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A weak Lefschetz result for Chow groups of complete intersections

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Abstract. We introduce a weak Lefschetz-type result on Chow groups of complete intersections. As an application, we can reproduce some of the results in [P]. The purpose of this paper is not to reproduce all of [P] but rather illustrate why the aforementioned weak Lefschetz result is an interesting idea worth exploiting in itself. We hope the reader agrees.

1 Introduction

Let $X/\mathbb{C} := V(F_1, \dots, F_r) \subset \mathbb{P}^{n+r} \ni [z_0, \dots, z_{n+r}]$ be a smooth complete intersection of multidegree $(d_1, \dots, d_r)$ such that $1 \leq d_1 \leq \cdots \leq d_r$ and $Z = V(G_1, \dots, G_r) \subset \mathbb{P}^{n+1+r}$ be the complete intersection of the same multidegree, where

$$G_j = \begin{cases} F_j & \text{if } j < r \\ F_r + z_{n+1+r}^{d_r} & \text{if } j = r \end{cases}. \tag{1}$$

Evidently, Z is also smooth and $X = Z \cap V(z_{n+1+r})$, provided that

$$W := V(F_1, \dots, F_{r-1}) \subset \mathbb{P}^{n+r},$$

is likewise smooth, which we will assume. Observe that $X = W \cap V(F_r)$. There is a commutative diagram, where i and j are the inclusion maps, and v is the projection from $[0, \dots, 0, 1] \in \mathbb{P}^{n+1+r}$:

$$\begin{array}{ccccc} X & -\!\!- & \overset{j}{-\!\!-} & \to & Z \\ & \searrow\, i & & v\, \swarrow & \\ & & W & & \end{array}$$

Let $\mathrm{CH}^r(X, m)$ be the higher Chow group [Blo]. Then, the following is proved in [T] and [S] (also see [Lew3]).

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Proposition 1.1

$$i^* \circ v_* = d_r \cdot j^* : CH^\ell(Z, m) \to CH^\ell(X, m).$$

Note that in the case $r = 1$, $W = \mathbb{P}^{n+1}$, and when $m = 0$, one can view this as a sort of weak Lefschetz theorem for $X \hookrightarrow Z$. In general, W is a complete intersection of "lower order" than X. So using an induction process, we arrive at the following:

Theorem 1.2 *Let $X \subset \mathbb{P}^{n+r}$ be a general complete intersection and $\ell \geq 0$, a given integer. Then, for n sufficiently large, $CH^\ell(X, m; \mathbb{Q}) \simeq CH^\ell(\mathbb{P}^{n+r}, m; \mathbb{Q})$.*

We arrive at a homological analog of Theorem 1.2 as well. In the case $m = 0$, Theorem 1.2 provides some evidence in support of a conjecture of Nori. Indeed, the results of this paper are very similar to those in [P]. The purpose of this paper is *not* to reproduce all of the results in [P] but rather illustrate why Proposition 1.1 is an interesting idea worth exploiting in itself. We hope the reader agrees.

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2 Some ideas of Bloch, Beilinson, and Nori

Throughout this paper, X will be a projective algebraic manifold (= smooth projective variety over $\mathbb{C}$) of dimension n. We expect the results of this paper to be true, if we replace $\mathbb{C}$ by any algebraically closed field. This section draws inspiration from [J2].

Let k be a field and W/k be a quasi-projective variety. We recall Bloch's higher Chow groups $\mathrm{CH}^r(W, m)$ [Blo]. We point out that these higher Chow groups have proper push-forwards, flat pullbacks (contravariance for arbitrary maps between smooth varieties), homotopy invariance, localization sequence, as well as a projective bundle formula (Bloch, *op. cit.*). Note that $\mathrm{CH}^r(W, 0) = \mathrm{CH}^r(W)$ are the classical Chow groups [F]. We put $\mathrm{CH}^r(W, m; \mathbb{Q}) = \mathrm{CH}^r(W, m) \otimes \mathbb{Q}$. It turns out that if W is smooth, then $\mathrm{CH}^r(W, m) \simeq H^{2r-m}_{\mathcal{M}}(W, \mathbb{Z}(r))$, where the latter is Voevodsky's motivic cohomology [MVW].

Let $\mathcal{MM} = \mathcal{MM}(\mathbb{C})$ be the conjectural category of mixed motives, and let us recall Beilinson's conjectural formula:

$$\mathrm{Ext}^v_{\mathcal{MM}}(\mathbf{1}, h^{2r-v-m}(X)(r)) \simeq \mathrm{Gr}^v_F \mathrm{CH}^r(X, m; \mathbb{Q}). \tag{2}$$

(See [J2].) Actually, Beilinson's formula is when $m = 0$. This is an obvious generalization. Here, $\mathbf{1} := \mathrm{Sp}(\mathbb{C})$ is the trivial motive, $h^\bullet$ is motivic cohomology, and $F^\bullet \mathrm{CH}^r(X, m; \mathbb{Q})$ is the conjectured Bloch–Beilinson filtration [As, J1, Lew2] . Besides functoriality, it has the property that $F^N = 0$ for $N >> 1$. Out of simplicity, we will assume that Beilinson's conjectural formula in (2) holds in this section. Let L_X be the operation of taking the cup product with a hyperplane section of X. The hard Lefschetz conjecture states that the inverse to the hard Lefschetz isomorphism is cycle induced. A precise formulation is given in [K].

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Proposition 2.1 *Assume the hard Lefschetz conjecture and the aforementioned filtration* $F^\bullet CH^r(X,m;\mathbb{Q})$ *satisfying (2) with* $F^N = 0$ *for* $N >> 1$. *Then,* $F^{r+1}CH^r(X,m;\mathbb{Q}) = 0$.

Proof Since $F^N \mathrm{CH}^r(X,m;\mathbb{Q}) = 0$ for $N >> 1$, it suffices to show that $\mathrm{Gr}_F^v \mathrm{CH}^r(X,m;\mathbb{Q}) = 0$ for all $v > r$. So, let $v > r$ be given. Note that by dimension reasons alone, $\mathrm{CH}^r(X,m) = 0$ for $r > n + m$. Thus, we can assume that $r \leq n + m$. And so $2r - v - m < r - m \leq n$. The hard Lefschetz conjecture implies the hard Lefschetz isomorphism on the level of motivic cohomology, viz.,

$$L_X^{n-(2r-v-m)} : h^{2r-v-m}(X)(r) \xrightarrow{\sim} h^{2(n-r+v+m)-v-m}(X)(n-r+v+m).$$

Applying $\mathrm{Ext}^v_{\mathcal{MM}}(\mathbf{1}, -)$ and Beilinson's formula, we arrive at

$$L_X^{n-(2r-v-m)} : \mathrm{Gr}_F^v \mathrm{CH}^r(X,m;\mathbb{Q}) \xrightarrow{\sim} \mathrm{Gr}_F^v \mathrm{CH}^{n-r+v+m}(X,m;\mathbb{Q}).$$

But

$$n - r + v + m > n + m;$$

hence,

$$\mathrm{CH}^{n-r+v+m}(X,m;\mathbb{Q}) = 0,$$

a fortiori, $\mathrm{Gr}_F^v \mathrm{CH}^r(X,m;\mathbb{Q}) = 0$, and we are done. ∎

Now, let $j : Y \hookrightarrow X$ be a smooth hyperplane section.

Conjecture 2.2 (Nori)

$$j^* : CH^r(X,m;\mathbb{Q}) \xrightarrow{\sim} CH^r(Y,m;\mathbb{Q}),$$

if $2r < \dim Y + m$.

(Actually, Nori's formulation is when $m = 0$. This is an obvious extension.)

Proposition 2.3 *Same setting as in Proposition 2.1 above. Then, Nori's conjecture holds.*

Proof Note that $2r < \dim Y + m$ implies that $2r - v - m < \dim Y$. Therefore, by the weak Lefschetz, together with the hard Lefschetz conjecture (and making use of Corollary 2.11 in [K], which implies that $(j^*)^{-1}$ is cycle induced), one has an isomorphism

$$j^* : h^{2r-v-m}(X)(r) \xrightarrow{\sim} h^{2r-v-m}(Y)(r),$$

and hence by Beilinson,

$$j^* : \mathrm{Gr}_F^v \mathrm{CH}^r(X,m;\mathbb{Q}) \xrightarrow{\sim} \mathrm{Gr}_F^v \mathrm{CH}^r(Y,m;\mathbb{Q}).$$

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Next, there is a commutative diagram:

$$\begin{array}{ccccc} F^rCH^r(X,m;\mathbb{Q}) & \hookrightarrow & F^{r-1}CH^r(X,m;\mathbb{Q}) & \twoheadrightarrow & \mathrm{Gr}_F^{r-1}CH^r(X,m;\mathbb{Q}) \\ \downarrow\wr & & \downarrow & & \downarrow\wr \\ F^rCH^r(Y,m;\mathbb{Q}) & \hookrightarrow & F^{r-1}CH^r(Y,m;\mathbb{Q}) & \twoheadrightarrow & \mathrm{Gr}_F^{r-1}CH^r(Y,m;\mathbb{Q}) \end{array}$$

The left vertical arrow is an isomorphism, because $F^r = \mathrm{Gr}_F^r$ by Proposition 2.1. By the 5-lemma, the middle vertical arrow is an isomorphism. Now, argue by downward induction. ∎

Thus, assuming the hard Lefschetz conjecture and (2) above, we arrive at:

Corollary 2.4 *Same setting as in Proposition 2.1 above. Let $X \subset \mathbb{P}^{n+r}$ be a smooth complete intersection and $\ell \geq 0$, a given integer. Then, for n sufficiently large, $CH^\ell(X,m;\mathbb{Q}) \simeq CH^\ell(\mathbb{P}^{n+r},m;\mathbb{Q})$.*

We will prove Corollary 2.4 under the sole assumption that X is general. Namely, X is general if it corresponds to a point in a nonempty Zariski open subset of the parameter space of complete intersections of a given multidegree in $\mathbb{P}^{n+r}$, governed by certain generic properties. We should point out that our proof is entirely motivic.

3 Fano varieties

Let $G(k+1,N+1) = \{\mathbb{C}^{k+1}\text{'s} \subset \mathbb{C}^{N+1}\} = \{\mathbb{P}^k\text{'s} \subset \mathbb{P}^N\}$. It is a Grassmannian space of dimension $(k+1)(N-k)$.

Definition 3.1 Let $X \subset \mathbb{P}^n$ be a variety. Then, $\Omega_X(k) = \{\mathbb{P}^k\text{'s} \subset \mathbb{P}^n | \mathbb{P}^k \subset X\} \subset G(k+1,N+1)$ is called the Fano variety of $\mathbb{P}^k$'s in X.

Theorem 3.2 (Borcea [B]) *Let $X \subset \mathbb{P}^{n+r}$ be a general (complex) complete intersection of multidegree $(d_1,\ldots,d_r)$. Then, $\Omega_X(k)$ is nonempty and smooth of pure dimension $\delta = (k+1)(n+r-k) - \sum_{j=1}^r \binom{d_j+k}{k}$, provided $\delta \geq 0$ and X is not a quadric. In the case X is a quadric, we require $n \geq 2k$. Furthermore, if $\delta > 0$ or if in the case X is a quadric, $n > 2k$, then $\Omega_X(k)$ is connected (hence irreducible).*

Corollary 3.3 *Let $Z \subset \mathbb{P}^{n+r+1}$ be a general complete intersection of multidegree $(d_1,\ldots,d_r)$ and assume $l \geq 0$, where $l = k(n+1+r-k) + r - \sum_{j=1}^r \binom{d_j+k}{k}$. Then, through every point of Z, there passes a $\mathbb{P}^k \subset Z$. In particular, there is an l-dimensional family of $\mathbb{P}^k$'s in a general Z passing through a generic point $p \in Z$.*

Proof This is proved in [T] or [S]. Having said this, the case $r = 1$ is proved in [Lew3], and a simple generalization works for $r \geq 1$. ∎

Note that $l = l(n,k,d_1,\ldots,d_r)$. For fixed n and multidegree, we will simply write $l = l(k)$. Next, we introduce the following:

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Definition 3.4 [L-S].

$$P(X) = \{(c, x) \in \Omega_X(k) \times X \mid x \in \mathbb{P}^k_c\},$$

is called the cylinder correspondence. The corresponding homomorphism $\Phi_* := P(X)_* : \mathrm{CH}_{\bullet - k}(\Omega_X(k), m) \to \mathrm{CH}_\bullet(X, m)$ is called the cylinder homomorphism.

Remark 3.5 The question now is how large a value of k can we expect to entertain here. Recall that $1 \le d_1 \le d_2 \le \cdots \le d_r$. We know from [De] that the Hodge level of $H^n(X)$ is given by $n - 2\underline{k}$, where $\underline{k} = \left[\left(n + 1 - \sum_{j=1}^{r-1}(d_j - 1)\right)/d_r\right] = [(n + r - \sum_{j=1}^{r-1} d_j)/d_r]$. Thus, $\underline{k}$ is the largest integer for which $H^n(X, \mathbb{C}) = F^{\underline{k}} H^n(X, \mathbb{C})$. It turns out from some cohomological considerations that $k \le \underline{k}$, and that $l \ge 0$ in Corollary 3.3 does not always hold for $k = \underline{k}$ [Lew3]. However, for any given k subject to $0 \le k \le \underline{k}$, one can find n such that $l \ge 0$.

Let $X \subset \mathbb{P}^{n+r}$ be a general complete intersection of multidegree $(d_1, \ldots, d_r)$ and let $Z \subset \mathbb{P}^{n+r+1}$ be a general complete intersection of multidegree $(d_1, \ldots, d_r)$ such that $X = Z \cap \mathbb{P}^{n+r}$. We can view X as a hyperplane section of Z with the inclusion map $j : X \hookrightarrow Z$. Provided that $l \ge 0$, we have the following:

(i) $\Omega_X(k)$ is nonempty, smooth, and of pure dimension $(n - 2k) + l =: \delta$.
(ii) $\Omega_Z(k)$ is nonempty, smooth, and of pure dimension $(n + 1 - k) + l$.
(iii) Through a generic point $p \in Z$, there passes an l-dimensional family of $\mathbb{P}^k$'s.

Now, let Ω_Z be a subvariety cut out by l general hyperplane sections of $\Omega_Z(k)$ and define $\Omega_X = \Omega_X(k) \cap \Omega_Z$. By Bertini's theorem, we can assume:

(iv) Ω_Z is smooth and irreducible of dimension $n + 1 - k$.
(v) Ω_X is smooth and of pure dimension $n - 2k$.

We have a commutative diagram, where the π's and ρ's are projections:

(3)

$$\begin{array}{ccccc}
P(X) & \overset{\pi_X}{\dashrightarrow} & & \to & X \\
\big\downarrow & \searrow & & \pi \nearrow & \searrow j \\
\big\downarrow & & \tilde{X} & & Z \\
\rho_X \big\downarrow & & \big\downarrow & \searrow & \pi_Z \nearrow \\
\big\downarrow & & \rho \big\downarrow & & P(Z) \\
\big\downarrow & & \big\downarrow & \swarrow \rho_Z & \\
\Omega_X & \hookrightarrow & \Omega_Z & &
\end{array}$$

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We have:

(a) $\tilde{X} := \pi_Z^{-1}(X)$ is smooth by the second theorem of Bertini. ($\tilde{X}$ is a general member of a linear system on $P(Z)$ with no base points, obtained by the pullback of a linear system on Z to $P(Z)$.)

(b) π and π_Z are generically finite to one and onto of degree q say.

(c) $\rho_X : P(X) \to \Omega_X$ and $\rho_Z : P(Z) \to \Omega_Z$ are $\mathbb{P}^k$ bundles. (Observe that $\rho_X^{-1}(c) = \mathbb{P}_c^k$, since $\rho_X^{-1}(c) = \{(c, p) \mid p \in \mathbb{P}_c^k\}$.)

(d) $\dim X = \dim \tilde{X} = n$, $\dim Z = \dim P(Z) = n+1$, $\dim P(X) = n-k$, $\dim \Omega_X = n-2k$,and $\dim \Omega_Z = n-k+1$. (Note that all varieties here are smooth.)

We are interested in the cylinder homomorphism $\Phi_* : \mathrm{CH}^{\bullet-k}(\Omega_X, m; \mathbb{Q}) \to \mathrm{CH}^{\bullet}(X, m; \mathbb{Q})$. Our *key* point is that it turns out [T, S, Lew3] by an analysis of (3) that

$$(4) \qquad \Phi_* = \pi_{X,*} \circ \rho_X^* : \mathrm{CH}^{\bullet-k}(\Omega_X, m; \mathbb{Q}) \to \mathrm{CH}^{\bullet}(X, m; \mathbb{Q}) \big/ j^* \mathrm{CH}^{\bullet}(Z, m; \mathbb{Q})$$

is *surjective*. What we need to know is $j^*\mathrm{CH}^{\bullet}(Z, m; \mathbb{Q})$. To this end, what we really need is some sort of weak Lefschetz result for j^* of Chow groups, which brings us back to Proposition 1.1. This will require an appropriate choice of Z, following the prescription in (1).

4 Prescription for Z

Again, we are given $X = V(F_1, \ldots, F_r) \subset \mathbb{P}^{n+r}$, a general complete intersection of multidegree $(d_1, \ldots, d_r)$, viz., F_j homogeneous of degree d_j. Let us put

$$Z := V(G_1, \ldots, G_r) \subset \mathbb{P}^{n+1+r},$$

where $G_i = F_i$ for all $i = 1, \ldots, r-1$ and $G_r = F_r + z_{n+r+1}^{d_r}$. Obviously, $X = Z \cap V(z_{n+1+r})$. It is easy to check that Z is smooth.

Proposition 4.1 [T,S,Lew3] *$\Omega_Z(k)$ is smooth and of pure dimension $(n+1-k)+l$, where we assume that $l \geq 0$ (l being defined earlier). Similarly, $\tilde{X}$ is smooth of dimension n.*

Thus, we may now assume $\dim \Omega_Z = n-k+1$. Now, consider the projection from $p := [0, \ldots, 0, 1] \in \mathbb{P}^{n+1+r}$ and $\nu : Z \to \mathbb{P}^{n+r}$ (explicit, $\nu[p_0, \ldots, p_{n+r+1}]) = [p_0, \ldots, p_{n+r}]$). Let $W = \nu(Z)$. Note that $p = [p_0, \ldots, p_{n+r+1}] \in Z$ means $F_i(p_0, \ldots, p_{n+r}) = 0$ for $i = 1, \ldots, r-1$ and $F_r(p_0, \ldots, p_{n+r}) + p_{n+r+1}^{d_r} = 0$. There are two possibilities:

1. $p_{n+r+1} = 0 \Rightarrow [p_0, \ldots, p_{n+r}] \in X$.
2. $p_{n+r+1} \neq 0 \Rightarrow [p_0, \ldots, p_{n+r+1}] \notin X$.

Hence, $X \subset W$ and $W = V(F_1, \ldots, F_{r-1})$. Observe that in the case $r = 1$, $W = \mathbb{P}^{n+1}$. Now, consider the inclusion maps $j : X \hookrightarrow Z$ and $i : X \hookrightarrow W$.

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Proposition 4.2 *Let X, Z, and W be given as above. Then, the following diagram is commutative:*

$$\begin{array}{ccc} CH^\bullet(Z,m) & \overset{d_r\cdot j^*}{\dashrightarrow} & CH^\bullet(X,m) \\ & {\scriptstyle \nu_*}\searrow \quad \nearrow{\scriptstyle i^*} & \\ & CH^\bullet(W,m). & \end{array}$$

Proof We have to show that $i^*\nu_* = d_r \cdot j^*$: Let $p \in W$, we can see that $\nu^{-1}(p) = p$ if $p \in i(X)$ and d_r distinct points if $p \notin i(X)$ so $\deg(\nu) = d_r$. Note that $(gr(i))^T \circ gr(\nu) = (gr(j))^T$ as sets in $Z \times X$ and $(gr(j))^T$ is irreducible since X is. So, as varieties, there are multiplicities: $i^* \circ \nu_* = \ell \cdot j^*$ for some $\ell \in \mathbb{N}$. To show that $\ell = d_r$, let us consider

$$\ell \cdot X = \langle \ell \cdot j^*(Z), X\rangle_X = \langle i^* \circ \nu_*(Z), X\rangle_X$$

$$= \langle i^*(\deg(\nu)W), X\rangle_X = d_r\langle i^*W, X\rangle_X = d_r \cdot X.$$

Thus, $\ell = d_r$. ∎

5 The main application

We prove the following.

Theorem 5.1 *Let $X \subset \mathbb{P}^{n+r}$ be a general complete intersection and $\ell \geq 0$, a given integer. Then, for n sufficiently large, $CH^\ell(X,m;\mathbb{Q}) \simeq CH^\ell(\mathbb{P}^{n+r},m;\mathbb{Q})$.*

Proof First of all, we are given fixed choices of $r \geq 1$, multidegree, and $\ell \geq 0$. We will proceed by induction on r. We choose k such that $\ell - k < 0$. Now, given W in Proposition 4.2, we choose n large enough so that $\mathrm{CH}^\ell(W,m;\mathbb{Q}) \simeq \mathrm{CH}^\ell(\mathbb{P}^{n+r},m;\mathbb{Q})$ (using induction on r), and well as $l \geq 0$ (see Remark 3.5). Next, combining (4) with Proposition 4.2, we arrive at

$$\Phi_* : \mathrm{CH}^{\ell-k}(\Omega_X,m;\mathbb{Q}) \to \mathrm{CH}^\ell(X,m;\mathbb{Q})/i^*\mathrm{CH}^\ell(W,m;\mathbb{Q})$$

is surjective. For a complex variety $S/\mathbb{C}$, let $\lambda_S : S \to \mathrm{Sp}(\mathbb{C})$ be the structure map. Now, since n is sufficiently large, we can certainly assume that $n > \ell$. We have the projective bundle formula [Blo]

$$\mathrm{CH}^\ell(\mathbb{P}^{n+r},m) = \bigoplus_{q=0}^{\ell} \lambda^*_{\mathbb{P}^{n+r}}\mathrm{CH}^{\ell-q}(\mathrm{Sp}(\mathbb{C}),m) \cup L^q_{\mathbb{P}^{n+r}} \simeq \bigoplus_{q=0}^{\ell} \mathrm{CH}^{\ell-q}(\mathrm{Sp}(\mathbb{C}),m).$$

Applying i^*, we arrive at

$$\sum_{q=0}^{\ell} \lambda^*_X \mathrm{CH}^{\ell-q}(\mathrm{Sp}(\mathbb{C}),m) \cup L^q_X.$$

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Now, let $f : X \to \mathbb{P}^n$ be a finite morphism. Notice that $\lambda_X = \lambda_{\mathbb{P}^n} \circ f$. Tensoring with $\mathbb{Q}$ and applying f_*, together with $\lambda_X^* = f^* \circ \lambda_{\mathbb{P}^n}^*$ and the projection formula, we end up with

$$f_*\left(\sum_{q=0}^{\ell} \lambda_X^* \mathrm{CH}^{\ell-q}(\mathrm{Sp}(\mathbb{C}), m) \cup L_X^q\right) =$$

$$\mathrm{CH}^{\ell}(\mathbb{P}^n, m; \mathbb{Q}) = \bigoplus_{q=0}^{\ell} \lambda_{\mathbb{P}^n}^* \mathrm{CH}^{\ell-q}(\mathrm{Sp}(\mathbb{C}), m; \mathbb{Q}) \cup L_{\mathbb{P}^n}^q \simeq \bigoplus_{q=0}^{\ell} \mathrm{CH}^{\ell-q}(\mathrm{Sp}(\mathbb{C}), m; \mathbb{Q}).$$

We deduce then that i^* is injective. ∎

Remark 5.2 The above proof uses the fact that $\mathrm{CH}^{\bullet<0}(\Omega_X, m) = 0$. If we use the fact that $\mathrm{CH}^{\bullet>n+m-2k}(\Omega_X, m) = 0$, then we get a corresponding homological statement. In particular,

Proposition 5.3 *If n is sufficiently large, then*

$$CH_\ell(X, m; \mathbb{Q})/CH_{\ell+r}(\mathbb{P}^{n+r}, m; \mathbb{Q}) = 0.$$

Proof Let $\ell \geq 0 \in \mathbb{Z}$ be given. In this case, $\Phi_* : \mathrm{CH}_{\ell-k}(\Omega_X, m; \mathbb{Q}) \to \mathrm{CH}_\ell(X, m; \mathbb{Q})$ is the cylinder homomorphism. If we choose $k > \ell$, then $\mathrm{CH}_{\ell-k}(\Omega_X, m) = 0$. Next, if n is sufficiently large, then as in the proof of Theorem 5.1, $\mathrm{CH}_\ell(X, m; \mathbb{Q})/\mathrm{CH}_{\ell+r}(\mathbb{P}^{n+r}, m; \mathbb{Q}) = 0$. ∎

Remark 5.4 If k is given such that $l = l(k) \geq 0$ for X, then for that *same* k, $l(k) \geq 0$ for W. This brings us to our final result.

Proposition 5.5 *If k is given such that $l = l(k) \geq 0$, then $CH^{k+1}(X; \mathbb{Q})$ is representable and $CH^k(X; \mathbb{Q})$ is finite dimensional; moreover, $CH^{k+1}(X, 1; \mathbb{Q})$ is decomposable, viz., the intersection product*

$$CH^1(X, 1; \mathbb{Q}) \otimes CH^k(X, 0; \mathbb{Q}) \xrightarrow{\cup} CH^{k+1}(X, 1; \mathbb{Q})$$

is surjective.

Proof The former part of Proposition 5.5 is an exercise for the reader. The latter part hinges on:

Lemma 5.6 *Let $f : Z_1 \to Z_2$ be a morphism of smooth projective varieties over $\mathbb{C}$. Then, $f_*(CH^\ell_{dec}(Z_1, m))$ is decomposable. (Note that $f^*(CH^\ell_{dec}(Z_2, m))$ is trivially decomposable.)*

Proof Note that $\mathrm{CH}^1(Z_j, 1) = \mathbb{C}^\times$ and $f^* : \mathrm{CH}^1(Z_2, 1) \to \mathrm{CH}^1(Z_1, 1)$ is the identity under this identification. So, consider $\beta_1 \cup \gamma_1$, where $\gamma_1 \in \mathrm{CH}^1(Z_1, 1)$ and $\beta_1 \in \mathrm{CH}^{\ell-1}(Z_1, 0)$. Then, $\gamma_1 = f^*(\gamma_2)$ for some $\gamma_2 \in \mathrm{CH}^1(Z_2, 1)$. Then, by the projection formula,

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$$f_*(\beta_1 \cup \gamma_1) = f_*(\beta_1 \cup f^*(\gamma_2)) = f_*(\beta_1) \cup \gamma_2,$$

and we are done. ∎

Now, assuming $l = l(k) \geq 0$, we have a surjection

$$\Phi_* = \pi_{X,*} \circ \rho_X^* : \mathrm{CH}^1(\Omega_X, 1; \mathbb{Q}) \to \mathrm{CH}^{k+1}(X, 1; \mathbb{Q})/i^*\mathrm{CH}^{k+1}(\mathbb{P}^{n+r}, 1; \mathbb{Q}).$$

By the projective bundle formula, $i^*\mathrm{CH}^{k+1}(\mathbb{P}^{n+r}, 1; \mathbb{Q})$ is decomposable. Furthermore, $\mathrm{CH}^1(\Omega_X, 1; \mathbb{Q})$ is by definition decomposable, hence so is $\rho_X^*\mathrm{CH}^1(\Omega_X, 1; \mathbb{Q})$. We now apply the lemma to $\pi_{X,*}$. ∎

Remark 5.7 A natural question arises, and raised by the referee, as to what extent do we need to work with $\mathbb{Q}$-coefficients for the Chow groups, especially if one is willing to deal with torsion issues? The answer is, not so much, if one is willing to invert bounded denominators. From Proposition 4.2, there is the denominator value of d_r. But, this is dependent on an inductive process (as indicated in the proof of Theorem 5.1), and which will also involve $d_1, \ldots, d_{r-1}$. Also, there is the degree of π in (3)(b), which is easy to compute, and is also dependent on an inductive process. These ideas and more will be considered in future work, and thus this line of enquiry will not be pursued here.

References

[As] M. Asakura, *Motives and algebraic de Rham cohomology*. In: B. Gordon, J. Lewis, S. Müller-Stach, S. Saito, and N. Yui (eds.), The arithmetic and geometry of algebraic cycles, Proceedings of the CRM Summer School, June 7–19, 1998, Banff, AB, Canada, NATO Science Series, 548, Kluwer Academic Publishers, 2000.

[Blo] S. Bloch, *Algebraic cycles and higher K-theory*. Adv. Math. **61**(1986), 267–304.

[B] C. Borcea, *Deforming varieties of k-planes of projective complete intersections*. Pac. J. Math. **143**(1990), no. 1, 25–36.

[De] P. Deligne, *Cohomologie des intersections completes*. SGA 7 (II), Exp. XI, Lecture Notes in Mathematics, 340, Springer, Heidelberg-New York-Berlin. 1973, pp. 39–61.

[F] W. Fulton, *Intersection theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3 Folge, Band 2, 1984, 470 pp.

[J1] U. Jannsen, *Mixed motives and algebraic K-theory*. Lecture Notes in Mathematics, 1400, Springer-Verlag, Heidelberg-New York-Berlin, 1990.

[J2] U. Jannsen, *Motivic sheaves and filtrations on Chow groups*. In: Proceedings of Symposia in Pure Mathematics, 55, 1994, pp. 245–302.

[K] S. Kleiman, *Algebraic cycles and the Weil conjectures*. In: Dix Exposés sur la cohomologie des schemes. North-Holland, Amsterdam, Masson, Paris, 1968, pp. 359–386.

[Lew3] J. D. Lewis, *Cylinder homomorphisms and Chow groups*. Math. Nachr. **160**(1993), 205–221.

[Lew2] J. D. Lewis, *A filtration on the Chow groups of a complex projective variety*. Compos. Math. **128**(2001), 299–322.

[L-S] J. D. Lewis and A. S. Sertöz, *Motives of Fano varieties*. Math. Z. **261**(2009), 531–544.

[MVW] C. Mazza, V. Voedvodsky, and C. Weibel, *Lecture notes on motivic cohomology*. Clay Math. Monogr. **2**(2006), 216 pages.

[P] K. Paranjape, *Cohomological and cycle-theoretic connectivity*. Ann. Math. **139**(1994), 641–660.

[S] J. Shtayat, *Nori's conjecture revisited*. Ph.D. thesis, University of Alberta, Canada, 2018, 78 pp.

[T] S. Tuncer, *Representability of algebraic Chow groups of complex projective complete intersections and applications to motives*. Ph.D. thesis, University of Alberta, Canada, 2011, 78 pages.

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