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Plünnecke inequalities for measure graphs with applications

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Abstract. We generalize Petridis's new proof of Plünnecke's graph inequality to graphs whose vertex set is a measure space. Consequently, by a recent work of Björklund and Fish, this gives new Plünnecke inequalities for measure-preserving actions which enable us to deduce, via a Furstenberg correspondence principle, Banach density estimates in countable abelian groups that extend those given by Jin.

1. *Introduction*

1.1. *Background and summary of results.*

Given an abelian group G with subsets $A, B \subset G$, it is of great interest to estimate the size of the *product set* (commonly referred to as the *sumset* when additive notation is employed) defined by

$$AB = \{ab \mid a \in A, b \in B\}.$$

In particular, one is also interested in the sizes of iterated product sets B^k, which may be recursively defined by $B^1 = B$ and $B^k = B^{k-1}B$ for positive integers $k > 1$. General inequalities regarding the cardinalities of these were given by Plünnecke and Ruzsa, a comprehensive treatment of which may be found in [**8**].

THEOREM 1.1. [**7**] *For finite subsets $A, B \subset G$ of an abelian group G, define the magnification ratios*

$$D_k = D_k(A, B) = \min_{B' \subset B, B' \neq \emptyset} \frac{|A^k B'|}{|B'|}.$$

Then for positive integers $j < k$, we have

$$D_j^{1/j} \geq D_k^{1/k}.$$

Recently, Petridis [**6**] found a simplified proof of the Plünnecke inequalities. We use the ideas developed by Petridis, and by Björklund and the second author [**2**], and extend the Plünnecke inequalities to the setting of measure-preserving systems. As an application of these inequalities, we obtain new combinatorial consequences.

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Let us first recall some standard definitions.

Definition 1.2. (G acting on a measure space) We say that a group G acts on a measure space $(X, \mathcal{B}, \mu)$ if, for each $g \in G$, the map $x \mapsto g \cdot x$ is measure-preserving, that is, it is measurable and $\mu(gB) = \mu(B)$ for each $B \in \mathcal{B}$.

The simplest example of such an action is the action of a discrete group G on itself, where μ is just the counting measure. But in many cases we might be interested in genuinely non-discrete measure-preserving systems.

The following definition, which first appeared in [**2**], extends the notion of the magnification ratios from Theorem 1.1 to the setting of measure-preserving actions.

Definition 1.3. [**2**] Given a countable abelian group G acting on a measure space $(X, \mathcal{B}, \mu)$, let us define, for $A \subset G$ and $B \in \mathcal{B}$ of finite positive measure, the magnification ratio

$$c(A, B) = \inf\left\{ \frac{\mu(AB')}{\mu(B')} \;\middle|\; B' \subset B, \mu(B') > 0 \right\}.$$

Given these definitions, it is now natural to ask whether Plünnecke's inequalities generalize to the setting of measure-preserving systems. A special case of this was proven in [**2**].

THEOREM 1.4. [**2**] *Suppose that G is a countable abelian group acting on a probability space $(X, \mathcal{B}, \mu)$ with $L^2(X, \mathcal{B}, \mu)$ separable (for example, X is a compact metric space with Borel σ-algebra $\mathcal{B}$). Then for $A \subset G$, and $B \in \mathcal{B}$ with $\mu(B) > 0$, we have*

$$c(A, B) \geq c(A^k, B)^{1/k}.$$

The main purpose of this paper is to extend the remaining cases of Plünnecke's inequality. More precisely, we show the following.

THEOREM 1.5. (Plünnecke inequalities for measure-preserving actions) *Suppose that G is a countable abelian group acting on a probability space $(X, \mathcal{B}, \mu)$ with $L^2(X, \mathcal{B}, \mu)$ separable. Then for $A \subset G$ and $B \in \mathcal{B}$ with $\mu(B) > 0$, the sequence*

$$c(A^k, B)^{1/k}$$

is decreasing in k.

We now turn to some applications of these measure-theoretic Plünnecke inequalities to estimates of the Banach densities of sumsets. Let us first recall the basic construction of the upper and lower Banach densities.

Definition 1.6. (Følner sequence) Let G be a countable group. A (left) *Følner sequence* in G is a sequence $F_1, F_2, \ldots$, of finite subsets of G such that, for each fixed $g \in G$,

$$\lim_{n\to\infty} \frac{|F_n \cap gF_n|}{|F_n|} = 1.$$

A group which possesses a Følner sequence is called *amenable*. It is well known that all countable abelian groups are amenable. Each Følner sequence $(F_n)_{n\in\mathbb{N}}$ gives rise to corresponding upper and lower densities for $A \subset G$ given by

$$\overline{d}_{(F_n)}(A) = \limsup_{n\to\infty} \frac{|F_n \cap A|}{|F_n|}$$

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and

$$\underline{d}_{(F_n)}(A) = \liminf_{n\to\infty} \frac{|F_n \cap A|}{|F_n|},$$

respectively. For example, the Følner sequence $F_n = [1, n] \cap \mathbb{Z}$ in the additive group $\mathbb{Z}$ gives rise to the classical upper and lower *asymptotic* densities in the natural numbers, which are of particular interest in number theory. Of interest to us is the *upper* (respectively, *lower*) *Banach density*, denoted by $d^*(A)$ (respectively, $d_*(A)$), which may be defined as the maximum (respectively, minimum) of the set

$$\{\overline{d}_{(F_n)}(A) \mid (F_n)_{n\in\mathbb{N}} \text{ is Følner}\}.$$

The fact that these extrema are attained in any given countable amenable group may be verified by a simple diagonalization argument. In fact one can show (see [**1**, Lemma 3.3]) the stronger assertion that given any Følner sequence $(F_n)_{n\in\mathbb{N}}$ and $A \subset G$, there exist $t_n \in G$ such that

$$d^*(A) = \overline{d}_{(t_n F_N)}(A),$$

and likewise for lower Banach density. In particular, in the additive group $\mathbb{Z}^d$ one only needs to look at sequences of finite cubes of strictly increasing cardinality. One may now ask whether Banach densities of sumsets satisfy analogous Plünnecke–Ruzsa type inequalities. Indeed, Jin has proven in [**5**] such inequalities for the additive semi-group $\mathbb{Z}_{\geq 0}$.

THEOREM 1.7. (Jin) *Suppose that* $A, B \subset \mathbb{Z}_{\geq 0}$. *Then*

$$d^*(A + B) \geq d^*(kA)^{1/k} d^*(B)^{1-1/k}$$

and

$$d_*(A + B) \geq d^*(kA)^{1/k} d_*(B)^{1-1/k}.$$

Recently, Björklund and the second author [**2**] generalized Jin's theorem to all countable abelian groups. We extend this generalization as follows.

THEOREM 1.8. *Suppose that* G *is a countable abelian group and* $A, B \subset G$. *Then, for integers* $0 < j < k$,

$$d^*(A^j B) \geq d^*(A^k)^{j/k} d^*(B)^{1-j/k}$$

and

$$d_*(A^j B) \geq d^*(A^k)^{j/k} d_*(B)^{1-j/k}.$$

It seems unclear whether or not our results, even for just $G = (\mathbb{Z}, +)$, are attainable from an application of Jin's techniques to the inequality $D_k^{1/k} \leq D_j^{1/j}$. We are also able to obtain a result involving multiple different factors, which is a generalization of the $j = 1$ case of Theorem 1.8.

THEOREM 1.9. *Suppose that* G *is a countable abelian group and* $B, A_1, \ldots, A_k \subset G$. *Then*

$$d^*(A_1 \cdots A_k) \leq d^*(B)^{1-k} \prod_{i=1}^{k} d^*(A_i B)$$

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and

$$d^*(A_1 \cdots A_k) \leq d_*(B)^{1-k} \prod_{i=1}^{k} d_*(A_i B).$$

The deduction of these density estimates from the measure-theoretic Plünnecke inequalities uses the strategy developed by Björklund and the second author in [**2**]: a Furstenberg correspondence principle for product sets. This correspondence principle can be derived from the seminal work of Furstenberg [**4**]. Nevertheless, it was noticed much later in [**1**]. The following version of the correspondence principle (the third and fourth inequalities) for product sets is due to Björklund and the second author [**3**].

PROPOSITION 1.10. (Furstenberg's correspondence principle) *Suppose that G is a countable abelian group and $A, B \subset G$. Then there exists a compact metrizable space X on which G acts by homeomorphisms such that there exist G-invariant ergodic Borel probability measures μ, ν on X together with a clopen $\widetilde{B} \subset X$ such that*

$$\begin{aligned} d^*(B) &= \mu(\widetilde{B}), \\ d^*(AB) &\geq \mu(A\widetilde{B}), \\ d_*(B) &\leq \nu(\widetilde{B}), \\ d_*(AB) &\geq \nu(A\widetilde{B}). \end{aligned}$$

Note that $AB = \bigcup_{a \in A} aB$ is a union of translates, and so we may consider this correspondence principle as the classical Furstenberg correspondence principle appearing in the ergodic-theoretic proof of Szémeredi's theorem [**4**] with intersections replaced with unions. We must, however, in this case make the extra effort of ensuring the ergodicity of the measures. Another important nuisance is that, unlike in the case of intersections, we may need to consider the possibility of infinitely many translates.

Proof of Theorems 1.8 and 1.9. Let (X, μ) and $\widetilde{B}$ be as in the correspondence principle. Note that we may assume that $d^*(B) > 0$ as the result is trivial otherwise. Note also that in §7 (see Lemma 7.3) we show that $d^*(A^k) \leq \mu(A^k \widetilde{B})$. Altogether, this gives

$$\begin{aligned} \left(\frac{d^*(A^j B)}{d^*(B)}\right)^{1/j} &\geq \left(\frac{\mu(A^j \widetilde{B})}{\mu(\widetilde{B})}\right)^{1/j} \\ &\geq c(A^j, \widetilde{B})^{1/j} \\ &\geq c(A^k, \widetilde{B})^{1/k} \\ &\geq \left(\frac{d^*(A^k)}{d^*(B)}\right)^{1/k}, \end{aligned}$$

which shows the first inequality. The second one may be deduced from the same argument applied to the measure ν instead of μ from the correspondence principle. Moreover, Theorem 1.9 may be obtained by applying the correspondence principle to the Plünnecke inequality for different summands (Proposition 6.2).

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1.2. *Outline of the paper.* The main object introduced in this paper is what we call a *measure graph*. Section 2 provides all the relevant definitions and basic properties. Intuitively, a measure graph is a directed edge-labelled graph equipped with a measure on the vertex set that mimics certain elementary combinatorial properties of the classical graph-theoretic notion of a *matching*. The aim is to prove a measure-theoretic version of the classical Plünnecke inequality for commutative graphs. The classical approach employs Menger's theorem, which has no obvious measure-theoretic analogue. However, Petridis [**6**] has recently found a new proof of this inequality that avoids the use of Menger's theorem. In §3 we generalize this proof to measure graphs. Immediate corollaries concerning measure-preserving actions are given in §4. We then, in §5, turn to extending the results regarding $c(A, B)$ with A finite, to ones where A is countable. Section 6 is devoted to a proof of a measure-theoretic analogue of Ruzsa's Plünnecke inequality involving different summands. Finally, we prove the correspondence principle for product sets in §7.

2. *Definitions*

By a labelled directed graph we mean a tuple (V, E, A) where V and A are sets and $E \subset V \times V \times A$. We regard an element $(v, w, a) \in E$ as an edge directed from v to w and labelled a. For subsets $W \subset V$ and labels $a \in A$ the a-image and a-preimage are defined, respectively, as

$$\mathrm{Im}_a^+(W) = \{v \in V \mid (w, v, a) \in E \text{ for some } w \in W\}$$

and

$$\mathrm{Im}_a^-(W) = \{v \in V \mid (v, w, a) \in E \text{ for some } w \in W\}.$$

That is, the a-image of W consists of the vertices that may be approached from W by walking, in the direction of the orientation, along an edge labelled a. Moreover, we define for $W \subset V$ the (pre)image $\mathrm{Im}^{\pm}(W) = \bigcup_{a \in A} \mathrm{Im}_a^{\pm}(W)$. For each integer h we have the h-fold image $\mathrm{Im}^h(W)$ defined recursively by $\mathrm{Im}^0(W) = W$ and $\mathrm{Im}^h(W) = \mathrm{Im}^+(\mathrm{Im}^{h-1}(W))$ for $h > 0$ and $\mathrm{Im}^h(W) = \mathrm{Im}^-(\mathrm{Im}^{h+1}(W))$ for $h < 0$. In other words, $\mathrm{Im}^h(W)$ consists of all end points of walks with $|h|$ steps that begin at W and agree (respectively, disagree) with the orientation of each edge if $h > 0$ (respectively, $h < 0$). Define also the incoming and outgoing degrees of a vertex v as $d^-(v) = |\{(x, y, a) \in E \mid y = v\}|$ and $d^+(v) = |\{(x, y, a) \in E \mid x = v\}|$, respectively. Note that $|\mathrm{Im}^{\pm}(\{v\})| \le d^{\pm}(v)$ with strict inequality possible in the case of multiple edges between two vertices (of course any two such edges would have different labels). Given an edge e from v to w, we will call v the tail, denoted tail(e), and w the head, denoted by head(e). Let $E^+(v)$ denote the edges whose tail is v and $E^-(v)$ those edges whose head is v.

Definition 2.1. A *measure graph* is a tuple $\Gamma = (V, \mathcal{B}, \mu, A, E)$ where $(V, \mathcal{B}, \mu)$ is a finite measure space (that is, $\mu(V) < \infty$), A is a finite set and (V, E, A) is a labelled directed graph such that:

(1) for each $a \in A$ the sets

$$L_a^+(\Gamma) = L_a^+ = \{x \in V \mid \exists y \in V \text{ such that } (x, y, a) \in E\}$$

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and

$$L_a^-(\Gamma) = L_a^- = \{x \in V \mid \exists y \in V \text{ such that } (y, x, a) \in E\}$$

are measurable;

(2) for $a \in A$ and measurable $W \subset L_a^{\pm}$ we have that $\mathrm{Im}_a^{\pm}(W)$ is measurable and $\mu(W) = \mu(\mathrm{Im}_a^{\pm}(W))$;

(3) for each label $a \in A$ and vertex $x \in V$ we have at most one outgoing and at most one incoming a-labelled edge incident to x, that is, $|\mathrm{Im}_a^{\pm}(x)| \leq 1$.

Example 2.2. (The (A, Y, h)-orbit graph) Given an abelian group G acting on a measure space $(X, \mathcal{B}, \mu)$, one may form, for each integer $h > 0$, finite $A \subset G$ and $Y \in \mathcal{B}$ of finite measure, a measure graph whose underlying vertex set is $\bigsqcup_{k=0}^{h} A^k Y \times \{k\}$ with edge set

$$\{((x, k), (a \cdot x, k+1), a) \mid a \in A, x \in A^k Y, k = 0, 1, \ldots, h-1\}.$$

The measure is the restriction of the natural product measure on $X \times \{0, 1, \ldots, h\}$.

Given a labelled graph $\Gamma = (V, E, A)$ and $W \subset V$, the subgraph *induced* by W is the directed labelled graph (W, E_W, A) where $E_W = \{(w_1, w_2, a) \mid w_1, w_2 \in W, a \in A \text{ and } (w_1, w_2, a) \in E\}$. We say that a subgraph of Γ is an *induced subgraph* if it is induced by some subset of V.

Example 2.3. If $\Gamma = (V, \mathcal{B}, \mu, A, E)$ is a measure graph and $W \subset V$ is measurable then the subgraph of Γ induced by W is a measure graph (with the restricted measure, restricted σ-algebra and the same edge-label set A). Note that the set of vertices with an outgoing edge labelled $a \in A$ is $L_a^+ \cap W \cap \mathrm{Im}_a^-(L_a^- \cap W)$ and thus is measurable as required.

Note that the (A, Y, h)-orbit graph defined above is a generalization of the commutative addition graph studied in classical additive combinatorics; see, for example, [**8**, **9**]. It is also an example of what is known as a commutative, or Plünnecke, graph which may be defined as follows.

Definition 2.4. A *layered graph* is a directed labelled graph (V, E, A) together with a partition $V = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_h$ such that if $e = (x, y, a) \in E$ is a directed edge then $x \in V_i$ and $y \in V_{i+1}$ for some $i \in \{0, \ldots, h-1\}$. We call V_k the kth layer and we say that (V, E, A) is an h-*layered* graph (we regard the partition as part of the data of a layered graph). A *semi-commutative* (or *semi-Plünnecke*) graph is a layered graph (V, E, A) such that if $(x, y, a) \in E$ is an edge then there is an injection $\phi : E^+(y) \to E^+(x)$ such that $(\mathrm{head}(\phi(e)), \mathrm{head}(e), a) \in E$ for all $e \in E^+(y)$. A *commutative*, or *Plünnecke*, graph Γ is a directed layered graph such that both Γ and the dual graph (the layered graph obtained by reversing edges and the ordering of the layers) Γ^* are semi-commutative.

Example 2.5. The (A, Y, h)-orbit graph defined above is a commutative graph with layering $V = \bigsqcup_{j=0}^{h} A^j Y \times \{j\}$. To check semi-commutativity, take a typical edge $((x, j), (ax, j+1), a)$ running from $A^j Y \times \{j\}$ to $A^{j+1} Y \times \{j+1\}$ where $0 \leq j \leq h-1$. Then for edges

$$e = ((ax, j+1), (a'ax, j+2), a') \in E^+((ax, j+1))$$

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we may choose $\phi(e) = ((x, j), (a'x, j+1), a')$ since, by commutativity of G, $a \cdot (a' \cdot x) = (a'a \cdot x)$ and thus there is an a-labelled edge from head$(\phi(e))$ to head(e) as required. The semi-commutativity of the dual can be similarly verified.

The following is an easy exercise in commutative graphs (see [**8**]).

PROPOSITION 2.6. *Suppose that (V, E) is an h-layered commutative graph with layers $V = V_0 \sqcup \cdots \sqcup V_h$. Then for $S \subset V_j$ and $T \subset V_k$, where $0 \le j < k \le h$, we have that the channel between S and T (that is, the subgraph consisting of all directed paths from S to T) is a commutative graph. We denote this subgraph* $\mathrm{ch}(S, T)$.

We will be interested in studying channels of an (A, Y, h)-orbit graph. It turns out that these are measurable.

LEMMA 2.7. *Given an h-layered measure graph $\Gamma = (V, \mathcal{B}, \mu, A, E)$ with layering† $V = V_0 \sqcup \cdots \sqcup V_h$ and measurable $S \subset V_i$, $T \subset V_j$, where $0 \le i < j \le h$, we have that the channel* $\mathrm{ch}(S, T)$ *has measurable vertex set.*

Proof. Let us denote the vertex set of a subgraph Γ' as $V(\Gamma')$. We use induction on $j - i$. The base case $j = i + 1$ holds since then $\mathrm{ch}(S, T)$ has vertex set $V(\mathrm{ch}(S, T)) = S \cap \mathrm{Im}^-(T) \bigsqcup T \cap \mathrm{Im}(S)$. Now suppose that $j - i > 1$. Then by the induction hypothesis we have that $\mathrm{ch}(S, \mathrm{Im}^-(T))$ has measurable vertex set. By the base case, $\mathrm{ch}(\mathrm{Im}^- T, T)$ has measurable vertex set. Now let $U = V(\mathrm{ch}(S, \mathrm{Im}^- T)) \cap V(\mathrm{ch}(\mathrm{Im}^- T, T)) \subset V_{j-1}$. Finally, we have $V(\mathrm{ch}(S, T)) = V(\mathrm{ch}(S, U)) \cup V(\mathrm{ch}(U, T))$ which is measurable again by the induction hypothesis. □

Note that the previous lemma and Example 2.3 demonstrate that the channel between two measurable sets may be naturally viewed as a measure graph (as channels are induced subgraphs).

We now turn to generalizing the notion of the number of edges in a bipartite graph.

Definition 2.8. Fix a one-layered commutative measure graph $(U, \mathcal{B}, \mu, A, E)$ with layering $U = U_0 \sqcup U_1$. Define the *flow* of Γ to be the quantity

$$\mathrm{Flow}(\Gamma) = \int_{U_0} d^+(v)\, d\mu(v).$$

We now show that the flow behaves nicely and that $d^{\pm}$ is a measurable function.

PROPOSITION 2.9. *Under the setting of the previous definition, the map $d^+ : U \to \mathbb{R}$ is measurable and* $\mathrm{Flow}(\Gamma) = \mathrm{Flow}(\Gamma^*)$*, that is,*

$$\int_{U_0} d^+(v)\, d\mu(v) = \int_{U_1} d^-(v)\, d\mu(v).$$

Proof. Since (by definition of a measure graph) $|\mathrm{Im}_a^{\pm}(\{v\})| \le 1$, we may express

$$d^{\pm} = \sum_{a \in A} \chi_{L_a^{\pm}},$$

† We always assume implicitly that each layer is measurable.

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and thus $d^{\pm}$ is measurable. Consequently,

$$\mathrm{Flow}(\Gamma) = \sum_{a\in A} \mu(L_a^+) = \sum_{a\in A} \mu(\mathrm{Im}_a^+(L_a^+)) = \sum_{a\in A} \mu(L_a^-) = \mathrm{Flow}(\Gamma^*)$$

as required. □

3. *Plünnecke's inequality for measure graphs*

Definition 3.1. Given a commutative measure graph $\Gamma = (V, \mathcal{B}, \mu, A, E)$ with layering $V = V_0 \sqcup \cdots \sqcup V_h$, the magnification ratio of order j, where $j \in \{1, \ldots, h\}$, is

$$D_j = \inf_{Y\subset V_0, \mu(Y)>0} \frac{\mu(\mathrm{Im}^j(Y))}{\mu(Y)}.$$

Moreover, for $C > 0$, define the *weight* (corresponding to C) to be the measure on $\mathcal{B}$ given by

$$w(S) = \sum_{j=0}^{h} C^{-j}\mu(S \cap V_j)$$

for $S \in \mathcal{B}$. Furthermore, we say that $S \in \mathcal{B}$ is a *cutset* if any path from V_0 to V_h intersects S and that S is an *ϵ-minimal cutset* if S is a cutset such that

$$w(S) \le m_0 + \epsilon,$$

where $m_0 = \inf\{w(Y) \mid Y \in \mathcal{B} \text{ is a cutset}\}$.

LEMMA 3.2. *Fix a two-layered commutative measure graph $(U, \mathcal{B}, \mu, A, E)$ with layering $U = U_0 \sqcup U_1 \sqcup U_2$ and $C > 0$. Then if U_1 is an ϵ-minimal cutset (with respect to the weight corresponding to C), then U_0 is an $f(\epsilon)$-minimal cutset where*

$$f(\epsilon) = \epsilon + 4|A|^2C\epsilon + 4|A|^2\epsilon.$$

Proof. Let $m_0 = \inf\{w(S) \mid S \in \mathcal{B} \text{ is a cutset}\}$. Note, for measurable $S \subset U_1$, that $\mathrm{Im}(S) \sqcup (U_1\backslash S)$ is a cutset and thus $m_0 \le w(\mathrm{Im}(S)) + w(U_1\backslash S)$. On the other hand, since U_1 is ϵ-minimal, $w(S) + w(U_1\backslash S) - \epsilon \le m_0$ and thus $w(S) \le w(\mathrm{Im}(S)) + \epsilon$. A similar argument yields that $w(S) \le w(\mathrm{Im}^{-1}(S)) + \epsilon$. Thus

$$C\mu(S) \le \mu(\mathrm{Im}(S)) + C^2\epsilon \qquad (\star)$$

and

$$C^{-1}\mu(S) \le \mu(\mathrm{Im}^{-1}(S)) + \epsilon \qquad (\dagger)$$

for measurable $S \subset U_1$.

For each integer $i \ge 0$ let

$$X_i = \{u \in U_1 \mid d^-(v) = i\},$$
$$Y_i = \{u \in U_2 \mid d^-(v) = i\},$$
$$X_i' = \{u \in U_1 \mid d^+(v) = i\},$$
$$Y_i' = \{u \in U_0 \mid d^+(v) = i\}.$$

The X_i are measurable and partition U_1. Let $k = |A|$. Now define inductively $T_k = \mathrm{Im}(X_k)$ and $T_i = \mathrm{Im}(X_i)\backslash T_{i+1}$ for $i = k-1, k-2, \ldots, 1$. Note that the T_i partition the set of

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vertices in U_2 that have at least one incoming edge. Moreover, by the definition of commutativity we have that each vertex in T_i has inward degree at least i (specifically, this is by the semi-commutativity of the dual). Thus we obtain

$$\sum_{i=1}^{k} i\mu(T_i) \leq \sum_{i=1}^{k} \mathrm{Flow}(\mathrm{ch}(U_1, T_i)) = \mathrm{Flow}(\mathrm{ch}(U_1, U_2)), \tag{1}$$

where the right-hand side is well defined since induced subgraphs with measurable vertex sets are measure graphs. From now on we will use the shorthand notation $\mathrm{Flow}(U_i, U_j) := \mathrm{Flow}(\mathrm{ch}(U_i, U_j))$.

Moreover, since $\mathrm{Im}(X_j \sqcup \cdots \sqcup X_k) = T_j \sqcup \cdots \sqcup T_k$, we have by $(\star)$ that

$$C \sum_{i=j}^{k} \mu(X_i) \leq \sum_{i=j}^{k} \mu(T_i) + C^2\epsilon. \tag{2}$$

Adding these inequalities for $j = 1, \ldots, k$, we obtain

$$C \cdot \mathrm{Flow}(U_0, U_1) = C \sum_{i=1}^{k} i\mu(X_i) \leq \sum_{i=1}^{k} i\mu(T_i) + kC^2\epsilon \tag{3}$$

which implies, together with the preceding inequality, that

$$C \cdot \mathrm{Flow}(U_0, U_1) \leq \mathrm{Flow}(U_1, U_2) + kC^2\epsilon. \tag{4}$$

We will now apply the same argument to the dual graph to obtain an inequality of the form

$$\mathrm{Flow}(U_1, U_2) \leq C \cdot \mathrm{Flow}(U_0, U_1) + O_{k,C}(\epsilon).$$

To do this, inductively define $T'_k = \mathrm{Im}^{-1}(X'_k)$ and $T'_i = \mathrm{Im}^{-}(X'_i) \backslash T'_{i+1}$ for $i = k-1$, $k-2, \ldots, 1$. This time we have, by the duals of the previous arguments (with $(\dagger)$ in place of $(\star)$) and the fact that flows are the same for duals (Proposition 2.9), that

$$\sum_{i=1}^{k} i\mu(T'_i) \leq \mathrm{Flow}(\mathrm{ch}_{\Gamma^*}(U_1, U_0)) = \mathrm{Flow}(U_0, U_1) \tag{5}$$

and, for each $\ell \in \{1, 2, \ldots, k\}$,

$$C^{-1} \sum_{i=\ell}^{k} \mu(X'_i) \leq \sum_{i=\ell}^{k} \mu(T'_i) + \epsilon \tag{6}$$

which, by summing as before, gives

$$C^{-1}\,\mathrm{Flow}(U_1, U_2) \leq \mathrm{Flow}(U_0, U_1) + k\epsilon.$$

Thus $\mathrm{Flow}(U_1, U_2)$ is close to $C \cdot \mathrm{Flow}(U_0, U_1)$. This means that the inequalities above must have been close to being equalities. We will now explicitly estimate how close. Let us start with inequality (6). From it, (4), and (5) we obtain that, for each $j \in \{1, \ldots, k\}$,

$$\begin{aligned} C^{-1} \sum_{i=j}^{k} \mu(X'_i) - \sum_{i=j}^{k} \mu(T'_i) + (k-1)\epsilon &\geq C^{-1} \sum_{i=1}^{k} i\mu(X'_i) - \sum_{i=1}^{k} i\mu(T'_i) \\ &\geq C^{-1} \cdot \mathrm{Flow}(U_1, U_2) - \mathrm{Flow}(U_0, U_1) \\ &\geq -kC\epsilon, \end{aligned}$$

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where the first inequality is obtained by summing (6) for $\ell \in \{1, 2, \ldots, k\}\backslash\{j\}$. This finally gives

$$C^{-1}\sum_{i=j}^{k}\mu(X_i') - \sum_{i=j}^{k}\mu(T_i') \geq -kC\epsilon - (k-1)\epsilon \tag{7}$$

and so

$$\left|C^{-1}\sum_{i=j}^{k}\mu(X_i') - \sum_{i=j}^{k}\mu(T_i')\right| \leq \max\{kC\epsilon + (k-1)\epsilon, \epsilon\} \leq kC\epsilon + k\epsilon. \tag{8}$$

Thus the triangle inequality gives

$$|C^{-1}\mu(X_i') - \mu(T_i')| \leq 2kC\epsilon + 2k\epsilon. \tag{9}$$

We now wish to show that T_i' is approximately Y_i' (that is, they have small symmetric difference). Note that $T_j' \sqcup \cdots \sqcup T_k' \subset Y_j' \sqcup \cdots \sqcup Y_k'$ since, as before, the definition of a commutative graph implies that each vertex in T_i' has outward degree at least i. Thus

$$\sum_{i=j}^{k}\mu(T_i') - \sum_{i=j}^{k}\mu(Y_i') \leq 0. \tag{10}$$

Combining this with (6) and (4), we have, for each $j \in \{1, \ldots, k\}$, that

$$\begin{aligned}\sum_{i=j}^{k}\mu(T_i') - \sum_{i=j}^{k}\mu(Y_i') &\geq \sum_{i=1}^{k} i\mu(T_i') - \sum_{i=1}^{k} i\mu(Y_i')\\ &\geq C^{-1}\sum_{i=1}^{k} i\mu(X_i') - k\epsilon - \sum_{i=1}^{k} i\mu(Y_i')\\ &= C^{-1}\cdot \mathrm{Flow}(U_1, U_2) - \mathrm{Flow}(U_0, U_1) - k\epsilon\\ &\geq -kC\epsilon - k\epsilon.\end{aligned}$$

Thus

$$\left|\sum_{i=j}^{k}\mu(T_i') - \sum_{i=j}^{k}\mu(Y_i')\right| \leq kC\epsilon + k\epsilon,$$

from which the triangle inequality implies

$$|\mu(T_i') - \mu(Y_i')| \leq 2kC\epsilon + 2k\epsilon. \tag{11}$$

Combining this with (9) yields

$$|C^{-1}\mu(X_i') - \mu(Y_i')| \leq 4kC\epsilon + 4k\epsilon.$$

Finally, we get

$$|w(U_1) - w(U_0)| = \left|C^{-1}\sum_{i=1}^{k}\mu(X_i') - \sum_{i=1}^{k}\mu(Y_i')\right| \leq 4k^2C\epsilon + 4k^2\epsilon$$

and so in fact U_0 is $(\epsilon + 4k^2C\epsilon + 4k^2\epsilon)$-minimal. □

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We will now inductively apply the above lemma to construct ϵ-minimal cutsets that lie in the union of the top and bottom layers.

LEMMA 3.3. *Suppose that $\Gamma = (V, \mathcal{B}, \mu, A, E)$ is an h-layered commutative measure graph with layering $V = V_0 \sqcup \cdots \sqcup V_h$. Fix $C > 0$ and let w be the weight on Γ corresponding to C. Then for each $\epsilon > 0$ there exists an ϵ-minimal cutset $S \in \mathcal{B}$ such that $S \subset X_0 \sqcup X_h$.*

Proof. We will prove, by induction on $j \in \{h-1, \ldots, 1, 0\}$, that there exists an ϵ-minimal cutset contained in $V_0 \sqcup V_1 \sqcup \cdots \sqcup V_j \sqcup V_h$. The base case $j = h-1$ is clear. Therefore, fix $\delta > 0$ and suppose that $j \in \{1, \ldots, h-1\}$ and $S \subset V_0 \sqcup V_1 \sqcup \cdots \sqcup V_j \sqcup V_h$ is a δ-minimal cutset. Let $S_i = S \cap X_i$ for $i \in \{1, \ldots, h\}$. Let $U_0 \subset V_{j-1}$ be those vertices that may be approached to V_{j-1} from V_0 along a path that does not intersect S. Let $U_2 = V_{j+1} \cap \mathrm{ch}(V_{j+1}, V_h \backslash S_h)$ be the set of vertices in V_{j+1} that can approach a vertex in $V_h \backslash S_h$ by a directed path in the graph. We know that U_2 is measurable by the measurability of channels, and similarly U_0 is measurable by an application of the measurability of channels to the subgraph induced by $\bigsqcup_{i=0}^{h} V_i \backslash S_i$. Let $H = \mathrm{ch}(U_0, U_2)$ and let $U_1 \subset V_j$ by the vertices in H that lie in V_j. Thus H is a two-layered measure subgraph of Γ that is also commutative. Let us equip H with the measure $C^{-j+1}\mu$ instead of μ, since then the weight function on H corresponding to C agrees with that of Γ.

Subclaim: The middle layer U_1 is a δ-minimal cutset of H.

To see this, firstly note that $U_1 \subset S_j$ (see Figure 1). If U_1 is not δ-minimal, then there exists a cutset T in H of weight $w(T) < w(U_1) - \delta \leq w(S_j) - \delta$. But then the set $S' = S_0 \cup \cdots S_{j-1} \cup T \cup S_h$ is a cutset of Γ of weight

$$w(S') \leq \sum_{i=0}^{j-1} w(S_i) + w(T) + w(S_h) < \sum_{i=0}^{j-1} w(S_i) + w(S_j) - \delta + w(S_h) = w(S) - \delta,$$

contradicting S being δ-minimal. This proves the subclaim.

Hence we get by Lemma 3.2 that $(S \cup U_0) \backslash S_j$ is a $(\delta + f(\delta))$-minimal cutset, where f is as in the respective lemma (which we may take with the parameters of Γ, that is, we consider H as having labelling set A and thus this f does not depend on H). Taking $\delta \to 0$ finishes the induction step and hence the proof of this lemma. □

We are now ready to show that in the case $C = D_h^{1/h}$ the bottom layer is in fact a cutset of minimal weight.

COROLLARY 3.4. *Suppose that $\Gamma = (X, \mathcal{B}, \mu, A, E)$ is an h-layered commutative measure graph with layering $X = X_0 \sqcup \cdots \sqcup X_h$. Suppose that $D_h^{1/h} > 0$ and let w be the weight corresponding to $C = D_h^{1/h}$. Then X_0 is a cutset of minimal weight.*

Proof. We want to show that X_0 is ϵ-minimal for all $\epsilon > 0$. Choose $\epsilon > 0$ and by the above lemma an ϵ-minimal cutset $S \subset X_0 \sqcup X_h$. Write $S_i = X_i \cap S$. As S is a cutset we have $\mathrm{Im}^h(X_0 \backslash S_0) \subset S_h$ and so

$$\mu(S_h) \geq \mu(\mathrm{Im}^h(X_0 \backslash S_0)) \geq D_h \mu(X_0 \backslash S_0) = C^h \mu(X_0) - C^h \mu(S_0).$$

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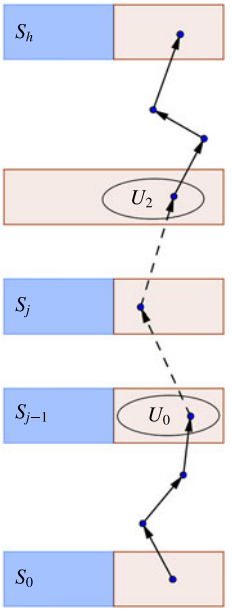


FIGURE 1. A dotted two-length path as shown cannot exist as this gives rise to a path from V_0 to V_h which avoids S, as shown, by the definition of U_0 and U_2. Thus $U_1 \subset S_j$. One can similarly argue that the S' given in the proof of the subclaim is a cutset.

Thus

$$w(S) = \mu(S_0) + C^{-h}\mu(S_h) \geq \mu(X_0) = w(X_0)$$

and so $w(X_0)$ is ϵ-minimal. □

We may now finally prove the Plünnecke inequality for measure graphs.

THEOREM 3.5. (Plünnecke inequality for measure graphs) *Suppose that* $\Gamma = (X, \mathcal{B}, \mu, A, E)$ *is an* h*-layered commutative measure graph with layering* $X = X_0 \sqcup \cdots \sqcup X_h$. *Then, for* $j \in \{1, \ldots, h\}$,

$$D_j^h \geq D_h^j.$$

Proof. If $D_h = 0$ then we are done. If $D_h > 0$ then we may set $C = D_h^{1/h}$ and apply the above corollary as follows. For each non-null measurable $Z \subset X_0$ we have that $(X_0 \backslash Z) \sqcup \mathrm{Im}^j(Z)$ is a cutset, and thus by minimality of X_0 we have

$$\mu(X_0) = w(X_0) \leq w((X_0 \backslash Z) \sqcup \mathrm{Im}^j(Z)) = \mu(X_0) - \mu(Z) + D_h^{-j/h}\mu(\mathrm{Im}^j(Z)),$$

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and so

$$D_h^{j/h} \leq \frac{\mu(\mathrm{Im}^j(Z))}{\mu(Z)},$$

which completes the proof as $Z \subset X_0$ was arbitrary. □

4. *Applications to measure-preserving systems*

First, we recall the notions of the magnification ratios for the dynamical setting introduced by Björklund and the second author in [**2**].

Definition 4.1. Suppose that G is a countable abelian group acting on a measure space $(X, \mathcal{B}, \mu)$. Define, for $A \subset G$ and $B \in \mathcal{B}$ of positive finite measure, the magnification ratio

$$c(A, B) = \inf\left\{ \frac{\mu(AB')}{\mu(B')} \;\middle|\; B' \subset B, \mu(B') > 0 \right\}.$$

Moreover, for $\delta > 0$ we may define the δ-heavy magnification ratio

$$c_\delta(A, B) = \inf\left\{ \frac{\mu(AB')}{\mu(B')} \;\middle|\; B' \subset B, \mu(B') \geq \delta \cdot \mu(B) \right\}.$$

Furthermore, if $E \subset X$ is measurable then we may define the restricted magnification ratio

$$c(A, B, E) = \inf\left\{ \frac{\mu(AB' \backslash E)}{\mu(B')} \;\middle|\; B' \subset B, \mu(B') > 0 \right\}.$$

By applying the Plünnecke inequality for measure graphs to the case of orbit graphs we obtain the following Plünnecke inequality for measure-preserving systems.

THEOREM 4.2. *Suppose that G is a countable abelian group acting on a measure space $(X, \mathcal{B}, \mu)$. Then, for $A \subset G$ finite and measurable $B \in \mathcal{B}$ of positive finite measure,*

$$c(A^j, B)^{1/j} \geq c(A^k, B)^{1/k}$$

for positive integers $j < k$.

We may also obtain the G-system analogue of a classical restricted addition result.

THEOREM 4.3. *Suppose that G is a countable abelian group acting on a measure space $(X, \mathcal{B}, \mu)$. For finite $A \subset G$, measurable $B \subset X$ of positive finite measure and measurable $E \subset X$,*

$$c(A^j, B, A^{j-1}E)^{1/j} \geq c(A^k, B, A^{k-1}E)^{1/k}$$

for positive integer $j < k$.

Proof. Consider the subgraph of the (A, B, k)-orbit graph induced by the subset

$$B \times \{0\} \sqcup \bigsqcup_{j=1}^{k} (A^j B \backslash A^{j-1} E) \times \{j\}.$$

One may check that this subgraph is indeed commutative (see [**8**]). □

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5. *Countable set of translates*

The inequalities established in §4 required the set of translates $A \subset G$ to be finite. We now turn to extending Theorem 4.2 to the case where A is countable. We use the techniques developed by Björklund and the second author in [**2**].

The following proposition is analogous to [**2**, Proposition 2.2].

PROPOSITION 5.1. *Suppose that G is an abelian group acting on a probability space $(X, \mathcal{B}, \mu)$ and fix a finite $A \subset G$ and non-null $B \in \mathcal{B}$ together with a $0 < \delta < 1$ and positive integers $j \leq k$. If $B' \subset B$ is measurable and satisfies*

$$\left(\frac{\mu(A^k B')}{\mu(B')}\right)^{1/k} \leq (1-\delta)^{-1/j}\left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j}, \tag{12}$$

then $\mu(B') \geq \delta \cdot \mu(B)$ or there exists $B' \subset B'' \subset B$ such that $\mu(B''\backslash B) > 0$ and B'' satisfies (12).

Proof. We note that if the hypothesis holds for $B' = B_1$ and $B' = B_2$ with B_1 and B_2 disjoint, then it holds for $B_1 \sqcup B_2$ since the hypothesis may be rewritten as the inequality

$$\mu(A^k B') \leq (1-\delta)^{-k/j}\left(\frac{\mu(A^j B)}{\mu(B)}\right)^{k/j}\mu(B').$$

By Theorem 4.2 we know that there exists non-null measurable $B' \subset B$ such that (12) is satisfied. Suppose that $\mu(B') < \delta \cdot \mu(B)$, thus we wish to construct a strictly larger $B'' \supset B$ that satisfies (12) and is contained in B. Set $B_0 = B\backslash B'$. We have that

$$\frac{\mu(B_0)}{\mu(B)}(1-\delta)^{-1} > 1$$

and thus there exists $B_0' \subset B_0$ such that

$$\begin{aligned}\left(\frac{\mu(A^k B_0')}{\mu(B_0')}\right)^{1/k} &\leq \left(\frac{\mu(B_0)}{\mu(B)}(1-\delta)^{-1}\right)^{1/j}\left(\frac{\mu(A^j B_0)}{\mu(B_0)}\right)^{1/j} \\ &= (1-\delta)^{-1/j}\left(\frac{\mu(A^j B_0)}{\mu(B)}\right)^{1/j} \\ &\leq (1-\delta)^{-1/j}\left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j},\end{aligned}$$

and thus we may set $B'' = B' \sqcup B_0'$. □

We will now apply the above lemma to construct a set $B' \subset B$ such that $\mu(B') \geq \delta \cdot \mu(B)$ and (12) holds. The idea is to choose a set $B' \subset B$ that satisfies (12) and that is maximal in the sense that B does not contain any measurable $B'' \supset B'$ of strictly larger measure that satisfies (12). Such a set would have to necessarily satisfy $\mu(B') \geq \delta \cdot \mu(B)$. The existence of such a maximal B' follows from the continuity of measure together with the following easy lemma on monotone classes.

LEMMA 5.2. *Suppose that $(X, \mathcal{B}, \mu)$ is a finite measure space and $\mathcal{M} \subset \mathcal{B}$ is non-empty and closed under countable nested unions (that is, if $M_i \in \mathcal{M}$ with $M_i \subset M_{i+1}$ then $\bigcup_{i=1}^{\infty} M_i \in \mathcal{M}$). Then there exists $M \in \mathcal{M}$ such that $\mu(M) = \mu(M')$ for all $M' \in \mathcal{M}$ with $M \subset M'$.*

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Proof. For $M \in \mathcal{M}$ let $s(M) = \sup\{\mu(M') \mid M' \in \mathcal{M}, M \subset M'\}$. Choose $M_1 \in \mathcal{M}$. Now inductively choose $M_{n+1} \in \mathcal{M}$ such that $M_n \subset M_{n+1}$ and $\mu(M_{n+1}) \geq (\mu(M_n) + s(M_n))/2$. Let $M = \bigcup_{n=1}^{\infty} M_n$. We claim that $\mu(M) = s(M)$. To see this, note that $\mu(M_n) \to \mu(M)$ and $s(M_n) \geq s(M)$. Thus

$$\begin{aligned}\mu(M) &= \lim_{n\to\infty} \mu(M_{n+1}) \geq \limsup_{n\to\infty} \frac{\mu(M_n) + s(M_n)}{2} \\ &\geq \limsup_{n\to\infty} \frac{\mu(M_n) + s(M)}{2} = \frac{\mu(M) + s(M)}{2}\end{aligned}$$

and thus $\mu(M) \geq s(M)$ as required. □

If we set $\mathcal{M} = \{B' \subset B \mid B' \text{ satisfies (12)}\}$ then we see that $\mathcal{M}$ is non-empty by Theorem 4.2 and is closed under countable nested unions by the continuity of measure. Thus by the discussion above we obtain a $B' \subset B$ such that $\mu(B') \geq \delta \cdot \mu(B)$ and (12) holds. Consequently we have shown the following lemma.

LEMMA 5.3. *Suppose that G is an abelian group acting on a probability space $(X, \mathcal{B}, \mu)$ and fix a finite $A \subset G$ and non-null $B \in \mathcal{B}$ together with a $0 < \delta < 1$ and positive integers $j \leq k$. Then*

$$c_\delta(A^k, B)^{1/k} \leq (1-\delta)^{-1/j} \left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j}.$$

We may now obtain our first result about the case where $A \subset G$ is not necessarily finite.

LEMMA 5.4. *Suppose that G is an abelian group acting on a probability space $(X, \mathcal{B}, \mu)$ and fix a (not necessarily finite) set $A \subset G$ and non-null $B \in \mathcal{B}$ together with a $0 < \delta < 1$ and positive integers $j \leq k$. Then*

$$\sup\{c_\delta(A', B)^{1/k} \mid A' \subset A^k, A' \text{ is finite}\} \leq (1-\delta)^{-1/j} \left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j}.$$

Proof. If $A' \subset A^k$ is finite then one may choose a finite $A_0 \subset A$ such that $A' \subset A_0^k$. Consequently,

$$c_\delta(A', B)^{1/k} \leq c_\delta(A_0^k, B)^{1/k} \leq (1-\delta)^{-1/j} \left(\frac{\mu(A_0^j B)}{\mu(B)}\right)^{1/j} \leq (1-\delta)^{-1/j} \left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j},$$

and so, as A' was arbitrary, this completes the proof. □

The next non-trivial result due to Björklund and Fish allows us to extend the Plünnecke inequalities for a finite set of translates (Theorem 4.2) to the case of an infinite set of translates.

THEOREM 5.5. [**2**, Proposition 4.1] *Suppose that G is a countable group acting on a probability space $(X, \mathcal{B}, \mu)$ such that $L^2(X, \mathcal{B}, \mu)$ is separable and fix a (not necessarily finite) set $A \subset G$ and non-null $B \in \mathcal{B}$ together with a $0 < \delta < 1$. Then*

$$c(A, B) \leq \sup\{c_\delta(A', B) \mid A' \subset A, A' \text{ is finite}\}.$$

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THEOREM 5.6. (Plünnecke inequalities for an infinite set of translates) *Suppose that G is a countable abelian group acting on a probability space $(X, \mathcal{B}, \mu)$ such that $L^2(X, \mathcal{B}, \mu)$ is separable and fix a (not necessarily finite) set $A \subset G$ and non-null $B \in \mathcal{B}$ together with a $0 < \delta < 1$ and positive integers $j \leq k$. Then*

$$c(A^k, B)^{1/k} \leq c(A^j, B)^{1/j} \leq \left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j}.$$

Proof. By the previous two results we obtain for each $\delta > 0$ the inequalities

$$c(A^k, B)^{1/k} \leq \sup\{c_\delta(A', B)^{1/k} \mid A' \subset A^k, A' \text{ is finite}\} \leq (1-\delta)^{-1/j}\left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j}.$$

Taking $\delta \to 0$ gives

$$c(A^k, B)^{1/k} \leq \left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j}.$$

Now applying this to non-null $B_i \subset B$ such that

$$\frac{\mu(A^j B_i)}{\mu(B_i)} \to c(A^j, B)$$

gives

$$c(A^k, B)^{1/k} \leq c(A^k, B_i)^{1/k} \leq \left(\frac{\mu(A^j B_i)}{\mu(B_i)}\right)^{1/j} \to c(A^j, B)^{1/j}$$

as desired. □

6. *Different summands*

Given measure-preserving actions $G \curvearrowright (X, \mathcal{B}, \mu)$ and $G' \curvearrowright (X', \mathcal{B}', \mu')$, one can form the measure-preserving product action $G \oplus G \curvearrowright (X \times X, \mathcal{B} \otimes \mathcal{B}', \mu \times \mu')$ given by $(g, g') \cdot (x, x') = (g \cdot x, g' \cdot x')$. We will now verify that the corresponding multiplication ratios are multiplicative.

LEMMA 6.1. *Suppose that G and G' are countable groups acting on probability spaces $(X, \mathcal{B}, \mu)$ and $(X', \mathcal{B}', \mu')$, respectively. Then for $A \subset G$, $A' \subset G'$ and non-null $B \subset X$, $B' \subset X'$,*

$$c(A, B)c(A', B') = c(A \times A', B \times B').$$

Proof. For non-null $B_0 \subset B$ and $B_0' \subset B'$,

$$\begin{aligned} c(A \times A', B \times B') &\leq \frac{\mu \times \mu'(A \times A' \cdot B_0 \times B_0')}{\mu \times \mu'(B_0 \times B_0')} \\ &= \frac{\mu(A \cdot B_0)}{\mu(B_0)} \frac{\mu(A' \cdot B_0')}{\mu(B_0')}. \end{aligned}$$

By selecting appropriate B_0 and B_0', the right-hand side may be made arbitrarily close to $c(A, B)c(A', B')$ and thus

$$c(A, B)c(A', B') \geq c(A \times A', B \times B').$$

We now aim to show the reverse inequality. For $U \subset X \times X'$ and $(x_0, x_0') \in X \times X'$, let

$$U_{x_0} = \{x' \in X' \mid (x_0, x') \in U\}$$

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and

$$U^{x_0'} = \{x \in X \mid (x, x_0') \in U\}.$$

Also, let $\mathcal{B}(V)$ denote the measurable subsets of V where V is any subset of a measurable space. Define

$$\phi : \mathcal{B}(X \times B') \to \mathcal{B}(X \times X')$$

by

$$\phi(U) = \bigsqcup_{x \in X} \{x\} \times A' U_x = (\{1_G\} \times A') \cdot U.$$

By Fubini's theorem,

$$\begin{aligned}\mu \times \mu'(\phi(U)) &= \int_X \mu'(A' U_x)\, d\mu(x) \\ &\geq \int_X c(A', B') \mu'(U_x)\, d\mu(x) \\ &= c(A', B') \int_X \mu'(U_x)\, d\mu(x) \\ &= c(A', B') \cdot \mu \times \mu'(U).\end{aligned}$$

Thus $\mu \times \mu'(\phi(U)) \geq c(A', B') \cdot \mu \times \mu'(U)$ for $U \subset X \times B'$. We may reverse the role of coordinates to obtain a similar inequality, from which we finally get that

$$\begin{aligned}\mu \times \mu'((A \times A') \cdot U) &= \mu \times \mu'((\{1_G\} \times A')((A \times \{1_{G'}\}) \cdot U)) \\ &\geq c(A', B') \cdot \mu \times \mu'((A \times \{1_{G'}\}) \cdot U) \\ &\geq c(A', B') c(A, B) \cdot \mu \times \mu'(U)\end{aligned}$$

for $U \subset B \times B'$. This implies that $c(A, B)c(A', B') \leq c(A \times A', B \times B')$, as required. □

PROPOSITION 6.2. *Suppose that G is a countable abelian group acting on a probability space $(X, \mathcal{B}, \mu)$. Then, for $A_1, A_2, \ldots, A_k \subset G$ and non-null $B \in \mathcal{B}$,*

$$c(A_1 \cdots A_k, B) \leq \prod_{i=1}^{k} \frac{\mu(A_i B)}{\mu(B)}.$$

Proof. Choose rational numbers

$$\alpha_i > \frac{\mu(A_i B)}{\mu(B)}$$

and choose $n \in \mathbb{Z}_{>0}$ such that, for each $i \in \{1, \ldots, k\}$,

$$n_i := \frac{n}{\alpha_i} \in \mathbb{Z}_{>0}.$$

Suppose that there exists $T_i \subset G$ with $|T_i| = n_i$ such that the map

$$\begin{aligned}T_1 \times \cdots \times T_k \times (A_1 A_2 \cdots A_k B) &\to X \\ (t_1, \ldots, t_k, y) &\mapsto t_1 \cdots t_k y\end{aligned}$$

is injective. We may assume, without loss of generality, that we are in this case by naturally embedding $G \hookrightarrow G \oplus \mathbb{Z}/N\mathbb{Z}$ and $X \hookrightarrow X \times \mathbb{Z}/N\mathbb{Z}$ and replacing the measure-preserving

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system $G \curvearrowright X$ with the product measure-preserving system $G \oplus \mathbb{Z}/N\mathbb{Z} \curvearrowright X \times \mathbb{Z}/N\mathbb{Z}$, for large enough N. Let $A = \bigcup_{i=1}^{k} A_i T_i$ and notice that

$$\mu(AB) \leq \sum_{i=1}^{k} \mu(A_i T_i B) \leq \sum_{i=1}^{k} n_i \cdot \mu(A_i B) < \mu(B) \sum_{i=1}^{k} n_i \alpha_i = k \cdot n \cdot \mu(B)$$

and thus, by Theorem 5.6, we obtain non-null $B' \subset B$ such that

$$\mu(A^k B') \leq (k \cdot n)^k \mu(B').$$

However, by the injection above,

$$\mu(A^k B') \geq \mu(T_1 \cdots T_k A_1 A_2 \cdots A_k B') = \left(\prod_{i=1}^{k} n_i \right) \mu(A_1 \cdots A_k B').$$

Combining the previous two inequalities gives

$$c(A_1 \cdots A_k, B) \leq \frac{\mu(A_1 \cdots A_k B')}{\mu(B')} \leq k^k \prod_{i=1}^{k} \alpha_i.$$

Since the $\alpha_i > \mu(A_i B)/\mu(B)$ were arbitrary rational numbers, we obtain

$$c(A_1 \cdots A_k, B) \leq k^k \prod_{i=1}^{k} \frac{\mu(A_i B)}{\mu(B)}. \tag{13}$$

We now wish to remove the k^k constant. This may be done by considering a large cartesian power as follows. Let us denote $V^{\times m} = V \times V \cdots \times V$ and $V^{\oplus m} = V \oplus V \cdots \oplus V$, where m factors are present. For each positive integer m, an application of (13) and Lemma 6.1 to the sets $A_1^{\oplus m} \cdots A_k^{\oplus m} = (A_1 A_2 \cdots A_k)^{\oplus m} \subset G^{\oplus m}$ and $B^{\times m} \subset X^{\times m}$ gives

$$c(A_1 \cdots A_k, B) \leq k^{k/m} \prod_{i=1}^{k} \frac{\mu(A_i B)}{\mu(B)}.$$

Taking the limit $m \to \infty$ gives the desired result. □

7. *Correspondence principle for product sets*

We will now establish a Furstenberg correspondence principle for product sets. The first appearance of the correspondence principle for product sets was in [**1**]. The principle appearing in this paper is due to Björklund and Fish [**3**]. Given a countable group G and $B \subset G$, we define the Furstenberg *G-system corresponding to B* to be the topological G-system $G \curvearrowright X$; that is, G acts on compact metric space X by homeomorphisms, given by the following construction. Let $X_0 = \{0, 1\}^G$ be the space of all sequences indexed by G equipped with the product topology. Let $z \in \{0, 1\}^G$ be the indicator function of B, that is, $z_g = 1$ if and only if $g \in B$. Note that there is a natural action of G on X_0 given by

$$(g \cdot x)_h = x_{gh}$$

for $g, h \in G$ and $(x_k)_{k \in G} \in X_0$.

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Let $X = \overline{Gz} = \overline{\{gz \mid g \in G\}}$ be the closure of the orbit of z. Note that X is G-invariant. This defines the system corresponding to B. Moreover, we define the *clopen set corresponding to* B to be the set

$$\widetilde{B} = \{x \in X \mid x_1 = 1\},$$

which is a clopen subset of X.

LEMMA 7.1. *Suppose that G is a countable abelian group, $B \subset G$ and X is the G-system corresponding to B. Suppose that μ is a G-invariant Borel probability measure on X. Then, for finite $A_0 \subset G$,*

$$d^*(A_0B) \geq \mu(A_0\widetilde{B}) \geq d_*(A_0B).$$

Proof. Note that it suffices to prove this for μ ergodic by either of the following arguments. Suppose that the result holds for all ergodic μ, and thus holds for all convex combinations of ergodic measures. It is a well-known fact that the extreme G-invariant measures are precisely the ergodic measures. The Krein–Milman theorem therefore implies that any G-invariant probability measure is in the weak* closure of the set of all convex combinations of ergodic measures. Since the map $\nu \mapsto \nu(A_0\widetilde{B})$ is weak* continuous (since $A_0\widetilde{B}$ is clopen), we obtain the result.

Alternatively, one may use Bauer's maximum principle (instead of the Krein–Milman) which says that the maximum (respectively, minimum) of the the map $\nu \mapsto \nu(A_0\widetilde{B})$ is attained at an extremal (hence ergodic) measure, say μ^* (respectively, μ_*), and thus, for any G-invariant μ,

$$d_*(A_0B) \leq \mu_*(A_0\widetilde{B}) \leq \mu(A_0\widetilde{B}) \leq \mu^*(A_0\widetilde{B}) \leq d^*(A_0B).$$

Now we turn to the proof of the lemma under the assumption that μ is ergodic. Given any Følner sequence $(F_n)_{n\in\mathbb{N}}$ and continuous $f \in C(X)$ we have, by the von Neumann mean ergodic theorem, that

$$\frac{1}{|F_N|}\sum_{g\in F_N} f \circ g \to \int f \, d\mu \tag{14}$$

in the L^2-norm. We may then, by the Borel–Cantelli lemma, pass to a subsequence of $(F_n)_{n\in\mathbb{N}}$ to obtain almost everywhere pointwise convergence in (14). In particular, we may apply this result to $f = \chi_{A_0\widetilde{B}}$ and get a Følner sequence $(F_n)_{n\in\mathbb{N}}$ such that

$$\frac{1}{|F_N|}\sum_{g\in F_N} \chi_{A_0\widetilde{B}}(gx) \to \mu(A_0\widetilde{B})$$

for some $x \in X$. Now fix $N \in \mathbb{N}$ and note that since $X = \overline{Gz}$ we have $h_i z \to x$ for some $h_i \in G$ and thus $\chi_{A_0\widetilde{B}}(gh_i z) \to \chi_{A_0\widetilde{B}}(gx)$ for each $g \in F_N$. Therefore, for some large M, we have for $q_N := h_M$ the equality

$$\frac{1}{|F_N|}\sum_{g\in F_N} \chi_{A_0\widetilde{B}}(gx) = \frac{1}{|F_N|}\sum_{g\in F_N} \chi_{A_0\widetilde{B}}(gq_N z) = \frac{|q_N F_N \cap A_0 B|}{|q_N F_N|},$$

where the second equality is obtained from the fact that, by construction of the corresponding system and clopen set, $gq_N z \in A_0\widetilde{B}$ if and only if $gq_N \in A_0B$. Since $(q_N F_N)_{N\in\mathbb{N}}$ is a Følner sequence, the limit (as $N \to \infty$) of this quantity must be between $d_*(A_0B)$ and $d^*(A_0B)$. □

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In fact, notice that the inequality $d^*(A_0B) \geq \mu(A_0\widetilde{B})$ in the previous lemma is also true for infinite A_0 since we can always write A_0 as an increasing union $A_1 \subset A_2 \subset \cdots$ of finite sets and thus

$$d^*(A_0B) \geq d^*(A_kB) \geq \mu(A_k\widetilde{B}) \to \mu(A\widetilde{B}) \quad \text{as } k \to \infty.$$

PROPOSITION 7.2. (Correspondence principle for product sets [**2**]) *Suppose that G is a countable abelian group and $A, B \subset G$. Then there exists a compact metrizable space X on which G acts by homeomorphisms such that there exist G-invariant ergodic Borel probability measures μ, ν on X together with a clopen $\widetilde{B} \subset X$ such that*

$$d^*(B) = \mu(\widetilde{B}),$$
$$d^*(AB) \geq \mu(A\widetilde{B}),$$
$$d_*(B) \leq \nu(\widetilde{B}),$$
$$d_*(AB) \geq \nu(A\widetilde{B}).$$

Proof. The space X and clopen set $\widetilde{B}$ will be those coming from the correspondence. Note that the second and third inequalities are satisfied for all μ, ν by the lemma above. Moreover, this lemma shows that $d^*(B) \geq \mu(\widetilde{B})$, for all μ. Therefore to construct μ satisfying the first equality, it is enough to construct a not necessarily ergodic μ and then apply Bauer's maximum principle. Let z be as in the construction of the correspondence. Choose a Følner sequence $F_n \subset G$ such that

$$\frac{|F_n \cap B|}{|F_n|} \to d^*(B).$$

Consider now the averages

$$\mu_n = \frac{1}{|F_n|} \sum_{g \in F_n} \delta_{g \cdot z}$$

of point mass measures, and let $\mu = \lim_{k\to\infty} \mu_{n_k}$ be a weak* limit of a subsequence of these. Since $(F_n)_{n\in\mathbb{N}}$ is Følner, the μ is G-invariant. Note that for $C \subset G$,

$$\mu_n(C\widetilde{B}) = \frac{|F_n \cap CB|}{|F_n|}.$$

In particular, the $C = \{1\}$ case shows that the choice of Følner sequence, together with the fact that $\widetilde{B}$ is clopen, implies that $d^*(B) = \mu(\widetilde{B})$. We now turn to dealing with the final inequality. To construct such a ν, it is enough to construct such a not necessarily ergodic ν by the following argument. The map $\nu \mapsto \nu(A\widetilde{B})$ is weak* lower semi-continuous since $A\widetilde{B}$ is open. Thus Bauer's minimum (but not maximum) principle applies and thus if at least one not necessarily ergodic ν satisfies the final inequality, then some ergodic minimizer does too. Write A as a union of an increasing sequence of finite sets $A_1 \subset A_2 \subset \cdots$ and choose a Følner sequence $E_m \subset G$ such that

$$\frac{|E_m \cap AB|}{|E_m|} \to d_*(AB) \quad \text{as } m \to \infty.$$

As before, the averages

$$\nu_m = \frac{1}{|E_m|} \sum_{g \in E_n} \delta_{g \cdot z}$$

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have a weak* convergent subsequence $\nu_{m_j} \to \nu$. Since each $A_k\widetilde{B}$ is clopen,

$$\begin{aligned} d_*(AB) &= \lim_{j\to\infty} \frac{|E_{m_j} \cap AB|}{|E_{m_j}|} \geq \lim_{j\to\infty} \frac{|E_{m_j} \cap A_k B|}{|E_{m_j}|} \\ &= \lim_{j\to\infty} \nu_{m_j}(A_k\widetilde{B}) = \nu(A_k B) \to \nu(A\widetilde{B}) \quad \text{as } k \to \infty. \end{aligned}$$ □

The following statement was proven in [**2**].

LEMMA 7.3. [**2**] *Suppose that G is a countable abelian group that acts ergodically on a probability space $(X, \mathcal{B}, \mu)$ such that $L^2 = L^2(X, \mathcal{B}, \mu)$ is separable (for instance, a Borel probability space). Then*

$$d^*(A) \leq \mu(AB)$$

for $A \subset G$ and non-null $B \in \mathcal{B}$.

Proof. Choose a Følner sequence $(F_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \frac{|F_n \cap A|}{|F_n|} = d^*(A).$$

Define

$$f_N = \frac{1}{|F_N|} \sum_{g\in F_N} \chi_{g^{-1}AB} \in L^2.$$

As $\|f_N\|_2 \leq 1$ and L^2 is separable (and thus has unit ball compact metrizable in the weak topology), we may pass to a subsequence of $(F_n)_{n\in\mathbb{N}}$ such that f_N converges weakly to some $f \in L^2$. But f is G-invariant and thus constant by ergodicity. Therefore,

$$\mu(AB) = \langle f_N, 1\rangle \to \langle f, 1\rangle = f,$$

and so in fact f is the constant function $\mu(AB)$.

Notice that, for $b \in B$,

$$\begin{aligned} f_N(b) &= \frac{1}{|F_N|}|\{g \in F_N \mid b \in g^{-1}AB\}| \\ &= \frac{1}{|F_N|}|\{g \in F_N \mid gb \in AB\}| \\ &\geq \frac{1}{|F_N|}|\{g \in F_N \mid g \in A\}|, \end{aligned}$$

and so $d^*(A) \leq \liminf_{N\to\infty} f_N(b)$. In other words,

$$\chi_B d^*(A) \leq \chi_B \liminf_{N\to\infty} f_N.$$

Integrating this inequality and applying Fatou's lemma yields

$$\begin{aligned} \mu(B)d^*(A) &\leq \int \chi_B \liminf_{N\to\infty} f_N \\ &\leq \liminf_{N\to\infty} \int \chi_B f_N \\ &= \langle \chi_B, f\rangle \\ &= \mu(B)\mu(AB). \end{aligned}$$

As $\mu(B) > 0$, we obtain the desired inequality. □

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