we see that

$$f(x_1, x_2, x_3, x_4) = \sum_{i=1}^{3} \tau_{2i-1}\tau_{2i},$$

where $\tau_1, \ldots, \tau_6$ are linear forms in $x_1, \ldots, x_4$ with rational coefficients. From this it follows readily that f represents 0 non-trivially.

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CUBIC FORMS REPRESENTING ARITHMETIC PROGRESSIONS

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1. The following result has recently been proved by Lewis (3), Davenport (2) and Birch (1).

There exists an integer n_0 such that every cubic form with rational coefficients and at least n_0 integral variables represents zero non-trivially.

The arguments of Lewis and Birch are simple, and yield also various generalizations of this result. Davenport's proof is complicated, but it shows that the minimal n_0 satisfies

$$10 \leqslant n_0 \leqslant 32. \tag{1}$$

The object of this paper is to prove the

Theorem. *Every non-degenerate cubic form with rational coefficients in at least n_0+2 integral variables represents an arithmetic progression.*

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If the cubic form is denoted by $C = C(x_1, \ldots, x_n)$, with $n \geqslant n_0 + 2$, then the assertion means that for some rational a, b, with $b \neq 0$, and every integer $u \geqslant 0$ the Diophantine equation $C(x_1, \ldots, x_n) = a + bu$ has a solution in integers x_i depending on u. We shall make frequent use of the fact that this assertion is rationally invariant. That is, if C, representing the progression $a + bu$, is taken by a rational, non-singular, linear transformation of the variables into $C' = C'(y_1, \ldots, y_n)$ then C' also represents a progression. Indeed C' represents $q^3a + q^3bu$, for any integer q such that $qy_1, \ldots, qy_n$, regarded as linear forms in the x_i, have integral coefficients.

C is said to be degenerate if some transformation of this kind takes it into a form C' which is independent of y_n. It is clearly necessary to exclude such forms.

2. We simplify the problem in three stages; the first stage is

LEMMA 1. *Suppose that $C = C(x_1, \ldots, x_n)$ with $n \geqslant n_0$ has rational coefficients and is non-degenerate, but does not represent an arithmetic progression. Then C is rationally related to a form of the shape*

$$x_1^2 L(x_2, \ldots, x_n) + C_1(x_2, \ldots, x_n), \tag{2}$$

where $L(x_2, \ldots, x_n)$ is a linear form in $x_2, \ldots, x_n$, and the $(n-1)$ary cubic form C_1 is non-degenerate and does not represent an arithmetic progression.

Proof. By the Lewis–Davenport–Birch result, the equation $C(x_1, \ldots, x_n) = 0$ has a solution in integers, not all 0. By a preliminary equivalence transformation we may suppose this solution to be $x_1, \ldots, x_n = 1, 0, \ldots, 0$. That is, we may suppose that the coefficient of x_1^3 in C is zero.

Suppose first that the terms in $x_1^2x_2, \ldots, x_1^2x_n$ are also missing. Then C is a form of the shape

$$x_1 Q(x_2, \ldots, x_n) + C_1(x_2, \ldots, x_n), \tag{3}$$

where Q is a quadratic form. Clearly this form represents a progression unless Q is identically 0, in which case (3) is a degenerate form.

So we may suppose that the terms in $x_1^2x_2, \ldots, x_1^2x_n$ are not all missing. By a rational transformation on $x_2, \ldots, x_n$, we may suppose that these terms reduce to $x_1^2x_n$. An obvious parallel transformation will now remove all terms in $x_1x_ix_n$, $i = 2, \ldots, n$. So we may suppose that

$$C = x_1^2x_n + x_1 Q(x_2, \ldots, x_{n-1}) + C_1(x_2, \ldots, x_n),$$

where Q is an $(n-2)$ary quadratic and C_1 an $(n-1)$ary cubic form.

Now putting $x_1 = 0$ we see that C_1 cannot represent a progression, or C would do so. Putting $x_n = 0$, we see that Q is identically 0, otherwise the argument used for (3) shows that C would represent a progression. This, however, gives us (2), with $L = x_n$.

The assumption $L = x_n$, though legitimate, prevents us from further transforming C_1. Dropping it, we prove the remaining assertion that C_1 is non-degenerate. For if C_1 is degenerate, we may suppose it independent of x_n. Now if L also is independent of x_n, the form (2) is degenerate; if not, (2) takes the shape (3) on interchanging x_1, x_n, and C represents a progression, contrary to hypothesis.

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3. The next step is

LEMMA 2. *Assume the hypotheses of Lemma* 1, *but with* $n \geqslant n_0+1$ *in place of* $n \geqslant n_0$. *Then* C *is rationally related to a form of the shape*

$$(a_1x_1^2+a_2x_2^2)x_n+C_2(x_3, \ldots, x_n), \tag{4}$$

where $a_1a_2 \neq 0$ *and* C_2 *is non-degenerate and does not represent an arithmetic progression.*

Proof. With the hypothesis $n \geqslant n_0+1$, we can apply Lemma 1 twice. First, we appeal to Lemma 1 to show that we may suppose C to be the form (2). Then the conclusion of Lemma 1 shows that the hypotheses of Lemma 1 hold good with C_1 for C. We may therefore suppose $C_1 = x_2^2L_1(x_3, \ldots, x_n)+C_2(x_3, \ldots, x_n)$. There is, however, no loss of generality, as we saw in the proof of Lemma 1, in taking $L_1 = x_n$. Thus we may suppose

$$C = x_1^2L(x_2, \ldots, x_n)+x_2^2x_n+C_2(x_3, \ldots, x_n), \tag{5}$$

with a C_2 having the required properties.

We prove that L is independent of $x_2, \ldots, x_{n-1}$. First, if L is not independent of x_2 then it is clear that $C(1, x_2, 0, \ldots, 0)$ represents a progression. Next, if L is independent of x_2 but not of all of $x_3, \ldots, x_{n-1}$, we may clearly suppose $L = x_{n-1}+cx_n$, by a suitable transformation of the variables $x_3, \ldots, x_{n-1}$, for some constant c. By a parallel transformation, not affecting the term $x_2^2x_n$, we may suppose $L = x_{n-1}$. Now, since $x_1^2-x_2^2$ factorizes, we see that $C(x_1, \ldots, x_{n-1}, -x_{n-1})$ is rationally related to a form of the shape

$$x_1x_2x_{n-1}+C_2,$$

which clearly represents a progression.

Now we must have $L = a_1x_n$, with $a_1 \neq 0$ since C is non-degenerate. Hence, the form (5) takes the shape (4) (with $a_2 = 1$).

4. Now at the third and last stage we make full use of the hypothesis $n \geqslant n_0+2$ of the theorem. This enables us to apply Lemma 2 to the form C_1 in (2). Then an argument like that of Lemma 2 shows that we may suppose

$$C = (a_1x_1^2+a_2x_2^2+x_3^2)x_n+C_3(x_4, \ldots, x_n) \quad (a_1a_2 \neq 0).$$

The theorem now follows at once (putting $x_4, \ldots, x_n = 0, \ldots, 0, 1$) if we can show that the ternary quadratic

$$Q = Q(x_1, x_2, x_3) = a_1x_1^2+a_2x_2^2+x_3^2$$

represents an arithmetic progression. Here we may suppose that a_1, a_2 are integers, as the property asserted is rationally invariant.

The desired result on quadratic forms is well known, though it is not easy to find a convenient reference. It follows at once from two others which are specializations of results I have given elsewhere. The first of these (cf. (4), Lemma 1) is:

There exist a_0, m_0 $(m_0 \neq 0)$ *such that the congruence* $Q \equiv a \pmod m$ *is soluble in integers for every* $m \neq 0$ *and every* $a \equiv a_0 \pmod{m_0}$. (It is not difficult to see that for the Q here considered we can take $a_0 = 1$, $m_0 = 8a_1a_2$.)

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The other result we need (cf. (5), Lemma 2) is:

Given Q and an integer $m_0 \neq 0$, there exists a non-zero integer $N = N(Q, m_0)$, prime to m_0, such that Q represents every positive integer a which is divisible by N^2 and such that $Q \equiv a \pmod{m}$ is soluble in integers for every $m \neq 0$.

These two results are valid for every quadratic form, with integral coefficients, which is non-degenerate and either indefinite or positive and has at least three variables. With them the proof of the theorem is complete.

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CUMULANTS OF SOME DISTRIBUTIONS ARISING FROM A TWO-STATE MARKOFF CHAIN

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The asymptotic values of the variances and covariances for the distribution of the transition numbers AA, AB, BA and BB for a two-state Markoff chain have been given by Bartlett (1). Whittle (3) obtained the probability distribution and their factorial moments for the k-state chain under the restriction that the first and the last observations belong to the rth and the sth state. But the expressions for the factorial moments according to Whittle himself are of little immediate use. He obtains useful results for the low order factorial moments by approximating in a grosser fashion, and the evaluation of the cumulants from these factorial moments is rather cumbersome. The object of this note is to give the first four cumulants and product cumulants which are exact to the order $n^r (p_{11} - p_{21})^{n-r}$, where r is the order of the cumulant, for the distribution of the transition numbers of a two-state Markoff chain. They have been calculated by using the method developed by Iyer and Kapur (2). This method can equally be used for evaluating the asymptotic values of the cumulants for the k-state also. This is done by differentiating directly the determinantal characteristic equation.

For a Markoff chain of n observations involving two states A and B in which the probabilities for the transitions from i to j are given by the matrix (p_{ij}), $i, j = 1$ or 2,

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