INTERPOLATION AND THE EXCHANGE RULE

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Abstract. It was proved by Maksimova in 1977 that exactly eight varieties of Heyting algebras have the amalgamation property, and hence exactly eight axiomatic extensions of intuitionistic propositional logic have the deductive interpolation property. The prevalence of these properties for substructural logics and varieties of pointed residuated lattices (their algebraic semantics) is far less well understood. Taking as our starting point a formulation of intuitionistic propositional logic as the full Lambek calculus with exchange, weakening, and contraction, we investigate the role of the exchange rule—algebraically, the commutativity law—in determining the scope of these properties. First, we show that there are continuum-many varieties of idempotent semilinear residuated lattices that have the amalgamation property and contain non-commutative members, and hence continuum-many axiomatic extensions of the corresponding logic that have the deductive interpolation property in which exchange is not derivable. We then show that, in contrast, exactly 60 varieties of commutative idempotent semilinear residuated lattices have the amalgamation property, and hence exactly 60 axiomatic extensions of the corresponding logic with exchange have the deductive interpolation property. From this latter result, it follows also that there are exactly 60 varieties of commutative idempotent semilinear residuated lattices whose first-order theories have a model completion.

§1. Introduction. By a remarkable result of Maksimova [24], precisely eight axiomatic extensions of intuitionistic propositional logic IPC have the following *deductive interpolation property*: For any formulas α, β of such a logic L, if $\alpha \vdash_{\mathsf{L}} \beta$, then there exists a formula γ, whose variables occur in both α and β, such that $\alpha \vdash_{\mathsf{L}} \gamma$ and $\gamma \vdash_{\mathsf{L}} \beta$.[1] Maksimova's proof was essentially algebraic. First, she proved that an axiomatic extension of IPC has deductive interpolation if and only if the associated variety (equational class) of Heyting algebras has the amalgamation property, and subsequently that there are precisely eight such varieties. This result was later strengthened by Ghilardi and Zawadowski [18], who, building on Pitts' theorem establishing uniform deductive interpolation for IPC [37], proved that all eight axiomatic extensions of IPC with deductive interpolation have this stronger property, and that the first-order theories of all eight varieties of Heyting algebras with amalgamation have a model completion.

Received March 25, 2024.

2020 *Mathematics Subject Classification.* 03G10, 03G25, 03B47, 08B05, 03C40.

Key words and phrases. deductive interpolation, amalgamation, substructural logic, full Lambek calculus, residuated lattice, model completion.

[1] Deductive interpolation is equivalent in this setting to *Craig interpolation*: For any formulas α, β of L, if $\vdash_{\mathsf{L}} \alpha \to \beta$, then there exists a formula γ, whose variables occur in both α and β, such that $\vdash_{\mathsf{L}} \alpha \to \gamma$ and $\vdash_{\mathsf{L}} \gamma \to \beta$. However, in other settings—in particular, for axiomatic extensions of modal logics or substructural logics—these properties may diverge.

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0022-4812/00/0000-0000
DOI:10.1017/jsl.2026.10198

https://doi.org/10.1017/jsl.2026.10198 Published online by Cambridge University Press

$$\frac{\Gamma_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi_2, \Pi_1, \Gamma_2 \Rightarrow \Delta}\ (\mathrm{e}) \qquad \frac{\Gamma_1, \Pi, \Pi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta}\ (\mathrm{c})$$

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta}\ (\mathrm{i}) \qquad \frac{\Pi \Rightarrow}{\Gamma_1, \Pi, \Gamma_2 \Rightarrow \Delta}\ (\mathrm{o})$$

$$\frac{\Gamma_1, \Pi_1, \Gamma_2 \Rightarrow \Delta \quad \Gamma_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Pi_1, \Pi_2, \Gamma_2 \Rightarrow \Delta}\ (\mathrm{m})$$

FIGURE 1. Basic structural rules.

Similar results have been established for normal modal logics. In particular, Maksimova proved that between 43 and 49 axiomatic extensions of S4 have deductive interpolation [25], while among axiomatic extensions of Gödel–Löb logic GL, there are continuum-many [26] that have this property. Less well-understood, however, is the prevalence of deductive interpolation among *substructural logics*. These logics are typically viewed as axiomatic extensions of the *full Lambek calculus* FL—a sequent calculus presented in a language with binary operation symbols $\wedge, \vee, \cdot, \backslash, /$ and constant symbols e, f—with algebraic semantics provided by varieties of pointed residuated lattices (see Appendix A and [14, 31]). Interpolation results have been obtained for a wide range of substructural logics (see, e.g., [15, 20, 23, 27–29, 32, 35]), but Maksimova-style descriptions of the axiomatic extensions of a logic that have deductive interpolation are known only for a few cases. Notably, nine axiomatic extensions of the logic R-Mingle with unit have deductive interpolation (equivalently, nine varieties of Sugihara monoids have the amalgamation property) [28], and, using the fact that a variety of MV-algebras has the amalgamation property if and only if it is generated by a single totally ordered algebra [34], countably infinitely many axiomatic extensions of Łukasiewicz logic have deductive interpolation.

Extending FL with the structural rules of exchange (e), weakening (i) and (o), and contraction (c) depicted in Figure 1 yields a sequent calculus for IPC, where the operations $\wedge$ and $\cdot$, and likewise $\backslash$ and $/$, can be identified. From an algebraic perspective, adding combinations of (e), (i), (o), and (c) to FL yields sequent calculi for varieties of pointed residuated lattices that are commutative, integral, bounded, or square-increasing, respectively, i.e., satisfy $x \cdot y \approx y \cdot x$, $x \leq \mathrm{e}$, $\mathrm{f} \leq x$, or $x \leq x \cdot x$, respectively. In particular, pointed residuated lattices satisfying all these equations are term-equivalent to Heyting algebras.

Examples of axiomatic extensions of the full Lambek calculus with exchange, $\mathsf{FL_e}$, that have deductive interpolation are well known and abundant; indeed, continuum-many such extensions have been described in [10]. For many years, however, it was an open question as to whether exchange is derivable in *every* axiomatic extension of FL that has deductive interpolation (see [20, Problem 5]). A counterexample showing that this is not the case was given in [19]. Motivated by this result, we examine

https://doi.org/10.1017/jsl.2026.10198 Published online by Cambridge University Press

here the critical role played by exchange in determining the scope of deductive interpolation in substructural logics.

As a natural starting point, we consider a sequent calculus in which exchange is not derivable that deviates as little as possible from IPC. Since exchange is already derivable in the presence of weakening and contraction, we replace the weakening rules (i) and (o) with a less powerful variant, the mingle rule (m) (see Figure 1), obtaining the full Lambek calculus with mingle and contraction, $\mathsf{FL}_{\mathsf{cm}}$, corresponding to the variety of idempotent pointed residuated lattices. Not only is the exchange rule (e) not derivable in $\mathsf{FL}_{\mathsf{cm}}$, a growing body of work demonstrates many semantic similarities between IPC and axiomatic extensions of this logic (see., e.g., [6–8, 16, 17, 22]). The logic presented in [19] that has deductive interpolation but does not derive exchange is an axiomatic extension of $\mathsf{FL}_{\mathsf{cm}}$, and we show here that there are in fact continuum-many such logics.

Our proof is algebraic, following the approach of Maksimova. We use the fact that an axiomatic extension of FL has deductive interpolation if the associated variety of pointed residuated lattices has the amalgamation property, noting that the converse holds in the presence of a local deduction theorem for the logic, or, equivalently, the congruence extension property for the variety [29]. The proof therefore amounts to exhibiting continuum-many varieties of idempotent pointed residuated lattices that have the amalgamation property and contain non-commutative members. In fact, these varieties have a rather special form. Their members satisfy $\mathrm{e} \approx \mathrm{f}$—and are therefore referred to simply as residuated lattices—and are *semilinear*, that is, representable as subdirect products of totally ordered algebras. Let $\mathsf{SemRL}_{\mathsf{cm}}$ denote the axiomatic extension of $\mathsf{FL}_{\mathsf{cm}}$ associated with the variety of idempotent semilinear residuated lattices, noting that this logic can also be presented as a hypersequent calculus (see, e.g., [30]). The first main result of this article is as follows.

THEOREM A.

(i) *Continuum-many varieties of idempotent semilinear residuated lattices have the amalgamation property and contain non-commutative members.*
(ii) *Continuum-many axiomatic extensions of* $\mathsf{SemRL}_{\mathsf{cm}}$, *in which exchange is not derivable, have the deductive interpolation property.*

As a natural next step, we consider how the picture presented in Theorem A changes when exchange (algebraically, commutativity) is reinstated. To this end, let $\mathsf{SemRL}_{\mathsf{ecm}}$ denote the axiomatic extension of $\mathsf{FL}_{\mathsf{cm}}$ associated with the variety of commutative idempotent semilinear residuated lattices (corresponding also to the hypersequent calculus for $\mathsf{SemRL}_{\mathsf{cm}}$ extended with exchange [30]). The following result demonstrates that in this setting, reinstating exchange reduces the number of axiomatic extensions having deductive interpolation (algebraically, amalgamation) from continuum-many to finitely many.

THEOREM B.

(i) *Exactly* 60 *varieties of commutative idempotent semilinear residuated lattices have the amalgamation property.*
(ii) *Exactly* 60 *axiomatic extensions of* $\mathsf{SemRL}_{\mathsf{ecm}}$ *have the deductive interpolation property.*

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Since the first-order theory of a locally finite variety has a model completion[2] if and only if the variety has the amalgamation property [39, p. 319, Corollary 1], we also obtain the following result.

THEOREM C. *There are exactly* 60 *varieties of commutative idempotent semilinear residuated lattices whose first-order theories have a model completion*.

Theorems B and C raise the obvious question as to whether we may cast our net still wider and obtain similar results for a weaker logic than $\mathsf{SemRL}_{\mathsf{ecm}}$. With respect to omitting semilinearity, the problem of whether only finitely many varieties of commutative idempotent residuated lattices have the amalgamation property is open and appears to be quite challenging. On the other hand, we show here that we can drop the requirement that members of the variety satisfy $\mathrm{e} \approx \mathrm{f}$. Despite the additional combinatorial complexities involved in dealing with the constant f, we show that there are still only finitely many varieties of commutative idempotent semilinear pointed residuated lattices that have the amalgamation property, and hence finitely many axiomatic extensions of the corresponding logic that have the deductive interpolation property. We do not give the precise number of such varieties (equivalently, axiomatic extensions), but show that these number more than 12,000,000.

The article is structured as follows. In Section 2, we recall basic facts about pointed residuated lattices and the relationship between deductive interpolation and amalgamation in this setting. In Section 3, we present structure theory for idempotent residuated lattices, in particular, the nested sum structure of *-involutive idempotent residuated chains. In Section 4, we prove Theorem A, using the existence of continuum-many pairwise incomparable minimal bi-infinite words over $\{0, 1\}$ to construct corresponding varieties of idempotent semilinear residuated lattices that both have the amalgamation property and contain non-commutative members. In Section 5, we prove Theorems B and C, using the structure theory of commutative idempotent residuated chains to classify the varieties of commutative idempotent semilinear residuated lattices that have the amalgamation property, and consider also the (logic corresponding to) the variety of commutative idempotent semilinear pointed residuated lattices.

§2. Preliminaries. In this section, we give a brief account of the relationship between deductive interpolation and amalgamation in the setting of pointed residuated lattices, referring to [2, 4, 29, 31, 36] for a more general presentation.

Let us first recall some basic facts regarding (pointed) residuated lattices, referring to [14, 31] for further details. A *pointed residuated lattice* is an algebraic structure $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}, \mathrm{f})$ such that:

1. $(A, \wedge, \vee)$ is a lattice with induced order $x \leq y \ :\Longleftrightarrow\ x \wedge y = x$ for $x, y \in A$;
2. $(A, \cdot, \mathrm{e})$ is a monoid;

[2] A theory T^* is a *model companion* of a theory T if every model of T embeds into a model of T^* and vice-versa, and every embedding between models of T^* is elementary; it is a *model completion* of T if additionally the theory of T^* together with the diagram of any model of T is complete. A theory has at most one model companion and if this exists, it is a model completion if and only if the class of models of the theory has the amalgamation property.

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3. $\backslash$ and $/$ are left and right residuals of $\cdot$ with respect to $\leq$, i.e., for $x, y, z \in A$,

$$y \leq x\backslash z \iff x \cdot y \leq z \iff x \leq z/y.$$

A *residuated lattice* is a pointed residuated lattice satisfying the equation $\mathrm{f} \approx \mathrm{e}$. Because the constant f is redundant in residuated lattices, they are frequently formulated in the definitionally equivalent language without f.

For readability, we often suppress the multiplication operation $\cdot$ and write xy in place of $x \cdot y$, adopting the convention that multiplication has priority over $\backslash, /$ and that the latter have priority over $\wedge, \vee$, dropping parentheses accordingly. Also, in order to make our discussion more transparent and compact, we make frequent use of several definable operations; in particular, we define for any pointed residuated lattice $\mathbf{A}$ and $x \in A$,

$$x^r := x\backslash \mathrm{e}, \quad x^\ell := \mathrm{e}/x, \quad \text{and} \quad x^\star := x^\ell \wedge x^r.$$

Let $\mathbb{I}$, $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$, and $\mathbb{P}_\mathsf{U}$ denote the class operators of taking isomorphic images, homomorphic images, subalgebras, products, and ultraproducts, respectively. A class of pointed residuated lattices is a *variety* if it is closed under $\mathbb{H}$, $\mathbb{S}$, and $\mathbb{P}$, or, equivalently, by Birkhoff's theorem, if it is the class of models of a set of equations. In particular, the class of all pointed residuated lattices is a variety. Given any class K of pointed residuated lattices, $\mathbb{V}(\mathsf{K}) := \mathbb{HSP}(\mathsf{K})$ is the smallest variety of pointed residuated lattices containing K, called the *variety generated by* K. The set of subvarieties of a variety V (i.e., varieties contained in V) forms a lattice, ordered by inclusion.

A pointed residuated lattice is called *commutative*, *idempotent*, or **-involutive*[3] if it satisfies $xy \approx yx$, $xx \approx x$, or $x^{\star\star} \approx x$, respectively; it is called *totally ordered* if its underlying lattice order is total, and *semilinear* if it is isomorphic to a subalgebra of a product of totally ordered pointed residuated lattices. Since every commutative pointed residuated lattice satisfies $x\backslash y \approx y/x$, the common value of $x\backslash y$ and y/x is in this case denoted by $x \to y$, and the common value of $x^\star$, x^ℓ, and x^r is denoted by $x^\star$. For the sake of brevity, we also refer to a totally ordered residuated lattice as a *residuated chain*.

The set of congruences of a pointed residuated lattice $\mathbf{A}$ forms a distributive lattice $\mathrm{Con}(\mathbf{A})$, ordered by inclusion. Residuated lattices are also e-regular in the sense that their congruences are determined by the congruence classes of e, and these congruence classes form subalgebras satisfying certain conditions. A subalgebra $\mathbf{C}$ of a residuated lattice $\mathbf{A}$ is called *convex* if whenever $x, y \in C$, $a \in A$, and $x \leq a \leq y$, then $a \in C$, and *normal* if whenever $x \in C$ and $a \in A$, then $a\backslash xa \wedge \mathrm{e}, ax/a \wedge \mathrm{e} \in A$. The convex normal subalgebras of a residuated lattice $\mathbf{A}$ form a lattice under inclusion, denoted by $\mathrm{NC}(\mathbf{A})$, that is isomorphic to $\mathrm{Con}(\mathbf{A})$, as witnessed by the mutually inverse order-preserving maps

$$\mathrm{Con}(\mathbf{A}) \to \mathrm{NC}(\mathbf{A});\ \Theta \mapsto \{x \in A \mid (x, \mathrm{e}) \in \Theta\},$$
$$\mathrm{NC}(\mathbf{A}) \to \mathrm{Con}(\mathbf{A});\ \mathbf{C} \mapsto \{(x, y) \in A^2 \mid x\backslash y \wedge y\backslash x \in C\}.$$

[3] In [8], the notation $x^\wedge$ is used instead of $x^\star$, and *-involutivity is called $^\wedge$-involutivity.

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Since the congruences of a pointed residuated lattice are the congruences of its f-free reduct, the above characterization also applies to the pointed setting.

A pointed residuated lattice $\mathbf{A}$ is called *simple* if $\mathrm{Con}(\mathbf{A})$ contains only the least congruence $\Delta_A := \{(a, a) \mid a \in A\}$ and the greatest congruence A^2; if $\mathbf{A}$ is simple and has no non-trivial proper subalgebras, it is called *strictly simple*. A pointed residuated lattice $\mathbf{A}$ is called *locally finite* if every finitely generated subalgebra of $\mathbf{A}$ is finite; it has the *congruence extension property* if for any subalgebra $\mathbf{B}$ of $\mathbf{A}$ and $\Theta \in \mathrm{Con}(\mathbf{B})$, there exists a $\Phi \in \mathrm{Con}(\mathbf{A})$ such that $\Phi \cap B^2 = \Theta$. A class of pointed residuated lattices is said to have one of these properties if all of its members have the property.

A *span* in a class K of pointed residuated lattices is a 5-tuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i_B, i_C)$ consisting of algebras $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{K}$ and embeddings $i_B \colon \mathbf{A} \to \mathbf{B}$, $i_C \colon \mathbf{A} \to \mathbf{C}$. A *one-sided amalgam in* K of this span is a triple $(\mathbf{D}, j_B, j_C)$ consisting of some $\mathbf{D} \in \mathsf{K}$, an embedding $j_B \colon \mathbf{B} \to \mathbf{D}$, and a homomorphism $j_C \colon \mathbf{C} \to \mathbf{D}$ such that $j_B i_B = j_C i_C$; it is called an *amalgam in* K of the span if also j_C is an embedding. The class K is said to have the *(one-sided) amalgamation property* if every span in K has a (one-sided) amalgam in K.

PROPOSITION 2.1 [9, Corollary 3.5]. *Let* V *be a variety of semilinear pointed residuated lattices that has the congruence extension property. Then* V *has the amalgamation property if and only if the class of* (*finitely generated*) *totally ordered members of* V *has the one-sided amalgamation property.*

Next, let $\mathbf{Fm}$ denote the algebra of formulas constructed for the language of pointed residuated lattices over a countably infinite set of variables. We write $\alpha(\overline{x})$ or $T(\overline{x})$ to denote that the variables of a formula α or set of formulas T, respectively, belong to the set $\overline{x}$. For convenience, we assume that $\overline{x}$, $\overline{y}$, etc. denote disjoint sets, writing $\overline{x}, \overline{y}$ to denote their union.

A consequence relation is associated with a variety V of pointed residuated lattices by defining for any $T \cup \{\alpha\} \subseteq \mathrm{Fm}$,

$$T \models_{\mathsf{V}} \alpha \;:\Longleftrightarrow\; \text{for every } \mathbf{A} \in \mathsf{V} \text{ and homomorphism } h \colon \mathbf{Fm} \to \mathbf{A},\\ \mathrm{e} \leq h(\beta) \text{ for all } \beta \in T \implies \mathrm{e} \leq h(\alpha).$$

Every axiomatic extension L of the full Lambek calculus FL (see Appendix A) corresponds to a variety V_L of pointed residuated lattices such that the deductive system $\vdash_\mathsf{L}$ coincides with $\models_{\mathsf{V}_\mathsf{L}}$. Indeed, this correspondence gives a dual lattice isomorphism between the lattice of axiomatic extensions of FL and the lattice of varieties of pointed residuated lattices (see, e.g., [14, 31]). We say that L has the *deductive interpolation property* if for any $\alpha(\overline{x}, \overline{y}), \beta(\overline{y}, \overline{z}) \in \mathrm{Fm}$ satisfying $\{\alpha\} \models_{\mathsf{V}_\mathsf{L}} \beta$, there exists a $\gamma(\overline{y}) \in \mathrm{Fm}$ satisfying $\{\alpha\} \models_{\mathsf{V}_\mathsf{L}} \gamma$ and $\{\gamma\} \models_{\mathsf{V}_\mathsf{L}} \beta$.

The following well-known results relate the deductive interpolation property for an axiomatic extension L of the full Lambek calculus to the amalgamation property for the corresponding variety V_L of pointed residuated lattices.

PROPOSITION 2.2 (cf. [29, Theorem 22]). *Let* L *be any axiomatic extension of the full Lambek calculus.*

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(i) *If* V_{L} *has the amalgamation property, then* L *has the deductive interpolation property.*
(ii) *If* V_{L} *has the congruence extension property and* L *has the deductive interpolation property, then* V_{L} *has the amalgamation property.*

§3. Idempotent residuated lattices. In this section, we present basic facts and structure theory for idempotent (semilinear) residuated lattices, referring to [3, 7, 8, 17, 19, 38] for further details.

For convenience, we first give a useful criterion for homomorphisms between residuated chains to be injective.

LEMMA 3.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be residuated chains and let* $a \in A$ *be any subcover of* e, *i.e.*, $a < e$ *and* $a < b \leq e$ *implies* $b = e$. *Then a homomorphism* $f : \mathbf{A} \to \mathbf{B}$ *is injective if and only if* $f(a) < f(e)$.

PROOF. The left-to-right direction is immediate. For the converse, suppose contrapositively that f is not injective. Let $\mathbf{C}$ be the convex normal subalgebra of $\mathbf{A}$ with universe $C := \{x \in A \mid f(x) = \mathrm{e}\}$. Since f is not injective, there exists a $b \in C \backslash \{\mathrm{e}\}$. Moreover, we may assume that $b < \mathrm{e}$, since otherwise $\mathrm{e} < b$, so $b \backslash \mathrm{e} < \mathrm{e}$ and $b \backslash \mathrm{e} \in C$. But then, since a is a subcover of e and C is convex, $a \in C$, yielding $f(a) = \mathrm{e}$. ⊣

Not every pointed residuated lattice has the congruence extension property, but it is known that this property is satisfied under the assumptions of either commutativity—reflecting the fact that every axiomatic extension of the full Lambek calculus with exchange has a local deduction theorem (see, e.g., [31])—or, as recorded below for convenience, idempotency, and semilinearity.

LEMMA 3.2 [7, Corollary 4.4]. *Every idempotent semilinear pointed residuated lattice has the congruence extension property.*

We will make free use of basic properties of idempotent residuated lattices (chains) in performing computations, summarized in the following lemmas (for a detailed account, see, e.g., [7, Section 3]).

LEMMA 3.3. *Let* $\mathbf{A}$ *be any idempotent residuated lattice. For any* $x, y \in A$,

(a) $x \wedge y \leq xy \leq x \vee y$;
(b) *if* $xy \leq e$, *then* $xy = x \wedge y$;
(c) *if* $e \leq xy$, *then* $xy = x \vee y$.

LEMMA 3.4. *Let* $\mathbf{A}$ *be any idempotent residuated chain. For any* $x, y \in A$,

$$xy = \begin{cases} x & \text{if } y \in (x^r, x] \text{ or } y \in [x, x^r] \\ y & \text{if } x \in (y^\ell, y] \text{ or } x \in [y, y^\ell] \end{cases},$$

$$x \backslash y = \begin{cases} x^r \vee y & \text{if } x \leq y \\ x^r \wedge y & \text{if } y < x \end{cases}, \qquad y / x = \begin{cases} x^\ell \vee y & \text{if } x \leq y \\ x^\ell \wedge y & \text{if } y < x \end{cases}.$$

A subset of A *is therefore a subuniverse of* $\mathbf{A}$ *if and only if it contains* e *and is closed under the operations* $x \mapsto x^\ell$ *and* $x \mapsto x^r$.

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LEMMA 3.5. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be any idempotent residuated chains. An injective map* $h\colon A \to B$ *is an embedding if and only if it is order-preserving and satisfies* $h(e) = e$, $h(x^\ell) = h(x)^\ell$, *and* $h(x^r) = h(x)^r$, *for all* $x \in A$.

If an idempotent residuated chain is also *-involutive, then its normal convex subalgebras and congruences have a special form.

PROPOSITION 3.6. *Let* $\mathbf{A}$ *be a* **-involutive idempotent residuated chain, let* $\mathbf{C}$ *be a convex normal subalgebra of* $\mathbf{A}$, *and let* $\Theta := \{(x, y) \in A^2 \mid x\backslash y \wedge y\backslash x \in C\}$ *be the congruence corresponding to* $\mathbf{C}$. *Then* $[x]_\Theta = \{x\}$ *for all* $x \notin C$. *In particular, every quotient of* $\mathbf{A}$ *is isomorphic to a subalgebra of* $\mathbf{A}$.

PROOF. Suppose toward a contradiction that $(x, y) \in \Theta$ with $x, y \notin C$ and $x \neq y$. Assume further, without loss of generality, that $x < y$. From $(x, y) \in \Theta$, we have $x\backslash y \wedge y\backslash x \in C$, and, by direct computation, $x\backslash y \wedge y\backslash x = (x^r \vee y) \wedge (y^r \wedge x)$. Since $\mathbf{A}$ is totally ordered, either $x^r \vee y \in C$ (in which case $y < x^r \in C$) or $y^r \wedge x \in C$ (in which case $x > y^r \in C$). Hence either $x^r \in C$ or $y^r \in C$. Suppose that $x^r \in C$. Then also $x^{r\ell} \in C$. By [8, Corollary 4.5], either $x^{r\ell} = x$ or $x^\star = x^r$. But $x = x^{r\ell} \in C$ contradicts $x \notin C$. On the other hand, $x^\star = x^r$ implies $x^\star \in C$. In this case, $x^{\star\star} = x \in C$, since $\mathbf{A}$ is *-involutive, again a contradiction. The assumption that $y^r \in C$ similarly leads to a contradiction.

Clearly, the quotient $\mathbf{A}/\Theta$ is obtained from $\mathbf{A}$ by collapsing the elements in $\mathbf{C}$ and leaving the remaining elements uncollapsed. Using [8, Corollary 4.5], it is then easy to see that the map $h\colon \mathbf{A}/\Theta \to \mathbf{A}$ defined by $h([x]_\Theta) := x$ if $x \notin C$ and $h([x]_\Theta) := \mathrm{e}$ if $x \in C$, is an embedding. ⊣

We now introduce an operator for combining a family of residuated chains that is especially well behaved for *-involutive idempotent residuated chains, and will play a central role in our investigations below.

Let us say first that a residuated chain $\mathbf{A}$ is *admissible* if $x\backslash \mathrm{e}, \mathrm{e}/x \notin \{\mathrm{e}\}$ for each $x \in A\backslash\{\mathrm{e}\}$. Given a totally ordered set $(I, \leq)$ whose greatest element (if any) is denoted by $\top$, we say that an indexed family $(\mathbf{A}_i)_{i\in I}$ of residuated chains is *admissible* if $\mathbf{A}_i$ is admissible for all $i \in I\backslash\{\top\}$.

Let $(I, \leq)$ be a non-empty totally ordered set and $\mathbf{A}_i = (A_i, \wedge_i, \vee_i, \cdot_i, \backslash_i, /_i, \mathrm{e})$ an admissible residuated chain for each $i \in I\backslash\{\top\}$, assuming for simplicity of notation that $A_i \cap A_j = \{\mathrm{e}\}$ for $i \neq j$. We define an algebraic structure $\boxplus_{i\in I}\mathbf{A}_i$ with universe $A := \bigcup_{i\in I} A_i$ as follows. First, we let $\leq$ be the smallest partial order on A satisfying:

1. $\leq$ extends $\leq_i$ for each $i \in I$;
2. if $i < j$, $x \in A_i$, $y \in A_j$, and $x <_i \mathrm{e}$, then $x \leq y$;
3. if $i < j$, $x \in A_i$, $y \in A_j$, and $\mathrm{e} <_i x$, then $y \leq x$.

It is easy to see that $\leq$ is a total order with lattice operations $\wedge$ and $\vee$. Next, for $* \in \{\cdot, \backslash, /\}$, we let $x * y := x *_i y$ if $x, y \in A_i$, and for $x \in A_i$, $y \in A_j$ with $i < j$, let $x * y := x *_i \mathrm{e}$ and $y * x := \mathrm{e} *_i x$. The resulting algebraic structure $(A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e})$ is denoted by $\boxplus_{(I,\leq)}\mathbf{A}_i$, and called a *nested sum* of $(\mathbf{A}_i)_{i\in I}$. Note that we can always assume that the chain $(I, \leq)$ has a top element, since we can add $\top$ to I and set $\mathbf{A}_\top$ to be a trivial algebra. Moreover, we write $\boxplus_{i=1}^k \mathbf{A}_i$ and $\mathbf{A}_1 \boxplus \mathbf{A}_2$ for, respectively, $I = \{1, \ldots, k\}$ and $I = \{1, 2\}$ with the standard total order. We also stipulate that $\boxplus_{i=1}^k \mathbf{A}_i$ is a trivial algebra for $k < 1$. When each $\mathbf{A}_i$ is integral, the nested sum is often called the *ordinal sum* and has been used extensively in the study of totally ordered

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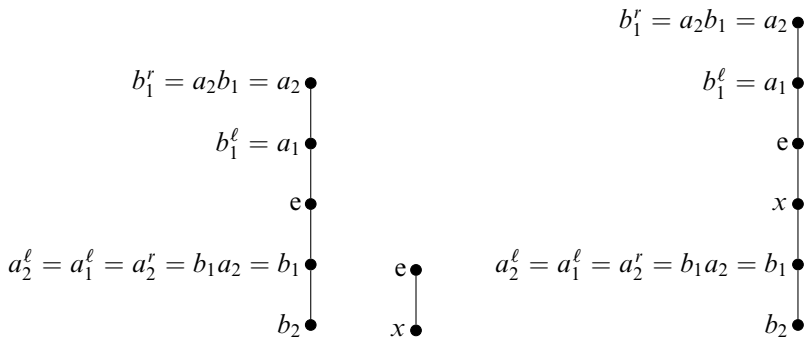


FIGURE 2. Labeled Hasse diagrams for idempotent residuated chains $\mathbf{A}$ (left), $\mathbf{B}$ (middle), and their nested sum $\mathbf{A} \boxplus \mathbf{B}$ (right). Note that the algebra $\mathbf{A}$ is *-involutive.

integral residuated chains (see, e.g., [1, 12]). The binary nested sum is associative and has the trivial algebra as an identity. Intuitively, the nested sum $\mathbf{A} \boxplus \mathbf{B}$ results from 'replacing' the unit of $\mathbf{A}$ by the algebra $\mathbf{B}$. See Figure 2 for an example depicting a nested sum.

The following structural description is fundamental to the development of ideas in subsequent sections. Parts (i) and (ii) follow from [8, Lemma 4.18] and part (iii) is [8, Lemma 4.22].

LEMMA 3.7.

(i) *Every *-involutive idempotent residuated chain is admissible.*
(ii) *The class of *-involutive idempotent residuated chains is closed under nested sums.*
(iii) *Every *-involutive idempotent residuated chain is the nested sum of its* 1*-generated subalgebras.*

The following result shows that embeddings between components of nested sums lift appropriately to embeddings between the nested sums themselves.

PROPOSITION 3.8. *Let $\mathbf{A} = \boxplus_{(I,\leq)} \mathbf{A}_i$ and $\mathbf{B} = \boxplus_{(J,\leq)} \mathbf{B}_j$ be nested sums of idempotent residuated chains and suppose that $f : I \to J$ is an order-embedding with $f(\top) = \top$ and $g_i : A_i \to B_{f(i)}$ is an embedding for each $i \in I$. Then the map $g : A \to B$, defined by $g(a) := g_i(a)$ for $a \in A_i$ is an embedding. In particular, the inclusion map is an embedding of $\mathbf{B}_j$ into $\mathbf{B}$ for each $j \in J$.*

PROOF. The map g is clearly well-defined, injective, and satisfies $g(\mathrm{e}) = \mathrm{e}$. Hence, by Lemma 3.5, it suffices to show that g is order-preserving and preserves the operations $x \mapsto x^\ell$ and $x \mapsto x^r$. That g preserves $^\ell$ and ^{r} follows from the fact that $g(x^\bullet) = g_i(x^\bullet) = g_i(x)^\bullet = g(x)^\bullet$, for any $i \in I$, $x \in A_i$, and $\bullet \in \{\ell, r\}$, since the operations of $\mathbf{A}$ and $\mathbf{B}$ extend the operations on $\mathbf{A}_i$ and $\mathbf{B}_{f(i)}$, respectively, and g_i is a homomorphism. Finally, suppose that $x, y \in A$ with $x \leq y$ and $x \in A_i$, $y \in A_j$ for $i, j \in I$. There are several cases. If $i = j$, then $g(x) = g_i(x) \leq g_i(y) = g(y)$, since the order on $\mathbf{B}$ extends the order on $\mathbf{B}_{f(i)}$ and g_i is order-preserving. Otherwise,

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$i \neq j$. If $x \leq \mathrm{e} \leq y$, then clearly $g(x) = g_i(x) \leq \mathrm{e} \leq g_j(y) = g(y)$. If $x \leq y \leq \mathrm{e}$, then $i < j$, yielding $f(i) < f(j)$. Moreover, $g_i(x) \leq \mathrm{e}$, and $g_j(y) \leq \mathrm{e}$. Hence, by the definition of the nested sum, $g(x) = g_i(x) \leq g_j(y)$. Similarly, if $\mathrm{e} \leq x \leq y$, then $g(x) \leq g(y)$. ⊣

In Section 5, we will make use of a structural description of finite commutative idempotent residuated chains that involves a number of basic building blocks. An *odd Sugihara monoid* is a commutative idempotent semilinear *-involutive residuated lattice; a *Brouwerian algebra* is a residuated lattice satisfying $xy \approx x \wedge y$, and a *Heyting algebra* is a pointed residuated lattice satisfying $xy \approx x \wedge y$ and $\mathrm{f} \leq x$; *relative Stone algebras* and *Gödel algebras* are semilinear Brouwerian algebras and semilinear Heyting algebras, respectively.

Let us also fix some useful notation:

- For $n \in \mathbb{N}$, we denote by $\mathbf{G}_n$ the $(n+1)$-element relative Stone algebra with universe $G_n = \{c_n < \cdots < c_1 < \mathrm{e}\}$, i.e., the $(n+1)$-element commutative idempotent residuated chain with $x \cdot y := x \wedge y$.
- For $m, n \in \mathbb{N}$, we define a commutative idempotent residuated chain $\mathbf{C}^n_m = (C^n_m, \wedge, \vee, \cdot, \backslash, /, \mathrm{e})$ as follows. The universe and underlying order of this algebra is given by $C^n_m := \{b_m < \cdots < b_0 < \mathrm{e} < a_n < \cdots < a_0\}$. For multiplication, we let $a_i a_j := a_{\min\{i,j\}} = a_i \vee a_j$, $b_k b_l := b_{\max\{k,l\}} = b_k \wedge b_l$, and $a_i b_k = b_k a_i := b_k$ for $i, j \in \{0, \ldots, n\}$, $k, l \in \{0, \ldots, m\}$, where e is the unit. The residual is uniquely determined by the order and definition of the multiplication by setting $x \to z := \max\{y \in C^n_m \mid xy \leq z\}$.

Note that, up to isomorphism, $\mathbf{C}^0_0$ is the three-element odd Sugihara monoid and $\boxplus^k_{i=1}\mathbf{C}^0_0$ is the totally ordered $(2k+1)$-element odd Sugihara monoid. In what follows we will always assume that in the nested sum $\boxplus^k_{i=1}\mathbf{C}^{n_i}_{m_i}$ we have $C^{n_i}_{m_i} = \{b^i_{m_i} < \cdots < b^i_0 < \mathrm{e} < a^i_{n_i} < \cdots < a^i_0\}$.

LEMMA 3.9 [19, Proposition 3.4]. *Let* $\mathbf{A}$ *be a finite commutative idempotent residuated chain. Then there exists a* $k \in \mathbb{N}$ *such that* $\mathbf{A}$ *contains an isomorphic copy* $\mathbf{B}$ *of* $\boxplus^k_{i=1}\mathbf{C}^0_0$ *with* $B = \{a \in A \mid a^{\star\star} = a\}$. *Moreover,* $\mathbf{A}$ *is partitioned by the family of intervals* $\{B_b\}_{b \in B}$, *where* $B_b := \{a \in A \mid a^{\star\star} = b\}$.

We call the algebra $\mathbf{B}$ in the previous lemma the *Sugihara skeleton* of $\mathbf{A}$.

LEMMA 3.10. *Let* $\mathbf{A}$ *be a finite commutative idempotent residuated chain. Then* $\mathbf{A}$ *is isomorphic to a nested sum* $(\boxplus^k_{i=1}\mathbf{C}^{n_i}_{m_i}) \boxplus \mathbf{G}_p$ *with* $k, p \in \mathbb{N}$ *and* $m_i, n_i \in \mathbb{N}$ *for each* $i \in \{1, \ldots, k\}$.

PROOF. Let $\mathbf{B} = \boxplus^k_{i=1}\mathbf{C}^0_0$ be the Sugihara skeleton of $\mathbf{A}$, with universe $B = \{b^1_0 < \cdots < b^k_0 < \mathrm{e} < a^k_0 < \cdots < a^1_0\}$. Define $m_i := |B_{b^i_0}| - 1$ and $n_i := |B_{a^i_0}| - 1$ for $i \in \{1, \ldots, k\}$ and $p := |B_{\mathrm{e}}| - 1$. Also, for $i \in \{1, \ldots, k\}$, let $A_i := B_{b^i_0} \cup \{\mathrm{e}\} \cup B_{a^i_0}$. Then $A_i = \{b^i_{m_i} < \cdots < b^i_0 < \mathrm{e} < a^i_{n_i} < \cdots < a^i_0\}$. It is not hard to see that the map $f_i \colon C^{n_i}_{m_i} \to A_i$ defined by $f(\mathrm{e}) := \mathrm{e}$, $f_i(a_j) := a^i_j$, $f_i(b_j) := b^i_j$ is an embedding. Moreover, for $B_{\mathrm{e}} = \{x_p < \cdots < x_1 < \mathrm{e}\}$ the map $f_0 \colon G_p \to B_{\mathrm{e}}$, defined by $f_0(\mathrm{e}) := \mathrm{e}$, and $f_0(c_j) := x_j$ for each $j \in \{1, \ldots, p\}$ is an embedding. Defining $f : (\boxplus^k_{i=1}\mathbf{C}^{n_i}_{m_i}) \boxplus \mathbf{G}_p \to \mathbf{A}$, by

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$$f(x) = \begin{cases} f_i(x) & \text{if } x \in C_{m_i}^{n_i} \\ f_0(x) & \text{if } x \in G_p, \end{cases}$$

yields the desired isomorphism. Moreover, the embedding f is surjective, since $\mathbf{A}$ is partitioned by $\{B_b\}_{b\in B}$. ⊣

LEMMA 3.11. *Suppose that* $(\boxplus_{i=1}^{k}\mathbf{C}_{m_i}^{n_i}) \boxplus \mathbf{G}_p \cong (\boxplus_{j=1}^{l}\mathbf{C}_{r_j}^{s_j}) \boxplus \mathbf{G}_q$. *Then* $k = l$ *and* $p = q$, *and* $r_i = m_i$ *and* $s_i = n_i$ *for each* $i \in \{1, \dots, k\}$.

PROOF. Let $\mathbf{A} \cong (\boxplus_{i=1}^{k}\mathbf{C}_{m_i}^{n_i}) \boxplus \mathbf{G}_p \cong (\boxplus_{j=1}^{l}\mathbf{C}_{r_j}^{s_j}) \boxplus \mathbf{G}_q$. Let $\mathbf{B}$ be the Sugihara skeleton of $\mathbf{A}$. Note first that $2k + 1 = |\{a \in A \mid a^{\star\star} = a\}| = 2l + 1$. Hence $k = l$. Similarly we have for $B = \{b_0^1 < \cdots < b_0^k < \mathrm{e} < a_0^k < \cdots < a_0^1\}$ that $m_i = |B_{b_0^i}| - 1 = r_i$, $n_i = |B_{a_0^i}| - 1 = s_i$, and $p = |B_{\mathrm{e}}| - 1 = q$. ⊣

We have therefore established the following structural description of finite commutative idempotent residuated chains.

PROPOSITION 3.12 (cf. [8, 19, 38]). *Let* $\mathbf{A}$ *be any finite commutative idempotent residuated chain. Then* $\mathbf{A}$ *is isomorphic to a unique nested sum* $(\boxplus_{i=1}^{k}\mathbf{C}_{m_i}^{n_i}) \boxplus \mathbf{G}_p$ *with* $k, p \in \mathbb{N}$ *and* $m_i, n_i \in \mathbb{N}$ *for each* $i \in \{1, \dots, k\}$.

§4. Interpolation without exchange. Our main aim in this section is to prove the following result.

THEOREM A.

(i) *Continuum-many varieties of idempotent semilinear residuated lattices have the amalgamation property and contain non-commutative members.*

(ii) *Continuum-many axiomatic extensions of* $\mathsf{SemRL}_{\mathsf{cm}}$ *in which exchange is not derivable have the deductive interpolation property.*

The second part of Theorem A follows from the first part, using Proposition 2.2 and the fact that the variety of idempotent semilinear residuated lattices is an equivalent algebraic semantics for $\mathsf{SemRL}_{\mathsf{cm}}$. To prove the first part, we make use of a construction of continuum-many atoms in the lattice of varieties of idempotent semilinear residuated lattices given in [13]. Each of these atoms is generated by a single infinite non-commutative algebra $\mathbf{A}_S$, and, as we show here, has the amalgamation property.

For any $S \subseteq \mathbb{Z}$, let $A_S := \{a_i \mid i \in \mathbb{Z}\} \cup \{b_j \mid j \in \mathbb{Z}\} \cup \{\mathrm{e}\}$, and totally order the elements of this set by stipulating $b_i < b_j < \mathrm{e} < a_j < a_i$, for any $i, j \in \mathbb{Z}$ with $i < j$. For the multiplication, let e be the multiplicative unit and, for any $i, j \in \mathbb{Z}$, let $a_i a_j := a_{\min\{i,j\}}$, $b_i b_j := b_{\min\{i,j\}}$, and

$$a_i b_j := \begin{cases} a_i & \text{if } i < j \text{ or } i = j \in S \\ b_j & \text{if } i > j \text{ or } i = j \notin S \end{cases}, \qquad b_j a_i := \begin{cases} b_j & \text{if } j < i \text{ or } i = j \in S \\ a_i & \text{if } j > i \text{ or } i = j \notin S. \end{cases}$$

It is straightforward to check that this multiplication is residuated, and hence that the residual operations $\backslash$ and $/$ satisfy for all $x, y \in A_S$,

$$x\backslash y = \max\{z \in A_S \mid xz \le y\}, \qquad y/x = \max\{z \in A_S \mid zx \le y\}.$$

We denote the residuated chain obtained in this way by $\mathbf{A}_S$.

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LEMMA 4.1 [13, Corollary 5.2]. *Let $S \subseteq \mathbb{Z}$. Then $\mathbf{A}_S$ is a strictly simple idempotent residuated chain, and hence generated by any element $x \in A_S \setminus \{e\}$.*

The algebras $\mathbf{A}_S$ encode some of the dynamics of bi-infinite words. A *word* over $\{0,1\}$ is a function $w\colon W \to \{0,1\}$, where W is an interval of $\mathbb{Z}$; it is said to be *finite* if $|W|$ is finite and *bi-infinite* if $W = \mathbb{Z}$. A finite word $v\colon W \to \{0,1\}$ is a *subword* of a word w if there exists a $k \in \mathbb{Z}$ such that $v(i) = w(i+k)$ for all $i \in W$. For any $S \subseteq \mathbb{Z}$, we will often consider the bi-infinite word given by the characteristic function of S,

$$w_S\colon \mathbb{Z} \to \{0,1\}; \quad i \mapsto \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$

Indeed, every bi-infinite word is of the form w_S for some $S \subseteq \mathbb{Z}$. We define a pre-order $\sqsubseteq$ on the set of all bi-infinite words by setting $w_1 \sqsubseteq w_2$ if and only if every finite subword of w_1 is a subword of w_2. For bi-infinite words w_1, w_2, we write $w_1 \cong w_2$ if and only if $w_1 \sqsubseteq w_2$ and $w_2 \sqsubseteq w_1$. A bi-infinite word w is said to be *minimal* if it is minimal with respect to the pre-order $\sqsubseteq$, i.e., if $w' \sqsubseteq w$ for some bi-infinite word w', then $w' \cong w$.

EXAMPLE 4.2. Let $\mathbb{E}$ be the set of even integers and $\mathbb{O}$ the set of odd integers. Clearly, $w_{\mathbb{E}}(i) = w_{\mathbb{O}}(i+1)$ for any integer i, and it follows easily that $w_{\mathbb{E}} \cong w_{\mathbb{O}}$. The algebra $\mathbf{A}_{\mathbb{E}}$ is depicted in Figure 3.

The following lemma collects some facts about the algebras $\mathbf{A}_S$ that will be needed in what follows, summarizing portions of Lemma 5.1, Theorem 5.4, and Corollary 5.5 of [13].

LEMMA 4.3 [13]. *Let $S, T \subseteq \mathbb{Z}$.*

(i) *$\mathbb{V}(\mathbf{A}_S) \subseteq \mathbb{V}(\mathbf{A}_T)$ if and only if $w_S \sqsubseteq w_T$ if and only if $\mathbf{A}_S$ embeds into the ultrapower $\mathbf{A}_T^{\mathbb{N}}/U$ for every non-principal ultrafilter U over $\mathbb{N}$.*
(ii) *Every non-trivial 1-generated totally ordered member of $\mathbb{V}(\mathbf{A}_S)$ is isomorphic to an algebra of the form $\mathbf{A}_{S'}$ for some $S' \subseteq \mathbb{Z}$ with $w_{S'} \sqsubseteq w_S$.*
(iii) *If w_S is minimal, then $\mathbb{V}(\mathbf{A}_S)$ is an atom in the lattice of varieties of idempotent semilinear residuated lattices.*

By part (i) of this lemma, $\mathbb{V}(\mathbf{A}_S) = \mathbb{V}(\mathbf{A}_T)$ if and only if $w_S \cong w_T$. Combining part (iii) with the fact that there exist continuum-many pairwise incomparable minimal bi-infinite words (see [33]) therefore yields the following result.

PROPOSITION 4.4 [13, Corollary 5.6]. *There are continuum-many atoms in the lattice of varieties of idempotent semilinear residuated lattices of the form $\mathbb{V}(\mathbf{A}_S)$ for some $S \subseteq \mathbb{Z}$ such that w_S is minimal.*

Hence it remains to show that $\mathbb{V}(\mathbf{A}_S)$ has the amalgamation property for any $S \subseteq \mathbb{Z}$ such that w_S is minimal. To this end, we prove several technical lemmas.

LEMMA 4.5. *Let $S \subseteq \mathbb{Z}$. Then $\mathbf{A}_S$ is a *-involutive idempotent residuated chain; in particular, each $\mathbf{A}_S$ is admissible.*

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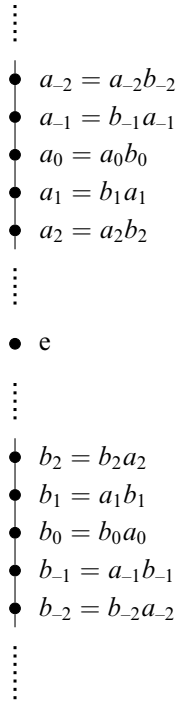


FIGURE 3. Labeled Hasse diagrams for the algebra $\mathbf{A}_{\mathbb{E}}$.

PROOF. Direct computation shows that, for each $i \in \mathbb{Z}$,

$$a_i^\ell = \begin{cases} b_i & \text{if } i \in S \\ b_{i-1} & \text{if } i \notin S \end{cases}, \qquad a_i^r = \begin{cases} b_i & \text{if } i \notin S \\ b_{i-1} & \text{if } i \in S \end{cases},$$

$$b_i^\ell = \begin{cases} a_i & \text{if } i \notin S \\ a_{i+1} & \text{if } i \in S \end{cases}, \qquad b_i^r = \begin{cases} a_i & \text{if } i \in S \\ a_{i+1} & \text{if } i \notin S. \end{cases}$$

As a consequence, $b_i^\star = a_i \wedge a_{i+1} = a_{i+1}$ and $a_i^\star = b_i \wedge b_{i-1} = b_{i-1}$ for all $i \in S$. It follows that $b_i^{\star\star} = a_{i+1}^\star = b_{(i+1)-1} = b_i$ and $a_i^{\star\star} = b_{i-1}^\star = a_{(i-1)+1} = a_i$. Hence $x^{\star\star} = x$ for all x and $\mathbf{A}_S$ is *-involutive. That $\mathbf{A}_S$ is admissible is immediate from part (i) of Lemma 3.7. ⊣

Now, for $S \subseteq \mathbb{Z}$, let $\mathsf{V}_S := \mathbb{V}(\mathbf{A}_S)$ be the variety generated by $\mathbf{A}_S$ and define

$$\mathsf{K}_S := \mathbb{I}(\{\mathbf{A}_T \mid w_T \sqsubseteq w_S\}).$$

It follows from parts (i) and (ii) of Lemma 4.3 that K_S consists of the non-trivial 1-generated algebras of V_S. Moreover, since V_S satisfies $x^{\star\star} \approx x$, by Lemma 4.5, each totally ordered member of V_S is isomorphic to a nested sum of algebras from K_S by part (iii) of Lemma 3.7.

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LEMMA 4.6. *Let $(I, \leq)$ be a chain, $(\mathbf{A}_i)_{i\in I}$ an admissible indexed family of idempotent residuated chains, and U an ultrafilter over some set X. Then the map $\iota\colon \boxplus_{(I,\leq)} (\mathbf{A}_i^X/U) \to (\boxplus_{(I,\leq)}\mathbf{A}_i)^X/U$; $[a]_U \mapsto [a]_U$ is an embedding.*

PROOF. First note that if an idempotent residuated chain $\mathbf{A}$ is admissible, then so is every ultrapower of $\mathbf{A}$, since admissibility can be expressed by the quasiequations $x^\ell \approx \mathrm{e} \Rightarrow x \approx \mathrm{e}$ and $x^r \approx \mathrm{e} \Rightarrow x \approx \mathrm{e}$. Hence $(\mathbf{A}_i^X/U)_{i\in I}$ is also admissible. The map ι is well defined, since for each $i \in I$ and any $a = (a_x)_{x\in X}, b = (b_x)_{x\in X} \in A_i^X$,

$$\begin{aligned}[a]_U = [b]_U \text{ in } \mathbf{A}_i^X/U &\iff \{x \in X \mid a_x = b_x\} \in U\\ &\iff [a]_U = [b]_U \text{ in } (\boxplus_{(I,\leq)}\mathbf{A}_i)^X/U.\end{aligned}$$

Moreover, ι is clearly injective. To see that ι is order-preserving, consider any $[a]_U, [b]_U \in \boxplus_{(I,\leq)}(A_i^X/U)$ with $[a]_U \leq [b]_U$. There are three cases: either $a, b \in A_i^X$, or $a \in A_i^X$ and $b \in A_j^X$ with $i < j$, or $a \in A_i^X$ and $b \in A_j^X$ with $i > j$.

If $a, b \in A_i^X$ for some $i \in I$, then $\{x \in X \mid a_x \leq b_x\} \in U$, so also $\iota([a]_U) \leq \iota([b]_U)$. Otherwise we may assume that $[a]_U, [b]_U \neq [\mathrm{e}]_U$.

Suppose next that $a \in A_i^X$ and $b \in A_j^X$ with $i < j$. Then $[a]_U < [b]_U < [\mathrm{e}]_U$ or $[a]_U < [\mathrm{e}]_U < [b]_U$. If $[a]_U < [b]_U < [\mathrm{e}]_U$, then $\{x \in X \mid a_x < \mathrm{e}\}, \{x \in X \mid b_x < \mathrm{e}\} \in U$. But then also $\{x \in X \mid a_x < \mathrm{e}, b_x < \mathrm{e}\} \in U$. Hence, since $i < j$, by definition of the nested sum, $\{x \in X \mid a_x < \mathrm{e}, b_x < \mathrm{e}\} \subseteq \{x \in X \mid a_x < b_x\}$, i.e., $\{x \in X \mid a_x < b_x\} \in U$, so $\iota([a]_U) \leq \iota([b]_U)$. If $[a]_U < [\mathrm{e}]_U < [b]_U$, then $\{x \in X \mid a_x < \mathrm{e}\}, \{x \in X \mid \mathrm{e} < b_x\} \in U$. Hence clearly also $\iota([a]_U) < \iota([\mathrm{e}]_U) < \iota([b]_U)$.

Suppose finally that $a \in A_i^X$ and $b \in A_j^X$ with $i > j$. Then, similarly to the previous case, $\iota([a]_U) \leq \iota([b]_U)$ in $(\boxplus_{(I,\leq)}\mathbf{A}_i)^X/U$. Also $\iota([\mathrm{e}]_U) = [\mathrm{e}]_U$. To show that ι preserves $^\ell$ and ^{r}, consider any $a = (a_x)_{x\in X} \in A_i$ for some $i \in I$. Then $\iota([a]_U^\ell) = \iota([(a_x^\ell)_{x\in X}]_U) = [(a_x^\ell)_{x\in X}]_U = [a]_U^\ell$, and similarly for ^{r}. Hence the claim follows from Lemma 3.5. ⊣

LEMMA 4.7. *Let $S \subseteq \mathbb{Z}$.*

(i) *$\mathbb{HSP}_{\mathsf{U}}(\mathbf{A}_S)$ is the class of totally ordered members of V_S.*

(ii) *If w_S is minimal, then $\mathbb{HSP}_{\mathsf{U}}(\mathbf{A}_S)$ is closed under nested sums, and, in particular, the totally ordered members of V_S are exactly the nested sums of members of K_S.*

PROOF. (i) Each member of $\mathbb{HSP}_{\mathsf{U}}(\mathbf{A}_S)$ is totally ordered by Łoś' Theorem and Proposition 3.6. Conversely, every totally ordered member of V_S is finitely subdirectly irreducible, and hence contained in $\mathbb{HSP}_{\mathsf{U}}(\mathbf{A}_S)$ by Jónsson's lemma.

(ii) Suppose that w_S is minimal. Since every member of $\mathbb{HSP}_{\mathsf{U}}(\mathbf{A}_S)$ is a nested sum of members of K_S, every nested sum of members of $\mathbb{HSP}_{\mathsf{U}}(\mathbf{A}_S)$ is a nested sum of members of K_S. It therefore suffices to prove that $\mathbb{HSP}_{\mathsf{U}}(\mathbf{A}_S)$ contains every nested sum of members of K_S. Since every algebra embeds into an ultraproduct of its finitely generated subalgebras, every nested sum of members of K_S embeds into an ultraproduct of finite nested sums of members of K_S. Hence it suffices to show that every finite nested sum of members of $\{\mathbf{A}_T \mid w_T \sqsubseteq w_S\}$ (recalling that $\mathsf{K}_S := \mathbb{I}(\{\mathbf{A}_T \mid w_T \sqsubseteq w_S\})$) is contained in $\mathbb{HSP}_{\mathsf{U}}(\mathbf{A}_S)$.

Consider first any non-principal ultrafilter U over $\mathbb{N}$. Since $\mathbf{A}_S^{\mathbb{N}}/U \in \mathbb{HSP}_{\mathsf{U}}(\mathbf{A}_S)$, there exists a chain $(I, \leq)$ and an indexed family $(\mathbf{A}_{T_i})_{i\in I}$ of members of K_S such that $\mathbf{A}_S^{\mathbb{N}}/U \cong \boxplus_{(I,\leq)}\mathbf{A}_{T_i}$. Note that, since $\mathbf{A}_S$ is infinite, $\mathbf{A}_S^{\mathbb{N}}/U$ is uncountable, by [5, Theorem 1.28], and hence, since each $\mathbf{A}_{T_i}$ is countable, I must be infinite.

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Now let $S_1, \dots, S_n$ be any subsets of $\mathbb{Z}$ such that $w_{S_1}, \dots, w_{S_n} \sqsubseteq w_S$. The chain $\{1 < 2 < \cdots < n\}$ can be considered as a subchain of $(I, \leq)$ and $\boxplus_{i=1}^n \mathbf{A}_{T_i} \in \mathbb{HSP}_{\mathsf{U}}(\mathbf{A}_S)$. By assumption, w_S is minimal. So $w_{S_i} \cong w_S \cong w_{T_i}$ and there exists an embedding $f_i \colon \mathbf{A}_{S_i} \to \mathbf{A}_{T_i}^{\mathbb{N}}/U$, for each $i \in \{1, \dots n\}$, by part (i) of Lemma 4.3. Hence there exists an embedding $f \colon \boxplus_{i=1}^n \mathbf{A}_{S_i} \to \boxplus_{i=1}^n (\mathbf{A}_{T_i}^{\mathbb{N}}/U)$, by Proposition 3.8. But also, by Lemma 4.6, there exists an embedding $\iota \colon \boxplus_{i=1}^n (\mathbf{A}_{T_i}^{\mathbb{N}}/U) \to (\boxplus_{i=1}^n \mathbf{A}_{T_i})^{\mathbb{N}}/U$ and composing the two maps yields an embedding $\iota \circ f \colon \boxplus_{i=1}^n \mathbf{A}_{S_i} \to (\boxplus_{i=1}^n \mathbf{A}_{T_i})^{\mathbb{N}}/U$, showing that $\boxplus_{i=1}^n \mathbf{A}_{S_i} \in \mathbb{HSP}_{\mathsf{U}}(\mathbf{A}_S)$. ⊣

LEMMA 4.8. *Let S be any subset of $\mathbb{Z}$ such that w_S is minimal. Then V_S has the amalgamation property.*

PROOF. Since V_S has the congruence extension property, by Lemma 3.2, it suffices, using Proposition 2.1, to show that the class of totally ordered members of V_S has the one-sided amalgamation property. In fact, we show that the class of totally ordered members of V_S has the amalgamation property.

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be totally ordered members of V_S, and assume without loss of generality that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$, considering the span $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ with inclusion maps implicit. Then $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are nested sums of 1-generated totally ordered members of V_S, by Lemma 4.7, and we write

$$\mathbf{B} = \boxplus_{(I, \leq_I)} \mathbf{B}_i, \qquad \mathbf{C} = \boxplus_{(J, \leq_J)} \mathbf{C}_j,$$

where each $\mathbf{B}_i, \mathbf{C}_j$ is 1-generated. Further, we may assume without loss of generality that $I \cap J$ consists of exactly those indices where $\mathbf{A}_i = \mathbf{B}_i = \mathbf{C}_i$ is a 1-generated subalgebra of $\mathbf{A}$. Take any total order $\leq$ on $K := I \cup J$ extending $\leq_I \cup \leq_J$. For each $k \in K$, define

1. $\mathbf{D}_k := \mathbf{A}_k$ if $k \in I \cap J$;
2. $\mathbf{D}_k := \mathbf{B}_k$ if $k \in I \backslash J$;
3. $\mathbf{D}_k := \mathbf{C}_k$ if $k \in J \backslash I$.

Then $\mathbf{D} = \boxplus_{(K, \leq)} \mathbf{D}_k$ is an amalgam (again, with inclusion maps implicit) of the span $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, and $\mathbf{D} \in \mathsf{V}_S$, by Lemma 4.7. ⊣

PROOF OF THEOREM A. (i) By Proposition 4.4, there are continuum-many atoms in the lattice of varieties of idempotent semilinear residuated lattices of the form $\mathbb{V}(\mathbf{A}_S)$ for some $S \subseteq \mathbb{Z}$ such that w_S is minimal. Moreover, each such variety $\mathbb{V}(\mathbf{A}_S)$ contains a non-commutative member $\mathbf{A}_S$, and, by Lemma 4.8, has the amalgamation property.

(ii) Immediate from (i). ⊣

We conclude this section by showing that there are continuum-many axiomatic extensions in which the exchange rule is not derivable that *lack* the deductive interpolation property.

PROPOSITION 4.9.

(i) *Continuum-many varieties of idempotent semilinear residuated lattices contain non-commutative members and do not have the amalgamation property.*
(ii) *Continuum-many axiomatic extensions of* $\mathsf{SemRL}_{\mathsf{cm}}$ *in which exchange is not derivable do not have the deductive interpolation property.*

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PROOF. The proof of [8, Theorem 5.2] provides a span $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i_B, i_C)$ of finite idempotent residuated chains $\mathbf{A}, \mathbf{B}, \mathbf{C}$ such that no idempotent semilinear residuated lattice is an amalgam for this span. For each $S \subseteq \mathbb{Z}$, let W_S be the variety generated by $\mathsf{V}_S \cup \{\mathbf{B}, \mathbf{C}\}$. It is not hard to see, using Jónsson's lemma and the existence of continuum-many pairwise incomparable minimal bi-infinite words, that there are continuum-many distinct varieties of the form W_S, where w_S is minimal. Each W_S is a variety of idempotent semilinear residuated lattices, so the span $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i_B, i_C)$ has no amalgam in W_S. This proves part (i), and part (ii) follows as in the proof of Theorem A. ⊣

§5. Interpolation with exchange. In this section, we prove the following theorem.

THEOREM B.

(i) *Exactly* 60 *varieties of commutative idempotent semilinear residuated lattices have the amalgamation property.*
(ii) *Exactly* 60 *axiomatic extensions of* $\mathsf{SemRL}_{\mathsf{ecm}}$ *have the deductive interpolation property.*

THEOREM C. *There are exactly* 60 *varieties of commutative idempotent semilinear residuated lattices whose first-order theories have a model completion.*

The second part of Theorem B follows from the first part, using Proposition 2.2 and the fact that the variety of commutative idempotent semilinear residuated lattices is an equivalent algebraic semantics for $\mathsf{SemRL}_{\mathsf{ecm}}$. Since commutative idempotent semilinear residuated lattices are locally finite [38, Theorem 18] and the first-order theory of a locally finite variety has a model completion if and only if and only if the variety has the amalgamation property [39, p. 319, Corollary 1], this result also implies Theorem C. The main challenge of this section is therefore to prove the first part of Theorem B.

For a class K of algebras, let $\mathsf{K}_{\mathrm{fin}}$ denote the class of finite members of K, and for a variety V of idempotent semilinear residuated lattices, let $\mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$ denote the class of finite totally ordered members of V. The following lemma allows us to restrict our attention to embeddings between finite commutative idempotent residuated chains.

LEMMA 5.1. *Let* V *be a locally finite variety of idempotent semilinear residuated lattices. Then* V *has the amalgamation property if and only if* $\mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$ *has the one-sided amalgamation property.*

PROOF. Note first that V has the congruence extension property by Lemma 3.2. Moreover, the class of finitely subdirectly irreducible members of V consists of the totally ordered members of V, and is closed under subalgebras. Hence V has the amalgamation property if and only if every span of finitely generated totally ordered members of V has a one-sided amalgam in the class of totally ordered members of V, by Proposition 2.1. Since V is locally finite, its finitely generated totally ordered members are exactly the members of $\mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$. ⊣

Our proof of part (i) of Theorem B will consist of two parts. First, we describe several classes of finite commutative idempotent residuated chains that have the one-sided amalgamation property, corresponding to generating classes for varieties of

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commutative idempotent semilinear residuated lattices that have the amalgamation property. Second, to show that these are *all* the varieties of commutative idempotent semilinear residuated lattices with the amalgamation property, we exhibit a number of closure properties—results guaranteeing that if a variety V of commutative idempotent semilinear residuated lattices has the amalgamation property and V contains certain algebras, then V must also contain certain other algebras.

REMARK 5.2. Instead of appealing to the one-sided amalgamation property, the arguments in this section could alternatively be framed in terms of the essential amalgamation property discussed in [11, 12], which is well-suited to the study of amalgamation in semilinear varieties of residuated lattices.

We first develop some technical tools that will be used in the proof. For the first part of our proof, the next lemma will be useful to characterize the classes $\mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$ for the varieties V of commutative idempotent semilinear residuated lattices that have the amalgamation property.

LEMMA 5.3 [21, Theorem 2.3]. *Let V be a locally finite variety and let $\mathsf{K} \subseteq \mathsf{V}$ be a class of finite algebras such that $\mathbb{HS}(\mathsf{K}) = \mathsf{K}$. Then $(\mathbb{HSP}_{\mathsf{U}}(\mathsf{K}))_{\mathrm{fin}} = \mathsf{K}$.*

Recall that the algebras $\mathbf{G}_n$ and $\mathbf{C}^n_m$ $(n, m \in \mathbb{N})$ are defined in Section 3.

LEMMA 5.4.

(i) *An injective map $f : G_p \to G_q$ is an embedding of $\mathbf{G}_p$ into $\mathbf{G}_q$ if and only if it is order-preserving and $f(e) = e$.*
(ii) *An injective map $f : C^n_m \to C^s_r$ is an embedding of $\mathbf{C}^n_m$ into $\mathbf{C}^s_r$ if and only if it is order-preserving, $f(a_0) = a_0$, $f(b_0) = b_0$, and $f(e) = e$.*

PROOF. (i) Note that $x^\star = \mathrm{e}$ and $y^\star = \mathrm{e}$ for all $x \in G_p$ and $y \in G_q$. The claim therefore follows from Lemma 3.5.

(ii) Note that $\{x \in C^n_m \mid x^{\star\star} = x\} = \{b_0 < \mathrm{e} < a_0\}$ for all $n, m \in \mathbb{N}$. Hence every embedding $f : \mathbf{C}^n_m \to \mathbf{C}^s_r$ satisfies $f(a_0) = a_0$ and $f(b_0) = b_0$, while the other required properties are immediate. Conversely, let $f : C^n_m \to C^s_r$ be an injective order-preserving map such that $f(a_0) = a_0$, $f(b_0) = b_0$, and $f(\mathrm{e}) = \mathrm{e}$. Since f is order-preserving, $\mathrm{e} < f(a_i)$ for each $i \in \{1, \dots, n\}$. So $f(a_i{}^\star) = f(b_0) = b_0 = f(a_i)^\star$, and, similarly, $f(b_j{}^\star) = a_0 = f(b_j)^\star$ for each $j \in \{1, \dots, m\}$. Hence f is a homomorphism, by Lemma 3.5. ⊣

LEMMA 5.5. *Suppose that $\mathbf{A} = (\boxplus_{i=1}^k \mathbf{C}^{n_i}_{m_i}) \boxplus \mathbf{G}_p$ and $\mathbf{B} = (\boxplus_{j=1}^l \mathbf{C}^{s_j}_{r_j}) \boxplus \mathbf{G}_q$. Then there is a one-to-one correspondence between embeddings $h : \mathbf{A} \to \mathbf{B}$ and triples $(f, g, \{h_i\}_{i=1}^k)$ such that $f : \{1, \dots, k\} \to \{1, \dots, l\}$ is an order-embedding, $g : \mathbf{G}_p \to \mathbf{G}_q$ is an embedding, and $h_i : \mathbf{C}^{n_i}_{m_i} \to \mathbf{C}^{s_{f(i)}}_{r_{f(i)}}$ is an embedding for each $i \in \{1, \dots, k\}$.*

PROOF. Let $(f, g, \{h_i\}_{i=1}^k)$ be a triple such that $f : \{1, \dots, k\} \to \{1, \dots, l\}$ is an order-embedding, $g : \mathbf{G}_p \to \mathbf{G}_q$ is an embedding, and $h_i : \mathbf{C}^{n_i}_{m_i} \to \mathbf{C}^{s_{f(i)}}_{r_{f(i)}}$ is an embedding for each $i \in \{1, \dots, k\}$. By Proposition 3.8, the following map is an embedding of $\mathbf{A}$ into $\mathbf{B}$:

$$h : A \to B; \quad a \mapsto \begin{cases} h_i(a) & \text{if } a \in C^{n_i}_{m_i} \\ g(a) & \text{if } a \in G_p. \end{cases}$$

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Conversely, let $h\colon \mathbf{A} \to \mathbf{B}$ be an embedding. Then for each $i \in \{1, \dots, k\}$, there is a $f(j) \in \{1, \dots, l\}$ such that $h[C_{m_i}^{n_i}] \subseteq C_{r_{f(i)}}^{s_{f(i)}}$. So there is a map $f\colon \{1, \dots, k\} \to \{1, \dots, l\}$ that is injective, since h is an embedding, and order-preserving, since otherwise the definition of the nested sum would yield a contradiction to the fact that h is a homomorphism. Hence for each $i \in \{1, \dots, k\}$, we can define the embedding $h_i\colon \mathbf{C}_{m_i}^{n_i} \to \mathbf{C}_{r_{f(i)}}^{s_{f(i)}}$ to be the restriction of h to $C_{m_i}^{n_i}$. Note also that $h[G_p] \subseteq G_q$, so we can define the embedding $g\colon \mathbf{G}_p \to \mathbf{G}_q$ to be the restriction of h to G_p. Hence we obtain the desired triple $(f, g, \{h_i\}_{i=1}^k)$.

Finally it is straightforward to check that the two constructions are inverse to each other. ⊣

We define the following classes of commutative idempotent residuated chains for $m, n, p \in \{0, 1, \omega\}$:

$$\begin{aligned}
[\![\mathbf{G}_p]\!] &:= \mathbb{I}(\{\mathbf{G}_q \mid q \in \mathbb{N}, q \leq p\}),\\
[\![\mathbf{C}_m^n \mathbf{G}_p]\!] &:= \mathbb{I}(\{\mathbf{C}_r^s \boxplus \mathbf{G}_q \mid q, r, s \in \mathbb{N}, r \leq m, s \leq n, q \leq p\}) \cup [\![\mathbf{G}_p]\!],\\
[\![(\mathbf{C}_m^n)^* \mathbf{G}_p]\!] &:= \mathbb{I}(\{(\boxplus_{i=1}^k \mathbf{C}_{r_i}^{s_i}) \boxplus \mathbf{G}_q \mid k, q, r_i, s_i \in \mathbb{N}, r_i \leq m, s_i \leq n, q \leq p\}).
\end{aligned}$$

Note that $\mathbb{V}([\![\mathbf{G}_\omega]\!])$ is the variety of relative Stone algebras, $\mathbb{V}([\![(\mathbf{C}_0^0)^* \mathbf{G}_0]\!])$ is the variety of odd Sugihara monoids, and $\mathbb{V}([\![(\mathbf{C}_\omega^\omega)^* \mathbf{G}_\omega]\!])$ is the variety of commutative idempotent semilinear residuated lattices.

The next two lemmas follow easily from the definitions of the respective classes.

LEMMA 5.6. *Suppose that* $\mathbf{B} = (\boxplus_{j=1}^l \mathbf{C}_{r_j}^{s_j}) \boxplus \mathbf{G}_q$ *is a homomorphic image of* $\mathbf{A} = (\boxplus_{i=1}^k \mathbf{C}_{m_i}^{n_i}) \boxplus \mathbf{G}_p$. *Then* $l \leq k$, $r_j \leq \max\{m_1, \dots, m_k\}$, $s_j \leq \max\{n_1, \dots, n_k\}$, *and* $q \leq \max\{m_1, \dots, m_k, p\}$ *for each* $j \in \{1, \dots, l\}$.

LEMMA 5.7.

(i) $\mathbb{HS}([\![\mathbf{G}_p]\!]) = [\![\mathbf{G}_p]\!]$ *for each* $p \in \{0, 1, \omega\}$.
(ii) $\mathbb{HS}([\![\mathbf{C}_m^n \mathbf{G}_p]\!]) = [\![\mathbf{C}_m^n \mathbf{G}_p]\!]$ *and* $\mathbb{HS}([\![(\mathbf{C}_m^n)^* \mathbf{G}_p]\!]) = [\![(\mathbf{C}_m^n)^* \mathbf{G}_p]\!]$ *for any* $m, n, p \in \{0, 1, \omega\}$ *with* $p \geq m$.
(iii) $\mathbb{HS}([\![\mathbf{C}_m^n \mathbf{G}_0]\!] \cup [\![\mathbf{G}_p]\!]) = [\![\mathbf{C}_m^n \mathbf{G}_0]\!] \cup [\![\mathbf{G}_p]\!]$ *and* $\mathbb{HS}([\![(\mathbf{C}_0^n)^* \mathbf{G}_0]\!] \cup [\![\mathbf{G}_p]\!]) = [\![(\mathbf{C}_0^n)^* \mathbf{G}_0]\!] \cup [\![\mathbf{G}_p]\!]$ *for any* $m, n, p \in \{0, 1, \omega\}$ *with* $p \geq m$.

The following lemma will be helpful in amalgamating components appearing in nested sum decompositions of the pertinent algebras. We will often abbreviate a span $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i_B, i_C)$ by (i_B, i_C), or, more explicitly, $(i_B\colon \mathbf{A} \to \mathbf{B}, i_C\colon \mathbf{A} \to \mathbf{C})$.

LEMMA 5.8.

(i) *Every span* $(i_1\colon \mathbf{G}_{q_1} \to \mathbf{G}_{q_2}, i_2\colon \mathbf{G}_{q_1} \to \mathbf{G}_{q_3})$ *in* $[\![\mathbf{G}_p]\!]$ *has an amalgam in* $[\![\mathbf{G}_p]\!]$, *for any* $p \in \{0, 1, \omega\}$.
(ii) *Every span* $(i_1\colon \mathbf{C}_{r_1}^{s_1} \to \mathbf{C}_{r_2}^{s_2}, i_2\colon \mathbf{C}_{r_1}^{s_1} \to \mathbf{C}_{r_3}^{s_3})$ *in* $[\![\mathbf{C}_m^n \mathbf{G}_p]\!]$ *has an amalgam in* $[\![\mathbf{C}_m^n \mathbf{G}_p]\!]$, *for any* $m, n, p \in \{0, 1, \omega\}$.

PROOF. Note first that, by [19, Lemma 6.5], the class of finite commutative idempotent residuated chains has the amalgamation property.

(i) For $p = 0$, the claim is trivial. If $p = 1$, then, up to isomorphism, $[\![\mathbf{G}_p]\!]$ contains just $\mathbf{G}_0$ and $\mathbf{G}_1$, and the only embeddings are $\mathbf{G}_0 \hookrightarrow \mathbf{G}_0$, $\mathbf{G}_0 \hookrightarrow \mathbf{G}_1$, $\mathbf{G}_1 \hookrightarrow \mathbf{G}_1$, so the

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claim follows. Finally suppose that $p = \omega$. By [19, Lemma 6.5], the span (i_1, i_2) has an amalgam $((\boxplus_{i=1}^{k} \mathbf{C}_{m_i}^{n_i}) \boxplus \mathbf{G}_{q_4}, j_1, j_2)$ in the class of finite commutative idempotent residuated chains. Hence $j_1[G_{q_2}] \subseteq G_{q_4}$ and $j_2[G_{q_3}] \subseteq G_{q_4}$, by Lemma 5.5, so $(\mathbf{G}_{q_4}, j_1, j_2)$ is an amalgam of the span (i_1, i_2) in $[\![\mathbf{G}_p]\!]$.

(ii) If $m \leq 1$ and $n \leq 1$, then $r_1, r_2, r_3 \leq 1$ and $s_1, s_2, s_3 \leq 1$. In this case, $\mathbf{C}_m^n$ together with the inclusion maps $j_1 \colon \mathbf{C}_{r_2}^{s_2} \hookrightarrow \mathbf{C}_m^n$, $j_2 \colon \mathbf{C}_{r_3}^{s_3} \hookrightarrow \mathbf{C}_m^n$ is an amalgam of the span (i_1, i_2) in $[\![\mathbf{C}_m^n \mathbf{G}_p]\!]$.

If $m \leq 1$ and $n = \omega$, define $f_1 := i_1 \restriction_{[\mathrm{e}, a_0]} \colon [\mathrm{e}, a_0] \to [\mathrm{e}, a_0] \subseteq C_{r_2}^{s_2}$ and $f_2 := i_2 \restriction_{[\mathrm{e}, a_0]} \colon [\mathrm{e}, a_0] \to [\mathrm{e}, a_0] \subseteq C_{r_3}^{s_3}$. Let $(\{\mathrm{e} < a_k < \cdots < a_0\}, g_1, g_2)$ be an amalgam of (f_1, f_2) in the class of finite chains. Now define the maps $j_1 \colon \mathbf{C}_{r_2}^{s_2} \to \mathbf{C}_m^k$, $j_1(b_i) := b_i$, $j_1(a_i) := f_1(a_i)$ and $j_2 \colon \mathbf{C}_{r_3}^{s_3} \to \mathbf{C}_m^k$, $j_2(b_i) := b_i$, $j_2(a_i) := f_2(a_i)$. By Lemma 5.4, these maps are embeddings. Hence $(\mathbf{C}_m^k, j_1, j_2)$ is an amalgam of the span (i_1, i_2) in $[\![\mathbf{C}_m^n \mathbf{G}_p]\!]$. The case where $m = \omega$ and $n \leq 1$ is very similar.

Finally, suppose that $m = n = \omega$. Then, by [19, Lemma 6.5], the span (i_1, i_2) has an amalgam $((\boxplus_{j=1}^{k} \mathbf{C}_{m_j}^{n_j}) \boxplus \mathbf{G}_q, j_1, j_2)$ in the class of finite commutative idempotent residuated chains. Hence, by Lemma 5.5, there is an $i \in \{1, \ldots, k\}$ such that $(\mathbf{C}_{m_j}^{n_j}, j_1, j_2)$ is an amalgam of the span (i_1, i_2) in $[\![\mathbf{C}_m^n \mathbf{G}_p]\!]$. ⊣

We have now assembled the necessary ingredients to prove the positive part of our result. The following proposition shows that several families of varieties considered in this section have the amalgamation property. Subsequently, we will show that these are the only varieties of commutative idempotent residuated lattices that have the amalgamation property.

PROPOSITION 5.9. *The following varieties have the amalgamation property*:

(a) $\mathbb{V}([\![\mathbf{G}_p]\!])$ *for any* $p \in \{0, 1, \omega\}$;
(b) $\mathbb{V}([\![\mathbf{C}_m^n \mathbf{G}_p]\!])$ *for any* $m, n, p \in \{0, 1, \omega\}$ *with* $p \geq m$;
(c) $\mathbb{V}([\![(\mathbf{C}_m^n)^* \mathbf{G}_p]\!])$ *for any* $m, n, p \in \{0, 1, \omega\}$ *with* $p \geq m$;
(d) $\mathbb{V}([\![\mathbf{C}_m^n \mathbf{G}_0]\!] \cup [\![\mathbf{G}_p]\!])$ *for any* $m, n \in \{0, 1, \omega\}$, $p \in \{1, \omega\}$ *with* $p \geq m$;
(e) $\mathbb{V}([\![(\mathbf{C}_0^n)^* \mathbf{G}_0]\!] \cup [\![\mathbf{G}_p]\!])$ *for any* $n \in \{0, 1, \omega\}$, $p \in \{1, \omega\}$.

PROOF. Let K be one of the generating sets of finite chains from (a) to (e). Then $\mathbb{HS}(\mathsf{K}) = \mathsf{K}$, by Lemma 5.7, and $(\mathbb{HSP}_{\mathrm{U}}(\mathsf{K}))_{\mathrm{fin}} = \mathsf{K}$, by Lemma 5.3. Hence K consists of the finite chains of $\mathbb{V}(\mathsf{K})$, and, by Lemma 5.1, it suffices in each case to show that every span in K has a one-sided amalgam in K.

(a) Immediate from Lemma 5.8(i).

(b) Let $m, n, p \in \{0, 1, \omega\}$ with $p \geq m$ and let $(i_1 \colon \mathbf{A} \to \mathbf{B}, i_2 \colon \mathbf{A} \to \mathbf{C})$ be a span in $[\![\mathbf{C}_m^n \mathbf{G}_p]\!]$. Since $[\![\mathbf{G}_p]\!]$ has the amalgamation property, we can assume that $\mathbf{B}, \mathbf{C} \notin [\![\mathbf{G}_p]\!]$, i.e., $\mathbf{B} = \mathbf{C}_{r_2}^{s_2} \boxplus \mathbf{G}_{q_2}$ and $\mathbf{C} = \mathbf{C}_{r_3}^{s_3} \boxplus \mathbf{G}_{q_3}$ with $r_2, r_3 \leq m$, $s_2, s_3 \leq n$, $q_1 \leq p$. If $\mathbf{A} = \mathbf{G}_{q_1}$, then, since $[\![\mathbf{G}_p]\!]$ has the amalgamation property, the span $(i_1 \colon \mathbf{G}_{q_1} \to \mathbf{G}_{q_2}, i_2 \colon \mathbf{G}_{q_1} \to \mathbf{G}_{q_3})$ has an amalgam $(\mathbf{G}_{q_4}, f_1, f_2)$. Also, by Lemma 5.8, the span $(g_1 \colon \mathbf{C}_0^0 \hookrightarrow \mathbf{C}_{r_2}^{s_2}, g_2 \colon \mathbf{C}_0^0 \hookrightarrow \mathbf{C}_{r_3}^{s_3})$ has an amalgam $(\mathbf{C}_{r_4}^{s_4}, h_1, h_2)$ in $[\![\mathbf{C}_m^n \mathbf{G}_p]\!]$. Now, by Lemma 5.5, the maps $j_1 \colon B \to C_{r_4}^{s_4} \boxplus G_{q_4}$, $j_1(x) = f_1(x)$ for $x \in G_{q_2}$, $j_1(x) = h_1(x)$ for $x \in C_{r_2}^{s_2}$, $j_2 \colon B \to C_{r_4}^{s_4} \boxplus G_{q_4}$, $j_2(x) = f_2(x)$ for $x \in G_{q_3}$, $j_2(x) = h_2(x)$ for $x \in C_{r_3}^{s_3}$ are embeddings and hence $(\mathbf{C}_{r_4}^{s_4} \boxplus \mathbf{G}_{q_4}, j_1, j_2)$ is an amalgam of the span (i_1, i_2) in $[\![\mathbf{C}_m^n \mathbf{G}_p]\!]$.

If $\mathbf{A} = \mathbf{C}_{r_1}^{s_1} \boxplus \mathbf{G}_{q_1}$, then, since $[\![\mathbf{G}_p]\!]$ has the amalgamation property, the span $(i_1 \restriction_{G_{q_1}} \colon \mathbf{G}_{q_1} \to \mathbf{G}_{q_2}, i_2 \restriction_{G_{q_1}} \colon \mathbf{G}_{q_1} \to \mathbf{G}_{q_3})$ has an amalgam $(\mathbf{G}_{q_4}, f_1, f_2)$. Also, by

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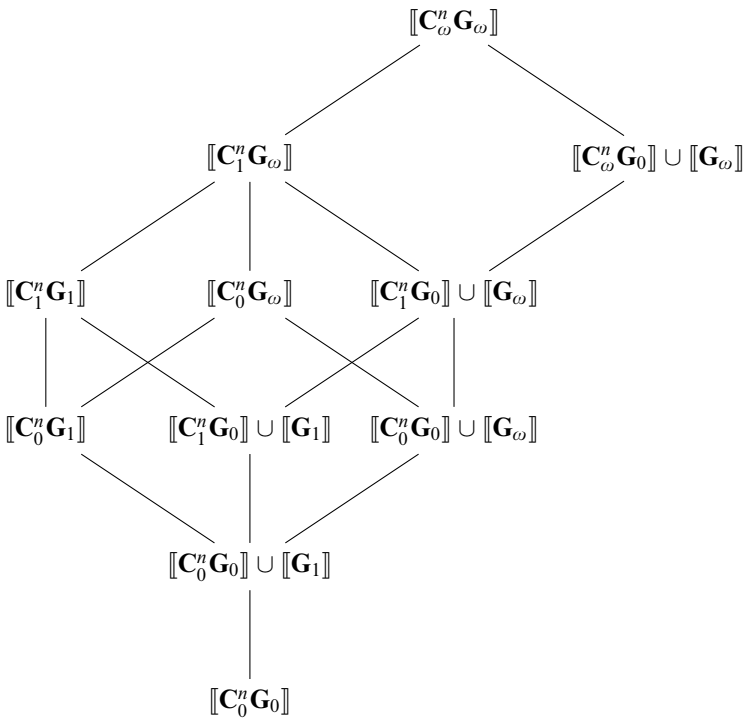


FIGURE 4. Classes of finite chains from (b) and (d) of Proposition 5.9.

Lemma 5.8, the span $(i_1{\restriction}_{C_{r_1}^{s_1}} : \mathbf{C}_{r_1}^{s_1} \to \mathbf{C}_{r_2}^{s_2}, i_2{\restriction}_{C_{r_1}^{s_1}} : \mathbf{C}_{r_1}^{s_1} \to \mathbf{C}_{r_3}^{s_3})$ has an amalgam $(\mathbf{C}_{r_4}^{s_4}, g_1, g_2)$ in $[\![\mathbf{C}_m^n \mathbf{G}_p]\!]$. Now, by Lemma 5.5, the maps $j_1 : B \to C_{r_4}^{s_4} \boxplus G_{q_4}$, $j_1(x) = f_1(x)$ for $x \in G_{q_2}$, $j_1(x) = g_1(x)$ for $x \in C_{r_2}^{s_2}$, $j_2 : B \to C_{r_4}^{s_4} \boxplus G_{q_4}$, $j_2(x) = f_2(x)$ for $x \in G_{q_3}$, $j_2(x) = g_2(x)$ for $x \in C_{r_3}^{s_3}$ are embeddings and hence $(\mathbf{C}_{r_4}^{s_4} \boxplus \mathbf{G}_{q_4}, j_1, j_2)$ is an amalgam of the span (i_1, i_2) in $[\![\mathbf{C}_m^n \mathbf{G}_p]\!]$.

(c) Very similar to part (b) by first amalgamating the summands and the index sets and then using Lemma 5.5.

(d) and (e) Let $m, n \in \{0, 1, \omega\}$, $p \in \{1, \omega\}$. The only spans that are not covered by parts (b) and (c) are of the form $(i_1 : \mathbf{G}_0 \to \boxplus_{i=1}^k \mathbf{C}_{r_i}^{s_i}, i_2 : \mathbf{G}_0 \to \mathbf{G}_p)$ and $(i_1' : \mathbf{G}_0 \to \mathbf{G}_p, i_2' : \mathbf{G}_0 \to \boxplus_{i=1}^k \mathbf{C}_{r_i}^{s_i})$. For these cases, define the one-sided amalgam $(\boxplus_{i=1}^k \mathbf{C}_{r_i}^{s_i}, j_1, j_2)$ with $j_1 : \boxplus_{i=1}^k \mathbf{C}_{r_i}^{s_i} \to \boxplus_{i=1}^k \mathbf{C}_{r_i}^{s_i}$, $j_1(x) = x$, $j_2 : \mathbf{G}_p \to \boxplus_{i=1}^k \mathbf{C}_{r_i}^{s_i}$, $j_2(x) = \mathrm{e}$, and the one-sided amalgam $(\mathbf{G}_p, f_1, f_2)$ with $f_1 : \mathbf{G}_p \to \mathbf{G}_p$, $f_1(x) = x$, $f_2 : \boxplus_{i=1}^k \mathbf{C}_{r_i}^{s_i} \to \mathbf{G}_p$, $f_2(x) = \mathrm{e}$, respectively. ⊣

Note that in the previous proposition, case (a) accounts for three varieties, each one of cases (b) and (c) accounts for an additional 18 varieties, case (d) accounts for 15 varieties, and case (e) accounts for six varieties. Hence, Proposition 5.9 identifies a total of $3 + 18 + 18 + 15 + 6 = 60$ varieties of commutative idempotent semilinear residuated lattices with the amalgamation property. For a fixed $n \in \{0, 1, \omega\}$, the corresponding posets of the classes of finite chains from (b) and (d), and (c) and (e), are displayed in Figures 4 and 5, respectively.

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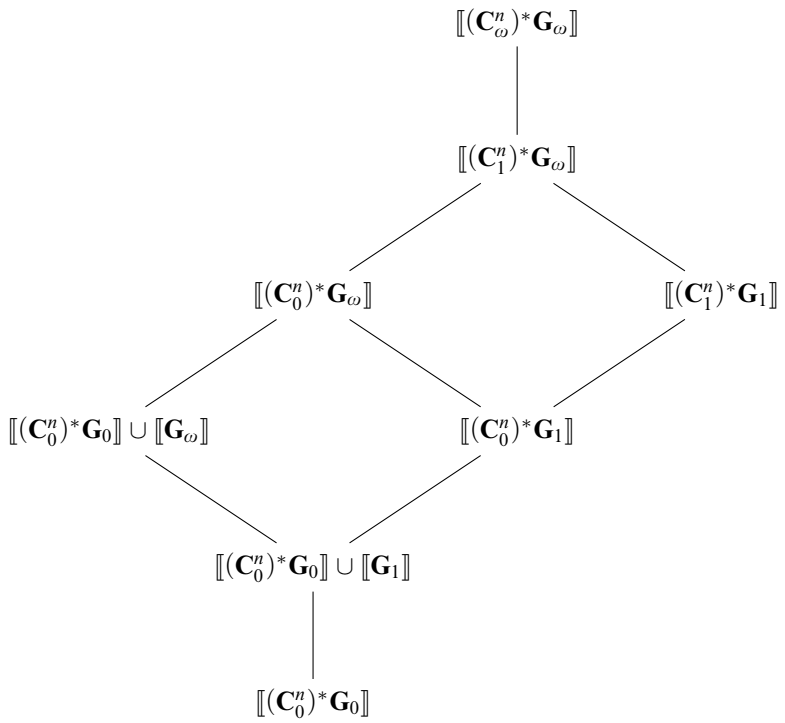


FIGURE 5. Classes of finite chains from (c) and (e) of Proposition 5.9.

We now turn our attention to the closure properties used to show that the list of varieties with the amalgamation property given in Proposition 5.9 is exhaustive.

LEMMA 5.10. *Let* V *be a variety of commutative idempotent semilinear residuated lattices and let* $\mathbf{A} \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$. *If* $\mathbf{A} \boxplus \mathbf{C}_m^n \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$, *then* $\mathbf{A} \boxplus \mathbf{G}_m \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$.

PROOF. Clearly, every convex normal subalgebra of a $\mathbf{C}_m^n$ is also a convex normal subalgebra of $\mathbf{A} \boxplus \mathbf{C}_m^n$ by the definition of the nested sum. It is easy to see that the interval $[b_0, a_0]$ is a convex normal subalgebra of $\mathbf{C}_m^n$ and, by direct computation, that the congruence Θ corresponding to this convex normal subalgebra satisfies $(x, y) \in \Theta$ if and only if $x = y$ or $b_0 \leq x, y$, for any $x, y \in \mathbf{C}_m^n$. Now, if $x, y \in \mathbf{A} \boxplus \mathbf{C}_m^n$ with $(x, y) \in \Theta$, then $b_0 \leq x \to y \wedge y \to x \leq a_0$. It follows from the definition of the nested sum that $x \to y \notin \mathbf{C}_m^n$ if $x \notin \mathbf{C}_m^n$ or $y \notin \mathbf{C}_m^n$, so also $(x, y) \in \Theta$ if and only if $x = y$ or $b_0 \leq x, y \leq a_0$ for any $x, y \in \mathbf{A} \boxplus \mathbf{C}_m^n$. Hence we obtain $\mathbf{A} \boxplus \mathbf{G}_m \cong (\mathbf{A} \boxplus \mathbf{C}_m^n)/\Theta \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$. ⊣

LEMMA 5.11. *Let* V *be a variety of commutative idempotent semilinear residuated lattices, let* $\mathbf{A}, \mathbf{B} \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$, *and suppose that* V *has the amalgamation property.*

(i) *If* $\mathbf{A} \boxplus \mathbf{G}_2 \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$, *then* $\mathbf{A} \boxplus \mathbf{G}_n \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$ *for every* $n \in \mathbb{N}^{>0}$.
(ii) *If* $\mathbf{C}_0^0 \boxplus \mathbf{C}_0^0 \boxplus \mathbf{B} \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$, *then* $(\boxplus_{i=1}^k \mathbf{C}_0^0) \boxplus \mathbf{B} \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$ *for every* $k \in \mathbb{N}^{>0}$.
(iii) *If* $\mathbf{C}_2^n \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$ *for some* $n \in \mathbb{N}$, *then* $\mathbf{C}_m^n, \mathbf{G}_m \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$ *for every* $m \in \mathbb{N}$.
(iv) *If* $\mathbf{C}_m^2 \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$ *for some* $m \in \mathbb{N}$, *then* $\mathbf{C}_m^n \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$ *for every* $n \in \mathbb{N}$.

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(v) *If* $\mathbf{C}_m^0, \mathbf{C}_0^n \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$ *for some* $m, n \in \mathbb{N}$, *then* $\mathbf{C}_m^n \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$.
(vi) *If* $\mathbf{A} \boxplus \mathbf{C}_0^0 \boxplus \mathbf{B} \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$ *and* $\mathbf{C}_m^n \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$, *then* $\mathbf{A} \boxplus \mathbf{C}_m^n \boxplus \mathbf{B} \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$.
(vii) *If* $\mathbf{A} \boxplus \mathbf{G}_1 \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$ *and* $\mathbf{G}_p \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$, *then* $\mathbf{A} \boxplus \mathbf{G}_p \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$.

PROOF. Note first that, since V has the amalgamation property, by Lemma 5.1, $\mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$ has the one-sided amalgamation property. In the following, we will use this without mentioning it explicitly.

(i) Suppose that $\mathbf{A} \boxplus \mathbf{G}_2 \in \mathsf{V}$. We prove the claim by induction on n. For $n = 1$, note that $\mathbf{A} \boxplus \mathbf{G}_1$ is a subalgebra of $\mathbf{A} \boxplus \mathbf{G}_2$. Suppose that the claim holds for $n \in \mathbb{N}^{>0}$, i.e., $\mathbf{A} \boxplus \mathbf{G}_n \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$. Consider the span of embeddings $i_1 \colon \mathbf{A} \boxplus \mathbf{G}_1 \to \mathbf{A} \boxplus \mathbf{G}_2$, $i_1(c_1) = c_2$ and $i_1(x) = x$ for $x \in A$, and $i_2 \colon \mathbf{A} \boxplus \mathbf{G}_1 \to \mathbf{A} \boxplus \mathbf{G}_n$, $i_2(c_1) = c_1$ and $i_1(x) = x$ for $x \in A$. Then there exists an algebra $\mathbf{D} \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$, an embedding $j_1 \colon \mathbf{A} \boxplus \mathbf{G}_2 \to \mathbf{D}$, and a homomorphism $j_2 \colon \mathbf{A} \boxplus \mathbf{G}_n \to \mathbf{D}$ such that $j_1 \circ i_1 = j_2 \circ i_2$. Since c_1 covers e in $\mathbf{A} \boxplus \mathbf{G}_n$ and $j_2(c_1) = j_2(i_2(c_1)) = j_1(i_1(c_1)) < \mathrm{e}$, by Lemma 3.1, also j_2 is an embedding, so we have $j_2(c_n) < \cdots < j_2(c_1) = j_1(c_2) < j_1(c_1) < \mathrm{e}$. Let $S = \mathrm{im}(j_1) \cup \mathrm{im}(j_2)$. Then $\mathrm{e} \in S$ and S is closed under *, by construction. Hence, by Lemma 3.4, it is the universe of a subalgebra $\mathbf{S}$ of $\mathbf{D}$ which is clearly isomorphic to $\mathbf{A} \boxplus \mathbf{G}_{n+1}$.

(ii) Suppose that $\mathbf{C}_0^0 \boxplus \mathbf{C}_0^0 \boxplus \mathbf{B} \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$. We prove the claim by induction on k. For $k = 1$, note that $\mathbf{C}_0^0 \boxplus \mathbf{B}$ is a subalgebra of $\mathbf{C}_0^0 \boxplus \mathbf{C}_0^0 \boxplus \mathbf{B}$. Suppose that the claim holds for $k \in \mathbb{N}^{>0}$, i.e., $(\boxplus_{i=1}^k \mathbf{C}_0^0) \boxplus \mathbf{B} \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$, and consider the span of embeddings $i_1 \colon \mathbf{C}_0^0 \boxplus \mathbf{B} \to \mathbf{C}_0^0 \boxplus \mathbf{C}_0^0 \boxplus \mathbf{B}$, $i_1(x) = x$ for $x \in B$, $i_1(b_0) = b_0^1$, $i_1(a_0) = a_0^1$ and $i_2 \colon \mathbf{C}_0^0 \boxplus \mathbf{B} \to (\boxplus_{i=1}^k \mathbf{C}_0^0) \boxplus \mathbf{B}$, $i_2(x) = x$ for $x \in B$, $i_2(b_0) = b_0^k$, $i_2(a_0) = a_0^1$. Then there exists an algebra $\mathbf{D} \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$, an embedding $j_1 \colon \mathbf{C}_0^0 \boxplus \mathbf{C}_0^0 \boxplus \mathbf{B} \to \mathbf{D}$, and a homomorphism $j_2 \colon (\boxplus_{i=1}^k \mathbf{C}_0^0) \boxplus \mathbf{B} \to \mathbf{D}$ such that $j_1 \circ i_1 = j_2 \circ i_2$. Let $S = \mathrm{im}(j_1) \cup \mathrm{im}(j_2)$. Then $\mathrm{e} \in S$ and S is closed under *, by construction. Hence, by Lemma 3.4, it is the universe of a subalgebra $\mathbf{S}$ of $\mathbf{D}$. Since j_2 restricts to an embedding on $\mathbf{C}_0^0 \boxplus \mathbf{B}$, by Lemma 3.1, it is also an embedding. For each $x \in B$,

$$j_2(b_0^1) < \cdots < j_2(b_0^k) = j_1(b_0^1) < j_1(b_0^2) < j_1(x) = j_2(x)$$
$$j_2(x) < j_1(a_0^2) < j_1(a_0^1) = j_2(a_0^k) < \cdots < j_2(a_0^1),$$

and it is straightforward to check that $\mathbf{S}$ is isomorphic to $(\boxplus_{i=1}^{k+1} \mathbf{C}_0^0) \boxplus \mathbf{B}$.

(iii) Suppose that $\mathbf{C}_2^n \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$ for some $n \in \mathbb{N}$. We prove the claim by induction on m. First note that, since $\mathbf{G}_m$ is a homomorphic image of $\mathbf{C}_m^n$, it suffices to show that $\mathbf{C}_m^n \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$ for each $m \in \mathbb{N}$. For $m = 1$, note that $\mathbf{C}_1^n$ is a subalgebra of $\mathbf{C}_2^n$. Suppose that the claim holds for $m \in \mathbb{N}^{>0}$, i.e., $\mathbf{C}_m^n \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$. We consider the span of embeddings $i_1 \colon \mathbf{C}_1^n \to \mathbf{C}_2^n$, $i_1(x) = x$ and $i_2 \colon \mathbf{C}_1^n \to \mathbf{C}_m^n$, $i_2(b_1) = b_m$, $i_2(x) = x$ for $x \neq b_1$. Then there exists an algebra $\mathbf{D} \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$, an embedding $j_1 \colon \mathbf{C}_2^n \to \mathbf{D}$, and a homomorphism $j_2 \colon \mathbf{C}_m^n \colon \to \mathbf{D}$ such that $j_1 \circ i_1 = j_2 \circ i_2$. Let $S = \mathrm{im}(j_1) \cup \mathrm{im}(j_2)$. Then $\mathrm{e} \in S$ and S is closed under *, by construction. Hence, by Lemma 3.4, it is the universe of a subalgebra $\mathbf{S}$ of $\mathbf{D}$. Since j_2 restricts to an embedding on $\mathbf{C}_0^0$, by Lemma 3.1, it is also an embedding and we get $j_2(b_1) < \cdots < j_2(b_m) = j_1(b_1) < j_1(b_2)$ and $j_1(a_i) = j_2(a_i)$ for each $i \in \{1, \ldots, n\}$. It follows that $\mathbf{S}$ is isomorphic to $\mathbf{C}_{m+1}^n$.

(iv) The proof is very similar to (iii).

(v) Suppose that $\mathbf{C}_m^0, \mathbf{C}_0^n \in \mathsf{V}_{\mathrm{fin}}^{\mathrm{c}}$ for some $m, n \in \mathbb{N}$ and consider the span of inclusions $i_1 \colon \mathbf{C}_0^0 \to \mathbf{C}_m^0$, $i_1(x) = x$ and $i_2 \colon \mathbf{C}_0^0 \to \mathbf{C}_m^0$, $i_2(x) = x$. Then there exists an

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algebra $\mathbf{D} \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$, an embedding $j_1 \colon \mathbf{C}^0_m \to \mathbf{D}$, and a homomorphism $j_2 \colon \mathbf{C}^n_0 \to \mathbf{D}$ such that $j_1 \circ i_1 = j_2 \circ i_2$. Let $S = \mathrm{im}(j_1) \cup \mathrm{im}(j_2)$. Then $\mathrm{e} \in S$ and S is closed under *, by construction. Hence, by Lemma 3.4, it is the universe of a subalgebra $\mathbf{S}$ of $\mathbf{D}$. Since j_2 restricts to an embedding on $\mathbf{C}^0_0$, by Lemma 3.1, it is also an embedding, so $\mathbf{S}$ is isomorphic to $\mathbf{C}^n_m$.

(vi) Suppose that $\mathbf{A} \boxplus \mathbf{C}^0_0 \boxplus \mathbf{B} \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$ and $\mathbf{C}^n_m \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$.We consider the span of inclusions $i_1 \colon \mathbf{C}^0_0 \to \mathbf{A} \boxplus \mathbf{C}^0_0 \boxplus \mathbf{B}$, $i_1(x) = x$ and $i_2 \colon \mathbf{C}^0_0 \to \mathbf{C}^n_m$, $i_2(x) = x$. Then there exists an algebra $\mathbf{D} \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$, an embedding $j_1 \colon \mathbf{A} \boxplus \mathbf{C}^0_0 \boxplus \mathbf{B} \to \mathbf{D}$, and a homomorphism $j_2 \colon \mathbf{C}^n_m \to \mathbf{D}$ such that $j_1 \circ i_1 = j_2 \circ i_2$. Let $S = \mathrm{im}(j_1) \cup \mathrm{im}(j_2)$. Then $\mathrm{e} \in S$ and S is closed under *, by construction. Hence, by Lemma 3.4, it is the universe of a subalgebra $\mathbf{S}$ of $\mathbf{D}$. Since j_2 restricts to an embedding on $\mathbf{C}^0_0$, by Lemma 3.1, it is also an embedding and it is straightforward to check that $\mathbf{S}$ is isomorphic to $\mathbf{A} \boxplus \mathbf{C}^n_m \boxplus \mathbf{B}$.

(vii) Suppose that $\mathbf{A} \boxplus \mathbf{G}_1 \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$ and $\mathbf{G}_p \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$. For $p = 1$, there is nothing to prove, so we may assume that $p > 1$. Consider the span of embeddings $i_1 \colon \mathbf{G}_1 \to \mathbf{G}_p$, $i_1(x) = x$ and $i_2 \colon \mathbf{G}_1 \to \mathbf{A} \boxplus \mathbf{G}_1$, $i_2(x) = x$. Then there exists an algebra $\mathbf{D} \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$, an embedding $j_1 \colon \mathbf{G}_p \to \mathbf{D}$, and a homomorphism $j_2 \colon \mathbf{A} \boxplus \mathbf{G}_1 \to \mathbf{D}$ such that $j_1 \circ i_1 = j_2 \circ i_2$. Let $S = \mathrm{im}(j_1) \cup \mathrm{im}(j_2)$. Then $\mathrm{e} \in S$ and S is closed under *, by construction. Hence, by Lemma 3.4, it is the universe of a subalgebra $\mathbf{S}$ of $\mathbf{D}$. Moreover, since j_2 restricts to an embedding on $\mathbf{G}_1$, by Lemma 3.1, it is also an embedding and it is easy to check that $\mathbf{S}$ is isomorphic to $\mathbf{A} \boxplus \mathbf{G}_p$. ⊣

We finally arrive at the proof of part (i) of Theorem B, recalling that part (ii) and Theorem C follow directly from this result.

Proof of Theorem B(i). Let V be a variety of commutative idempotent semilinear residuated lattices that has the amalgamation property. It suffices to show that V is one of the varieties from Proposition 5.9. Let $\mathsf{K} = \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$. We make a case distinction on whether algebras from the set $\{\mathbf{G}_p, \mathbf{C}^n_0, \mathbf{C}^0_m, \mathbf{C}^0_0 \boxplus \mathbf{G}_p \mid m, n, p \in \{0, 1, 2\}\} \cup \{\mathbf{C}^0_0 \boxplus \mathbf{C}^0_0\}$ are contained in K, noting that $\mathbf{G}_1 \leq \mathbf{G}_2$, $\mathbf{C}^0_0 \leq \mathbf{C}^0_1 \leq \mathbf{C}^0_2$, $\mathbf{C}^0_0 \leq \mathbf{C}^1_0 \leq \mathbf{C}^2_0$, $\mathbf{C}^0_0 \leq \mathbf{C}^0_0 \boxplus \mathbf{C}^0_0$, $\mathbf{C}^0_0, \mathbf{G}_p \leq \mathbf{C}^0_0 \boxplus \mathbf{G}_p$, and $\mathbf{G}_m \in \mathbb{H}(\mathbf{C}^0_m)$, by Lemma 5.10.

If $\mathbf{C}^0_0 \notin \mathsf{K}$, then $\mathsf{K} \in \{[\![\mathbf{G}_p]\!] \mid p \in \{0, 1, \omega\}\}$, yielding $\mathsf{V} \in \{\mathbb{V}([\![\mathbf{G}_p]\!]) \mid p \in \{0, 1, \omega\}\}$. So we may assume that $\mathbf{C}^0_0 \in \mathsf{K}$ and let $n := \sup\{s \in \mathbb{N} \mid \mathbf{C}^s_0 \in \mathsf{K}\}$, assuming $\sup \mathbb{N} = \omega$. As n is fixed, our strategy is to show that K is one of the classes in the posets in Figures 4 and 5. The two subsequent case distinctions traverse these posets starting at the top.

For $\mathbf{C}^0_0 \boxplus \mathbf{C}^0_0 \notin \mathsf{K}$, there are 11 cases (see Figure 4):

1. If $\mathbf{C}^0_2, \mathbf{C}^0_0 \boxplus \mathbf{G}_1 \in \mathsf{K}$, then $\mathsf{K} = [\![\mathbf{C}^n_\omega \mathbf{G}_\omega]\!]$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}([\![\mathbf{C}^n_\omega \mathbf{G}_\omega]\!])$.
2. If $\mathbf{C}^0_2 \notin \mathsf{K}$ and $\mathbf{C}^0_1, \mathbf{C}^0_0 \boxplus \mathbf{G}_2 \in \mathsf{K}$, then $\mathsf{K} = [\![\mathbf{C}^n_1 \mathbf{G}_\omega]\!]$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}([\![\mathbf{C}^n_1 \mathbf{G}_\omega]\!])$.
3. If $\mathbf{C}^0_1 \notin \mathsf{K}$ and $\mathbf{C}^0_0 \boxplus \mathbf{G}_2 \in \mathsf{K}$, then $\mathsf{K} = [\![\mathbf{C}^n_0 \mathbf{G}_\omega]\!]$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}([\![\mathbf{C}^n_0 \mathbf{G}_\omega]\!])$.
4. If $\mathbf{C}^0_2, \mathbf{G}_2 \notin \mathsf{K}$ and $\mathbf{C}^0_1, \mathbf{C}^0_0 \boxplus \mathbf{G}_1 \in \mathsf{K}$, then $\mathsf{K} = [\![\mathbf{C}^n_1 \mathbf{G}_1]\!]$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}([\![\mathbf{C}^n_1 \mathbf{G}_1]\!])$.
5. If $\mathbf{C}^0_1, \mathbf{G}_2 \notin \mathsf{K}$ and $\mathbf{C}^0_0 \boxplus \mathbf{G}_1 \in \mathsf{K}$, then $\mathsf{K} = [\![\mathbf{C}^n_0 \mathbf{G}_1]\!]$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}([\![\mathbf{C}^n_0 \mathbf{G}_1]\!])$.

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6. If $\mathbf{C}_1^0, \mathbf{G}_1 \notin \mathsf{K}$, then $\mathsf{K} = \mathbb{I}\mathbf{C}_0^n\mathbf{G}_0\mathbb{I}$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}(\mathbb{I}\mathbf{C}_0^n\mathbf{G}_0\mathbb{I})$.
7. If $\mathbf{C}_0^0 \boxplus \mathbf{G}_1 \notin \mathsf{K}$ and $\mathbf{C}_2^0 \in \mathsf{K}$, then $\mathsf{K} = \mathbb{I}\mathbf{C}_\omega^n\mathbf{G}_0\mathbb{I} \cup \mathbb{I}\mathbf{G}_\omega\mathbb{I}$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}(\mathbb{I}\mathbf{C}_\omega^n\mathbf{G}_0\mathbb{I} \cup \mathbb{I}\mathbf{G}_\omega\mathbb{I})$.
8. If $\mathbf{C}_0^0 \boxplus \mathbf{G}_1, \mathbf{C}_2^0 \notin \mathsf{K}$ and $\mathbf{C}_1^0, \mathbf{G}_2 \in \mathsf{K}$, then $\mathsf{K} = \mathbb{I}\mathbf{C}_1^n\mathbf{G}_0\mathbb{I} \cup \mathbb{I}\mathbf{G}_\omega\mathbb{I}$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}(\mathbb{I}\mathbf{C}_1^n\mathbf{G}_0\mathbb{I} \cup \mathbb{I}\mathbf{G}_\omega\mathbb{I})$.
9. If $\mathbf{C}_0^0 \boxplus \mathbf{G}_1, \mathbf{C}_1^0 \notin \mathsf{K}$ and $\mathbf{G}_2 \in \mathsf{K}$, then $\mathsf{K} = \mathbb{I}\mathbf{C}_0^n\mathbf{G}_0\mathbb{I} \cup \mathbb{I}\mathbf{G}_\omega\mathbb{I}$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}(\mathbb{I}\mathbf{C}_0^n\mathbf{G}_0\mathbb{I} \cup \mathbb{I}\mathbf{G}_\omega\mathbb{I})$.
10. If $\mathbf{C}_0^0 \boxplus \mathbf{G}_1, \mathbf{C}_2^0, \mathbf{G}_2 \notin \mathsf{K}$ and $\mathbf{C}_1^0 \in \mathsf{K}$, then $\mathsf{K} = \mathbb{I}\mathbf{C}_1^n\mathbf{G}_0\mathbb{I} \cup \mathbb{I}\mathbf{G}_1\mathbb{I}$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}(\mathbb{I}\mathbf{C}_1^n\mathbf{G}_0\mathbb{I} \cup \mathbb{I}\mathbf{G}_1\mathbb{I})$.
11. If $\mathbf{C}_0^0 \boxplus \mathbf{G}_1, \mathbf{C}_1^0, \mathbf{G}_2 \notin \mathsf{K}$ and $\mathbf{G}_1 \in \mathsf{K}$, then $\mathsf{K} = \mathbb{I}\mathbf{C}_0^n\mathbf{G}_0\mathbb{I} \cup \mathbb{I}\mathbf{G}_1\mathbb{I}$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}(\mathbb{I}\mathbf{C}_0^n\mathbf{G}_0\mathbb{I} \cup \mathbb{I}\mathbf{G}_1\mathbb{I})$.

For $\mathbf{C}_0^0 \boxplus \mathbf{C}_0^0 \in \mathsf{K}$, there are another 8 cases (see Figure 5):

1. If $\mathbf{C}_2^0 \in \mathsf{K}$, then $\mathsf{K} = \mathbb{I}(\mathbf{C}_\omega^n)^*\mathbf{G}_\omega\mathbb{I}$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}(\mathbb{I}(\mathbf{C}_\omega^n)^*\mathbf{G}_\omega\mathbb{I})$.
2. If $\mathbf{C}_2^0 \notin \mathsf{K}$ and $\mathbf{C}_1^0, \mathbf{G}_2 \in \mathsf{K}$, then $\mathsf{K} = \mathbb{I}(\mathbf{C}_1^n)^*\mathbf{G}_\omega\mathbb{I}$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}(\mathbb{I}(\mathbf{C}_1^n)^*\mathbf{G}_\omega\mathbb{I})$.
3. If $\mathbf{C}_1^0 \notin \mathsf{K}$ and $\mathbf{C}_0^0 \boxplus \mathbf{G}_2 \in \mathsf{K}$, then $\mathsf{K} = \mathbb{I}(\mathbf{C}_0^n)^*\mathbf{G}_\omega\mathbb{I}$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}(\mathbb{I}(\mathbf{C}_0^n)^*\mathbf{G}_\omega\mathbb{I})$.
4. If $\mathbf{C}_2^0, \mathbf{G}_2 \notin \mathsf{K}$ and $\mathbf{C}_1^0 \in \mathsf{K}$, then $\mathsf{K} = \mathbb{I}(\mathbf{C}_1^n)^*\mathbf{G}_1\mathbb{I}$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}(\mathbb{I}(\mathbf{C}_1^n)^*\mathbf{G}_1\mathbb{I})$.
5. If $\mathbf{C}_1^0, \mathbf{G}_2 \notin \mathsf{K}$ and $\mathbf{C}_0^0 \boxplus \mathbf{G}_1 \in \mathsf{K}$, then $\mathsf{K} = \mathbb{I}(\mathbf{C}_0^n)^*\mathbf{G}_1\mathbb{I}$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}(\mathbb{I}(\mathbf{C}_0^n)^*\mathbf{G}_1\mathbb{I})$.
6. If $\mathbf{C}_1^0, \mathbf{G}_1 \notin \mathsf{K}$, then $\mathsf{K} = \mathbb{I}(\mathbf{C}_0^n)^*\mathbf{G}_0\mathbb{I}$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}(\mathbb{I}(\mathbf{C}_0^n)^*\mathbf{G}_0\mathbb{I})$.
7. If $\mathbf{C}_0^0 \boxplus \mathbf{G}_1 \notin \mathsf{K}$ and $\mathbf{G}_2 \in \mathsf{K}$, then $\mathsf{K} = \mathbb{I}(\mathbf{C}_0^n)^*\mathbf{G}_0\mathbb{I} \cup \mathbb{I}\mathbf{G}_\omega\mathbb{I}$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}(\mathbb{I}(\mathbf{C}_0^n)^*\mathbf{G}_0\mathbb{I} \cup \mathbb{I}\mathbf{G}_\omega\mathbb{I})$.
8. If $\mathbf{C}_0^0 \boxplus \mathbf{G}_1, \mathbf{G}_2 \notin \mathsf{K}$ and $\mathbf{G}_1 \in \mathsf{K}$, then $\mathsf{K} = \mathbb{I}(\mathbf{C}_0^n)^*\mathbf{G}_0\mathbb{I} \cup \mathbb{I}\mathbf{G}_1\mathbb{I}$, by Lemma 5.11, yielding $\mathsf{V} = \mathbb{V}(\mathbb{I}(\mathbf{C}_0^n)^*\mathbf{G}_0\mathbb{I} \cup \mathbb{I}\mathbf{G}_1\mathbb{I})$. ⊣

We conclude this section by considering amalgamation in the setting of *pointed* residuated lattices, i.e., without the additional assumption that $\mathrm{e} \approx \mathrm{f}$. There are many additional cases to consider in this more general setting, and we just sketch the key ideas here. Let V be a variety of commutative idempotent semilinear pointed residuated lattices. Again, it is enough to consider the class $\mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$ of the finite totally ordered members of V. For each algebra $(A, \wedge, \vee, \cdot, \to, \mathrm{e}, \mathrm{f}) = \mathbf{A} \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$ one of the following six conditions holds:

1. f is on the Sugihara skeleton of $\mathbf{A}$ and
   (a) $\mathrm{f} = \mathrm{e}$ in $\mathbf{A}$, or
   (b) $\mathrm{f} < \mathrm{e}$ in $\mathbf{A}$, or
   (c) $\mathrm{f} > \mathrm{e}$ in $\mathbf{A}$.
2. f is not on the Sugihara skeleton of $\mathbf{A}$ and
   (a) $\mathrm{f} < \mathrm{e}$ and $\mathrm{f}^\star = \mathrm{e}$ in $\mathbf{A}$, or
   (b) $\mathrm{f} < \mathrm{e}$ and $\mathrm{f}^\star > \mathrm{e}$ in $\mathbf{A}$, or
   (c) $\mathrm{f} > \mathrm{e}$ in $\mathbf{A}$.

For $i \in \{1, 2\}$ and $x \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$, we define the class

$$\mathsf{V}_{i(x)} := \{\mathbf{A} \in \mathsf{V}^{\mathrm{c}}_{\mathrm{fin}} \mid \mathbf{A} \text{ satisfies condition } i.(x)\}.$$

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Note that $\mathcal{P}_{\mathsf{V}} := \{\mathsf{V}_{i(x)} \mid i \in \{1,2\},\ x \in \{\mathrm{a},\mathrm{b},\mathrm{c}\}\}$ forms a partition of $\mathsf{V}^{\mathrm{c}}_{\mathrm{fin}}$ such that no algebra from one class embeds into an algebra from another class, since the conditions are preserved by embeddings. Hence, to check if V has the amalgamation property, it suffices to check if each of the classes in $\mathcal{P}_{\mathsf{V}}$ has the one-sided amalgamation property. In particular, to show that there are only finitely many varieties of commutative idempotent semilinear pointed residuated lattices with the amalgamation property, it suffices to show that there are only finitely many possibilities for $\mathcal{P}_{\mathsf{V}}$ such that each class in $\mathcal{P}_{\mathsf{V}}$ has the one-sided amalgamation property. Let $\mathbf{A}$ be a pointed residuated lattice and $\mathbf{B}$ a residuated lattice. We will write $\mathbf{A} \simeq \mathbf{B}$ if $\mathbf{B}$ is (isomorphic to) the f-free reduct of $\mathbf{A}$.

For $i \in \{1,2\}$ and $x \in \{\mathrm{a},\mathrm{b},\mathrm{c}\}$, we define $\mathbf{S}_{i(x)}$ to be the smallest commutative idempotent residuated chain satisfying $i.(x)$. Note that $\mathbf{S}_{i(x)}$ is well-defined and for each algebra $\mathbf{A}$ satisfying $i.(x)$ we have that $\mathbf{S}_{i(x)}$ is isomorphic to the subalgebra of $\mathbf{A}$ generated by f. In fact, the algebra $\mathbf{S}_{1(\mathrm{a})}$ is just a trivial algebra, $\mathbf{S}_{1(\mathrm{b})} \simeq \mathbf{C}^0_0$ with $\mathrm{f} = b_0$, and $\mathbf{S}_{1(\mathrm{c})} \simeq \mathbf{C}^0_0$ with $\mathrm{f} = a_0$. Moreover, $\mathbf{S}_{2(\mathrm{a})} \simeq \mathbf{G}_1$ with $\mathrm{f} \simeq c_1$, $\mathbf{S}_{2(\mathrm{b})} \simeq \mathbf{C}^0_1$ with $\mathrm{f} = b_1$, and $\mathbf{S}_{2(\mathrm{c})} \simeq \mathbf{C}^1_0$ with $\mathrm{f} = a_1$.

In what follows, the nested sums $\mathbf{A} \boxplus \mathbf{B}$ and $\mathbf{B} \boxplus \mathbf{A}$ of an idempotent pointed residuated chain $\mathbf{A}$ with an idempotent residuated chain $\mathbf{B}$ are understood in the obvious way, i.e., as the nested sum of the f-free reducts of $\mathbf{A}$ and $\mathbf{B}$ with constant f from $\mathbf{A}$.

LEMMA 5.12. *Let V be a variety of commutative idempotent semilinear pointed residuated lattices with the amalgamation property.*

(i) *Let $i \in \{1,2\}$, $x \in \{b,c\}$. If $\mathbf{A}$ is a commutative idempotent pointed residuated chain that satisfies condition $i.(x)$, then $\mathbf{A}$ is of the form $\mathbf{A}_1 \boxplus \mathbf{A}_2 \boxplus \mathbf{A}_3$ with $f \in A_2$ and $\mathbf{A}_2 \simeq \mathbf{C}^n_m$. Moreover, $\mathbf{A} \in \mathsf{V}_{i(x)}$ if and only if $\mathbf{A}_1 \boxplus \mathbf{S}_{i(x)} \in \mathsf{V}_{i(x)}$, $\mathbf{A}_2 \in \mathsf{V}_{i(x)}$, and $\mathbf{S}_{i(x)} \boxplus \mathbf{A}_3 \in \mathsf{V}_{i(x)}$.*

(ii) *If $\mathbf{A}$ is a commutative idempotent pointed residuated chain that satisfies condition $2.(a)$, then $\mathbf{A}$ is of the form $\mathbf{A}_1 \boxplus \mathbf{A}_2$ with $f \in A_2$ and $\mathbf{A}_2 \simeq \mathbf{G}_p$ for some $p > 0$. Moreover, $\mathbf{A} \in \mathsf{V}_{2(a)}$ if and only if $\mathbf{A}_1 \boxplus \mathbf{S}_{2(a)} \in \mathsf{V}_{2(a)}$ and $\mathbf{A}_2 \in \mathsf{V}_{2(a)}$.*

PROOF. The first parts of (i) and (ii) follow from Lemma 3.10 applied to the f-free reducts. For the second parts, the left-to-right directions are trivial and the proof of the right-to-left directions is very similar to the proof of Lemma 5.11. ⊣

Let $\mathsf{SemFL}_{\mathsf{ecm}}$ be the axiomatic extension of $\mathsf{FL}_{\mathsf{ecm}}$ associated with the variety of commutative idempotent semilinear pointed residuated lattices.

PROPOSITION 5.13.

(i) *There are only finitely many varieties of commutative idempotent semilinear pointed residuated lattices that have the amalgamation property.*

(ii) *There are only finitely many axiomatic extensions of $\mathsf{SemFL}_{\mathsf{ecm}}$ with the deductive interpolation property.*

PROOF SKETCH. (i) Let V be a variety of commutative idempotent semilinear pointed residuated lattices that has the amalgamation property. By the above discussion, it suffices to show that there are only finitely many possibilities for $\mathcal{P}_{\mathsf{V}} = \{\mathsf{V}_{i(x)} \mid i \in \{1,2\},\ x \in \{\mathrm{a},\mathrm{b},\mathrm{c}\}\}$ such that each class has the one-sided amalgamation property. For $\mathsf{V}_{1(\mathrm{a})}$, we have already classified all 60 possibilities.

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For $i \in \{1,2\}$ and $x \in \{\mathrm{b},\mathrm{c}\}$, by Lemma 5.12(i), it is enough to separately classify which algebras of the form $\mathbf{A}_1 \boxplus \mathbf{S}_{i(x)}\mathbf{A}_2 \simeq \mathbf{C}_m^n$, and $\mathbf{S}_{i(x)} \boxplus \mathbf{A}_3$ are contained in $\mathsf{V}_{i(x)}$. First we notice that, by adapting the proof of Lemma 5.11, there are essentially the same 60 possibilities for the class $\{\mathbf{A} \in \mathsf{V}_{i(x)} \mid \mathbf{A} \cong \mathbf{S}_{i(x)} \boxplus \mathbf{A}_3\}$ as in the case where $\mathrm{e} \approx \mathrm{f}$, and similarly for the class $\{\mathbf{A} \in \mathsf{V}_{i(x)} \mid \mathbf{A} \cong \mathbf{A}_1 \boxplus \mathbf{S}_{i(x)}\}$. Moreover, it can be shown by using similar[4] closure properties as in Lemma 5.11 that there are only finitely many possibilities for the class $\{\mathbf{A}_2 \in \mathsf{V}_{i(x)} \mid \mathbf{A}_2 \simeq \mathbf{C}_m^n\}$. Hence for $i \in \{1,2\}$ and $x \in \{\mathrm{b},\mathrm{c}\}$, there are only finitely many possibilities for $\mathsf{V}_{i(x)}$.

For $i = 2$ and $x = \mathrm{a}$, note that, by Lemma 5.12(ii), it is enough to separately classify which algebras of the form $\mathbf{A}_1 \boxplus \mathbf{S}_{2(\mathrm{a})}$ and which algebras $\mathbf{A}_2$ with $\mathbf{A}_2 \simeq \mathbf{G}_p$ and $p > 1$ are contained in $\mathsf{V}_{2(\mathrm{a})}$. As above, it follows that for the class $\{\mathbf{A} \in \mathsf{V}_{2(\mathrm{a})} \mid \mathbf{A} \cong \mathbf{A}_1 \boxplus \mathbf{S}_{2(\mathrm{a})}\}$, there are at most 60 possibilities and for the class $\{\mathbf{A} \in \mathsf{V}_{2(\mathrm{a})} \mid \mathbf{A} \simeq \mathbf{G}_p \text{ for some } p > 1\}$ it can again be shown using similar closure properties as in Lemma 5.11 that there are only finitely many possibilities. Hence there are only finitely many possibilities for $\mathsf{V}_{2(\mathrm{a})}$.

Part (ii) then follows from the fact that the variety of commutative idempotent semilinear pointed residuated lattices is an equivalent algebraic semantics for $\mathsf{SemFL}_{\mathsf{ecm}}$. ⊣

If we let $\mathsf{V}_{1(\mathrm{a})} = [\![\mathbf{G}_\omega]\!]$ with f interpreted as e and let $\mathsf{V}_{2(\mathrm{a})}$ be the class of all commutative idempotent pointed residuated chains $\mathbf{A}$ with $\mathbf{A} \simeq \mathbf{G}_p$ for some $p \in \mathbb{N}$ that satisfy 2.(a), then there are at least 60 choices for $\mathsf{V}_{1(\mathrm{b})}$, 60 choices for $\mathsf{V}_{1(\mathrm{c})}$, 60 choices for $\mathsf{V}_{2(\mathrm{b})}$, and 60 choices for $\mathsf{V}_{2(\mathrm{c})}$ such that the obtained classes together form $\mathcal{P}_\mathsf{V}$ for some variety V that has the amalgamation property. That there are 60 choices in each case follows by considering the cases where all chains in the respective class are of the form $\mathbf{S}_{i(x)} \boxplus \mathbf{A}$, where at least 60 suitable choices for classes of algebras $\mathbf{A}$ may be found, as in the case for $\mathrm{e} \approx \mathrm{f}$. That these choices actually form $\mathcal{P}_\mathsf{V}$ for some variety V follows from the fact that a homomorphic image of an algebra of the form $\mathbf{S}_{i(x)} \boxplus \mathbf{A}$ for $x \in \{\mathrm{b},\mathrm{c}\}$ will always be either contained again in $\mathsf{V}_{i(x)}$, in $\mathsf{V}_{1(\mathrm{a})}$, or in $\mathsf{V}_{2(\mathrm{a})}$. Finally, to show that these classes all have the one-sided amalgamation property follows analogously as in the case where $\mathrm{e} \approx \mathrm{f}$. Hence we obtain the following result:

PROPOSITION 5.14.

(i) *There are at least* $60^4 > 12,000,000$ *varieties of commutative idempotent semilinear pointed residuated lattices that have the amalgamation property.*

(ii) *There are at least* $60^4 > 12,000,000$ *axiomatic extensions of* $\mathsf{SemFL}_{\mathsf{ecm}}$ *with the deductive interpolation property.*

§A. The full Lambek calculus. We restrict our attention in this appendix to formulas constructed over a countably infinite set of variables using the binary operation symbols $\wedge, \vee, \cdot, \backslash, /$ and constants e, f, and define a *sequent* to be an ordered pair consisting of a finite sequence Γ of formulas and a sequence Δ of

[4]Roughly speaking, there are at most three places where extra points can be added to $\mathbf{S}_{i(x)}$: Below f in the 'same block', above f in the 'same block' (if f is not on the Sugihara skeleton), and below f* in the block of f*. In each of these places, the possibilities are to either add no point, at most one point, or two and more points, similarly to Lemma 5.11(i).

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Identity Axioms

$$\frac{}{\alpha \Rightarrow \alpha}\ (\text{ID})$$

Cut Rule

$$\frac{\Gamma_2 \Rightarrow \alpha \quad \Gamma_1, \alpha, \Gamma_3 \Rightarrow \Delta}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta}\ (\text{CUT})$$

Left Operation Rules

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \mathrm{e}, \Gamma_2 \Rightarrow \Delta}\ (\mathrm{e}\Rightarrow) \qquad \frac{}{\mathrm{f} \Rightarrow}\ (\mathrm{f}\Rightarrow)$$

$$\frac{\Gamma_2 \Rightarrow \alpha \quad \Gamma_1, \beta, \Gamma_3 \Rightarrow \Delta}{\Gamma_1, \beta/\alpha, \Gamma_2, \Gamma_3 \Rightarrow \Delta}\ (/\Rightarrow) \qquad \frac{\Gamma_2 \Rightarrow \alpha \quad \Gamma_1, \beta, \Gamma_3 \Rightarrow \Delta}{\Gamma_1, \Gamma_2, \alpha\backslash\beta, \Gamma_3 \Rightarrow \Delta}\ (\backslash\Rightarrow)$$

$$\frac{\Gamma_1, \alpha, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \cdot \beta, \Gamma_2 \Rightarrow \Delta}\ (\cdot\Rightarrow) \qquad \frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Delta}\ (\wedge\Rightarrow)_1 \qquad \frac{\Gamma_1, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Delta}\ (\wedge\Rightarrow)_2$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Delta \quad \Gamma_1, \beta, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \alpha \vee \beta, \Gamma_2 \Rightarrow \Delta}\ (\vee\Rightarrow)$$

Right Operation Rules

$$\frac{}{\Rightarrow \mathrm{e}}\ (\Rightarrow\mathrm{e}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathrm{f}}\ (\Rightarrow\mathrm{f})$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\Rightarrow/) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\Rightarrow\backslash)$$

$$\frac{\Gamma_1 \Rightarrow \alpha \quad \Gamma_2 \Rightarrow \beta}{\Gamma_1, \Gamma_2 \Rightarrow \alpha \cdot \beta}\ (\Rightarrow\cdot) \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)_1 \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)_2$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow\wedge)$$

FIGURE A1. The full Lambek calculus FL.

at most one formula, denoted by $\Gamma \Rightarrow \Delta$. Concatenations of finite sequences are denoted using commas, and empty sequences by an empty space. The *full Lambek calculus* FL displayed in Figure A1 consists of a set of sequent rules presented schematically, where the (indexed) symbols Γ and Π stand for arbitrary finite sequences of formulas, Δ for an arbitrary sequence of at most one formula, and α and β for arbitrary formulas. Further well-studied sequent calculi are obtained by adding the basic structural rules of exchange (e), weakening (i) and (o), contraction (c), and mingle (m), depicted in Figure 1.

A *derivation* in a sequent calculus C of a sequent S from a set of sequents R is a finite tree of sequents with root S such that each node is either a member of R or constructed from its parent nodes using a rule of C. Given a set of formulas $T \cup \{\alpha\}$, we write $T \vdash_{\mathsf{C}} \alpha$ if there exists a derivation of $\Rightarrow \alpha$ from the set of sequents

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$\{ \Rightarrow \beta \mid \beta \in T\}$. In the case of FL and its extensions with basic structural rules, the relation $\vdash_{\mathsf{C}}$ is a *deductive system*: a relation $\vdash$ between sets of formulas and formulas that satisfies, for any set of formulas $T \cup T' \cup \{\alpha\}$:

1. if $\alpha \in T$, then $T \vdash \alpha$;
2. if $T \vdash \alpha$ and $T' \vdash \beta$ for every $\beta \in T$, then $T' \vdash \alpha$;
3. if $T \vdash \alpha$, then $\sigma[T] \vdash \sigma(\alpha)$ for any substitution σ;
4. if $T \vdash \alpha$, then there exists a finite $T_0 \subseteq T$ such that $T_0 \vdash \alpha$.

Moreover, it is common, when no confusion may occur, to denote by C the deductive system $\vdash_{\mathsf{C}}$ induced by a sequent calculus C.

Funding. W.F. was supported by the Czech Science Foundation (GAČR), grant no. 25-18306M. G.M. and S.S. were supported by the Swiss National Science Foundation (SNSF), grant no. 200021_215157.

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