z'_1\}$, we have

$$\begin{aligned} |N_i^+(z, I_j)| &\geq d_i^+(z, A \setminus V(Q_1 \cup Q_2)) - |I_{3-j}| \\ &\geq d_i^+(z, A_i) - |A_1 \triangle A_i| - |X_A \setminus X_A^1| - |X'| - |V(Q_1 \cup Q_2)| - |I_{3-j}| \\ &\geq \left(\frac{1}{2} - 3\sqrt{\delta}\right) n - \left(2\delta + 4\delta + 4\eta + \sqrt{\epsilon}\right) n - \left\lceil \frac{n+r}{4} \right\rceil > \frac{n}{4} - 5\eta n. \end{aligned}$$

Similarly, we have $|N_i^-(z, I_j)| \geq \frac{n}{4} - 5\eta n$. This implies that

$$|N_i^+(z, I_j) \cap N_i^-(z, I_j)| \geq 2\left(\frac{n}{4} - 5\eta n\right) - |I_j| \geq \frac{n}{4} - 10\eta n.$$

Hence, for each colour $i \in \mathcal{C}_a \cup \{i_1, i_2\}$, one may view $D_i^{\pm}[I_1, I_2]$ as an undirected graph $G_i[I_1, I_2]$ with minimum degree at least $\frac{n}{4} - 10\eta n$, where two vertices are adjacent in $G_i[I_1, I_2]$ if and only if there are two opposite arcs between them. By Lemma 6.2, there exists a transversal undirected path $Q_a$ inside $\mathcal{D}_a$ with two endpoints $x_a \in N_{G_{i_1}}(z_1, I_1)$ and $y_a \in N_{G_{i_2}}(z'_1, I)$ (here $y_a \in I_2$ if and only if $|I_1| = |I_2|$).

Similarly, there exists a transversal undirected path $Q_b$ inside $\mathcal{D}_b$ with two endpoints $x_b \in N_{G_{i_3}}(z_s, J_1)$ and $y_b \in N_{G_{i_4}}(z'_t, J)$ (here $y_b \in J_2$ if and only if $|J_1| = |J_2|$). Thus, $\mathcal{D}$ contains a transversal directed Hamilton cycle $z_1 Q_1 z_s x_b Q_b y_b z'_t Q_2 z'_1 y_a Q_a x_a z_1$. □

Now, we proceed by considering the value of $q$.

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**Subcase 2.1.** $q$ is even and $q \geq 2$.

Using Lemma 6.3 (iii), we connect $P^0$ (with endpoints in the same part), all rainbow directed paths in $\mathbf{P}$, all edges in $M$ and $Q$ (from $A$ to $B$ with length $q-1$) into a single rainbow directed path $P$, whose length is at most $(8\sqrt{\epsilon}+3\sqrt{\delta}+16\sqrt{\delta}+\eta)n$ and endpoints are $x \in A \setminus (V_{\text{bad}} \cup X')$, $y \in B \setminus (V_{\text{bad}} \cup X')$. Choose $c \in (\mathcal{C}_2 \cup \mathcal{C}'_{\text{bad}}) \setminus \text{col}(P)$ and let $u_q v_q$ be an edge in $D_c[B \setminus V(P), A \setminus V(P)]$. Consequently, $P$ and $u_q v_q$ are two disjoint rainbow directed paths satisfying all conditions in Claim 6.9. Thus, $\mathcal{D}$ contains a transversal directed Hamilton cycle, a contradiction.

**Subcase 2.2.** $q$ is odd.

Using Lemma 6.3 (iii), we connect all rainbow paths in $\mathbf{P}$, all edges in $M$ and $Q$ (from $B$ to $A$ with length $q$) into a single rainbow directed path $P$, whose length is at most $(3\sqrt{\delta}+16\sqrt{\delta}+\eta)n$ and endpoints are $x \in B \setminus V_{\text{bad}}$, $y \in A \setminus V_{\text{bad}}$. Therefore, $P$ and $P^0$ (from $A$ to $B$) are two disjoint rainbow paths satisfying all conditions in Claim 6.9. It follows that $\mathcal{D}$ contains a transversal directed Hamilton cycle, a contradiction.

**Subcase 2.3.** $q = 0$.

If $|X'| \geq 2$, then choose $u \in A \cap X'$ if $|A \cap X'| > |B \cap X'|$, and choose $u \in B \cap X'$ if $|A \cap X'| = |B \cap X'|$. Delete $u$ from $X'$ when constructing $P^0$. Based on the definition of $X'$, there exists a rainbow directed copy of $P_3$ from $B$ to $A$ with centre $u$, whose vertices are disjoint from $P^0$. Applying Lemma 6.3 (iii) again to connect $P^0$ (from $A$ to $B$), all rainbow directed paths in $\mathbf{P}$, $M$ into a single rainbow path $P$, whose length is at most $(8\sqrt{\epsilon}+3\sqrt{\delta}+16\sqrt{\delta}+\eta)n$, with endpoints in different parts. Thus, $P$ and $P_x$ are two disjoint rainbow paths satisfying all conditions in Claim 6.9. Therefore, $\mathcal{D}$ contains a transversal directed Hamilton cycle, a contradiction.

If $|X'| \leq 1$, then together with the condition that $\delta^0(\mathcal{D}) \geq \frac{n}{2}$, it is easy to find two disjoint rainbow edges with opposite directions between $A$ and $B$. By Claim 6.9 and a similar discussion as above, we know that $\mathcal{D}$ contains a transversal directed Hamilton cycle, a contradiction.

**Case 3.** $|\mathcal{C}_1| \geq \eta n$ and $|\mathcal{C}_2| \geq \eta n$.

In this case, each vertex in $Y \setminus V_{\text{bad}}$ lies in $Y_i$ for at least $(1-13\sqrt{\delta})|\mathcal{C}_1 \cup \mathcal{C}_2|$ colours $i \in \mathcal{C}_1 \cup \mathcal{C}_2$. Denote $\bigcup_{k\in[2]}(X_A^k \cup X_B^k) =: \{v_1, v_2, \ldots, v_t\}$. Notice that $t < \frac{1}{2}\sqrt{\delta}n$. By the definition of $X_Y^k$, we obtain that for each $i \in [t]$, there exist $c_{4i-3}, c_{4i-2}, c_{4i-1}, c_{4i} \in (\mathcal{C}_1 \cup \mathcal{C}_2) \setminus \{c_1, c_2, \ldots, c_{4i-5}, c_{4i-4}\}$ and $v_i^1, v_i^2, v_i^3, v_i^4 \in (A \cup B) \setminus (V_{\text{bad}} \cup \{v_\ell^1, v_\ell^2, v_\ell^3, v_\ell^4 : \ell \in [i-1]\})$ such that

- if $v_i \in X_Y^1$, then $c_{4i-4+j} \in \mathcal{C}_1$, $v_i^j \in Y \cap Y_{c_{4i-4+j}}$, $v_i^j v_i \in E(D_{c_{4i-4+j}})$ for all $j \in [2]$ and $v_i v_i^j \in E(D_{c_{4i-4+j}})$ for all $j \in \{3,4\}$,
- if $v_i \in X_Y^2$, then $c_{4i-4+j} \in \mathcal{C}_2$, $v_i^j \in Z \cap Z_{c_{4i-4+j}}$, $v_i^j v_i \in E(D_{c_{4i-4+j}})$ for all $j \in [2]$ and $v_i v_i^j \in E(D_{c_{4i-4+j}})$ for all $j \in \{3,4\}$.

Therefore, there exists a set of disjoint rainbow directed paths $\mathbf{P}^0 = \{v_i^1 v_i v_i^2 : i \in [t]\}$ with $\text{col}(v_i^1 v_i v_i^2) = \{c_{4i-3}, c_{4i-2}\}$ for each $i \in [t]$.

Based on the definition of $X'$, we move a vertex from $X'$ to $Y$ whenever it is covered by a rainbow directed $P_3$ from $Z$ to $Y$ (in Step 2 of Case 1). Avoiding all vertices and colours appearing in $\mathbf{P}^0$, we then obtain a family of disjoint rainbow directed paths $\mathbf{P^1} = \{v^1 v v^2 : v \in X'\}$, where for each vertex $v \in Y \cap X'$ we have $v^1 \in Z \setminus (V_{\text{bad}} \cup X')$ and $v^2 \in Y \setminus (V_{\text{bad}} \cup X')$. Furthermore, for every such path at least one of the colours of $v^1 v$ and $v v^2$ can be chosen from both $\mathcal{C}_1$ and $\mathcal{C}_2$. Without loss of generality, assume that $|A| \leq |B|$.

Using colours in $\mathcal{C}_1 \cup \mathcal{C}_2$ and applying Lemma 6.3 (i)-(iii), we connect all rainbow directed paths in $\mathbf{P}^0 \cup \mathbf{P}^1$ into a single rainbow directed path $P^1$ with endpoints $u_1 \in A \setminus V_{\text{bad}}$ and $w_1 \in B \setminus V_{\text{bad}}$, whose length is at most $4t + 4|X'| \leq 2\sqrt{\delta}n + 4|X'|$.

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By Lemma 6.1, there exists a set of disjoint rainbow directed paths $\mathbf{P}^2 = \{x_i^1 x_i x_i^2 : i \in [s]\}$, each of which has colours $c_i^1, c_i^2$ for $i \in [s]$. Applying Lemma 6.3 (i)-(ii), we connect all these rainbow directed paths together with $P^1$ into a single rainbow directed path $P^2$, which has endpoints $u_1 \in A \setminus V_{\text{bad}}$ and $w_2 \in B \setminus V_{\text{bad}}$ and length at most $|E(P^1)| + 4|X \setminus X'| + 2 \le (2\sqrt{\delta} + 4\sqrt{\epsilon})n + 2$.

Choose a maximal rainbow matching, say $M$, inside $\{D_i[A \setminus V(P^2)] \cup D_i[B \setminus V(P^2)] : i \in \mathcal{C}_{\text{bad}}\}$ such that for each edge in $M[Y]$ with colour $j$, we have $D_j[Y \setminus V(P^2 \cup M)]$ contains a 2-matching. Denote $\mathcal{C}'_{\text{bad}} := \mathcal{C}_{\text{bad}} \setminus \text{col}(M)$. Note that $D_j$ is $(19\sqrt{\delta}, \text{EC2})$-extremal for all $j \in \mathcal{C}'_{\text{bad}}$. Hence, $\{D_i[A \setminus V(P^2 \cup M), B \setminus V(P^2 \cup M)] : i \in \mathcal{C}'_{\text{bad}}\}$ contains a transversal matching, say $M'$. By using Lemma 6.3 (i)-(iii) to connect $P^2$ and all rainbow edges in $M \cup M'$, we obtain a rainbow directed path $P^3$ with endpoints $u_1 \in A \setminus V_{\text{bad}}$ and $w_3 \in B \setminus V_{\text{bad}}$, whose length is at most $|E(P^2)| + 16\sqrt{\delta}n \le (18\sqrt{\delta} + 4\sqrt{\epsilon})n + 2$.

Since $u_1, w_3 \notin V_{\text{bad}}$, there exist $i_0, i_1 \in \mathcal{C}_1 \setminus \text{col}(P^3)$ such that $u_1 \in A_{i_0}$ and $w_3 \in B_{i_1}$. By the construction of $P^3$, one may assume that $v \notin V(P^3)$ unless $v \in V_{\text{bad}} \cup X'$, and $c \notin \text{col}(P^3)$ unless $c \in \mathcal{C}_{\text{bad}}$. Similar to the above two cases, we know that each digraph $D_i^{\pm}[A \setminus V(P^3), B \setminus V(P^3)]$ with $i \in \mathcal{C}_2$ can be viewed as an undirected graph with minimum degree at least $(\frac{1}{2} - 45\sqrt{\delta})n$; each digraph $D_i[A \setminus V(P^3)] \cup D_i[B \setminus V(P^3)]$ with $i \in \mathcal{C}_1$ can be viewed as the union of two disjoint undirected graphs with minimum degree at least $(\frac{1}{2} - 45\sqrt{\delta})n$. We first prove the following claim.

**Claim 6.10.** *Assume that $Q_1 = z_1 \dots z_k$ and $Q_2 = z'_t \dots z'_1$ are two disjoint rainbow directed paths inside $\mathcal{D}$ such that $z_1, z'_1 \in A \setminus V_{bad}$, $z_k, z'_t \in B \setminus V_{bad}$ and each of them has length at most $19\sqrt{\delta}n$. If $V_{bad} \subseteq V(Q_1 \cup Q_2)$, $\mathcal{C}_{bad} \subseteq col(Q_1 \cup Q_2)$ and $|\mathcal{C}_2 \setminus col(Q_1 \cup Q_2)|$ is even, then $\mathcal{D}$ contains a transversal directed Hamilton cycle.*

**Proof of Claim** 6.10. Note that $z_1, z'_1 \in A \setminus V_{\text{bad}}$ and $z_k, z'_t \in B \setminus V_{\text{bad}}$. There exist $\ell_1, \ell_2, \ell_3 \in \mathcal{C}_1 \setminus \text{col}(Q_1 \cup Q_2)$ and $\ell_4 \in \mathcal{C}_2 \setminus \text{col}(Q_1 \cup Q_2)$ such that $z_1 \in A_{\ell_1}$, $z_k \in B_{\ell_2}$, $z'_1 \in A_{\ell_4}$ and $z'_t \in B_{\ell_3}$.

Split $A \setminus V(Q_1 \cup Q_2)$ into $I_1$ and $I_2$ such that $|I_1| = \frac{|\mathcal{C}_2 \setminus \text{col}(Q_1 \cup Q_2)|}{2}$, and split $B \setminus V(Q_1 \cup Q_2)$ into $J_1$ and $J_2$ such that $|J_1| = \frac{|\mathcal{C}_2 \setminus \text{col}(Q_1 \cup Q_2)|}{2}$. Let $\mathcal{D}_1 := \{D_i^{\pm}[I_1, J_1] : i \in \mathcal{C}_2 \setminus (\text{col}(Q_1 \cup Q_2) \cup \{\ell_4\})\}$. By Lemma 6.2, $\mathcal{D}_1$ contains a transversal undirected path $Q$ with endpoints $z''_1 \in N^+_{\ell_4}(z'_1) \cap J_1$ and $y \in I_1$. Since $y \notin V_{\text{bad}}$, there exists $\ell_5 \in \mathcal{C}_1 \setminus (\text{col}(Q_1 \cup Q_2) \cup \{\ell_1, \ell_2, \ell_3\})$ such that $y \in A_{\ell_5}$.

Assume that $|I_2| = k_1$ and $|J_2| = k_2$. We further split $I_2$ into $I'_2$ and $I''_2$ such that $|I'_2| = \lceil \frac{k_1}{2} \rceil$, and split $J_2$ into $J'_2$ and $J''_2$ such that $|J'_2| = \lceil \frac{k_2}{2} \rceil$. Split the colour set $\mathcal{C}_1 \setminus (\text{col}(Q_1 \cup Q_2) \cup \{\ell_1, \ell_2, \ell_3, \ell_5\})$ into two subsets $\mathcal{C}_a$ and $\mathcal{C}_b$ such that $|\mathcal{C}_a| = k_1 - 1$ and $|\mathcal{C}_b| = k_2 - 1$. Let

$$\mathcal{D}_a := \{D_i^{\pm}[I'_2, I''_2] : i \in \mathcal{C}_a\} \text{ and } \mathcal{D}_b := \{D_i^{\pm}[J'_2, J''_2] : i \in \mathcal{C}_b\}.$$

Applying Lemma 6.2 again yields that

- $\mathcal{D}_a$ contains a transversal undirected path $Q_a$ with endpoints $x_a \in N^-_{\ell_1}(z_1) \cap I'_2$, $y_a \in N^+_{\ell_5}(y) \cap I_2$;
- $\mathcal{D}_b$ contains a transversal undirected path $Q_b$ with endpoints $x_b \in N^+_{\ell_2}(z_k) \cap J'_2$, $y_b \in N^-_{\ell_3}(z'_t) \cap J_2$.

Hence, $z_1 Q_1 z_k x_b Q_b y_b z'_t Q_2 z'_1 z''_1 Q y y_a Q_a x_a z_1$ is a transversal directed Hamilton cycle in $\mathcal{D}$. □

Assume that $|\mathcal{C}_2 \setminus \text{col}(P^3)|$ is odd. Notice that there exists an edge $z'_1 z'_2 \in E(D_{i_2}[A \setminus V(P^3), B \setminus V(P^3)])$ for some $i_2 \in \mathcal{C}_2 \setminus \text{col}(P^3)$. This edge together with the path $P^3$ form two rainbow directed paths that satisfy all conditions in Claim 6.10. It follows that $\mathcal{D}$ contains a transversal directed Hamilton cycle, a contradiction. Hence $|\mathcal{C}_2 \setminus \text{col}(P^3)|$ is even.

Since $\delta^0(\mathcal{D}) \ge \frac{n}{2}$ and $|A| \le |B|$, each vertex in $A$ has an in-neighbour in $B$ in each $D_i$ with $i \in \mathcal{C}_1 \cup \text{col}(M)$. Suppose that there exists an edge $y_2 y_1 \in E(D_{i_3}[B, A])$ for some $i_3 \in \mathcal{C}_1 \cup \text{col}(M)$,

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where $y_1 \in A \setminus (V(P^3) \cup V_{\mathrm{bad}})$ and $y_2 \in B$. If $i_3 \in \mathrm{col}(M)$, then assume $uv \in E(M[A]) \cap E(D_{i_3})$. Applying Lemma 6.3 (i)–(ii), we successively connect $P^2$, all rainbow edges in $M - uv$ and all rainbow edges in $M'$. This results in a rainbow directed path $\tilde{P}^3$ with endpoints $\tilde{u}_1 \in A \setminus V_{\mathrm{bad}}$ and $\tilde{w}_3 \in B \setminus V_{\mathrm{bad}}$. Therefore, $P^3$ can be written as $uvv'\tilde{u}_1\tilde{P}^3\tilde{w}_3$, where the colours of $vv'$ and $v'\tilde{u}_1$ are in $\mathcal{C}_2$. Since $|\mathcal{C}_2 \setminus \mathrm{col}(P^3)|$ is even, one has $|\mathcal{C}_2 \setminus \mathrm{col}(\tilde{P}^3)|$ is even. Hence one may let $P^3 := \tilde{P}^3$ if $i_3 \in \mathrm{col}(M)$ and further assume $i_3 \notin \mathrm{col}(P^3)$.

Notice that $y_2 \notin V(P^3)$ unless it is in $V_{\mathrm{bad}} \cup X'$. If $y_2 \notin V_{\mathrm{bad}} \cup X'$, then $P^3$ and $y_2y_1$ are two disjoint rainbow directed paths satisfying all conditions in Claim 6.10. Hence, $\mathcal{D}$ contains a transversal Hamilton cycle, a contradiction. Therefore, $y_2 \in V_{\mathrm{bad}} \cup X'$.

Recall that $y_2^1y_2y_2^2$ is the rainbow directed path that covers $y_2$. Assume $y_2^1y_2$ and $y_2y_2^2$ have colours $j_1$ and $j_2$ in $P^3$, respectively. Hence, there exists a rainbow directed $\hat{P}^3$ starting at $A$ and ending at $B$ such that $P^3$ is obtained by connecting $y_2^1y_2y_2^2$ and $\hat{P}^3$ if $y_2^1 \in A$, and $P^3$ is obtained by connecting $\hat{P}^3$ and $y_2^1y_2y_2^2$ otherwise.

If $y_2 \in X'$, then by adjusting the selections of $y_2^1, y_2^2$ and $j_1, j_2$, one may get a rainbow directed path $P^3$ such that $|\mathcal{C}_2 \setminus \mathrm{col}(P^3)|$ is odd. This situation therefore reduces to the case where $|\mathcal{C}_2 \setminus \mathrm{col}(P^3)|$ is odd.

If $y_2 \in V_{\mathrm{bad}}$, then $|\mathcal{C}_2 \setminus \mathrm{col}(\hat{P}^3)|$ is even. Let $\hat{P} := y_2^1y_2y_1$ if $y_2^1 \in B$, and let $\hat{P} := y'y_2^1y_2y_1$ with $y'y_2^1 \in E(D_{c'})$ for unused colour $c' \in \mathcal{C}_2$ and unused vertex $y' \in B$ if $y_2^1 \in A$. Consequently, $\hat{P}^3$ and $\hat{P}$ are two rainbow directed paths satisfying all conditions in Claim 6.10. Hence, $\mathcal{D}$ contains a transversal directed Hamilton cycle, a contradiction. □

## 7. Proof of Theorem 5.3

In this section we present the proof of Theorem 5.3. Since the overall strategy parallels that of Theorem 5.2, the argument itself is relatively short; therefore we do not separate out preliminary claims but instead give a complete, self-contained proof of Theorem 5.3.

**Proof of Theorem** 5.3. Choose constant $\epsilon, \delta$ such that $\frac{1}{n} \ll \epsilon \leq \delta^2 \ll 1$, let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a collection of digraphs on a common vertex set $V$ of size $n$ and $\delta^0(\mathcal{D}) \geq \frac{n}{2}$. Suppose that $\mathcal{D}$ does not contain transversal directed Hamilton cycles. For each $i \in [m]$, since $D_i$ is $(\epsilon, \mathrm{EC3})$-extremal, one may assume that $(C_i^1, C_i^2, C_i^3, C_i^4, L_i)$ is a $\epsilon$-characteristic partition of $D_i$. By Lemma 5.1, one has $|C_1^k \triangle C_i^k| \leq 2\delta n$ for all $k \in [4]$ and all $i \in [m]$. Expand $C_1^1 \cup C_1^2 \cup C_1^3 \cup C_1^4$ into a partition $C^1 \cup C^2 \cup C^3 \cup C^4$ of $V$ such that $|C^1| - |C^3| \leq 1$ and $|C^2| = |C^4|$. Hence, for $i \in [2, m]$ and $k \in [4]$, we have $|C^k \triangle C_i^k| \leq 2\delta n + 2\epsilon n < 3\delta n$.

Now, we define the set of bad vertices. Let $(Y, Z) \in \{(C^1, C^3), (C^2, C^4)\}$. Define

$$X := \left\{x \in V : x \in L_i \text{ for at least } 3\sqrt{\delta}|\mathcal{C}_3| \text{ colors } i \in \mathcal{C}_3\right\},$$

$$X_{YZ} := \left\{x \in (Y \cup Z) \setminus X : x \notin Y_i \cup Z_i \text{ for at least } 10\sqrt{\delta}|\mathcal{C}_3| \text{ colors } i \in \mathcal{C}_3\right\},$$

$$X'_{YZ} := \left\{x \in X_{YZ} : x \in W_i \text{ for some} W \in \{Y, Z\} \text{with at least } \frac{3}{2}\sqrt{\delta}n \text{ colors } i \in \mathcal{C}_3\right\}.$$

Recall that for all $i \in \mathcal{C}_3$, we have $|L_i| \leq 2\epsilon n$ and $|(Y_1 \cup Z_1) \triangle (Y_i \cup Z_i)| < 4\delta n$. Hence

$$3\sqrt{\delta}|\mathcal{C}_3||X| \leq 2\epsilon n|\mathcal{C}_3| \text{ and } 10\sqrt{\delta}|\mathcal{C}_3||X_{YZ}| \leq \sum_{i \in \mathcal{C}_3} |(Y \cup Z) \setminus (Y_i \cup Z_i)| \leq (4\delta + 2\epsilon)n|\mathcal{C}_3|.$$

It follows that $|X| \leq \sqrt{\epsilon}n$ and $|X_{YZ}| \leq \frac{9}{20}\sqrt{\delta}n < \frac{1}{2}\sqrt{\delta}n$. For a vertex $x \in X_{YZ} \setminus X'_{YZ}$, we know $x \in Y_i \cup Z_i$ for at most $3\sqrt{\delta}n$ colours $i \in \mathcal{C}_3$ and $x \in \bigcup_{k \in [4]} C_i^k$ for at least $(1 - 3\sqrt{\delta})|\mathcal{C}_3|$ colours $i \in \mathcal{C}_3$.

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Thus, $x \in (Y \cup Z) \setminus (Y_i \cup Z_i)$ for at least $(1-4\sqrt{\delta})|\mathcal{C}_3|$ colours $i \in \mathcal{C}_3$ and $x \in \bigcup_{k\in[4]} C_i^k \setminus (Y_i \cup Z_i)$ for at least $(1-3\sqrt{\delta})|\mathcal{C}_3| - 3\sqrt{\delta}n \geq (1-10\sqrt{\delta})|\mathcal{C}_3|$ colours $i \in \mathcal{C}_3$. Hence

$$\begin{aligned}|X_{YZ} \setminus X'_{YZ}|(1-4\sqrt{\delta})|\mathcal{C}_3| &\leq \sum_{i\in\mathcal{C}_3} |(Y \cup Z) \setminus (Y_i \cup Z_i)| \\ &\leq \sum_{i\in\mathcal{C}_3} (|(Y \cup Z) \setminus (Y_1 \cup Z_1)| + |(Y_1 \cup Z_1)\triangle(Y_i \cup Z_i)|) \\ &\leq (4\delta + 2\epsilon)n|\mathcal{C}_3|.\end{aligned}$$

It follows that $|X_{YZ} \setminus X'_{YZ}| \leq 8\delta n$. Then move vertices in $X_{YZ} \setminus X'_{YZ}$ to $V \setminus (Y \cup Z)$. Therefore, each vertex in $(Y \cup Z) \setminus (X \cup X'_{YZ})$ lies in $Y_i \cup Z_i$ for at least $(1-10\sqrt{\delta})|\mathcal{C}_3|$ colours $i \in \mathcal{C}_3$.

Let $\{Y, Z\} \in \big\{\{C^1, C^3\}, \{C^2, C^4\}\big\}$. Define

$$X_Y := \left\{x \in Y \setminus (X \cup X'_{YZ}) : x \notin Y_i \text{ for at least } 20\sqrt{\delta}|\mathcal{C}_3| \text{ colors } i \in \mathcal{C}_3\right\},$$

$$X'_Y := \left\{x \in X_Y : x \in Y_i \text{ for at least } 6\sqrt{\delta}n \text{ colors } i \in \mathcal{C}_3\right\}.$$

By a similar discussion as above, we know that $|X_Y| < \frac{1}{2}\sqrt{\delta}n$ and each vertex of $X_Y \setminus X'_Y$ lies in $Z_i$ for at least $(1-20\sqrt{\delta})|\mathcal{C}_3|$ colours $i \in \mathcal{C}_3$. Furthermore, $|X_Y \setminus X'_Y| \leq 4\delta n$. We then move vertices in $X_Y \setminus X'_Y$ to $Z$. Hence $\big||C^2| - |C^4|\big| \leq 24\delta n$.

Define $V_{\text{bad}} := X \cup \big(\bigcup_{(Y,Z)\in\{(C^1,C^3),(C^2,C^4)\}} X'_{YZ}\big) \cup \big(\bigcup_{k\in[4]} X'_{C^k}\big)$. Obviously, $|V_{\text{bad}}| < (\sqrt{\epsilon} + 3\sqrt{\delta})n$. Moreover, each vertex in $C^k \setminus V_{\text{bad}}$ lies in $C_i^k$ for at least $(1-20\sqrt{\delta})|\mathcal{C}_3|$ colours $i \in \mathcal{C}_3$.

The subsequent result allows us to connect two disjoint short rainbow directed paths into a single short rainbow directed path. Its proof follows the same lines as that of Lemma 6.3, and is therefore omitted.

**Claim 7.1** (Connecting tool). *Let $P = u_1u_2 \dots u_s$ and $Q = v_1v_2 \dots v_t$ be two disjoint rainbow directed paths inside $\mathcal{D}$ with $u_s \in Y \setminus V_{bad}$, $v_1 \in Z \setminus V_{bad}$, where $Y, Z \in \{C^1, C^2, C^3, C^4\}$. Assume that $|\mathcal{C}_3 \setminus col(P \cup Q)| > 21\sqrt{\delta}n$, and that for all $i \in [4]$, if $C^i$ is a candidate set for selecting $w_1$ or $w'_1$, then $|C^i \setminus (V(P \cup Q) \cup V_{bad})| > 25\delta n$.*

(i) *If $(Y, Z) \in \{(C^1, C^2), (C^3, C^4)\}$, then there are two colours $c_1, c_2 \in \mathcal{C}_3 \setminus col(P \cup Q)$ and a vertex $w_1 \in Y \setminus (V(P \cup Q) \cup V_{bad})$ such that $u_1Pu_sw_1v_1Qv_t$ is a rainbow directed path with colours $col(P \cup Q) \cup \{c_1, c_2\}$. A similar result holds for $(Y, Z) \in \{(C^2, C^3), (C^4, C^1)\}$.*

(ii) *If $(Y, Z) \in \{(C^1, C^1), (C^3, C^3)\}$, then there are two colours $c_1, c_2 \in \mathcal{C}_3 \setminus col(P \cup Q)$ and one vertex $w_1 \in Y \setminus (V(P \cup Q) \cup V_{bad})$ such that $u_1Pu_sw_1v_1Qv_t$ is a rainbow directed path with colours $col(P \cup Q) \cup \{c_1, c_2\}$.*

(iii) *If $\{Y, Z\} = \{C^2, C^4\}$, then there are three colours $c_1, c_2, c_3 \in \mathcal{C}_3 \setminus col(P \cup Q)$ and two vertices $w_1 \in Z \setminus (V(P \cup Q) \cup V_{bad})$, $w'_1 \in Y \setminus (V(P \cup Q) \cup V_{bad})$ such that $u_1Pu_sw_1w'_1v_1Qv_t$ is a rainbow directed path with colours $col(P \cup Q) \cup \{c_1, c_2, c_3\}$.*

(iv) *If $(Y, Z) = (C^i, C^j)$ with $(i, j) \in \{(1, 3), (1, 4), (4, 3)\}$, then there are two colours $c_1, c_2 \in \mathcal{C}_3 \setminus col(P \cup Q)$ and a vertex $w_1 \in C^2 \setminus (V(P \cup Q) \cup V_{bad})$ such that $u_1Pu_sw_1v_1Qv_t$ is a rainbow directed path with colours $col(P \cup Q) \cup \{c_1, c_2\}$. A similar result holds for $(Y, Z) = (C^i, C^j)$ with $(i, j) \in \{(3, 1), (3, 2), (2, 1)\}$.*

(v) *If $(Y, Z) = (C^i, C^i)$ with $i \in \{2, 4\}$, then there are two colours $c_1, c_2 \in \mathcal{C}_3 \setminus col(P \cup Q)$ and a vertex $w_1 \in C^{i+2} \setminus (V(P \cup Q) \cup V_{bad})$ such that $u_1Pu_sw_1v_1Qv_t$ is a rainbow directed path with colours $col(P \cup Q) \cup \{c_1, c_2\}$.*

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Delete vertices in $X$ from $\bigcup_{k\in[4]} C^k$ and choose a vertex $x \in X$. Then for some $j \in [4]$, we have $|N_i^-(x, C^j \setminus V_{\mathrm{bad}})| \geq 200\delta n$ for at least $200\delta n$ colours $i \in \mathcal{C}_3$. Recall that $|X| \leq \sqrt{\epsilon} n$. We first consider $j = 1$ and do the following operations.

(a) If $|N_i^+(x, C^1 \setminus V_{\mathrm{bad}})| \geq 200\delta n$ for at least $200\delta n$ colours $i \in \mathcal{C}_3$, then there is a rainbow directed copy of $P_3$ centred at $x$ with both endpoints in $C^1$. We then move such $x$ to $C^1$.

(b) Notice that each vertex in $C^1 \setminus V_{\mathrm{bad}}$ has at least $|C^4 \setminus V_{\mathrm{bad}}| - 16\delta n$ in-neighbours inside $C^4 \setminus V_{\mathrm{bad}}$ for all but at most $20\sqrt{\delta}|\mathcal{C}_3|$ digraphs $D_\ell$ with $\ell \in \mathcal{C}_3$. If $|N_i^+(x, C^2 \setminus V_{\mathrm{bad}})| \geq 200\delta n$ for at least $200\delta n$ colours $i \in \mathcal{C}_3$, then there is a rainbow directed copy of $P_4$ centred at $x$ with endpoints in $C^4, C^2$ respectively. We then move such $x$ to $C^1$.

Suppose (a)-(b) do not hold. Then for at least $(1 - 400\delta)|\mathcal{C}_3|$ digraphs $D_\ell$ with $\ell \in \mathcal{C}_3$, $x$ has at least $|C^3 \cup C^4| - 400\delta n - |V_{\mathrm{bad}}|$ out-neighbours in $(C^3 \cup C^4) \setminus V_{\mathrm{bad}}$. Observe that every vertex in $C^3 \setminus V_{\mathrm{bad}}$ has at least $|C^4 \setminus V_{\mathrm{bad}}| - 16\delta n$ out-neighbours inside $C^4 \setminus V_{\mathrm{bad}}$ for all but at most $20\sqrt{\delta}|\mathcal{C}_3|$ digraphs $D_\ell$ with $\ell \in \mathcal{C}_3$.

(c) If $|N_i^-(x, C^2 \setminus V_{\mathrm{bad}})| \geq 200\delta n$ for at least $200\delta n$ colours $i \in \mathcal{C}_3$, then we can find a rainbow directed copy of $P_3$ or $P_4$ centred at $x$ with endpoints in $C^2, C^4$ respectively; we then move such $x$ to $C^3$. The same conclusion holds when $|N_i^-(x, C^3 \setminus V_{\mathrm{bad}})| \geq 200\delta n$ for at least $200\delta n$ colours $i \in \mathcal{C}_3$; in that case we also move $x$ to $C^3$.

(d) Suppose (a)–(c) do not hold. Then in each $D_\ell$ with $\ell \in \mathcal{C}_3$, $x$ has at least $|(C^1 \cup C^4) \setminus V_{\mathrm{bad}}| - 400\delta n - |V_{\mathrm{bad}}|$ in-neighbours inside $(C^1 \cup C^4) \setminus V_{\mathrm{bad}}$ for all but at most $(1 - 400\delta)|\mathcal{C}_3|$ colours $i \in \mathcal{C}_3$. We then move $x$ to $C^2$.

We perform the above operation successively for $j = 2, 3, 4$. Let $X'$ denote the subset of $X$ formed by the vertices described in item (d). Hence, for the vertices in $X \setminus X'$, there exists a set of disjoint rainbow directed copies of $P_3$ or $P_4$ centred at those vertices, where each copy either has both endpoints in the same part $C^k$ with $k \in \{1, 3\}$, or has one endpoint in $C^2$ and the other in $C^4$. Moreover, $\big||C^2| - |C^4|\big| \leq 24\delta n + 2\sqrt{\epsilon} n$.

By the definition of $V_{\mathrm{bad}} \setminus X$, for vertices in $(V_{\mathrm{bad}} \setminus X) \cap Y$, we can find a set of disjoint rainbow directed copies of $P_3$ with centres in $(V_{\mathrm{bad}} \setminus X) \cap Y$ and endpoints in $Y \setminus V_{\mathrm{bad}}$ (if $Y \in \{C^1, C^3\}$) or in $(C^2 \cup C^4) \setminus (Y \cup V_{\mathrm{bad}})$ (if $Y \in \{C^2, C^4\}$). Denote the rainbow directed path with centre $x$ obtained above by $P_x$ for each $x \in V_{\mathrm{bad}} \setminus X'$.

In order to use the transversal blow-up lemma inside $\{D_i[C^2, C^4] : i \in \mathcal{C}_3\}$, we need to balance the sizes of $|C^2|$ and $|C^4|$. Without loss of generality, assume that $|C^2| - |C^4| = r$ with $0 \leq r \leq 25\delta n$. We now claim that by moving vertices in $C^2 \cap V_{\mathrm{bad}}$ or deleting rainbow directed paths inside $\mathcal{D}[C^2]$ and $\mathcal{D}[C^2, C^1 \cup C^3]$, we can make the number of remaining vertices in $C^2$ and $C^4$ differ by $\sigma$, where $\sigma = 1$ if $|C^2| + |C^4|$ is odd and $\sigma = 0$ otherwise.

Denote $s_1 := |X'_{C^2C^4} \cap C^2|$ and $s_2 := |X'_{C^2}|$. If $|C^2| - |C^4| - \sigma \leq s_1 + 2s_2$, then by moving $s'_1$ vertices in $X'_{C^2C^4} \cap C^2$ to $C^1 \cup C^3$ and $s'_2$ vertices in $X'_{C^2}$ to $C^4$ with $s'_1 + 2s'_2 = |C^2| - |C^4| - \sigma$, we can get our desired result. Hence, it suffices to consider $|C^2| - |C^4| - \sigma > s_1 + 2s_2$. We first move all vertices in $X'_{C^2C^4} \cap C^2$ to $C^1 \cup C^3$ and all vertices in $X'_{C^2}$ to $C^4$. Then $(V_{\mathrm{bad}} \setminus X) \cap C^2 = \emptyset$. Choose a set of disjoint maximal rainbow directed paths in $\mathcal{D}[C^2] \cup \mathcal{D}[C^2, C^1]$, denote such a set by $\{Q_1, \ldots, Q_t\}$. Assume that $\{Q'_1, \ldots, Q'_k\}$ is a set of disjoint rainbow directed copies of $P_3$ in $\mathcal{D}[C^2, C^3]$ with centres in $C^2$ and endpoints in $C^3$.

On the one hand, we consider that $|E(Q_1)| + \cdots + |E(Q_t)| + k \geq |C^2| - |C^4| - \sigma$. Then there exist an integer $k' \in [k]$ and a set of disjoint rainbow directed subpaths of $\{Q_1, \ldots, Q_t\}$, say $\{\tilde{Q}_1, \ldots, \tilde{Q}_{t'}\}$, such that $|E(\tilde{Q}_1)| + \cdots + |E(\tilde{Q}_{t'})| + k' = |C^2| - |C^4| - \sigma$. That is, $|C^4| - t' = |C^2| - (|V(Q'_1)| + \cdots + |V(Q'_{t'})|) - k' - \sigma$. If the end vertex of $\tilde{Q}_i$ with $i \in [t']$ (resp. the start vertex of $Q'_i$ with $i \in [k']$) lies in $V_{\mathrm{bad}} \setminus X'$, then we extend it by a vertex not in $V_{\mathrm{bad}}$. Applying Claim 7.1, we then connect the modified paths $\tilde{Q}_1, \ldots, \tilde{Q}_{t'}, Q'_1, \ldots, Q'_{k'}$ into a single rainbow

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path $P^0$ with endpoints in $C^2$ and $C^4$ respectively, whose length is at most $190\delta n$. Therefore, $|C^4 \setminus V(P^0)| = |C^2 \setminus V(P^0)| - \sigma$, as desired.

In each of the above cases, that is, either $|C^2| - |C^4| - \sigma \leq s_1 + 2s_2$, or $|C^2| - |C^4| - \sigma > s_1 + 2s_2$ and $|E(Q_1)| + \cdots + |E(Q_t)| + k \geq |C^2| - |C^4| - \sigma$, the number of the remaining vertices in $C^2$ and $C^4$ are balanced. Since $|E(P^0)| \leq 190\delta n$, by avoiding vertices in $V(P^0)$, one may choose a set of disjoint rainbow directed paths $\mathbf{P}^0 = \{P_x : x \in V_{\mathrm{bad}} \setminus (X' \cup P^0)\}$. For colours in $\mathcal{C}_{\mathrm{bad}} \setminus \mathrm{col}(P^0)$, we select a maximal rainbow matching $M$ inside $\mathcal{D}^{\pm}[C^2, C^4] \cup \mathcal{D}[C^1] \cup \mathcal{D}[C^3] \cup (\bigcup_{i\in[4]} \mathcal{D}[C^i, C^{i+1}])$ ensuring it avoids vertices in $V(P^0 \cup \mathbf{P}^0)$. Notice that if there exists a colour $c \in \mathcal{C}_{\mathrm{bad}} \setminus (\mathrm{col}(P^0 \cup M))$, then $D_c^{\pm}[C^1, C^3]$ is almost an undirected complete bipartite graph and $D_c[C^2], D_c[C^4]$ are almost undirected complete graphs. Consequently, for colours in $\mathcal{C}_{\mathrm{bad}} \setminus (\mathrm{col}(P^0 \cup M))$, we can choose a set of disjoint rainbow directed paths, say $\mathbf{P}^1$, with centres in $C^1$ and endpoints inside $C^3$, ensuring that at most one colour in $\mathcal{C}_{\mathrm{bad}}$ remains unused.

Let $\mathbf{P} := \mathbf{P}^0 \cup \mathbf{P}^1 \cup M \cup \{P^0\}$. By Claim 7.1 (ii) and (v), we connect all rainbow directed paths inside $\mathbf{P}$ with one endpoints in $C^2$ and the other in $C^4$ into $Q_0 = u_0 \dots v_0$, all rainbow directed paths inside $\mathbf{P}$ with both endpoints inside $C^1$ and $C^3$ into $Q_1 = u_1 \dots v_1$ and $Q_2 = u_2 \dots v_2$ respectively. Since $u_0, v_0, u_1, v_2 \notin V_{\mathrm{bad}}$, there are four distinct colours $i_1, i_2, i_3, i_4 \in \mathcal{C}_3 \setminus \mathrm{col}(Q_0 \cup Q_1 \cup Q_3)$ such that $u_0 \in C^2_{i_1}$, $v_0 \in C^4_{i_2}$, $u_1 \in C^1_{i_3}$ and $v_2 \in C^3_{i_4}$. We proceed by considering the following two cases.

• Suppose that $\sigma = 0$, i.e., $|C^4 \setminus V(P^0)| = |C^2 \setminus V(P^0)|$. We first assume that $\mathcal{C}_{\mathrm{bad}} \setminus \mathrm{col}(M \cup \mathbf{P}^0) = \emptyset$. By applying Lemma 6.2 to the digraph collection $\{D_i[C^1] \cup D_i[C^3] : i \in \mathcal{C}_3 \setminus \{i_1, i_2, i_3, i_4\}\}$, while avoiding vertices and colours in $Q_0 \cup Q_1 \cup Q_2$, we obtain two disjoint rainbow directed paths $P^1 = x_1 \dots y_1$ and $P^2 = x_2 \dots y_2$. Here $y_1 \in N^-_{i_3}(u_1, C^1)$, $x_2 \in N^+_{i_4}(v_2, C^3)$, $x_1 \in C^1_{c_1} \setminus V_{\mathrm{bad}}$ and $y_2 \in C^3_{c_2} \setminus V_{\mathrm{bad}}$ for two unused colours $c_1, c_2 \in \mathcal{C}_3$. We then connect $Q_1, Q_2$ via Claim 7.1 (iv) to form a rainbow directed path $P = x_1 \dots y_2$, which satisfies $V_{\mathrm{bad}} \setminus X' \subseteq V(P)$, $\mathcal{C}_{\mathrm{bad}} \subseteq \mathrm{col}(P)$ and $|C^4 \setminus V(P)| = |C^2 \setminus V(P)| + 1$.

Applying Lemma 6.2 to the digraph collection $\{D_i^{\pm}[C^2 \setminus V(P \cup Q_0), C^4 \setminus V(P \cup Q_0)] : i \in \mathcal{C}_3 \setminus (\mathrm{col}(P \cup Q_0) \cup \{i_1, i_2, c_1, c_2\})\}$, we obtain two disjoint rainbow directed paths $P^3 = x_3 \dots y_3$ and $P^4 = x_4 \dots y_4$, where $x_3 \in N^+_{c_1}(y_2, C^4 \setminus V_{\mathrm{bad}})$, $y_3 \in N^-_{i_1}(u_0, C^4 \setminus V_{\mathrm{bad}})$, $x_4 \in N^+_{i_2}(v_0, C^2 \setminus V_{\mathrm{bad}})$ and $y_4 \in N^-_{c_2}(x_1, C^4 \setminus V_{\mathrm{bad}})$, respectively. Hence $\mathcal{D}$ contains a transversal directed Hamilton cycle, a contradiction.

Now, assume that there exists a colour $c \in \mathcal{C}_{\mathrm{bad}} \setminus \mathrm{col}(M \cup \mathbf{P}^0)$. Since $\delta^0(\mathcal{D}) \geq \frac{n}{2}$, there exists an edge $uv \in E(D_c[C^2 \setminus V(Q_0 \cup Q_1 \cup Q_2), C^3 \cup C^4])$ if $|C^1| + |C^2| \leq |C^3| + |C^4|$, and $vu \in E(D_c[C^4 \setminus V(Q_0 \cup Q_1 \cup Q_2), C^1 \cup C^2])$ otherwise. By an argument analogous to that above, and considering whether $v \in V_{\mathrm{bad}}$, we obtain that $\mathcal{D}$ contains a transversal directed Hamilton cycle, a contradiction.

• Suppose that $\sigma = 1$, i.e., $|C^4 \setminus V(P^0)| = |C^2 \setminus V(P^0)| - 1$. We first consider that there exists a colour $c \in \mathcal{C}_{\mathrm{bad}} \setminus \mathrm{col}(M \cup \mathbf{P}^0)$. Choose a directed edge, say $xy$, in $E(D_c[C^3 \setminus V(Q_0 \cup Q_1 \cup Q_3), C^1 \setminus V(Q_0 \cup Q_1 \cup Q_3)])$. By applying Lemma 6.2 to the digraph collection $\{D_i[C^1 \setminus (V(Q_0 \cup Q_1 \cup Q_2) \cup \{x\})] \cup D_i[C^3 \setminus (V(Q_0 \cup Q_1 \cup Q_2) \cup \{y\})] : i \in \mathcal{C}_3 \setminus (\mathrm{col}(Q_0 \cup Q_1 \cup Q_2) \cup \{i_1, i_2, i_3, i_4, c\})\}$, we obtain two disjoint rainbow directed paths $P^1 = x_1 \dots y_1$ and $P^2 = x_2 \dots y_2$, where $y_1 \in N^-_{i_3}(u_1, C^1)$, $x_2 \in N^+_{i_4}(v_2, C^3)$, $x_1 \in C^1_{c_1} \setminus V_{\mathrm{bad}}$ and $y_2 \in C^3_{c_2} \setminus V_{\mathrm{bad}}$ for two unused colours $c_1, c_2 \in \mathcal{C}_3$. By sequentially connecting $Q_1, xy, Q_3$ via Claim 7.1 (iv), we obtain a rainbow directed path $P = x_1 \dots y_2$ such that $V_{\mathrm{bad}} \setminus X' \subseteq V(P)$, $\mathcal{C}_{\mathrm{bad}} \subseteq \mathrm{col}(P)$ and $|C^4 \setminus V(P)| = |C^2 \setminus V(P)| + 1$. Similar to the previous discussion, $\mathcal{D}$ contains a transversal directed Hamilton cycle, a contradiction.

Now, assume that $\mathcal{C}_{\mathrm{bad}} \setminus \mathrm{col}(M \cup \mathbf{P}^0) = \emptyset$. Since $|C^2| \geq |C^4|$, either $|C^1| + |C^2| \geq |C^3| + |C^4|$ or $|C^3| + |C^2| \geq |C^1| + |C^4|$. Hence, for each $c \in \mathcal{C}_3$, there exists an edge $uv \in E(D_c[C^2 \setminus V(Q_0 \cup Q_1 \cup Q_2), C^1 \cup C^2])$ or $uv \in E(D_c[C^2 \cup C^3, C^2 \setminus V(Q_0 \cup Q_1 \cup Q_2)])$. By a similar argument as above, and considering whether the vertex $v$ belongs to $V_{\mathrm{bad}}$ or not, we obtain that $\mathcal{D}$ contains a transversal directed Hamilton cycle, a contradiction.

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On the other hand, we consider that $|E(Q_1)|+\cdots+|E(Q_t)|+k<|C^2|-|C^4|-\sigma=r-s_1-2s_2-\sigma$. In this case, we move all vertices inside $Q_1,\dots,Q_t$, all vertices in $(V(Q_1')\cup\ldots\cup V(Q_k'))\cap C^2$ along with their in-neighbours in $C^3$ to $C^4$. Denote the resulting partition by $\tilde{C}^1\cup\tilde{C}^2\cup\tilde{C}^3\cup\tilde{C}^4$. Let $\tilde{\mathcal{C}}:=\mathcal{C}\setminus(\mathrm{col}(\bigcup_{i\in[t]}Q_i)\cup\mathrm{col}(\bigcup_{i\in[k]}Q_i'[C^3,C^2]))$. Therefore,

$$|\tilde{C}^4|\le\frac{|C^2|+|C^4|-r}{2}+s_2+2(r-s_1-2s_2-\sigma)\le\frac{|C^2|+|C^4|}{2}+50\delta n,$$
$$|\tilde{C}^2|\ge\frac{|C^2|+|C^4|+r}{2}-s_1-s_2-2(r-s_1-2s_2-\sigma)\ge\frac{|C^2|+|C^4|}{2}-50\delta n,$$
$$|\tilde{\mathcal{C}}|=|\mathcal{C}|-(|E(Q_1)|+\cdots+|E(Q_t)|)-k\ge(1-25\delta)n.$$

By the maximality of $\{Q_1,\dots,Q_t\}$, we know that $D_i[\tilde{C}^2]=D_i[\tilde{C}^2,\tilde{C}^1]=D_i[\tilde{C}^3,\tilde{C}^2]=\emptyset$ for all colour $i\in\tilde{\mathcal{C}}$. It follows from $\delta^0(\mathcal{D})\ge\lceil\frac{n}{2}\rceil$ that $\lceil\frac{n}{2}\rceil\le|\tilde{C}^3|+|\tilde{C}^4|$ and $\lceil\frac{n}{2}\rceil\le|\tilde{C}^1|+|\tilde{C}^4|$. Therefore,

$$|\tilde{C}^1|+|\tilde{C}^2|+|\tilde{C}^3|+|\tilde{C}^4|=n\le\left\lceil\frac{n}{2}\right\rceil+\left\lceil\frac{n}{2}\right\rceil\le|\tilde{C}^1|+|\tilde{C}^3|+2|\tilde{C}^4|,$$

which implies that $|\tilde{C}^2|\le|\tilde{C}^4|$. We complete our proof by using the subsequent lemma (i.e., Lemma 7.2), which leads to a contradiction by showing that $\mathcal{D}$ contains a transversal directed Hamilton cycle.

**Lemma 7.2.** *Suppose* $0<\frac{1}{n}\ll\delta\ll1$ *and* $0\le\gamma<\delta$. *Let* $\mathcal{C}$ *be a set of* $n$ *colours, and* $\mathcal{D}=\{D_i:i\in\mathcal{C}\}$ *be a collection of digraphs on a common vertex set* $V$ *of size* $n$ *such that* $\delta^0(\mathcal{D})\ge\lceil\frac{n}{2}\rceil$. *Assume that*

- $C^1\cup C^2\cup C^3\cup C^4$ *is a partition of* $V$ *with* $|C^2|\ge50\delta^{\frac{1}{4}}$, $\min\{|C^3|+|C^4|,|C^1|+|C^4|\}\ge\lceil\frac{n}{2}\rceil$ *and* $|C^4|=\left\lfloor\frac{|C^2|+|C^4|+1}{2}\right\rfloor+\gamma n$,
- $\mathcal{C}'\cup\mathcal{C}''$ *is a partition of* $\mathcal{C}$ *with* $|\mathcal{C}''|\le\delta n$,
- $D_i[C^2]=D_i[C^2,C^1]=D_i[C^3,C^2]=\emptyset$ *for all* $i\in\mathcal{C}'$.

*Then* $\mathcal{D}$ *contains a transversal directed Hamilton cycle.*

**Proof.** It is easy to see that $|C^3|+|C^4|+|C^1|+|C^4|=n+2\gamma n+\sigma$ and $\max\{|C^3|+|C^4|,|C^1|+|C^4|\}\le\lceil\frac{n}{2}\rceil+2\gamma n+\sigma$, where $\sigma=1$ if $|C^2|+|C^4|$ is odd and $\sigma=0$ otherwise. Without loss of generality, assume that $|C^3|+|C^4|=\lceil\frac{n}{2}\rceil+\gamma_1 n$ and $|C^1|+|C^4|=\lceil\frac{n}{2}\rceil+\gamma_2 n$ with $2\lceil\frac{n}{2}\rceil+\gamma_1 n+\gamma_2 n=n+2\gamma n+\sigma$. Define $Y_1:=\{v\in C^4:d_i^+(v,C^2)\le(1-\delta^{\frac{1}{4}})|C^2|$ for at least $\delta^{\frac{1}{4}}|\mathcal{C}'|$ colors $i\in\mathcal{C}'\}$, $Y_2:=\{v\in C^4{:}d_i^-(v,C^2)\le(1-\delta^{\frac{1}{4}})|C^2|$ for at least $\delta^{\frac{1}{4}}|\mathcal{C}'|$ colors $i\in\mathcal{C}'\}$, and $Y:=Y_1\cup Y_2$. It is routine to check that

$$\begin{aligned}\left\lceil\frac{n}{2}\right\rceil|C^2||\mathcal{C}'|&\le\sum_{i\in\mathcal{C}'}|E(D_i[C^2,C^3\cup C^4])|=\sum_{i\in\mathcal{C}'}|E(D_i[C^2,C^3])|+\sum_{i\in\mathcal{C}'}|E(D_i[C^2,C^4])|\\&\le|C^2||C^3||\mathcal{C}'|+|Y_2|(1-\delta^{\frac{1}{4}})|C^2|\delta^{\frac{1}{4}}|\mathcal{C}'|+|Y_2||C^2|(1-\delta^{\frac{1}{4}})|\mathcal{C}'|+(|C^4|-|Y_2|)|C^2||\mathcal{C}'|\\&=(|C^4|-\delta^{\frac{1}{2}}|Y_2|+|C^3|)|C^2||\mathcal{C}'|.\end{aligned}$$

Thus, $|Y_2|\le3\sqrt{\delta}n$. Similarly, $|Y_1|\le3\sqrt{\delta}n$. Notice that Lemma 6.2 and Claim 7.1 hold by setting $V_{\mathrm{bad}}:=Y$ and $\mathcal{C}_3:=\mathcal{C}'$.

For a vertex $v\in Y_1\setminus Y_2$ (resp. $v\in Y_2\setminus Y_1$), we have $d_i^-(v,C^2)\ge(1-\delta^{\frac{1}{4}})|C^2|$ (resp. $d_i^+(v,C^2)\ge(1-\delta^{\frac{1}{4}})|C^2|$) for at least $(1-\delta^{\frac{1}{4}})|\mathcal{C}'|$ colours $i\in\mathcal{C}'$. For a vertex $y\in Y_2$, one has $d_i^-(y,C^1\cup C^4)\ge40\sqrt{\delta}n$ for at least $\delta^{\frac{1}{4}}|\mathcal{C}'|-1$ colours $i\in\mathcal{C}'$. Otherwise, by the definition of $Y_2$, there exists

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a colour $c \in \mathcal{C}'$ such that $d_c^-(y, C^2) \leq (1-\delta^{\frac{1}{4}})|C^2|$ and $d_c^-(y, C^1 \cup C^4) \leq 40\sqrt{\delta}n$. Together with $|C^2| \geq 50\delta^{\frac{1}{4}}n$, one has

$$d_c^-(y) \leq 40\sqrt{\delta}n + (1-\delta^{\frac{1}{4}})|C^2| + |C^3| \leq \left\lceil \frac{n}{2} \right\rceil - \delta^{\frac{1}{4}}|C^2| + 40\sqrt{\delta}n < \left\lceil \frac{n}{2} \right\rceil,$$

a contradiction. Similarly, each vertex in $Y_1$ has at least $40\sqrt{\delta}n$ out-neighbours inside $C^3 \cup C^4$ for at least $\delta^{\frac{1}{4}}|\mathcal{C}'| - 1$ digraphs $D_i$ with $i \in \mathcal{C}'$.

Since $\delta^0(\mathcal{D}) \geq \frac{n}{2}$, we have $|E(D_i[C^2, C^3 \cup C^4])|, |E(D_i[C^1 \cup C^4, C^2])| \geq |C^2| \left\lceil \frac{n}{2} \right\rceil$ for each $i \in \mathcal{C}'$. This implies that for each $i \in \mathcal{C}'$, there are at most $|C^2|(\left\lceil \frac{n}{2} \right\rceil + \gamma_1 n) - |C^2| \left\lceil \frac{n}{2} \right\rceil = |C^2|\gamma_1 n$ non-edges in $D_i[C^2, C^3 \cup C^4]$ and at most $|C^2|\gamma_2 n$ non-edges in $D_i[C^1 \cup C^4, C^2]$. Therefore,

$$\sum_{i \in \mathcal{C}'} |E(D_i[Y_1, C^2])| \geq (|Y_1| - \gamma_1 n)|C^2||\mathcal{C}'| \text{ and } \sum_{i \in \mathcal{C}'} |E(D_i[C^2, Y_2])| \geq (|Y_2| - \gamma_2 n)|C^2||\mathcal{C}'|.$$

We proceed with the proof by considering the values of $|Y_1|$ and $|Y_2|$.

**Case 1.** $|Y_1| \geq \gamma_1 n,\ |Y_2| \geq \gamma_2 n$.

In view of Lemma 6.4, there exists a subset $Y_1' \subseteq Y_1$ with size $\gamma_1 n$ such that vertices in $Y_1 \setminus Y_1'$ can be covered by disjoint rainbow directed $P_3$ copies inside $\{D_i[Y_1, C^2] : i \in \mathcal{C}'\}$ with centres in $Y_1 \setminus Y_1'$; and a subset $Y_2' \subseteq Y_2$ with size $\gamma_2 n$ such that vertices in $Y_2 \setminus Y_2'$ can be covered by disjoint rainbow directed $P_3$ copies inside $\{D_i[C^2, Y_2] : i \in \mathcal{C}'\}$ with centres in $Y_2 \setminus Y_2'$. Therefore, by using colours in $\mathcal{C}'$,

- vertices in $(Y_1 \cup Y_2) \setminus (Y_1' \cup Y_2')$ can be covered by a set of disjoint rainbow directed $P_3$ copies with centres in $(Y_1 \cup Y_2) \setminus (Y_1' \cup Y_2')$ and endpoints in $C^2$,
- vertices in $Y_2' \setminus Y_1'$ can be covered by a set of disjoint rainbow directed $P_3$ or $P_4$ copies with centres in $Y_2' \setminus Y_1'$, starting at $C^4$ and ending at $C^2$,
- vertices in $Y_1' \setminus Y_2'$ can be covered by a set of disjoint rainbow directed $P_3$ or $P_4$ copies with centres in $Y_1' \setminus Y_2'$, starting at $C^2$ and ending at $C^4$,
- vertices in $Y_1' \cap Y_2'$ can be covered by a set of disjoint rainbow directed $P_3$, $P_4$ or $P_5$ copies with centres in $Y_1' \cap Y_2'$ and endpoints in $C^4$.

For each $y \in Y$, denote the rainbow directed path with centre $y$ by $P_y = y^1 * y * y^2$ with colours in $\mathcal{C}_3$. Let $\mathbf{P} := \{P_y : y \in Y\}$.

It is routine to verify that there exists a rainbow matching inside $D_i^{\pm}[C^2 \setminus V(\mathbf{P}), C^4 \setminus V(\mathbf{P})] \cup D_i[C^1] \cup D_i[C^3] \cup (\bigcup_{k \in [4]} D_i[C^k, C^{k+1}]) : i \in \mathcal{C}''\}$, say $M$. For all but at most one unused colour in $\mathcal{C}''$, we can find a rainbow directed copy of $P_3$ with centre in $Y$ and endpoints in $Z$, where $\{Y, Z\} = \{C^1, C^3\}$. If some colour $c \in \mathcal{C}''$ remains unused, then $D_c[C^1, C^3]$ is nearly a complete bipartite graph and $D_c[C^2], D_c[C^4]$ are almost complete graphs. Then, by a similar argument to that in Theorem 6.5, $\mathcal{D}$ contains a transversal directed Hamilton cycle.

**Case 2.** At most one of $|Y_1| \geq \gamma_1 n$ and $|Y_2| \geq \gamma_2 n$ holds.

We only consider the case that $|Y_1| \geq \gamma_1 n$ and $|Y_2| < \gamma_2 n$; other cases can be discussed similarly. Notice that vertices in $Y_1$ can be covered by a set of disjoint rainbow directed paths with centres in $Y_1$ and ending in $C^4 \setminus (Y_1 \cup Y_2)$. In particular, those rainbow directed paths can be chosen such that none of them contains edges inside $\mathcal{D}[C^4, C^3]$ or $\mathcal{D}[C^1, C^4]$. Furthermore, vertices in $Y_2 \setminus Y_1$ can be covered by a set of disjoint rainbow directed $P_3$ copies with centres in $Y_2 \setminus Y_1$, starting at $C^4 \setminus Y$, and using colours in $\mathcal{C}'$.

In the digraph collection $\mathcal{D}[C^4 \setminus Y]$, extend those rainbow directed paths or choose other disjoint rainbow directed paths into a set of disjoint maximal rainbow directed paths.

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Let $\mathbf{P} := \{Q_1, Q_2, \ldots, Q_t\}$ be a set consisting of all disjoint rainbow paths in the above, each of which has at least one endpoint in $C^4 \setminus Y$ and length $s_i$ $(1 \le i \le t)$ inside $C^4$. Let $\{e_1, e_2, \ldots, e_r\}$ be a set consisting of all disjoint rainbow edges in $\mathcal{D}[C^4 \setminus Y, C^3]$, and let $\{\tilde{Q}_1, \ldots, \tilde{Q}_k\}$ be a set of disjoint maximal rainbow directed $P_3$ in $\mathcal{D}^{\pm}[C^4 \setminus Y, C^1]$ with centres in $C^4$ and endpoints in $C^1$, each of which is chosen while avoiding previously used vertices and colours.

**Subcase 2.1.** $|C^2| \ge |C^4| - k - r - (s_1 + \cdots + s_t) - \sigma$.

In this subcase, there exist two integers $r' \in [r]$, $k' \in [k]$ and a set of disjoint rainbow directed paths $\mathbf{P}' = \{Q'_1, Q'_2, \ldots, Q'_\ell\}$ such that $|C^4 \setminus V(\mathbf{P}')| - r' - k' - \sigma = |C^2| - \ell$, where $Y'_1 \cup Y_2 \subseteq V(\mathbf{P}')$ and the endpoints of $\tilde{Q}_i$ are not in $Y$ for all $i \in [\ell]$. By Claim 7.1, we connect all the rainbow directed paths in $\mathbf{P}'$ into a single rainbow path $P^1$, whose endpoints lie in different parts. Since $|C^4| - |C^2| = 2\gamma n + \sigma$, we have $\ell \le |E(Q'_1)| + \cdots + |E(Q'_\ell)| = 2\gamma n + \sigma$. Therefore, $|E(P^1)| \le 6\gamma n + 3\sigma$ and $|C^4 \setminus V(P^1)| - \sigma = |C^2 \setminus V(P^1)|$. By a similar discussion as Subcase 2.1 of Theorem 6.5, $\mathcal{D}$ contains a transversal directed Hamilton cycle.

**Subcase 2.2.** $|C^2| < |C^4| - k - r - (s_1 + \cdots + s_t) - \sigma$.

In this subcase, $\sum_{i=1}^{t} s_i + k + r < 2\gamma n$. Assume that $V(Q_i[C^4]) := \{v_{s_1+\cdots+s_{i-1}+i}, \ldots, v_{s_1+\cdots+s_i+i}\}$ for each $i \in [t]$. Let $w$ be an unused vertex in $C^4$ and $c_1, c_2$ be two unused colours in $\mathcal{C}$. By the maximality of $\mathbf{P}$, in $D_{c_1}$ (resp. $D_{c_2}$), $w$ cannot be adjacent to the starting vertices of $Q_i[C^4]$ (resp. cannot be adjacent from the ending vertices of $Q_i[C^4]$) for all $i \in [t]$.

Define

$$I_1 := \{i : v_i \in N^-_{c_1}(w)\} \text{ and } I_2 := \{i : v_{i+1} \in N^+_{c_2}(w)\}.$$

Hence $I_1, I_2 \subseteq \bigcup_{i\in[t]} \left[\sum_{j\in[i-1]} s_j + i, \sum_{j\in[i]} s_j + i - 1\right]$ with $s_0 = 0$. By the maximality of $\mathbf{P}$, we know that

- $I_1 \cap I_2 = \emptyset$, which implies $|I_1| + |I_2| \le s_1 + \cdots + s_t$,
- each vertex in $N^+_{c_1}(w, C^3)$ must be an end vertex of some $e_i$ for $i \in [r]$ (hence $|N^+_{c_1}(w, C^3)| \le r$),
- each vertex in $N^-_{c_2}(w, C^1)$ must be a start vertex of some $\tilde{Q}_i$ for $i \in [k]$ (hence $|N^-_{c_2}(w, C^1)| \le k$).

Therefore,

$$\begin{aligned} n \le & d^+_{D_{c_1}}(w) + d^-_{D_{c_2}}(w) \le s_1 + \cdots + s_t + r + |C^3| + |C^2| + k + |C^1| + |C^2| \\ \le & 2\gamma n - 1 + n - |C^4| + |C^2| = n + 2\gamma n - 1 - (2\gamma n + \sigma) = n - 1 - \sigma, \end{aligned}$$

a contradiction. □

## 8. Concluding remarks

In this paper, we establish a transversal analogue of Ghouila-Houri's theorem [19] (Theorem 1.3), thereby resolving a problem proposed by Chakraborti, Kim, Lee, and Seo [6] for all sufficiently large $n$. As a consequence, we recover the transversal version of Dirac's theorem, previously obtained by Joos and Kim [21].

An *anti-directed cycle* is a digraph in which the underlying graph forms a cycle, and no pair of consecutive edges forms a directed path. DeBiasio and Molla [17] proved that the anti-directed Hamilton cycle is guaranteed to appear in a digraph $D$ if $\delta^0(D) \ge \frac{n}{2} + 1$. DeBiasio, Kühn, Molla,

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Osthus, and Taylor [16] showed that for sufficiently large $n$, every $n$-vertex digraph $D$ with $\delta^0(D) \geq \frac{n}{2}$ contains every orientation of a Hamilton cycle except, possibly, the anti-directed one. It would be interesting to generalise the above two results to a transversal version.

An *oriented graph* is a digraph with no cycle of length two. Keevash, Kühn, and Osthus [23] proved the oriented version of Dirac's theorem, which states that every $n$-vertex digraph with minimum semi-degree at least $\frac{3n-4}{8}$ contains a directed Hamilton cycle. It would be interesting to consider a transversal version of this theorem.

**Question 8.1.** *Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a collection of oriented graphs with common vertex set $V$ of size $n$. If $\delta^0(\mathcal{D}) \geq \lceil \frac{3n-4}{8} \rceil$, does $\mathcal{D}$ contain a transversal directed Hamilton cycle?*

We remark that our main technique used in this paper cannot be directly generalised to the above oriented case. This is because the absorbing structure we used does not exist in the above oriented graph collections since their semi-degree is much lower than $\frac{n}{2}$. To overcome this difficulty, one has to introduce new stable conditions and build an absorption structure using Lemma 3.4. It seems that new ideas are needed to prove the tight bound in Question 8.1.

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## Appendix A. Regularity for digraph collections

In this section, we prove the regularity lemma for digraph collections (i.e., Lemma 2.2). Before proceeding with the proof, we first make some preparations, including some necessary definitions and key theorems.

A *$k$-uniform hypergraph* (or *$k$-graph*) $\mathscr{F} = (V, E)$ consists of a vertex set $V$ and an edge set $E$ which is a family of $k$-element subsets of $V$, i.e., $E \subseteq \binom{V}{k}$. For a vertex $v \in V$, the degree of $v$ in $\mathscr{F}$, denoted by $\deg_{\mathscr{F}}(v)$, is the number of edges containing $v$. Given a $k$-graph $\mathscr{F}$, and $k$ disjoint sets $V_1, V_2, \ldots, V_k \subseteq V(\mathscr{F})$, define $\mathscr{F}[V_1, V_2, \ldots, V_k]$ to be the subhypergraph of $\mathscr{F}$ with edge set $\big\{\{v_1, v_2, \ldots, v_k\} \in E(\mathscr{F}) : v_i \in V_i \text{ for all } i \in [k]\big\}$. We call

$$d_{\mathscr{F}}(V_1, V_2, \ldots, V_k) := \frac{e(\mathscr{F}[V_1, V_2, \ldots, V_k])}{|V_1||V_2| \ldots |V_k|}$$

the *density* of the tuple $(V_1, V_2, \ldots, V_k)$ in $\mathscr{F}$. If $d_{\mathscr{F}}(V_1, V_2, \ldots, V_k) = 0$, then we say that $(V_1, V_2, \ldots, V_k)$ is empty in $\mathscr{F}$. Given $\epsilon > 0$ and $d \in [0, 1)$, we say that $(V_1, V_2, \ldots, V_k)$ is *$\epsilon$-regular* in $\mathscr{F}$ if for all $i \in [k]$, whenever $V_i' \subseteq V_i$ with $|V_i'| \geq \epsilon |V_i|$ one has

$$|d_{\mathscr{F}}(V_1', V_2', \ldots, V_k') - d_{\mathscr{F}}(V_1, V_2, \ldots, V_k)| \leq \epsilon,$$

and the tuple is *$(\epsilon, d)$-regular* if it is *$\epsilon$-regular* and $d_{\mathscr{F}}(V_1, V_2, \ldots, V_k) \geq d$. The following result was proved by Chung [14], and its proof follows the approach of the original Regularity Lemma for graphs [27].

**Theorem A.1** (Weak hypergraph regularity lemma). *For all integers $k \geq 2$, $L_0 \geq 1$, and every $\epsilon > 0$ there exists $N = N(\epsilon, L_0, k)$ such that if $\mathscr{F}$ is a $k$-graph on $n \geq N$ vertices, then $V(\mathscr{F})$ has a partition $V_0, V_1, \ldots, V_L$ such that the following hold:*

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- $L_0 \le L \le N$ *and* $|V_0| \le \epsilon n$,
- $|V_1| = \cdots = |V_L|$,
- *for all but at most* $\epsilon\binom{L}{k}$ *k-tuples* $\{i_1, \ldots, i_k\} \in \binom{[L]}{k}$, *we have that* $(V_{i_1}, \ldots, V_{i_k})$ *is* $\epsilon$*-regular in* $\mathscr{F}$.

Given two partitions $V_0, V_1, \ldots, V_k$ and $U_1, \ldots, U_\ell$ of a vertex set, we say that $V_0, V_1, \ldots, V_k$ *refines* $U_1, \ldots, U_\ell$ if for all $V_i$ with $i \in [k]$ there is $U_j$ for some $j \in [\ell]$ that contains $V_i$. For some integer $s > 0$, let $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_s$ be $s$ $K$-partite $k$-graphs sharing a common vertex set $V := V_1 \cup \ldots \cup V_K$. As a consequence of the proof of Theorem A.1, we can deduce the following corollary.

**Corollary A.2.** *For all integers* $K \ge k \ge 2$, $s \ge 1$, $L_0 \ge 1$, *and every* $\epsilon > 0$, *there exists* $N = N(\epsilon, s, L_0, K, k)$ *such that if* $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_s$ *are* $s$ $K$*-partite* $k$*-graphs sharing a common vertex set* $V := V_1 \cup \ldots \cup V_K$ *of size* $n \ge N$, *then* $V$ *can be refined into a partition* $U_0, U_1, \ldots, U_L$ *satisfying the following properties:*

- $L_0 \le L \le N$ *and* $|U_0| \le \epsilon n$,
- $|U_1| = \cdots = |U_L|$,
- *for each* $i \in [s]$ *and for all but at most* $\epsilon\binom{L}{k}$ *k-tuples* $\{i_1, \ldots, i_k\} \in \binom{[L]}{k}$, *we have that* $(U_{i_1}, \ldots, U_{i_k})$ *is* $\epsilon$*-regular in* $\mathscr{F}_i$.

The degree form of the weak hypergraph regularity lemma [29] is proved in the same way as the original degree form of the regularity lemma, which in fact can be derived from Theorem A.1 via some cleaning.

**Theorem A.3**. (Degree form of the weak hypergraph regularity lemma, [29]). *For all integers* $k \ge 2$, $L_0 \ge 1$ *and every* $\epsilon > 0$, *there exists* $N = N(\epsilon, L_0, k)$ *such that for every* $d \in [0, 1)$ *and for every* $k$*-graph* $\mathscr{F}$ *on* $n \ge N$ *vertices, there exists a partition of* $V(\mathscr{F})$ *into* $V_0, V_1, \ldots, V_L$ *and a spanning subhypergraph* $\mathscr{F}'$ *of* $\mathscr{F}$ *such that the following properties hold:*

- $L_0 \le L \le N$ *and* $|V_0| \le \epsilon n$,
- $|V_1| = \cdots = |V_L|$,
- $deg_{\mathscr{F}'}(v) \ge deg_{\mathscr{F}}(v) - (d + \epsilon)n^{k-1}$ *for all* $v \in V(\mathscr{F})$,
- *every edge of* $\mathscr{F}'$ *with more than one vertex in a single cluster* $V_i$ *for some* $i \in [L]$ *has at least one vertex in* $V_0$,
- *for all* $k$*-tuples* $\{i_1, i_2, \ldots, i_k\} \in \binom{[L]}{k}$, *we have that* $(V_{i_1}, V_{i_2}, \ldots, V_{i_k})$ *is either empty or* $(\epsilon, d)$*-regular in* $\mathscr{F}'$.

In fact, from the proof of Theorem A.3 in [29] and Corollary A.2 we can derive the following stronger version of the conclusion.

**Theorem A.4**. *For all integers* $K \ge k \ge 2$, $s \ge 1$, $L_0 \ge 1$, *and every* $\epsilon > 0$, *there exists* $N = N(\epsilon, s, L_0, K, k)$ *such that, for any* $s$ $K$*-partite* $k$*-graphs* $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_s$ *sharing a common vertex set* $V := V_1 \cup \ldots \cup V_K$ *of size* $n \ge N$, *we can find a refinement partition* $U_0, U_1, \ldots, U_L$ *of* $V$, *and a spanning subhypergraph* $\mathscr{F}_i' \subseteq \mathscr{F}_i$ *for each* $i \in [s]$ *such that the following properties hold:*

- $L_0 \le L \le N$ *and* $|U_0| \le \epsilon n$,
- $|U_1| = \cdots = |U_L|$,
- *for each* $i \in [s]$ *and each* $v \in V$, $deg_{\mathscr{F}_i'}(v) \ge deg_{\mathscr{F}_i}(v) - (d + \epsilon)n^{k-1}$,
- *for each* $i \in [s]$, *every edge of* $\mathscr{F}_i'$ *with more than one vertex in a single cluster* $V_j$ *for some* $j \in [L]$ *has at least one vertex in* $V_0$,

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- *for each $i \in [s]$ and all $k$-tuples $\{i_1, i_2, \dots, i_k\} \in \binom{[L]}{k}$, we have that $(U_{i_1}, U_{i_2}, \dots, U_{i_k})$ is either empty or $(\epsilon, d)$-regular in $\mathscr{F}_i'$.*

Since the proof of Theorem A.4 closely follows that of Theorem A.3, requiring only a repeated application of the cleaning operation on $s$ hypergraphs, we omit it here for brevity.

Let $\mathcal{D} = \{D_i : i \in \mathcal{C}\}$ be a collection of digraphs on a common vertex set $[n]$. Based on $\mathcal{D}$, we construct the auxiliary 4-graph $\mathscr{H}$ as follows. The vertex set is $V(\mathscr{H}) = [n] \cup \mathcal{C} \cup S_1 \cup S_2$, where $S_1$ and $S_2$ are new sets with $|S_1| = |S_2| = n$, and the four sets $[n], \mathcal{C}, S_1, S_2$ are pairwise disjoint. The edge set of $\mathscr{H}$ is defined by

$$E(\mathscr{H}) = \big\{\{i, j, c, x\} : i, j \in [n],\ c \in \mathcal{C},\ ij \in E(D_c); x \in S_1 \text{if } i < j \text{ and } x \in S_2 \text{ otherwise}\big\}.$$

Now, we prove Lemma 2.2.

**Proof of Lemma. 2.2** By increasing $L_0$ and decreasing $\epsilon$, $\delta$ as necessary, we may assume that $0 < \frac{1}{L_0} \ll \epsilon \ll \delta \ll 1$. Let $L_1 := \frac{24L_0}{\delta}$. Choose further positive constants $\epsilon', \alpha$ satisfying $\frac{1}{L_1} \ll \epsilon' \ll \alpha \ll \epsilon \ll \delta \ll 1$. Let $n_1$ be derived from Theorem A.3 by applying it with the parameters $(\epsilon', L_1, 4)$. Next, let $n_2$ be obtained from Theorem A.4 using the parameters $(\epsilon', 2, L_0, n_1, 3)$. Define $n_0 := \frac{4n_2}{\alpha}$. By increasing $n_0$ if necessary, we may assume that $L_1 \ll n_1 \ll n_2 \ll n_0$. Altogether,

$$0 < \frac{1}{n_0} \ll \frac{1}{n_2} \ll \frac{1}{n_1} \ll \frac{1}{L_1} \ll \epsilon' \ll \alpha \ll \epsilon \ll \delta \ll 1.$$

Let $n \geq n_0$ be an integer and $\mathcal{D}$ be a digraph collection on vertex set $V := [n]$ and with colour set $\mathcal{C}$, where $\delta n \leq |\mathcal{C}| \leq \frac{n}{\delta}$. Let $d \in [0, 1)$ and let $\mathscr{H}$ be the auxiliary 4-graph obtained from $\mathcal{D}$ with vertex set $U := V \cup \mathcal{C} \cup S_1 \cup S_2$. Theorem A.3 implies that there is a partition $U_0, U_1, \dots, U_K$ of $U$ and a spanning subhypergraph $\mathscr{H}_0$ of $\mathscr{H}$ such that

(a) $L_1 \leq K \leq n_1$ and $|U_0| \leq \epsilon'|U|$,
(b) $|U_1| = \cdots = |U_K| =: m'$,
(c) $\deg_{\mathscr{H}_0}(v) \geq \deg_{\mathscr{H}}(v) - (d + \epsilon')|U|^3$ for all $v \in U$,
(d) every edge of $\mathscr{H}_0$ with more than one vertex in a single cluster $U_i$ for some $i \in [K]$ has at least one vertex in $U_0$,
(e) for all 4-tuples $\{i_1, i_2, i_3, i_4\} \in \binom{[K]}{4}$, we have that $(U_{i_1}, U_{i_2}, U_{i_3}, U_{i_4})$ is either empty or $(\epsilon', d)$-regular in $\mathscr{H}_0$.

Delete all edges $e \in E(\mathscr{H}_0)$ satisfying $|e \cap U_i| \geq 2$ for some $i \in [K]$ and $e \cap (\mathcal{C} \cap U_0) = \emptyset$. Denote the resulting spanning subhypergraph by $\mathscr{H}'$. Note that for every $v \in U$, the number of edges incident to $v$ that are deleted is fewer than

$$\epsilon'|U|^3 + Km'^2|U| \leq \epsilon'|U|^3 + m'|U|^2 \leq \epsilon'|U|^3 + \frac{|U|^3}{L_1} \leq 2\epsilon'|U|^3.$$

Combining $(c)$, for each $v \in U$ we have

$$\deg_{\mathscr{H}'}(v) \geq \deg_{\mathscr{H}}(v) - (d + 3\epsilon')|U|^3.$$

Partition each cluster $U_i$, $i \in [K]$, into $\frac{1}{\alpha}$ subclusters of size at most $m_0$ so that all but at most 4 subclusters of $U_i$ have size exactly $m_0$ and the property that they lie entirely within one of $V, \mathcal{C}$, $S_1$ and $S_2$. If a subcluster does not have this property, then add it to $U_0$. The new exceptional set has size at most $|U_0| + 4\alpha m'K$, and let $V_0$ be its intersection with $V$, $\mathcal{C}_0$ be its intersection with $\mathcal{C}$. Relabel the subclusters so that those which are subsets of $V$ are $V_1, \dots, V_L$ and those which are subsets of $\mathcal{C}$ are $\mathcal{C}_1, \dots, \mathcal{C}_M$. For each $c \in \mathcal{C}$, let $D_c'$ be the spanning sub-digraph of $D_c$ with vertex set $V =: [n]$ and edge set

$$\big\{uv : u < v,\ x \in S_1 \text{ and } \{u, v, c, x\} \in \mathscr{H}'\big\} \cup \big\{vu : u < v,\ y \in S_2 \text{ and } \{u, v, c, y\} \in \mathscr{H}'\big\}.$$

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We claim that the properties of Lemma 2.2 (i)-(iv) are satisfied.

**Property (i)**. Note that $m'K \le |U|$. Since $\epsilon' \ll \epsilon$, one has

$$|V_0| + |\mathcal{C}_0| \le |U_0| + 4\alpha m'K \le (\epsilon' + 4\alpha)|U| = (\epsilon' + 4\alpha)\left(3n + \frac{n}{\delta}\right) \le \frac{\epsilon n}{2}. \tag{A.1}$$

Also,

$$L + M \le \frac{2K}{\alpha} \le \frac{2n_1}{\alpha} \le \frac{n_0}{2},$$

proving the required upper bound for both $L$ and $M$. Furthermore, $m_0(L+M) \le n + |\mathcal{C}|$ and

$$\frac{L_1}{6} \le \frac{3n + |\mathcal{C}|}{6m_0} \le \frac{n + |\mathcal{C}|}{2m_0} \le \frac{n + |\mathcal{C}| - \epsilon n}{m_0} \le L + M,$$

so

$$L = \frac{n - |V_0|}{m_0} \ge \frac{(n - |V_0|)(L+M)}{n + |\mathcal{C}|} \ge \frac{(n - |V_0|)L_1}{6(n + \frac{n}{\delta})} \ge \frac{\delta L_1}{24} = L_0, \tag{A.2}$$

and similarly for $M$.

**Property (ii)**. It follows by construction.

**Property (iii)**. By the construction of $\mathscr{H}'$, for all $v \in V$ we have

$$\sum_{c \in \mathcal{C}} \left(d^+_{D'_c}(v) + d^-_{D'_c}(v)\right) \ge \frac{\deg_{\mathscr{H}'}(v)}{n},$$

and $e(D'_c) \ge \frac{\deg_{\mathscr{H}'}(c)}{n}$ for all $c \in \mathcal{C}$. Furthermore,

$$(d + 3\epsilon')(3n + |\mathcal{C}|)^3 \le \left(\frac{8d}{\delta^3} + \frac{24\epsilon'}{\delta^3}\right) n^3 \le \left(\frac{8d}{\delta^3} + \frac{\epsilon}{4}\right) n^3. \tag{A.3}$$

Notice that $\sum_{c \in \mathcal{C}} \left(d^+_{D_c}(v) + d^-_{D_c}(v)\right) = \frac{\deg_{\mathscr{H}}(v)}{n}$ for all $v \in V$. Thus,

$$\begin{aligned}
\sum_{c \in \mathcal{C}} d^+_{D'_c}(v) &\ge \frac{\deg_{\mathscr{H}'}(v)}{n} - \sum_{c \in \mathcal{C}} d^-_{D'_c}(v) \\
&\ge \frac{\deg_{\mathscr{H}}(v)}{n} - \sum_{c \in \mathcal{C}} d^-_{D_c}(v) - \left(\frac{8d}{\delta^3} + \frac{\epsilon}{4}\right) n^2 \\
&= \sum_{c \in \mathcal{C}} d^+_{D_c}(v) - \left(\frac{8d}{\delta^3} + \frac{\epsilon}{4}\right) n^2
\end{aligned} \tag{A.4}$$

for all $v \in V$, and similarly for $\sum_{c \in \mathcal{C}} d^-_{D'_c}(v)$. Additionally, observe that $e(D_c) = \frac{\deg_{\mathscr{H}}(c)}{n}$ for each $c \in \mathcal{C}$. It follows that for each $c \in \mathcal{C}$,

$$e(D'_c) \ge \frac{\deg_{\mathscr{H}'}(c)}{n} \ge \frac{\deg_{\mathscr{H}}(c)}{n} - \left(\frac{8d}{\delta^3} + \frac{\epsilon}{4}\right) n^2 = e(D_c) - \left(\frac{8d}{\delta^3} + \frac{\epsilon}{4}\right) n^2. \tag{A.5}$$

**Property (iv).** It follows by construction.

**Property (v).** Suppose that $(V_0 \cup \mathcal{C}_0, V_1, \ldots, V_L, \mathcal{C}_1, \ldots, \mathcal{C}_M)$ does not satisfy property (v), we are to refine it and get a desired partition. Now, define the following auxiliary 3-graphs $\mathscr{H}_1$ and

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$\mathscr{H}_2$ on the common vertex set $U' := ([n] \cup \mathcal{C}) \setminus (V_0 \cup \mathcal{C}_0)$ and

$$E(\mathscr{H}_1) = \left\{\{u, v, c\} : uv \in D'_c[V_i, V_j], u, v \in V \setminus V_0, c \in \mathcal{C} \setminus \mathcal{C}_0, 1 \le i < j \le L\right\},$$

$$E(\mathscr{H}_2) = \left\{\{u, v, c\} : uv \in D'_c[V_j, V_i], u, v \in V \setminus V_0, c \in \mathcal{C} \setminus \mathcal{C}_0, 1 \le i < j \le L\right\}.$$

Notice that the partition $P = (V_1, \dots, V_L, \mathcal{C}_1, \dots, \mathcal{C}_M)$ can be viewed as two distinct partitions $P_1, P_2$ for the hypergraphs $\mathscr{H}_1, \mathscr{H}_2$ (both partitions divide $V$ and $\mathcal{C}$ in the same way, but the sets of edges among the partition sets are different). By increasing $n_2$, we may assume that $n_2 \gg N(\epsilon', 2, L_0, L+M, 3)$ (which is defined in Theorem A.4). Consequently, $n_0 \gg N(\epsilon', 2, L_0, L+M, 3)$ as well. Applying Theorem A.4, we obtain a refinement $U'_0, U'_1, \dots, U'_{K'}$ of the partition $P$ together with spanning subhypergraphs $\mathscr{H}'_1 \subseteq \mathscr{H}_1$ and $\mathscr{H}'_2 \subseteq \mathscr{H}_2$ that satisfy the following:

(a′) $L_0 \le K' \le n_2$ and $|U'_0| \le \epsilon'|U'|$,

(b′) $|U'_1| = \cdots = |U'_{K'}|$,

(c′) for each $i \in [2]$ and each $v \in V \setminus V_0$, $\deg_{\mathscr{H}'_i}(v) \ge \deg_{\mathscr{H}_i}(v) - (d+\epsilon')|U'|^2$,

(d′) for each $i \in [2]$, every edge of $\mathscr{H}'_i$ with more than one vertex in a single cluster $U'_j$ for some $j \in [K']$ has at least one vertex in $U'_0$,

(e′) for each $i \in [2]$ and all 3-tuples $\{i_1, i_2, i_3\} \in \binom{[K']}{3}$, we have that $(U'_{i_1}, U'_{i_2}, U'_{i_3})$ is either empty or $(\epsilon', d)$-regular in $\mathscr{H}'_i$.

Following the same operation as earlier, let $V'_0 := V \cap U'_0$ and $\mathcal{C}'_0 := \mathcal{C} \cap U'_0$. Relabel the subclusters such that those which are subsets of $V$ are $V'_1, \dots, V'_{L'}$ and those which are subsets of $\mathcal{C}$ are $\mathcal{C}'_1, \dots, \mathcal{C}'_{M'}$. For each $c \in \mathcal{C} \setminus (\mathcal{C}_0 \cup \mathcal{C}'_0)$, let $D''_c$ be the spanning sub-digraph of $D_c$ with vertex set $V$ and edge set

$$\left\{uv : \{u, v, c\} \in \mathscr{H}'_1 \text{ or } \{u, v, c\} \in \mathscr{H}'_2\right\} \cup E(D'^{\pm}_c[V_0, V \setminus (V_0 \cup V'_0)]).$$

Let $D^*_i := D'_i$ for $i \in \mathcal{C}_0 \cup \mathcal{C}'_0$ and $D^*_i := D''_i$ for $i \in \mathcal{C} \setminus (\mathcal{C}_0 \cup \mathcal{C}'_0)$. Let

$$P^* := (V_0 \cup \mathcal{C}_0 \cup V'_0 \cup \mathcal{C}'_0, V'_1, \dots, V'_{L'}, \mathcal{C}'_1 \dots, \mathcal{C}'_{M'})$$

be a partition of $V \cup \mathcal{C}$. Next, we will re-verify that $P^*$ and digraph collection $\mathcal{D}^* := \{D^*_i : i \in \mathcal{C}\}$ satisfies (i)–(v) of Lemma 2.2. By the construction of $\mathscr{H}_1$ and $\mathscr{H}_2$, it is easy to see that $P^*$ and $\mathcal{D}^*$ satisfy the property (v). Thus, we only need to re-verify properties (i)-(iv).

**Property (i).** By (5), (6) and (a′), we have

$$|V_0| + |\mathcal{C}_0| + |V'_0| + |\mathcal{C}'_0| \le \frac{\epsilon n}{2} + \epsilon'|U'| \le \epsilon n,$$

and

$$L_0 \le L \le L' \le n_2 \le n_0 \text{ and } L_0 \le M \le M' \le n_2 \le n_0.$$

**Property (ii).** It follows from (b') that $|V'_1| = \cdots = |V'_{L'}| = |\mathcal{C}'_1| = \cdots = |\mathcal{C}'_{M'}|$, as required.

**Property (iii).** For each vertex $v \in V$ and each colour $c \in \mathcal{C}$, by the construction of $D^*_c$, (c'), (7), (8) and (9), we have

$$e(D^*_c) \ge e(D_c) - \left(\frac{8d}{\delta^3} + \frac{\epsilon}{4}\right) n^2 - 2\left(\frac{8d}{\delta^3} + \frac{\epsilon}{4}\right) n^2 - \frac{4\epsilon' n^2}{\delta} \ge e(D_c) - \left(\frac{24d}{\delta^3} + \epsilon\right) n^2$$

and

$$\sum_{c \in \mathcal{C}} d^+_{D^*_c}(v) \ge \sum_{c \in \mathcal{C}} d^+_{D_c}(v) - \left(\frac{8d}{\delta^3} + \frac{\epsilon}{4}\right) n^2 - 2\left(\frac{8d}{\delta^3} + \frac{\epsilon}{4}\right) n^2 - \frac{2\epsilon' n^2}{\delta^2} \ge \sum_{c \in \mathcal{C}} d^+_{D_c}(v) - \left(\frac{24d}{\delta^3} + \epsilon\right) n^2.$$

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Analogously, the same inequality holds for $\sum_{c\in\mathcal{C}} d^-_{D^*_c}(v)$.

**Property (iv).** Notice that $\mathscr{H}_1$ and $\mathscr{H}_2$ are $(L+M)$-partite hypergraphs. Thus, there is no edge of $\mathscr{H}'_i$ with more than one vertex in a single cluster $U'_j$ for some $j\in[K']$. This implies that $P^*$ and $\mathcal{D}^*$ satisfy (iv).

This completes the proof of Lemma 2.2. □

## Appendix B. Transversal perfect matching in bipartite graph collections

### B.1 Proof of Lemma 2.7

In this subsection we prove Lemma 2.7, which provides a characteristic partition of each balanced bipartite graph whose minimum degree is just below $\frac{n}{2}$, allowing a tiny fraction of vertices that may have even smaller degrees.

**Proof of Lemma 2.7.** We first add vertices with degree at most $\left(\frac{1}{2}-\epsilon^5\right)n$ to $C_1\cup C_2$. Let $\mu=\epsilon^5$. Since $G$ is $\epsilon$-extremal, there are $X\subseteq V_1$, $Y\subseteq V_2$ each of size at least $\left(\frac{1}{2}-\mu\right)n$ such that $e_G(X,Y)\le\mu n^2$. Thus, it is easy to see that for all but at most $\sqrt{\mu}n$ vertices $v$ in $X$ we have $d_G(v,Y)\le\sqrt{\mu}n$. Similarly, for all but at most $\sqrt{\mu}n$ vertices $v$ in $Y$ we have $d_G(v,X)\le\sqrt{\mu}n$. We add these exceptional vertices from $X$ and from $Y$ to $C_1$ and $C_2$, respectively. Now, the new sets $X$ and $Y$ each has size at least $\left(\frac{1}{2}-\mu\right)n-\sqrt{\mu}n\ge\left(\frac{1}{2}-2\sqrt{\mu}\right)n$. Choose $X_1\subseteq X$, $Y_1\subseteq Y$ such that $|X_1|=|Y_1|=\left(\frac{1}{2}-2\sqrt{\mu}\right)n$.

Notice that $|C_i|\le(d+\sqrt{\mu})n\le 2\sqrt{\mu}n$ by $d\ll\epsilon$. Thus, we can choose $Z_1\subseteq V_1\setminus(X_1\cup C_1)$ and $Z_2\subseteq V_2\setminus(Y_1\cup C_2)$ such that $|Z_1|=|Z_2|=(\frac{1}{2}-2\sqrt{\mu})n$. Recall that $d_G(v,Y_1)\le\sqrt{\mu}n$ for each vertex $v\in X_1$. This implies that for each $v\in X_1$, one has

$$d_G(v,Z_2)\ge\left(\frac{1}{2}-\mu-5\sqrt{\mu}\right)n-|C_2|\ge\left(\frac{1}{2}-8\sqrt{\mu}\right)n.$$

Similarly, $d_G(v,Z_1)\ge\left(\frac{1}{2}-8\sqrt{\mu}\right)n$ for each $v\in Y_1$.

Since the number of edges between $X_1$ and $Z_2$ is at least $(\frac{1}{2}-2\sqrt{\mu})n\times\left(\frac{1}{2}-8\sqrt{\mu}\right)n$ and $|Z_2|=(\frac{1}{2}-2\sqrt{\mu})n$, there are at most $50\mu^{\frac{1}{4}}n$ vertices $v$ in $Z_2$ such that $d_G(v,X_1)\le(\frac{1}{2}-10\mu^{\frac{1}{4}})n$. Similarly, for all but at most $50\mu^{\frac{1}{4}}n$ vertices $v$ in $Z_1$ we have $d_G(v,Y_1)\ge(\frac{1}{2}-10\mu^{\frac{1}{4}})n$. Thus, we can choose $A_1\subseteq X_1$, $B_1\subseteq Z_1$, $A_2\subseteq Y_1$, $B_2\subseteq Z_2$ such that $|A_1|=|B_1|=|A_2|=|B_2|=(\frac{1}{2}-\epsilon)n$ and $d_G(v,X_i)\ge(\frac{1}{2}-2\epsilon)n$ for $v\in X_{3-i}$ and $X\in\{A,B\}$, $i\in[2]$. Let $C_1=V_1\setminus(A_1\cup B_1)$ and $C_2=V_2\setminus(A_2\cup B_2)$. Hence, $(A_1,B_1,C_1,A_2,B_2,C_2)$ is the desired partition of $G$. □

### B.2 Proof of Theorem 2.10

In this subsection we prove Theorem 2.10, giving a sufficient condition for the existence of a transversal perfect matching in bipartite graph collections. Before proceeding with the proof, we first present some key lemmas that will be used in the proof.

Assume that $\mathcal{G}=\{G_1,\dots,G_n\}$ is a collection of bipartite graphs on a common vertex partition $V_1\cup V_2$ with $|V_1|=|V_2|=n$. For $\mu>0$, we say that $\mathcal{G}$ is *$\mu$-nice* if for every $A\subseteq V_1$ and $B\subseteq V_2$ of size $\lfloor\frac{n}{2}\rfloor$, we have $e_{\mathcal{G}}(A,B):=\sum_{i\in[n]}e_{G_i}(A,B)\ge\mu n^3$.

**Lemma B.1.** *Suppose that $0<\frac{1}{n}\ll\mu'\ll\alpha\ll\gamma,\epsilon\ll\delta\ll 1$. Let $\mathcal{G}=\{G_1,\dots,G_n\}$ be a collection of bipartite graphs on a common vertex partition $V=V_1\cup V_2$ with $|V_1|=|V_2|=n$. If $\mathcal{G}$ is $(\gamma,\alpha,\epsilon,\delta)$-stable, then $\mathcal{G}$ is $\mu'$-nice.*

**Proof.** For the sake of contradiction, suppose that $\mathcal{G}$ is not $\mu'$-nice. Then there are sets $A\subseteq V_1$ and $B\subseteq V_2$, each of size $\lfloor\frac{n}{2}\rfloor$, such that $e_{\mathcal{G}}(A,B)\le\mu' n^3$. Therefore, for all but at most $\sqrt{\mu'}n$ colours

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$i \in [n]$, we have $e_{G_i}(A,B) < \sqrt{\mu'}n^2$, implying that such $G_i$ is not $\sqrt{\mu'}$-nice. This further implies that $\mathcal{G}$ is not $(1-\sqrt{\mu'}, \sqrt{\mu'})$-strongly stable and hence not $(\gamma,\alpha)$-strongly stable, since $\mu' \ll \gamma, \alpha$.

Without loss of generality, assume that for each $i \in [(1-\sqrt{\mu'})n]$, $G_i$ is not $\sqrt{\mu'}$-nice. Since $\epsilon > \mu'^{1/10}$, $G_i$ is $\epsilon$-extremal for all $i \in [(1-\sqrt{\mu'})n]$. By Lemma 2.7, the graph $G_i$ has a characteristic partition, denoted as $(A_1^i, B_1^i, C_1^i, A_2^i, B_2^i, C_2^i)$. It follows that, after possible relabelling $A_1^i, B_1^i, A_2^i, B_2^i$, for all but at most $\delta^2 n$ colours $i \in [(1-\sqrt{\mu'})n]$ we have

$$|A \triangle A_1^i| \le \frac{\delta n}{2} \text{ and } |B \triangle B_2^i| \le \frac{\delta n}{2}.$$

Otherwise, $e_{\mathcal{G}}(A,B) \ge \frac{\delta^3 n^3}{8} > \mu' n^3$ by $\mu' \ll \epsilon \ll \delta$, a contradiction. Delete such colours from $[(1-\sqrt{\mu'})n]$ and let $\mathcal{C}$ be the remaining colour set. Thus, for every $i,j \in \mathcal{C}$ the graphs $G_i$ and $G_j$ are not $\delta$-crossing. Therefore,

$$e(C_{\mathcal{G}}^{\epsilon,\delta}) \le \sqrt{\mu'}n^2 + \delta^2 n^2 < \delta n^2 \text{ since } \mu' \ll \delta.$$

Thus, $\mathcal{G}$ is not $(\epsilon,\delta)$-weakly stable, which contradicts the assumption that $\mathcal{G}$ is $(\gamma,\alpha,\epsilon,\delta)$-stable. □

The following result shows that if a bipartite graph collection $\mathcal{G}$ is $\mu$-nice under the minimum degree constraint, then we can find a large rainbow matching in $\mathcal{G}$.

**Lemma B.2.** *Let $0 < \frac{1}{n} \ll d \ll \mu \ll 1$. Suppose that $\mathcal{G} = \{G_1, \dots, G_n\}$ is a collection of bipartite graphs on a common vertex partition $V = V_1 \cup V_2$ with $|V_1| = |V_2| = n$, and for each vertex $v \in V$ we have $d_{G_i}(v) \ge \left(\frac{1}{2} - \mu\right) n$ for all but at most $dn$ colours $i \in [n]$. If $\mathcal{G}$ is $6\mu$-nice, then $\mathcal{G}$ contains a rainbow matching $\mathcal{M}$ with $e(\mathcal{M}) \ge n - dn - 2$.*

**Proof.** Let $\mathcal{M}$ be a maximum rainbow matching in $\mathcal{G}$, and suppose for a contradiction that $e(\mathcal{M}) \le n - dn - 3$. Since for each vertex $v \in V$ we have $d_{G_i}(v) \ge \left(\frac{1}{2} - \mu\right) n$ for all but at most $dn$ colours $i \in [n]$, there are vertices $v_1 \in V_1 \setminus V(\mathcal{M})$, $v_2 \in V_2 \setminus V(\mathcal{M})$ and distinct colours $c_1, c_2 \notin \mathrm{col}(\mathcal{M})$ such that $d_{G_{c_\ell}}(v_\ell) \ge \left(\frac{1}{2} - \mu\right) n$ for $\ell \in \{1,2\}$. Put $N_\ell := N_{G_{c_\ell}}(v_\ell)$. Thus, $|N_\ell| \ge \left(\frac{1}{2} - \mu\right) n$. Let $\mathcal{C} := [n] \setminus (\mathrm{col}(\mathcal{M}) \cup \{c_1, c_2\})$. For $x \in V(\mathcal{M})$, let $x^+$ denote the unique neighbour of $x$ in $\mathcal{M}$, and for $A \subseteq V(\mathcal{M})$, let $A^+ := \{x^+ : x \in A\}$.

By the maximality of $\mathcal{M}$, we have $N_1 \cap N_2^+ = \emptyset$, $N_2 \cap N_1^+ = \emptyset$ and $N_1, N_2 \subseteq V(\mathcal{M})$. This implies that

$$e(\mathcal{M}) \ge |N_1| + |N_2| \ge 2\left(\frac{1}{2} - \mu\right) n = n - 2\mu n.$$

Moreover, for each $c \in \mathcal{C}$, we have $e_{G_c}(N_1^+, N_2^+) = 0$ by the maximality of $\mathcal{M}$. Thus, it suffices to show that for all $c \in \mathrm{col}(\mathcal{M}[N_1, N_1^+] \cup \mathcal{M}[N_2, N_2^+])$, we have $e_{G_c}(N_1^+, N_2^+) \le 4\mu n^2$. If this holds, then

$$e_{\mathcal{G}}(N_1^+, N_2^+) \le 4\mu n^2 (n - 2\mu n) + 2\mu n^3 < 6\mu n^3,$$

contradicting the assumption that $\mathcal{G}$ is $6\mu$-nice. Next, we consider the case $c \in \mathrm{col}(\mathcal{M}[N_1, N_1^+])$; the proof for the case $c \in \mathrm{col}(\mathcal{M}[N_2, N_2^+])$ is similar and is omitted.

Let $c \in \mathrm{col}(\mathcal{M}[N_1, N_1^+])$ and $w \in N_1$ such that $\mathrm{col}(ww^+) = c$. Now let $\mathcal{M}' = \mathcal{M} - ww^+ + v_1 w$ be a rainbow matching with $\mathrm{col}(\mathcal{M}') = (\mathrm{col}(\mathcal{M}) \setminus \{c\}) \cup \{c_1\}$. Clearly, $\mathcal{M}'$ is a maximum rainbow matching. Notice that there exists a colour $c_3 \in \mathcal{C} \setminus \{c_1, c_2\}$ such that $d_{G_{c_3}}(w^+) \ge \left(\frac{1}{2} - \mu\right) n$. Let $N_3 := N_{G_{c_3}}(w^+)$. Consider the new vertex pair $(w^+, v_2)$ and colour pair $(c_3, c_2)$. By the maximality of $\mathcal{M}'$, we have $N_3^+ \cap N_2 = \emptyset$. Recall that $N_1^+ \cap N_2 = \emptyset$ and $|N_i| \ge \left(\frac{1}{2} - \mu\right) n$ for each $i \in [3]$. Thus,

$$|N_3^+ \triangle N_1^+| = |N_3^+ \cup N_1^+| - |N_3^+ \cap N_1^+| \le (n - |N_2|) - (|N_1| + |N_3| - (n - |N_2|)) \le 4\mu n. \quad \text{(A.6)}$$

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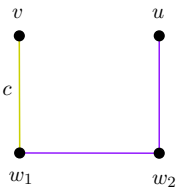



**Figure 5.** $w_1w_2$ is a $c$-absorbing edge of $(u, v)$.

On the other hand, by the maximality of $\mathcal{M}'$, we have $e_{G_c}(N_3^+, N_2^+) = 0$. Thus, together with (10), we have $e_{G_c}(N_1^+, N_2^+) \le 4\mu n^2$, as desired. □

Let $n$ be a positive integer and let $d, \epsilon > 0$ such that $0 < \frac{1}{n} \ll d \ll \epsilon \le 1$. Consider a bipartite graph collection $\mathcal{G} = \{G_1, \dots, G_n\}$ on a common vertex partition $V = V_1 \cup V_2$ with $|V_1| = |V_2| = n$ such that $d_G(x) \ge (\frac{1}{2} - \epsilon^5)n$ for all but at most $dn$ vertices $x \in V$. Lemma 2.7 implies that for each $i \in [n]$ if $G_i$ is $\epsilon$-extremal, then we can fix a characteristic partition

$$(A_1^i, B_1^i, C_1^i, A_2^i, B_2^i, C_2^i) \text{of} G_i.$$

A vertex $v \in V$ is called *$i$-good* if either $G_i$ is $\epsilon$-extremal and $v \in V \setminus (C_i^1 \cup C_i^2)$ or $G_i$ is not $\epsilon$-extremal and $d_{G_i}(v) \ge (\frac{1}{2} - \epsilon^3)n$. Note that for each $i \in [n]$ and $j \in \{1, 2\}$, there are at most $2\epsilon n$ vertices in $V_j$ that are not $i$-good.

**Definition B.3** (absorbing edge, absorbing matching). *Let $\mathcal{G} = \{G_1, \dots, G_n\}$ be a collection of bipartite graphs on a common vertex partition $V = V_1 \cup V_2$ with $|V_1| = |V_2| = n$. Given two vertices $u \in V_1$ and $v \in V_2$, an edge $e = w_1w_2$ with $u, v \notin \{w_1, w_2\}$ is called a $c$-absorbing edge of $(u, v)$ if $c \in L(w_1v)$ and $\mathrm{col}(w_1w_2) \in L(uw_2)$ (see Figure 5).*

*Given $\delta, \delta', \gamma, \gamma' \ge 0$, a rainbow matching $\mathcal{M}$ is an absorbing matching with parameters $(\delta, \delta', \gamma, \gamma')$ if $|E(\mathcal{M})| \le \gamma n$ and there exists a colour set $\mathcal{C}$ of size at least $\delta n$ such that given any colour $c \in \mathcal{C}$ and any $c$-good vertex $v \in V_2$, for all but at most $\delta' n$ vertices $u \in V_1$, there are at least $\gamma' n$ $c$-absorbing edges of $(u, v)$ within $\mathcal{M}$.*

The following two lemmas establish the existence of absorbing matchings when $\mathcal{G}$ is stable. We first consider the case that $\mathcal{G}$ is strongly stable.

**Lemma B.4.** *Let $0 < \frac{1}{n} \ll d \ll \lambda, \mu \ll \gamma, \alpha \ll 1$. Suppose that $\mathcal{G} = \{G_1, \dots, G_n\}$ is a collection of bipartite graphs on a common vertex partition $V = V_1 \cup V_2$ with $|V_1| = |V_2| = n$, satisfying the following conditions:*

- *for every $i \in [n]$, $d_{G_i}(x) \ge \left(\frac{1}{2} - \mu\right) n$ for all but at most $dn$ vertices $x \in V_1 \cup V_2$,*
- *for every $x \in V_1 \cup V_2$, $d_{G_i}(x) \ge \left(\frac{1}{2} - \mu\right) n$ for all but at most $dn$ colours $i \in [n]$.*

*If $\mathcal{G}$ is $(\gamma, \alpha)$-strongly stable, then there exists an absorbing matching with parameters $(1, 0, \lambda, \lambda^2)$.*

**Proof.** Without loss of generality, assume that $G_1, \dots, G_{\gamma n}$ are $\alpha$-nice. For each $i \in [\lambda n]$, define

$$\mathcal{F}_i := \{(u, v) \in V_1 \times V_2 : i \in L(uv)\}.$$

The degree condition of $G_i$ implies that $|\mathcal{F}_i| = e(G_i) \ge (n - dn)\left(\frac{1}{2} - \mu\right) n > \frac{1}{4}n^2$. Fix a colour $c \in [n]$ and two vertices $u \in V_1, v \in V_2$ such that $v$ is $c$-good, let

$$Z_i(c, uv) := \left\{(w, w') \in \mathcal{F}_i : ww' \text{ is a } c\text{-absorbing edge of } (u, v)\right\} \text{ and } Z(c, uv) := \bigcup_{i \in [\lambda n]} Z_i(c, uv).$$

Since $G_i$ is $\alpha$-nice, $e(G_i[N_{G_i}(u), N_{G_c}(v)]) \ge \frac{\alpha n^2}{2}$ for all but at most $dn$ indices $i \in [\gamma n]$. This means that for all but at most $dn$ indices $i \in [\gamma n]$, we have $|Z_i(c, uv)| \ge \frac{\alpha n^2}{2}$. Thus, we have $|Z(c, uv) \cap$

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$\mathcal{F}_i| \geq \frac{\alpha n^2}{2}$ for at least $(\lambda - 2d)n$ indices $i \in [\lambda n]$. Applying Lemma 3.4 with

$$\mathbf{H} := \{\mathcal{F}_i : i \in [\lambda n]\}, \quad \mathbf{Z} := \{Z(c, uv) : c \in [n], (u, v) \in V_1 \times V_2 \text{ and } v \text{ is } c\text{-good}\},$$

and parameters $t := \lambda n$, $\epsilon := \frac{\alpha}{2}$, to obtain a rainbow matching $\mathcal{M}$ inside $\mathbf{H}$ of size at least $(1 - \frac{\alpha^2}{16})\lambda n$ such that $|E(Z(c, uv)) \cap E(\mathcal{M})| \geq \frac{\alpha^2 \lambda n}{16}$ for all $Z(c, uv) \in \mathbf{Z}$. Therefore, there exists a subset $I \subseteq [\lambda n]$ of size at least $(1 - \frac{\alpha^2}{16})\lambda n$ such that for each $i \in I$, there is an edge $w_i w_i'$ in $\mathcal{M}$ with $i \in L(w_i w_i')$. For every $c \in [n]$ and $(u, v) \in V_1 \times V_2$ satisfying $v$ is $c$-good, there are at least $\frac{\alpha^2 \lambda n}{16}$ matching edges in $\mathcal{M}$ which are $c$-absorbing edges of $(u, v)$. By construction, $\mathcal{M}$ is an absorbing matching with parameters $(1, 0, \lambda, \frac{\alpha^2 \lambda}{16})$ and hence with parameters $(1, 0, \lambda, \lambda^2)$ since $\lambda \ll \alpha$. □

Next, we demonstrate that an absorbing matching can be found when $\mathcal{G}$ is weakly stable.

**Lemma B.5.** *Let $0 < \frac{1}{n} \ll d \ll \lambda, \mu \ll \epsilon \ll \delta < 1$. Suppose that $\mathcal{G} = \{G_1, \ldots, G_n\}$ is a collection of bipartite graphs on a common vertex partition $V = V_1 \cup V_2$ with $|V_1| = |V_2| = n$ such that the following conditions hold:*

- *for every $i \in [n]$, $d_{G_i}(x) \geq \left(\frac{1}{2} - \mu\right) n$ for all but at most $dn$ vertices $x \in V_1 \cup V_2$,*
- *for every $x \in V_1 \cup V_2$, $d_{G_i}(x) \geq \left(\frac{1}{2} - \mu\right) n$ for all but at most $dn$ colours $i \in [n]$.*

*If $\mathcal{G}$ is $(\epsilon, \delta)$-weakly stable, then there exists an absorbing matching with parameters $(\frac{\delta}{2}, \sqrt{\epsilon}, \lambda, \lambda^2)$.*

**Proof.** For each $i \in [n]$, we define

$$\mathcal{F}_i := \{(u, v) \in V_1 \times V_2 : i \in L(uv)\}.$$

Then $|\mathcal{F}_i| > \frac{n^2}{4}$. We first prove the following claim, which only requires the degree condition on $\mathcal{G}$.

**Claim B.6.** *Assume $ij \in E(C_{\mathcal{G}}^{\epsilon,\delta})$. Let $u \in V_1$ be a $j$-good vertex and $v \in V_2$ be an $i$-good vertex. Define*

$$Z_j(i, uv) := \{(w, w') \in \mathcal{F}_j : ww' \text{ is an } i\text{-absorbing edge of } (u, v)\}.$$

*Then $|Z_j(i, uv)| \geq 2^{-4}\delta n^2$.*

**Proof of Claim** B.6. For each $\ell \in [n]$, if $G_\ell$ is $\epsilon$-extremal, then by Lemma 2.7, there is a characteristic partition $(A_1^\ell, B_1^\ell, C_1^\ell, A_2^\ell, B_2^\ell, C_2^\ell)$ corresponding to $G_\ell$. Fix any such indices $i, j$ and vertices $u, v$. Since $v$ is $i$-good, we have

$$d_{G_i}(v, A_1^i) \geq \left(\frac{1}{2} - 2\epsilon\right) n \text{ or } d_{G_i}(v, B_1^i) \geq \left(\frac{1}{2} - 2\epsilon\right) n.$$

Assume, without loss of generality, that the former case holds (the latter case is analogous). Since $G_i$ and $G_j$ are $\delta$-crossing and $\epsilon \ll \delta$, one has

$$|A_1^i \cap A_1^j| \geq \frac{\delta n}{2}, |A_1^i \cap B_1^j| \geq \frac{\delta n}{2}, |A_1^j \cap B_1^i| \geq \frac{\delta n}{2}, \text{ and } |B_1^i \cap B_1^j| \geq \frac{\delta n}{2}.$$

Combining $|A_1^i \cap A_1^j| \geq \frac{\delta n}{2}$ with $d_{G_i}(v, A_1^i) \geq (\frac{1}{2} - 2\epsilon)n$, we obtain

$$d_{G_i}(v, A_1^j) \geq |A_1^i \cap A_1^j| - \left(|A_1^i| - d_{G_i}(v, A_1^i)\right) \geq \frac{\delta n}{2} - \epsilon n \geq \frac{\delta n}{3}.$$

Similarly, we have $d_{G_i}(v, B_1^j) \geq \frac{\delta n}{3}$. Recall that $u \in V_1$ and $u$ is $j$-good, which implies that $u \in X_1^j$ for some $X \in \{A, B\}$. Since $d_{G_j}(u, X_2^j) \geq (\frac{1}{2} - 2\epsilon)n$, there are at least $(\frac{1}{2} - 2\epsilon)n$ choices for $w' \in X_2^j$. Notice that in $G_j$, every vertex in $X_2^j$ is adjacent to all but at most $\epsilon n$ vertices in $X_1^j$. Fixing

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$w' \in X_2^j$, and using $d_{G_i}(v, X_1^j) \geq \frac{\delta n}{3}$, we conclude that there are at least $\frac{\delta n}{4}$ choices for $w \in X_1^j$. Thus,

$$|Z_j(i, uv)| \geq \frac{\delta n}{4}\left(\frac{1}{2} - 2\epsilon\right) n \geq 2^{-4}\delta n^2,$$

as desired. □

For any vertex $v \in V$, let

$$\mathcal{C}_v := \{i \in [n] : v \text{ is } i\text{-good}\}.$$

Since $e(C_{\mathcal{G}}^{\epsilon,\delta}) \geq \delta n^2$, there exists a subgraph $H$ of $C_{\mathcal{G}}^{\epsilon,\delta}$ such that $|V(H)| \geq \delta n$ and $\delta(H) \geq \delta n$. For each $i \in V(H)$, define the set

$$T_i := \left\{v \in V : |N_H(i) \setminus \mathcal{C}_v| \geq \frac{\delta n}{2}\right\}.$$

Observe that there are at most $4\epsilon n^2$ pairs $(u, i) \in V \times [n]$ such that $u$ is not $i$-good. This implies that $\frac{\delta n}{2}|T_i| \leq 4\epsilon n^2$. Hence, by $\epsilon \ll \delta$, we have $|T_i| \leq \frac{\sqrt{\epsilon} n}{2}$.

Let $\overline{T_i} := V \setminus T_i$. Clearly, $|\overline{T_i}| \geq 2n - \frac{\sqrt{\epsilon} n}{2}$. For each $i \in V(H)$ and $u \in \overline{T_i}$, we have $|\mathcal{C}_u \cap N_H(i)| \geq \frac{\delta n}{2}$. Now we independently and randomly select vertices from $V(H)$ with probability $\kappa := \frac{\lambda}{14}$ to obtain a set $\mathcal{U}$ of colours. A Chernoff bound implies that, with high probability,

(i) $\frac{\kappa|V(H)|}{2} \leq t := |\mathcal{U}| \leq 2\kappa|V(H)|$,

(ii) for every $i \in V(H)$ and $u \in \overline{T_i}$, we have $|\mathcal{C}_u \cap N_H(i, \mathcal{U})| \geq \frac{\delta\kappa n}{4}$.

Fix such a set $\mathcal{U}$ and let $\overline{\mathcal{U}} := V(H) \setminus \mathcal{U}$. By relabelling colours, one may assume $\mathcal{U} =: [t]$. Clearly,

$$|\overline{\mathcal{U}}| = |V(H)| - |\mathcal{U}| \geq \frac{\delta n}{2}, \text{ and } \delta(H[\overline{\mathcal{U}}]) \geq \delta(H) - |\mathcal{U}| \geq \frac{\delta n}{2}.$$

Define the multi-graph collection

$$\mathbf{Z} := \left\{Z(i, uv) := \bigcup_{j \in \mathcal{C}_u \cap N_H(i,\mathcal{U})} Z_j(i, uv) : i \in \overline{\mathcal{U}}, u \in \overline{T_i} \text{ and } v \text{ is } i\text{-good}\right\}.$$

Given $i$ and $u$, the number of choices for $j$ is at least $\frac{\delta\kappa n}{4}$ by (ii). Since $ij \in E(H)$, $u$ is $j$-good and $v$ is $i$-good, Claim B.6 implies that there are at least $2^{-4}\delta n^2$ $i$-absorbing edges of $(u, v)$ whose ordered vertex set is in $\mathcal{F}_j$. Thus, $|Z(i, uv) \cap \mathcal{F}_j| \geq 2^{-4}\delta n^2$ for at least $\frac{\delta\kappa n}{4} \geq 2^{-3}t$ indices $j \in [t]$. Applying Lemma 3.4 with

$$\mathbf{H} := \{\mathcal{F}_j : j \in [t]\}, \mathbf{Z} \text{ and } \epsilon := 2^{-4}\delta,$$

we can find a rainbow matching $\mathcal{M}$ in $\mathbf{H}$ of size at least $(1 - 2^{-10}\delta^2)t$ such that $|E(Z(i, uv)) \cap E(\mathcal{M})| \geq 2^{-10}\delta^2 t$ for all $Z(i, uv) \in \mathbf{Z}$. That is, there is $I \subseteq [t]$ with $|I| \geq (1 - 2^{-10}\delta^2)t$ such that for each $j \in I$ there is an edge $e_j \in E(\mathcal{M})$ with $j \in L(e_j)$. For every $i \in \overline{\mathcal{U}}$, $u \in \overline{T_i}$ and $v \in V_2$ is $i$-good, there are at least $2^{-10}\delta^2 t$ indices $j \in I$ such that $e_j$ is an $i$-absorbing edge of $(u, v)$. Thus, $\mathcal{M}$ is an absorbing matching with parameters $(\frac{\delta}{2}, \sqrt{\epsilon}, (1 - 2^{-10}\delta^2)t, 2^{-10}\delta^2 t)$ and hence with $(\frac{\delta}{2}, \sqrt{\epsilon}, \lambda, \lambda^2)$. □

Now, we are ready to prove Theorem 2.10.

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**Proof of Theorem** 2.10. Choose additional parameters $\beta, \lambda$ and $\mu'$ such that

$$1 < \frac{1}{n} \ll d \ll \mu \ll \beta \ll \lambda \ll \mu' \ll \alpha \ll \gamma, \epsilon \ll \delta \ll 1,$$

where the previous lemmas in this section hold with these suitable parameters.

By Lemmas B.4 and B.5, $\mathcal{G}$ contains an absorbing matching $\mathcal{M}$ with parameters $(\frac{\delta}{2}, \sqrt{\epsilon}, \lambda, \lambda^2)$. For any colour $c \in [n]$ and vertices $u \in V_1, v \in V_2$, we define the triple $(c, u, v)$ as absorbable if there are at least $\lambda^2 n$ disjoint $c$-absorbing edges of $(u, v)$ inside $\mathcal{M}$. By the definition of absorbing matching, one has $|\mathcal{M}| \leq \lambda n$. Furthermore, there exists a colour set $\mathcal{C} \subseteq [n] \setminus \mathrm{col}(\mathcal{M})$ of size at least $\frac{\delta n}{2}$ such that, for any given colour $c \in \mathcal{C}$ and $c$-good vertex $v \in V_2$, the triple $(c, u, v)$ is absorbable for all but at most $\sqrt{\epsilon} n$ vertices $u \in V_1$.

**Claim B.7.** *There exists an integer $r$ with $(2 - 2^{-10})\beta n \leq r \leq 2\beta n$. For each $i \in [r]$, there is a tuple $q_i := (c_i^1, c_i^2, w_i^1, w_i^{1'}, w_i^2, w_i^{2'})$ where $c_i^1, c_i^2 \in \mathcal{C}$ are distinct colours, $w_i^1, w_i^2 \in V_1$ and $w_i^{1'}, w_i^{2'} \in V_2$ are distinct vertices, satisfying the following properties:*

(i) *$(c_i^1, w_i^1, w_i^{1'})$ and $(c_i^2, w_i^2, w_i^{2'})$ are absorbable for all $i \in [r]$,*

(ii) *for every pair $(u, v) \in V_1 \times V_2$ and every colour $c \in [n]$, there are at least $2^{-20}\beta n$ indices $i \in [r]$ such that $c_i^1 \in L(uw_i^{1'})$, $c_i^2 \in L(vw_i^2)$ and $c \in L(w_i^1 w_i^{2'})$.*

**Proof of Claim** B.7. For every two distinct colours $c_i^1, c_i^2 \in \mathcal{C}$, and vertex pair $(u, v) \in V_1 \times V_2$ such that $d_{G_{c_i^1}}(u) \geq \left(\frac{1}{2} - \mu\right) n$ and $d_{G_{c_i^2}}(v) \geq \left(\frac{1}{2} - \mu\right) n$, and for every colour $c \in [n]$, let $S(c_i^1, c_i^2, u, v, c)$ be the collection of tuples

$$(w_i^1, w_i^{1'}, w_i^2, w_i^{2'}) \in V_1 \times V_2 \times V_1 \times V_2$$

that satisfy the following properties: $(c_i^1, w_i^1, w_i^{1'})$ and $(c_i^2, w_i^2, w_i^{2'})$ are absorbable, $c_i^1 \in L(uw_i^{1'})$, $c_i^2 \in L(vw_i^2)$, and $c \in L(w_i^1 w_i^{2'})$. We will prove that $|S(c_i^1, c_i^2, u, v, c)| \geq 2^{-7} n^4$. For this, we count the number of possible choices for $w_i^{1'}$, $w_i^{2'}$, $w_i^1$ and $w_i^2$ in turn.

• Recall that all but at most $2\epsilon n$ vertices in $V_2$ are $c_i^1$-good and $d_{G_{c_i^1}}(u) \geq \left(\frac{1}{2} - \mu\right) n$. Therefore, there are at least $\left(\frac{1}{2} - \mu\right) n - 2\epsilon n - 1 \geq \frac{n}{3}$ choices for $w_i^{1'}$ such that $w_i^{1'}$ is $c_i^1$-good and $c_i^1 \in L(uw_i^{1'})$.

• For vertices in $V_2$, there are at most $2\epsilon n + dn$ vertices $w$ that are either not $c_i^2$-good or $d_{G_c}(w) < \left(\frac{1}{2} - \mu\right) n$. Thus, we have at least $n - 2\epsilon n - dn - 2 \geq \frac{n}{2}$ choices for $w_i^{2'}$ such that $w_i^{2'}$ is $c_i^2$-good and $d_{G_c}(w_i^{2'}) \geq \left(\frac{1}{2} - \mu\right) n$.

• Fix $w_i^{1'}$ and $w_i^{2'}$. Since $w_i^{1'}$ is $c_i^1$-good and $c_i^1 \in \mathcal{C}$, we obtain that $(c_i^1, w, w_i^{1'})$ is absorbable for all but at most $\sqrt{\epsilon} n$ vertices $w \in V_1$. Recall that $d_{G_c}(w_i^{2'}) \geq \left(\frac{1}{2} - \mu\right) n$. Therefore, there are at least $\left(\frac{1}{2} - \mu\right) n - \sqrt{\epsilon} n - 1 \geq \frac{n}{3}$ choices for $w_i^1$ such that $(c_i^1, w_i^1, w_i^{1'})$ is absorbable and $c \in L(w_i^1 w_i^{2'})$.

• Similarly, since $d_{G_{c_i^2}}(v) \geq \left(\frac{1}{2} - \mu\right) n$, $c_i^2 \in \mathcal{C}$ and $w_i^{2'}$ is $c_i^2$-good, there are at least $\left(\frac{1}{2} - \mu\right) n - \sqrt{\epsilon} n - 2 \geq \frac{n}{3}$ choices for $w_i^2$ such that $c_i^2 \in L(vw_i^2)$ and $(c_i^2, w_i^2, w_i^{2'})$ is absorbable.

Therefore, $|S(c_i^1, c_i^2, u, v, c)| \geq 2^{-7} n^4$, as desired.

Let $\{c_i^1, c_i^2 : i \in [2\beta n]\} \subseteq \mathcal{C}$ be a set of distinct colours. Define

$$\mathcal{F}_i := \{(w_i^1, w_i^{1'}, w_i^2, w_i^{2'}) \in V_1 \times V_2 \times V_1 \times V_2 : (c_i^1, w_i^1, w_i^{1'}) \text{ and } (c_i^2, w_i^2, w_i^{2'}) \text{ are absorbable}\}.$$

Since for each colour $c \in \mathcal{C}$, there are at most $2\epsilon n$ vertices in each part that are not $c$-good. Hence, there exist at least $(n - 2\epsilon n)(n - \sqrt{\epsilon} n) \geq \frac{n^2}{2}$ vertex pairs $(x, y) \in V_1 \times V_2$ such that $(c, x, y)$ is absorbable. Thus, $|\mathcal{F}_i| \geq 2^{-4} n^4$. Let $\mathbf{H} := \{\mathcal{F}_i : i \in [2\beta n]\}$ be the collection of directed 4-graphs.

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For a vertex pair $(u, v) \in V_1 \times V_2$, let

$$I_{u,v} := \left\{ i \in [2\beta n] : d_{G_{c_i^1}}(u) \geq \left(\frac{1}{2} - \mu\right) n \text{ and } d_{G_{c_i^2}}(v) \geq \left(\frac{1}{2} - \mu\right) n \right\}.$$

Define the collection of directed multi-4-graphs as follows:

$$\mathbf{Z} := \left\{ S(u, v, c) := \bigcup_{i \in I_{u,v}} S(c_i^1, c_i^2, u, v, c) : (u, v) \in V_1 \times V_2,\ c \in [n] \right\}.$$

Since for each vertex $w \in V$, there are at most $dn$ indices $i \in [n]$ such that $d_{G_i}(w) < \left(\frac{1}{2} - \mu\right) n$. Hence for every $S := S(u, v, c) \in \mathbf{Z}$, we have $|E(S) \cap E(\mathcal{F}_i)| \geq 2^{-7} n^4$ for all but at most $2dn$ indices $i \in [2\beta n]$. By applying Lemma 3.4 with $t := 2\beta n$ and $\epsilon := 2^{-7}$, we deduce that there is a rainbow matching $M$ in $\mathbf{H}$ of size at least $(2 - 2^{-10})\beta n$ (and at most $2\beta n$) and $|E(S) \cap E(M)| \geq 2^{-20}\beta n$ for all $S \in \mathbf{Z}$. □

Let

$$V_{\text{abs}} := \bigcup_{i \in [r]} \{w_i^1, w_i^{1'}, w_i^2, w_i^{2'}\},\ \mathcal{C}_{\text{abs}} := \bigcup_{i \in [r]} \{c_i^1, c_i^2\},\ \mathcal{C}_{\text{rem}} := [n] \setminus (\text{col}(\mathcal{M}) \cup \mathcal{C}_{\text{abs}}),$$
$$U := V \setminus (V(\mathcal{M}) \cup V_{\text{abs}}),\ \mathcal{J} := \{J_i : i \in \mathcal{C}_{\text{rem}}\} \text{ where } J_i = G_i[U].$$

Note that $|[n] \setminus \mathcal{C}_{\text{rem}}| = \frac{|V \setminus U|}{2} \leq \lambda n + 4\beta n$. For each vertex $v \in U$, for all but at most $dn$ colours $c \in \mathcal{C}_{\text{rem}}$ we have

$$d_{J_c}(v) \geq \left(\frac{1}{2} - \mu\right) n - \lambda n - 4\beta n \geq \frac{|U|}{4} - \mu' \frac{|U|}{12}.$$

Since $\beta, \lambda, \mu' \ll \gamma, \alpha, \epsilon, \delta$, it is easy to see that $\mathcal{J}$ is $(\frac{\gamma}{2}, \frac{\alpha}{2}, 2\epsilon, \frac{\delta}{2})$-stable. By Lemmas B.1 and B.2, we deduce that $\mathcal{J}$ is $\mu'$-nice and there exists a rainbow matching $\mathcal{M}_1$ in $\mathcal{J}$ with size at least $|\mathcal{C}_{\text{rem}}| - dn - 2$. Assume that

$$(U \setminus V(\mathcal{M}_1)) \cap V_1 = \{u_1, \ldots, u_t\}, (U \setminus V(\mathcal{M}_1)) \cap V_2 = \{v_1, \ldots, v_t\}, \mathcal{C}_{\text{rem}} \setminus \text{col}(\mathcal{M}_1) = \{c_1, \ldots, c_t\}.$$

Do the following for each $(u_i, v_i, c_i)$ with $i = 1, 2, \ldots, t$ in turn. Choose an unused 6-tuple $q_{j_i} = (c_{j_i}^1, c_{j_i}^2, w_{j_i}^1, w_{j_i}^{1'}, w_{j_i}^2, w_{j_i}^{2'})$ for some $j_i \in [r]$, such that $c_{j_i}^1 \in L(u_i w_{j_i}^{1'})$, $c_{j_i}^2 \in L(v_i w_{j_i}^2)$, and $c_i \in L(w_{j_i}^1 w_{j_i}^{2'})$. This is possible since Claim B.7 implies that there are at least $2^{-20}\beta n$ choices for $q_{j_i}$, of which at most $dn + 2$ have been used. Thus, all vertices in $U \setminus V(\mathcal{M}_1)$ and all colours in $\mathcal{C}_{\text{rem}} \setminus \text{col}(\mathcal{M}_1)$ are contained in a rainbow matching of $\mathcal{G}$.

At the end of this process, there remains a set $I \subseteq [r]$ such that the $q_i$ with $i \in I$ are precisely the 6-tuples that were not chosen. Recall that $(c_i^1, w_i^1, w_i^{1'})$ and $(c_i^2, w_i^2, w_i^{2'})$ are absorbable for each $i \in I$. It follows that there are at least $\lambda^2 n$ disjoint $c_i^j$-absorbing edges of $(w_i^j, w_i^{j'})$ inside $\mathcal{M}$ for $i \in I$ and $j \in [2]$. Together with $\beta \ll \lambda$, we obtain a transversal perfect matching inside $\mathcal{G}$. □

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