J. K-Theory **12** (2013), 69–98
doi:10.1017/is013004029jkt226
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K-cycles for twisted K-homology

Dedicated to Aderemi Kuku
on the occasion of his seventieth birthday.

by

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Abstract

We summarise the construction of geometric cycles and their use in describing the Kasparov K-homology of a CW-complex X. When Kasparov K-homology is twisted by a degree three element of the Čech cohomology of X then there is a corresponding construction of twisted geometric cycles for the case where X is a smooth manifold however the method that was employed does not apply in the case of CW-complexes. In this article we propose a new approach to the construction of twisted geometric cycles for CW-complexes motivated by the study of D-branes in string theory.

Key Words: D-branes, twisted K-theory

Mathematics Subject Classification 2000: Primary 19L50, 19K35, 19K36.

1. Introduction

K-homology is the dual to K-theory [1, 2]. For a locally finite CW-complex X, there are three equivalent ways to define its (compactly supported) K-homology:

(1) *homotopy theory.* K-homology is the homology theory determined by the Bott (i.e., K-theory) spectrum [2, 17, 25].

(2) *functional analysis.* K-homology is the compactly-supported Kasparov group $KK_c^*(C_0(X),\mathbb{C})$ [2, 14, 15, 16].

(3) *K-cycles.* K-homology is the group $K_*^{top}(X)$ of K-cycles modulo equivalence on X [4, 5, 6].

In this note a 'twist' will be interpreted as a locally trivial bundle $\mathcal{A}$ of C^*-algebras over X with fiber an elementary C^*-algebra or equivalently as a principal $P\mathcal{U}(H)$-bundle $\mathcal{P}_\alpha$ over X, where H is a infinite dimensional complex Hilbert

*PB was partially supported by NSF grant DMS-07011184. ALC and BLW thank the Australian Research Council for support. ALC also thanks the Alexander von Humboldt Stiftung and colleagues at the University of Muenster.

https://doi.org/10.1017/is013004029jkt226 Published online by Cambridge University Press

space and α is the Dixmier-Douady invariant of $\mathcal{A}$. The relationship between these points of view is that given $\mathcal{P}_\alpha$ there is an associated bundle of compact operators, denoted by $\mathcal{P}_\alpha(\mathcal{K}(H))$, with structure group the automorphism group of the compact operators, namely $P\mathcal{U}(H)$.

There are three definitions of twisted K-homology in [24]:

(1) *homotopy theory.* Topological twisted K-homology $K^{hom}(X,\mathcal{P}_\alpha)$ is the homology theory determined by the bundle of Bott (i.e. K-theory) spectra associated to $\mathcal{P}_\alpha$.

(2) *functional analysis.* Analytic twisted K-homology is the Kasparov group $KK^*(\Gamma(X,\mathcal{P}_\alpha(\mathcal{K}(H))),\mathbb{C})$ where $\Gamma(X,\mathcal{P}_\alpha(\mathcal{K}(H)))$ is the C^*-algebra of continuous sections of the bundle $\mathcal{P}_\alpha(\mathcal{K}(H))$.

(3) *geometric K-cycles.* Geometric twisted K-homology is the group $K_*^{geo}(X,\mathcal{P}_\alpha)$ of $\mathcal{P}_\alpha$-twisted K-cycles modulo equivalence on X.

There is a natural homomorphism

$$\mu : K_*^{geo}(X,\mathcal{P}_\alpha) \to KK^*(\Gamma(X,\mathcal{P}_\alpha(\mathcal{K}(H))),\mathbb{C})$$

defined by the twisted version of Kasparov's abstract index [14]. The main theorem in [24] is that μ is an isomorphism for any closed *smooth* manifold X. The proof in [24] hinges on establishing the existence of a commutative diagram

$$\begin{array}{ccc} & K_*^{hom}(X,\mathcal{P}_\alpha) & \\ {\scriptstyle\Psi}\swarrow & & \searrow{\scriptstyle\Phi}\ {\scriptstyle\cong} \\ K_*^{geo}(X,\mathcal{P}_\alpha) & \xrightarrow{\quad\mu\quad} & KK^*(\Gamma(X,\mathcal{P}_\alpha(\mathcal{K}(H))),\mathbb{C}) \end{array}$$

and showing that the map Ψ is surjective and the map Φ is an isomorphism. However the arguments in [24] only hold when X is a closed smooth manifold, as the construction of Ψ requires X to be smooth. We remark that Φ can be shown to be an isomorphism for any finite CW-complex X without using Poincaré duality as in [24].

In this note, we study twisted K-homology of any locally finite CW-complex X with a twisting given by a locally trivial bundle $\mathcal{A}$ of elementary C^*-algebras. In Section 4, we introduce K-cycles for the twisted K-homology of X, and define the corresponding twisted K-homology, denoted $K_*^{top}(X,\mathcal{A})$. The first main theorem, stated in Section 4, is to establish the isomorphism of $K_*^{top}(X,\mathcal{A})$ with Kasparov twisted K-homology $KK_c^*(\Gamma(X,\mathcal{A}),\mathbb{C})$ (this notation is all explained in later Sections). The proof of this theorem occupies Sections 5 and 6. Our isomorphism of (twisted) K-cycle K-homology with (twisted) Kasparov K-homology is based on the notion of *normal bordism* which is introduced in Section 5. Even in the

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untwisted case normal bordism appears to give the most simple and direct proof of this isomorphism. Compare [4, 5, 6] and previous use of this notion in [13, 20, 10].

Twisted K-cycles are very closely connected to the D-branes of string theory. In Section 7, we define a notion of D-cycles for $(X,\mathcal{A})$, from which we define the twisted geometric K-homology, denoted $K_*^{geo}(X,\mathcal{A})$. We view D-cycles as a mathematical formulation of the string theoretic notion of a D-brane and hence in Section 8 we introduce the charge of a D-cycle as the element in the twisted K-homology of X determined by the D-cycle. We ask here the question of whether this charge map $h : K_*^{geo}(X,\mathcal{A}) \longrightarrow K_*^{top}(X,\mathcal{A})$ is an isomorphism for any locally finite CW-complex X? Note that $K_*^{geo}(X,\mathcal{A})$ agrees with the geometric K-cycle group defined in [24]. Thus our question amounts to whether the result of [24] for manifolds extends to the general case of CW-complexes. We mention that other authors are considering extensions of the geometric cycle approach to twisted K-homology, see for example [11] where the result in [24] for manifolds is extended to finite CW-complexes with torsion twistings.

We explain in addition the notion of properly supported (rather than compactly supported) D-cycles and twisted K-cycles in the final sections of this paper, and establish the isomorphism between the group of properly supported twisted K-cycles and Kasparov twisted K-homology $KK^j(\Gamma(X,\mathcal{A}),\mathbb{C})$. There we also ask the question: is the group of properly supported D-cycles isomorphic to the group of properly supported twisted K-cycles?

We will return elsewhere to an examination of the answers to the questions posed in these later Sections of the paper.

Acknowledgements. We are grateful to the referee for reading the preliminary version of this note and for valuable suggestions to improve the exposition in the previous version. PB was partially supported by NSF grant DMS-07011184 and AC and BLW were supported by the Australian Research Council. AC also thanks colleagues at the University of Muenster for their hospitality and the Alexander von Humboldt Stiftung for support.

2. Review of the untwisted case

Let X be a locally finite CW-complex. Then X is locally compact and $C_0(X)$ denotes, as usual, the commutative C^*-algebra consisting of all continuous vanishing-at-infinity functions $f : X \to \mathbb{C}$. The compactly-supported Kasparov group, denoted $KK_c^*(C_0(X),\mathbb{C})$, is the direct limit over all compact subsets $\Delta \subset X$ of

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$KK^*(C(\Delta),\mathbb{C})$ and we write this as:

$$KK_c^j(C_0(X),\mathbb{C}) := \varinjlim_{\substack{\Delta\subset X \\ \Delta \text{ compact}}} KK^j(C(\Delta),\mathbb{C}) \qquad j = 0,1.$$

Since X is a CW-complex, this is equal to the direct limit over the finite sub-CW-complexes of X.

Notation. For a complex Hilbert space H, $\mathcal{L}(H)$ is the C^*-algebra of all bounded operators on H.

Definition. 1) Any element in the Kasparov K-homology group $KK^0(C(\Delta),\mathbb{C})$ is given by a 5-tuple $(H_0,\psi_0,H_1,\psi_1,T)$ such that :

- H_0 and H_1 are separable Hilbert spaces.
- $\psi_0 : C(\Delta) \longrightarrow \mathcal{L}(H_0)$ and $\psi_1 : C(\Delta) \longrightarrow \mathcal{L}(H_1)$ are unital $*$-homomorphisms.
- $T : H_0 \longrightarrow H_1$ is a (bounded) Fredholm operator.
- For every $\alpha \in C(\Delta)$ the commutator $T \circ \psi_0(\alpha) - \psi_1(\alpha) \circ T \in \mathcal{L}(H_0,H_1)$ is compact.

2) Any element in the Kasparov K-homology group $KK^1(C(\Delta),\mathbb{C})$ is given by a 3-tuple (H,ψ,T) such that :

- H is a separable Hilbert space.
- $\psi : C(\Delta) \longrightarrow \mathcal{L}(H)$ is a unital $*$-homomorphism.
- $T : H \longrightarrow H$ is a (bounded) self-adjoint Fredholm operator.
- For every $\alpha \in C(\Delta)$ the commutator $T \circ \psi(\alpha) - \psi(\alpha) \circ T \in \mathcal{L}(H)$ is compact.

3) A K-cycle on X is a triple (M,E,φ) with

- M is a compact Spin^c Riemannian manifold without boundary.
- E is a complex vector bundle on M.
- $\varphi : M \longrightarrow X$ is a continuous map.

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Remarks 1. Given an oriented real Euclidean vector bundle F of rank k over a paracompact Hausdorff topological space Y, a Spin^c structure on F is a lift of the oriented frame bundle $P_{SO}(F)$ to a principal $Spin^c(k)$-bundle $P_{Spin^c}(F)$ where

$$1 \to U(1) \to Spin^c(k) \to SO(k) \to 1$$

is the unique (for $k > 2$) non-trivial central extension of $SO(k)$ by $U(1)$. See Appendix D in [19] for an equivalent definition of Spin^c structures (and note that there is a well known modification needed for $k \leq 2$). A real vector bundle with a Spin^c structure is called a Spin^c vector bundle. Spin^c structures are orientation conditions for complex K-theory in the sense that a Spin^c vector bundle is a real vector vector bundle with a given complex spinor bundle or a K-theory Thom class. See Section 4 in [5] and Theorem C.12 in [19] for more discussions of this. In particular, a spinor bundle for F determines an orientation of F.

2. (**Two-out-of-three principle**) Let $0 \to F' \to F \to F'' \to 0$ be a short exact sequence of oriented real vector bundles on a paracompact Hausdorff topological space Y. Then Spin^c structures for any two of the vector bundles determine a Spin^c structure for the third vector bundle.

3. A Spin^c Riemannian manifold is a Riemannian manifold W (perhaps with boundary) whose tangent bundle TW is a Spin^c vector bundle. If W has a boundary ∂W, at each boundary point, the outward normal vector defines a trivial rank 1 real vector bundle $N_{\partial W}$ over ∂W and

$$0 \to T(\partial W) \to TW|_{\partial W} \to N_{\partial W} \to 0$$

is an exact sequence of real Euclidean vector bundles over ∂W. Therefore, the two-out-of-three principle implies that if a Spin^c manifold W has a boundary, then this boundary, ∂W, is a Spin^c manifold in a canonical way.

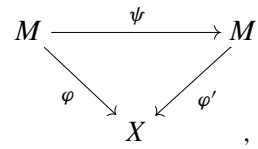
Keeping X fixed, denote by $\{(M, E, \varphi)\}$ the collection of all K-cycles on X. On this collection impose the equivalence relation $\sim$ generated by the three elementary steps:

- bordism,
- direct sum - disjoint union,
- vector bundle modification.

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We remind the reader of the definitions of these notions.

Isomorphism. We say (M,E,φ) is isomorphic to (M',E',φ') if and only if there exists a diffeomorphism $\psi\colon M \to M'$ preserving the Spin^c-structures such that $\psi^*(E') \cong E$ and with commutativity in the diagram



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here ψ is called a Spin^c-preserving diffeomorphism if the differential map $d\psi$: $TM \to TM'$ preserves the Spin^c structures.
Bordism. We say (M_0,E_0,φ_0) is **bordant** to (M_1,E_1,φ_1) if and only if there exists a (W,E,φ) such that:

1. W is a compact Spin^c Riemannian manifold with boundary,
2. E is a complex vector bundle on W,
3. φ is a continuous map from W to X,
4. $(\partial W, E|_{\partial W}, \varphi|_{\partial W}) \cong (M_0,E_0,\varphi_0) \sqcup (-M_1,E_1,\varphi_1)$.

Here $-M_1$ is M_1 with the Spin^c structure reversed (Cf. Section 3 in [6]).
Direct sum - disjoint union. Let E,E' be two complex vector bundles on M, then

$$(M,E,\varphi) \sqcup (M,E',\varphi) \sim (M,E \oplus E',\varphi).$$

Vector bundle modification. Let (M,E,φ) be a K-cycle on X, and let F be a Spin^c vector bundle on M. Assume that

$$\dim_{\mathbb{R}}(F_p) \equiv 0 \mod 2 \quad p \in M$$

for every fiber F_p of F. Let $\mathbf{1}_{\mathbb{R}}$ denote the trivial real line bundle on M. The unit sphere bundle of $F \oplus \mathbf{1}_{\mathbb{R}}$ will be denoted $S(F \oplus \mathbf{1}_{\mathbb{R}})$ and we introduce the projection:

$$\pi : S(F \oplus \mathbf{1}_{\mathbb{R}}) \to M$$

which is a fibration with oriented even-dimensional spheres as fibers. Then there is a complex vector bundle β whose restriction to each fiber of π is the Bott generator vector bundle of that oriented even dimensional sphere (Cf. Section 3 in [6]). Then

$$(M,E,\varphi) \sim (S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^*E, \varphi \circ \pi).$$

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Note that since M is a Spin^c manifold and F is a Spin^c vector bundle, $S(F \oplus \mathbf{1}_{\mathbb{R}})$ is (in a canonical way) a Spin^c manifold.

Two K-cycles (M,E,φ) and (M',E',φ') on X are equivalent if and only if it is possible to pass from (M,E,φ) to (M',E',φ') by a finite sequence of the three elementary steps. The K-cycle (or topological) K-homology of X, denoted by $K_*^{top}(X)$, is the set of equivalence classes of K-cycles:

$$K_*^{top}(X) := \{(M,E,\varphi)\}/\sim .$$

Addition in $K_*^{top}(X)$ is defined by disjoint union of K-cycles :

$$(M,E,\varphi) + (M',E',\varphi') = (M \sqcup M', E \sqcup E', \varphi \sqcup \varphi').$$

The bordism relation implies that the additive inverse of (M,E,φ) is $(-M,E,\varphi)$.

Remark Let (M,E,φ) be a K-cycle on X. We say (M,E,φ) *bounds* if there exists an $(W,\widetilde{E},\widetilde{\varphi})$ such that:

1. W is a compact Spin^c manifold with boundary,
2. $\widetilde{E}$ is a complex vector bundle on W,
3. $\widetilde{\varphi}\colon W \to X$ is a continuous map,
4. $(\partial W, \widetilde{E}|_{\partial W}, \widetilde{\varphi}|_{\partial W}) \cong (M,E,\varphi)$.

Then (M,E,φ) defines the zero element in $K_*(X)$ if and only if $(M,E,\varphi) \sim (M',E',\varphi')$ where (M',E',φ') bounds.

The equivalence relation $\sim$ on K-cycles (M,E,φ) preserves the dimension of M modulo 2. Therefore, as an abelian group, $K_*^{top}(X)$ is the direct sum

$$K_*^{top}(X) = K_0^{top}(X) \oplus K_1^{top}(X)$$

where $K_j^{top}(X)$ is the subgroup of $K_*^{top}(X)$ generated by those K-cycles (M,E,φ) such that every connected component of M has dimension $\equiv j \bmod 2$, $j = 0,1$.

Theorem [4, 5]. *Let X be a locally finite CW-complex. Then the natural map (explained below)*

$$\eta : K_j^{top}(X) \longrightarrow KK_c^j(C_0(X),\mathbb{C}) \qquad j = 0,1$$

is an isomorphism of abelian groups.

The natural map in the previous theorem, $\eta : K_j^{top}(X) \longrightarrow KK_c^j(C_0(X),\mathbb{C})$, is defined as follows. Given a K-cycle (M,E,φ) on X, denote by D_E the Dirac

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operator of M tensored with E. Quite directly D_E determines an element $[D_E] \in KK^*(C(M),\mathbb{C})$. Denote by

$$\varphi_* : KK^*(C(M),\mathbb{C}) \longrightarrow KK_c^*(C_0(X),\mathbb{C})$$

the map of abelian groups induced by $\varphi : M \to X$. Then $\eta(M,E,\varphi) = \varphi_*[D_E]$.

The General Index Problem. With X a locally finite CW-complex, the general index problem [4] is: *Given an element* $\xi \in KK_c^*(C(X),\mathbb{C})$ *explicitly construct a* K*-cycle* (M,E,φ)*on* X *with* $\eta(M,E,\varphi) = \xi$.

Given $\xi \in KK^0(C(X),\mathbb{C})$, suppose that the general index problem has been solved so that a K-cycle (M,E,φ) has been constructed with $\eta(M,E,\varphi) = \xi$. A corollary of such a construction is the following topological formula for Index$(F \otimes \xi)$:

$$\text{Index}(F \otimes \xi) = \epsilon_*(ch(F) \cap \varphi_*(ch(E) \cup Td(M) \cap [M])).$$

The notation in this formula is as follows. First, F can be any complex vector bundle on X and Index$(F \otimes \xi)$ is the integer obtained by tensoring F with ξ as KK cycles and then taking the index of the resulting Fredholm operator. Also, $\epsilon : X \to \{\bullet\}$ is the map of X to a point $\{\bullet\}$ so that we have $\epsilon_* : H_*(X;\mathbb{Q}) \to H_*(\{\bullet\};\mathbb{Q}) = \mathbb{Q}$ is the induced map in rational homology. Second, ch is the Chern character while $\cup$ and $\cap$ are the standard cup and cap products of algebraic topology [22]. Finally, $Td(M)$ is the Todd class of the Spin^c manifold M. The Spin^c structure of M orients M, so $[M] \in H_*(M;\mathbb{Z})$ is the orientation cycle of M. Note that $\varphi_* : H_*(M;\mathbb{Q}) \to H_*(X;\mathbb{Q})$ is the map of rational homology induced by $\varphi : M \to X$.

Equivalently, set

$$H_{ev}(X;\mathbb{Q}) = \bigoplus_l H_{2l}(X;\mathbb{Q})$$

then for each $\xi \in KK^0(C(X),\mathbb{C})$ there exists a unique $\mathcal{I}(\xi) \in H_{ev}(X;\mathbb{Q})$ such that whenever F is a complex vector bundle on X

$$\text{Index}(F \otimes \xi) = \epsilon_*(ch(F) \cap \mathcal{I}(\xi)).$$

To solve the index problem for ξ *rationally* is to explicitly calculate $\mathcal{I}(\xi)$. Using $KK^0(C(X),\mathbb{C})$ alone, however, there is no topological solution to the problem of calculating $\mathcal{I}(\xi)$. The above formula states that if a K-cycle (M,E,φ) on X has been constructed with $\eta(M,E,\varphi) = \xi$, then

$$\mathcal{I}(\xi) = \varphi_*(ch(E) \cup Td(M) \cap [M])$$

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which is a topological formula for $\mathcal{I}(\xi)$.

3. Twisting Data

If H is a complex Hilbert space, $\mathcal{K}(H)$ will denote the C^*-algebra of all compact operators on H. Hilbert spaces will be assumed to be separable, i.e., to admit a finite or countable orthonormal basis. Recall that a C^*-algebra A is an elementary C^*-algebra A if there exists a complex Hilbert space H and an isomorphism of C^*-algebras $A \cong \mathcal{K}(H)$.

Definition Let X be a second countable locally compact Hausdorff topological space. A *twisting datum* on X is a locally trivial bundle $\mathcal{A}$ of elementary C^*-algebras on X, that is for each fiber of $\mathcal{A}$ is an elementary C^*-algebra and with structure group the automorphism group of $\mathcal{K}(H)$ for some complex Hilbert space.

If E is a (locally trivial) bundle of Hilbert spaces on X, then $\mathcal{K}(E)$ is the twisting datum defined by

$$\mathcal{K}(E)_x = \mathcal{K}(E_x), \qquad x \in X.$$

Any twisting datum $\mathcal{A}$ is locally of the form $\mathcal{K}(E)$, i.e., for any $x \in X$ there exists an open set U in X with $x \in U$ and a (locally trivial) Hilbert space vector bundle E on U with $\mathcal{A}|U \cong \mathcal{K}(E)$.

For a C^*-algebra A, A^{op} denotes the opposite C^*-algebra. As Banach spaces $A = A^{op}$, and $*$ remains unchanged. Thus ab in A^{op} is ba in A. If $\mathcal{A}$ is a twisting datum on X, then $\mathcal{A}^{op}$ is the twisting datum obtained by replacing each fiber $\mathcal{A}_x$ by $\mathcal{A}_x^{op}$. If $\mathcal{A}$ and $\mathcal{B}$ are twisting data on X, then $\mathcal{A} \otimes \mathcal{B}$ is the twisting datum on X whose fiber at $x \in X$ is the C^*-algebra $\mathcal{A}_x \otimes \mathcal{B}_x$.

Let $\mathcal{U}(H)$ denote the group of unitary operators on the Hilbert space H in the strong operator topology. If H is not finite dimensional, then as a topological space, $\mathcal{U}(H)$ is contractible. This implies that the projective unitary group $P\mathcal{U}(H) = \mathcal{U}(H)/S^1$ is an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$.

Let $\mathcal{A}$ be a twisting datum on X, and assume that the fiber of $\mathcal{A}$ is infinite dimensional. Fix a infinite dimensional separable complex Hilbert space H. We use the notation $\mathcal{P}$ to denote the principal $P\mathcal{U}(H)$ bundle on X whose fiber at $x \in X$ is:

$$\mathcal{P}_x = \{C^*\text{-algebra isomorphisms} : \mathcal{K}(H) \to \mathcal{A}_x\}.$$

There is then the canonical isomorphism of twisting data on X

$$\mathcal{A} \cong \mathcal{P} \times_{\mathcal{U}(H)} \mathcal{K}(H).$$

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Note that the principal $P\mathcal{U}(H)$ bundle $\mathcal{P}$ is classified by a continuous map

$$X \longrightarrow BP\mathcal{U}(H).$$

Here $BP\mathcal{U}(H)$, the classifying space of $P\mathcal{U}(H)$, is an Eilenberg-MacLane space $K(\mathbb{Z},3)$. Hence the isomorphism class of principal $P\mathcal{U}(H)$ bundle $\mathcal{P}$ over X is classified by the homotopy class of a map $X \to BP\mathcal{U}(H)$ which in turn gives rise to an element in $H^3(X,\mathbb{Z})$. This element of $H^3(X,\mathbb{Z})$ is called the Dixmier-Douady invariant of $\mathcal{A}$ and will be denoted $DD(\mathcal{A})$.

On the other hand if $\mathcal{A}$ has finite dimensional fibers, we let $\mathcal{B}$ be any twisting datum on X with every fiber of $\mathcal{B}$ infinite dimensional and then $DD(\mathcal{A})$ is defined by:

$$DD(\mathcal{A}) =: DD(\mathcal{A}\otimes\mathcal{B}) - DD(\mathcal{B}).$$

For any two twisting data $\mathcal{A},\mathcal{B}$ on X,

$$DD(\mathcal{A}\otimes\mathcal{B}) = DD(\mathcal{A}) + DD(\mathcal{B})$$

and

$$DD(\mathcal{A}^{op}) = -DD(\mathcal{A}).$$

Definition. A *spinor bundle* for $\mathcal{A}$ is a vector bundle S of Hilbert spaces on X together with a given isomorphism of twisting data

$$\mathcal{A} \cong \mathcal{K}(\mathcal{S}).$$

A spinor bundle for $\mathcal{A}$ exists if and only if $DD(\mathcal{A}) = 0$.

Lemma 1 *Let $\mathcal{A}$ be any twisting datum on X. Then there is a canonical spinor bundle for $\mathcal{A}\otimes\mathcal{A}^{op}$.*

Proof: Let H be a Hilbert space. The set of Hilbert-Schmidt operators on H, denoted $\mathcal{L}_{\text{H-S}}(H)$, is an ideal in $\mathcal{K}(H)$. The $\mathbb{C}$-valued inner product

$$\langle T_1, T_2\rangle = Trace(T_1 T_2^*)$$

makes $\mathcal{L}_{\text{H-S}}(H)$ into a Hilbert space. Now let A be an elementary C^*-algebra. Choose an isomorphism of C^*-algebras

$$\psi : A \longrightarrow \mathcal{K}(H).$$

Now, $\psi^{-1}(\mathcal{L}_{\text{H-S}}(H))$ is an ideal in A, and is independent of the choice of ψ because the Hilbert-Schmidt operators are invariant under $Aut\mathcal{K}(H) = P\mathcal{U}(H)$. Denote this ideal by $A_{\text{H-S}}$. The left-multiplication and right-multiplication actions of A on

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$A_{\text{H-S}}$ combine to give a left action of $A \otimes A^{op}$ on $A_{\text{H-S}}$ which identifies $A \otimes A^{op}$ with the compact operators on the Hilbert space $A_{\text{H-S}}$:

$$A \otimes A^{op} \cong \mathcal{K}(A_{\text{H-S}}).$$

If $\mathcal{A}$ is a twisting datum on X, let $\mathcal{S}$ be the vector bundle of Hilbert spaces on X whose fiber at $x \in X$ is $(\mathcal{A}_x)_{\text{H-S}}$. Then $\mathcal{S}$ is a spinor bundle for $\mathcal{A} \otimes \mathcal{A}^{op}$ and is well-defined as our construction is independent of ψ. □

Notation. If $\mathcal{A}$ is a twisting datum on X, then $\Gamma_0(X,\mathcal{A})$ denotes the C^*-algebra of all continuous vanishing-at-infinity sections of $\mathcal{A}$.

Recall that the compactly-supported Kasparov group $KK_c^*(\Gamma_0(X,\mathcal{A}),\mathbb{C})$ is

$$KK_c^j(\Gamma_0(X,\mathcal{A}),\mathbb{C}) := \varinjlim_{\substack{\Delta \subset X \\ \Delta \text{ compact}}} KK^j(\Gamma(\Delta,\mathcal{A}),\mathbb{C}) \qquad j = 0,1$$

where $\Gamma(\Delta,\mathcal{A})$ is the C^*-algebra of all continuous sections of $\mathcal{A}$ restricted to Δ. Since X is a CW-complex, this is equal to the direct limit over the finite sub-CW-complexes of X. We will refer to $KK_c^*(\Gamma_0(X,\mathcal{A}),\mathbb{C})$ as the twisted Kasparov K-homology of X.

Lemma 2 *A spinor bundle $\mathcal{S}$ for $\mathcal{A} \otimes \mathcal{B}^{op}$ determines a stable isomorphism between $\Gamma_0(X,\mathcal{A})$ and $\Gamma_0(X,\mathcal{B})$, and thus gives an isomorphism*

$$\Psi_{\mathcal{S}} : KK_c^j(\Gamma_0(X,\mathcal{A}),\mathbb{C}) \cong KK_c^j(\Gamma_0(X,\mathcal{B}),\mathbb{C}). \qquad j = 0,1$$

Proof: Let $\mathcal{B}_{\text{H-S}}$ be the spinor bundle for $\mathcal{B}^{op} \otimes \mathcal{B}$. Lemma 1 implies

$$\mathcal{A} \otimes \mathcal{B}^{op} \otimes \mathcal{B} \cong \mathcal{A} \otimes \mathcal{K}(\mathcal{B}_{\text{H-S}}).$$

If $\mathcal{S}$ is a spinor bundle for $\mathcal{A} \otimes \mathcal{B}^{op}$, then

$$\mathcal{A} \otimes \mathcal{B}^{op} \otimes \mathcal{B} \cong \mathcal{K}(\mathcal{S}) \otimes \mathcal{B}.$$

Therefore $\mathcal{A} \otimes \mathcal{K}(\mathcal{B}_{\text{H-S}}) \cong \mathcal{K}(\mathcal{S}) \otimes \mathcal{B}$. Note that both Dixmier-Douady invariants of $\mathcal{K}(\mathcal{B}_{\text{H-S}})$ and $\mathcal{K}(\mathcal{S})$ are zero. For any compact subspace $\Delta \subset X$, we have

$$\begin{aligned} KK^j(\Gamma(\Delta,\mathcal{A}),\mathbb{C}) &\cong KK^j(\Gamma(\Delta,\mathcal{A} \otimes \mathcal{K}(\mathcal{B}_{\text{H-S}})),\mathbb{C}) \\ &\cong KK^j(\Gamma(\Delta,\mathcal{K}(\mathcal{S}) \otimes \mathcal{B}),\mathbb{C}) \\ &\cong KK^j(\Gamma(\Delta,\mathcal{B}),\mathbb{C}). \end{aligned}$$

Here the first and the third isomorphisms are provided by Morita equivalence bimodules $\mathcal{B}_{\text{H-S}}$ and $\mathcal{S}$ respectively. Passing to the direct limit, we get the desired isomorphism $\Psi_{\mathcal{S}}$. □

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4. K-cycles for twisted K-homology

As above, X is a locally finite CW-complex and $\mathcal{A}$ is a twisting datum on X.

Definition. A twisted-by-$\mathcal{A}$ K-cycle on X is a triple (M,σ,φ) where

- M is a compact Spin^c manifold without boundary,
- $\varphi : M \longrightarrow X$ is a continuous map,
- $\sigma \in K_0(\Gamma(M,\varphi^*\mathcal{A}^{op}))$ (the K_0-group of the C^*-algebra $\Gamma(M,\varphi^*\mathcal{A}^{op})$).

Remark 1. The twisted K-cycles defined here are closely related to the cycles in the original formulation of the Baum-Connes conjecture. In Section 7, we will define the notion of D-cycles, motivated by D-branes in string theory, which may be regarded as representing geometric cycles for $(X,\mathcal{A})$. In Section 8, we show that in fact the two resulting twisted K-homologies are isomorphic.

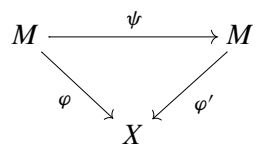
2. When $DD(\mathcal{A}) = 0$, then $\mathcal{A} \otimes \mathcal{K}(H)$ is isomorphic to a trivial bundle of C^*-algebras with fiber $\mathcal{K}(H)$. In this case, the K_0-group of the C^*-algebra $\Gamma(M,\varphi^*\mathcal{A}^{op})$ is isomorphic to the K-theory with coefficients in $\mathcal{K}(H)$ which is defined to be the Grothendieck group of isomorphism classes of finitely generated projective Hilbert $\mathcal{K}(H)$-module bundles over M, see Proposition 2.17 in [21].

Keeping $X,\mathcal{A}$ fixed, denote by $\{(M,\varphi,\sigma)\}$ the collection of all twisted-by-$\mathcal{A}$ K-cycles on X. On this collection impose the equivalence relation $\sim$ generated by the two elementary steps:

- bordism,
- vector bundle modification.

Next, we elaborate on these two steps for the case of twisted K-cycles.

Isomorphism. We say (M,φ,σ) is isomorphic to (M',φ',σ') if and only if there exists a diffeomorphism $\psi : M \to M'$ preserving the Spin^c-structures and with commutativity in the diagram



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and in addition $\psi^*(\sigma') = \sigma$ where $\psi^*\colon K_0\Gamma(M',\varphi'^*\mathcal{A}) \to K_0\Gamma(M,\varphi^*\mathcal{A})$ is the isomorphism of K-theory determined by the isomorphism of C^*-algebras

$$\Gamma(M',\varphi'^*\mathcal{A}^{op}) \cong \Gamma(M,\varphi^*\mathcal{A}^{op}).$$

Bordism. We say (M_0,φ_0,σ_0) is **bordant** to (M_1,φ_1,σ_1) if and only if there exists (W,φ,σ) such that:

1. W is a compact Spin^c manifold with boundary,
2. φ is a continuous map from W to X,
3. $\sigma \in K_0(\Gamma(W,\varphi^*\mathcal{A}^{op}))$,
4. $(\partial W,\varphi|_{\partial W},\sigma|_{\partial W}) \cong (M_0,\varphi_0,\sigma_0) \sqcup (-M_1,\varphi_1,\sigma_1)$.

Vector bundle modification. Let (M,φ,σ) be a twisted-by-$\mathcal{A}$ K-cycle on X, and let F be a Spin^c vector bundle on M of even rank. As in the untwisted case, $\mathbf{1}_{\mathbb{R}}$ denotes the trivial real line bundle on M, $S(F \oplus \mathbf{1}_{\mathbb{R}})$ is the unit sphere bundle of $F \oplus \mathbf{1}_{\mathbb{R}}$ and $\pi : S(F \oplus \mathbf{1}_{\mathbb{R}}) \to M$ is a fibration. Let $s : M \to S(F \oplus 1_{\mathbb{R}})$ be the canonical unit section of $\mathbf{1}_{\mathbb{R}}$. Then the given Spin^c structure for F determines a Gysin homomorphism (Cf. [8])

$$s_*\colon K_0(\Gamma(M,\varphi^*\mathcal{A}^{op})) \longrightarrow K_0(\Gamma(S(F \oplus \mathbf{1}_{\mathbb{R}}),(\varphi \circ \pi)^*\mathcal{A}^{op})).$$

Here we use the fact that $s^*(\varphi \circ \pi)^*\mathcal{A}^{op} = \varphi^*\mathcal{A}^{op}$. Then

$$(M,\varphi,\sigma) \sim (S(F \oplus \mathbf{1}_{\mathbb{R}}),\varphi \circ \pi,s_*\sigma).$$

In the following $(S(F \oplus \mathbf{1}_{\mathbb{R}}),\varphi \circ \pi,s_*\sigma)$ will be denoted $F\#(M,\varphi,\sigma)$ and will be referred to as the modification of (M,φ,σ) by F.

Composition Lemma. Let (M,φ,σ) be a twisted-by-$\mathcal{A}$ K-cycle on X, and let F be an even rank Spin^c vector bundle on M. Let F_1 be an even rank Spin^c vector bundle on $S(F \oplus \mathbf{1}_{\mathbb{R}})$. Then $F_1\#(F\#(M,\varphi,\sigma))$ is (in a canonical way) bordant to $(s^*F_1 \oplus F)\#(M,\varphi,\sigma)$.

Proof: We first construct a bordant manifold W between $S(F_1 \oplus \mathbf{1}_{\mathbb{R}})$ and $S(s^*F_1 \oplus F \oplus \mathbf{1}_{\mathbb{R}})$. Let $D(F_1 \oplus \mathbf{1}_{\mathbb{R}})$ be the unit ball bundle of $F_1 \oplus \mathbf{1}_{\mathbb{R}}$, then

$$S(F_1 \oplus \mathbf{1}_{\mathbb{R}}) = \partial(D(F_1 \oplus \mathbf{1}_{\mathbb{R}}))$$

is the underlying manifold for $F_1\#(F\#(M,\varphi,\sigma))$. Note that $s^*F \oplus F \oplus \mathbf{1}_{\mathbb{R}}$ is isomorphic to the normal bundle ν_ι for the the inclusion map

$$\iota : M \longrightarrow D(F_1 \oplus \mathbf{1}_{\mathbb{R}})$$

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where ι is defined by the composition of $s : M \to S(F \oplus \mathbf{1}_{\mathbb{R}})$ and the zero section of the bundle $F_1 \oplus \mathbf{1}_{\mathbb{R}}$ over $S(F \oplus \mathbf{1}_{\mathbb{R}})$. This ensures that we can identify the sphere bundle of $s^*F_1 \oplus F \oplus \mathbf{1}_{\mathbb{R}}$ with the boundary of the ball bundle of radius $1/4$ in the normal bundle ν_ι. Define

$$\widetilde{W} = D(F_1 \oplus \mathbf{1}_{\mathbb{R}}) - D_{1/4}(\nu_\iota),$$

then $\partial\widetilde{W} \cong S(F_1 \oplus \mathbf{1}_{\mathbb{R}}) \sqcup -S(s^*F_1 \oplus F \oplus \mathbf{1}_{\mathbb{R}})$. Let

$$\widetilde{\varphi}: \qquad \widetilde{W} \subset D(F_1 \oplus \mathbf{1}_{\mathbb{R}}) \xrightarrow{\pi} S(F \oplus \mathbf{1}_{\mathbb{R}}) \xrightarrow{\pi_F} M,$$

be the obvious projection and $\widetilde{s}$ be the composition of the canonical unit sections s and s_1 for $F \oplus \mathbf{1}_{\mathbb{R}}$ and $F_1 \oplus \mathbf{1}_{\mathbb{R}}$ respectively. Then $(\widetilde{W},\widetilde{\varphi},\widetilde{s}_*\sigma)$ provides the bordism between $F_1\#(F\#(M,\varphi,\sigma))$ and $(s^*F_1 \oplus F)\#(M,\varphi,\sigma)$. Here we applied the facts that the pushforward map $\widetilde{s}_*$ is functorial and is also a homotopy invariant. □

Two twisted-by-$\mathcal{A}$ K-cycles (M,φ,σ) and (M',φ',σ') on X are equivalent if and only if it is possible to pass from (M,φ,σ) to (M',φ',σ') by a finite sequence of the two elementary steps. The K-cycle (or topological) twisted-by-$\mathcal{A}$ K-homology of X, denoted, $K_*^{top}(X,\mathcal{A})$, is the set of equivalence classes of twisted-by-$\mathcal{A}$ K-cycles:

$$K_*^{top}(X,\mathcal{A}) := \{(M,\varphi,\sigma)\}/\sim .$$

Addition in $K_*^{top}(X,\mathcal{A})$ is defined by disjoint union of twisted-by-$\mathcal{A}$ K-cycles:

$$(M,\varphi,\sigma) + (M',\varphi',\sigma') = (M \sqcup M',\varphi \sqcup \varphi',\sigma \oplus \sigma').$$

Definition. Let (M,φ,σ) be a twisted-by-$\mathcal{A}$ K-cycle on X. We say (M,φ,σ) *bounds* if and only if there exists $(W,\widetilde{\varphi},\widetilde{\sigma})$ where:

1. W is a compact Spin^c manifold with boundary,
2. $\widetilde{\varphi}$: $W \to X$ is a continuous map,
3. $\widetilde{\sigma} \in K_0(\Gamma(W,\widetilde{\varphi}^*\mathcal{A}^{op}))$,
4. $(\partial W,\widetilde{\varphi}|_{\partial W},\widetilde{\sigma}|_{\partial W}) \cong (M,\varphi,\sigma)$.

As in the untwisted case the additive inverse of (M,φ,σ) is $(-M,\varphi,\sigma)$. The equivalence relation $\sim$ on twisted-by-$\mathcal{A}$ K-cycles (M,φ,σ) preserves the dimension of M modulo 2. Therefore, as an abelian group, $K_*^{top}(X,\mathcal{A})$ is the direct sum

$$K_*^{top}(X,\mathcal{A}) = K_0^{top}(X,\mathcal{A}) \oplus K_1^{top}(X,\mathcal{A})$$

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where $K_j^{top}(X,\mathcal{A})$ is the subgroup of $K_*^{top}(X,\mathcal{A})$ generated by those twisted-by-$\mathcal{A}$ K-cycles (M,φ,σ) such that every connected component of M has dimension $\equiv j \bmod 2$, $j = 0,1$.

There is a natural map $\eta_X : K_j^{top}(X,\mathcal{A}) \longrightarrow KK_c^j(\Gamma_0(X,\mathcal{A}),\mathbb{C})$ defined as follows. Given a twisted-by-$\mathcal{A}$ K-cycle (M,φ,σ) on X, then by applying Poincaré duality in twisted K-theory (Cf. [12] and [23]),

$$KK_0(\mathbb{C},\Gamma(M,\varphi^*\mathcal{A}^{op})) \cong KK^j(\Gamma(M,\varphi^*\mathcal{A}),\mathbb{C}),$$

we have

$$PD(\sigma) \in KK^j(\Gamma(M,\varphi^*\mathcal{A}),\mathbb{C}).$$

Denote by

$$\varphi_* : KK^*(\Gamma(M,\varphi^*\mathcal{A}),\mathbb{C}) \longrightarrow KK_c^*(\Gamma_0(X,\mathcal{A}),\mathbb{C})$$

the map of abelian groups induced by $\varphi : M \to X$. Then the natural map η_X is given by

$$\eta_X(M,\varphi,\sigma) = \varphi_*(PD(\sigma)).$$

It is routine to check that η_X is well-defined on $K_j^{top}(X,\mathcal{A})$ and is functorial in the following sense. Let $f : Y \to X$ be a continuous map and X be equipped with a twisting datum $\mathcal{A}$, then the following diagram commutes

$$\begin{array}{ccc} K_j^{top}(Y, f^*\mathcal{A}) & \xrightarrow{f_*} & K_j^{top}(X,\mathcal{A}) \\ \downarrow \eta_Y & & \downarrow \eta_X \\ KK_c^j(\Gamma_0(Y, f^*\mathcal{A}),\mathbb{C}) & \xrightarrow{f_*} & KK_c^j(\Gamma_0(X,\mathcal{A}),\mathbb{C}). \end{array}$$

Here $f_* : K_j^{top}(Y, f^*\mathcal{A}) \to K_j^{top}(X,\mathcal{A})$ is defined by

$$(M,\varphi,\sigma) \mapsto (M, f \circ \varphi,\sigma)$$

on the level of K-cycles.
Our first main result of this paper is:

Theorem 3 *Let X be a locally finite CW-complex with a given twisting datum $\mathcal{A}$. Then the natural map*

$$\eta_X : K_j^{top}(X,\mathcal{A}) \longrightarrow KK_c^j(\Gamma_0(X,\mathcal{A}),\mathbb{C}) \qquad j = 0,1$$

is an isomorphism of abelian groups.

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The General Twisted Index Problem. By analogy with our previous untwisted general index problem there is a twisted version that applies when X is a locally finite CW-complex and $\mathcal{A}$ is a twisting datum on X. This general twisted index problem is: *Given an element* $\xi \in KK_c^*(\Gamma_0(X,\mathcal{A}), \mathbb{C})$ *explicitly construct a twisted-by-*$\mathcal{A}$ *K-cycle* (M,φ,σ) *for* $(X,\mathcal{A})$ *with* $\eta(M,\varphi,\sigma) = \xi$.

5. Normal Bordism

Let (M,φ,σ) be a twisted-by-$\mathcal{A}$ K-cycle on X. A normal bundle for M is an real vector bundle ν on M together with a given short exact sequence

$$0 \to TM \to M \times \mathbb{R}^q \to \nu \to 0$$

of real vector bundles on M. Here q is a positive integer and $M \times \mathbb{R}^q$ is the trivial real vector bundle with fiber $\mathbb{R}^q$. The two-out-of-three principle implies that ν is a Spin^c vector bundle. If ν has even dimensional fibers, then the modification of (M,φ,σ) by ν will be denoted $\nu\#(M,\varphi,\sigma)$.

Definition. Two twisted-by-$\mathcal{A}$ K-cycles (M,φ,σ) and (M',φ',σ') are *normally bordant*, denoted by

$$(M,\varphi,\sigma) \sim_N (M',\varphi',\sigma'),$$

if there exist normal bundles with even dimensional fibers ν and ν' for M and M' respectively, such that $\nu\#(M,\varphi,\sigma)$ is bordant to $\nu'\#(M',\varphi',\sigma')$.

Lemma 4 *Normal bordism is an equivalence relation.*

Proof: Reflexivity and symmetry are evident. Transitivity needs more discussion. Let $(M_1,\varphi_1,\sigma_1) \sim_N (M_2,\varphi_2,\sigma_2)$, and $(M_2,\varphi_2,\sigma_2) \sim_N (M_3,\varphi_3,\sigma_3)$. Then there exist normal bundles with even dimensional fibers ν_1,ν_2 for M_1,M_2, and normal bundles with even dimensional fibers ν_2',ν_3 for M_2,M_3 such that

1. $\nu_1\#(M_1,\varphi_1,\sigma_1)$ is bordant to $\nu_2\#(M_2,\varphi_2,\sigma_2)$ using the bordism (W,φ,σ),
2. $\nu_2'\#(M_2,\varphi_2,\sigma_2)$ is bordant to $\nu_3\#(M_3,\varphi_3,\sigma_3)$ using the bordism (W',φ',σ').

As ν_2 and ν_2' are stably isomorphic, there exist *trivial* real bundles F_2 and F_2' over M_2, both of of even rank, such that $\nu_2 \oplus F_2 \cong \nu_2' \oplus F_2'$. Then the Composition Lemma implies that

1. $(F_2 \oplus \nu_1)\#(M_1,\varphi_1,\sigma_1)$ is bordant to $(F_2 \oplus \nu_2)\#(M_2,\varphi_2,\sigma_2)$ using the bordism $F_2\#(W,\varphi,\sigma)$,

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2. $(F_2' \oplus \nu_2)\#(M_2,\varphi_2,\sigma_2)$ is bordant to $(F_2' \oplus \nu_3)\#(M_3,\varphi_3,\sigma_3)$ using the bordism $F_2'\#(W',\varphi',\sigma')$.

By transitivity of the bordism relation, we obtain $(M_1,\varphi_1,\sigma_1) \sim_N (M_3,\varphi_3,\sigma_3)$. □

Remark The content of the next lemma is that for normal bordism there is no need to use elementary steps etc. The idea of normal bordism was first constructed by Jakob in [13] for an alternative definition of generalized homology theory, and further applied in [20] and [10] in the study of various versions of geometric K-homology.

Lemma 5 *Two twisted-by-$\mathcal{A}$ K-cycles (M,φ,σ) and (M',φ',σ') are equivalent if and only if they are normally bordant.*

Proof: The implication "normal bordism" $\Longrightarrow$ "equivalence" is immediate. Since normal bordism is an equivalence relation, the implication "equivalence" $\Longrightarrow$ "normal bordism" will follow from the two statements:
Statement 1 : If two twisted-by-$\mathcal{A}$ K-cycles (M,φ,σ) and (M',φ',σ') are bordant, then they are normally bordant.
Statement 2 : If (M',φ',σ') is a vector bundle modification of (M,φ,σ), then (M,φ,σ) and (M',φ',σ') are normally bordant.

To prove Statement 1, let (M_0,φ_0,σ_0) , (M_1,φ_1,σ_1) and (W,φ,σ) be as in the above definition of bordism. Choose an even rank normal bundle ν for W such that the restriction to its boundary, denoted by $\nu|\partial W$, is a normal bundle for ∂W. Then $\nu\#(W,\varphi,\sigma)$ is bordism between $(\nu|M_0)\#(M_0,\varphi_0,\sigma_0)$ and $(\nu|M_1)\#(M_1,\varphi_1,\sigma_1)$.

Statement 2 is proved by applying the composition lemma. Let (M,φ,σ) be a twisted-by-$\mathcal{A}$ K-cycle, and let F be an even rank Spin^c vector bundle on M. Choose an even rank normal bundle ν for $S(F \oplus \mathbf{1}_{\mathbb{R}})$. Then $s^*\nu \oplus F \oplus \mathbf{1}_{\mathbb{R}} \cong \nu_M$ is a normal bundle ν_M for M, here the normal bundle ν_M is obtained by taking the normal bundle to the inclusion $s_0 \circ s : M \to \nu$ with s_0 the zero section of ν. According to the composition lemma, $\nu\#(F\#(M,\varphi,\sigma))$ is bordant to $(s^*\nu \oplus F)\#(M,\varphi,\sigma)$. This proves Statement 2. □

Corollary. A twisted-by-$\mathcal{A}$ K-cycle (M,φ,σ) is zero in $K_*^{top}(X,\mathcal{A})$ if and only if (M,φ,σ) is normally bordant to a twisted-by-$\mathcal{A}$ K-cycle which bounds.

6. Proof of Theorem 3

This Section is devoted entirely to the proof of the main theorem in Section 4.

It suffices to consider the case when X is a finite CW-complex with a given twisting datum $\mathcal{A}$. Let Y be a sub-CW-complex of X. Within the C^*-algebra $\Gamma(X,\mathcal{A})$, let I be the ideal generated by all $s \in \Gamma(X,\mathcal{A})$ such that $s(p) = 0$ for

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all $p \in Y$. There is then the short exact sequence of C^*-algebras

$$0 \to I \to \Gamma(X,\mathcal{A}) \to \Gamma(Y,\mathcal{A}|_Y) \to 0.$$

In fact, $I = \Gamma_0(X - Y,\mathcal{A})$. In Kasparov K-homology this gives the six-term exact sequence

$$\begin{array}{ccccc} KK^0(\Gamma(Y,\mathcal{A}|_Y),\mathbb{C}) & \longrightarrow & KK^0(\Gamma(X,\mathcal{A}),\mathbb{C}) & \longrightarrow & KK^0(I,\mathbb{C}) \\ \uparrow & & & & \downarrow \\ KK^1(I,\mathbb{C}) & \longleftarrow & KK^1(\Gamma(X,\mathcal{A}),\mathbb{C}) & \longleftarrow & KK^1(\Gamma(Y,\mathcal{A}|Y),\mathbb{C}). \end{array}$$

Theorem 3 is proved by considering finite CW pairs (X,Y), that is, X is a finite CW-complex and Y is a finite sub-CW-complex. To this end, $K_*^{top}(X,Y;\mathcal{A})$ is defined as follows.

Definition A twisted-by-$\mathcal{A}$ K-cycle on (X,Y) is a triple (M,σ,φ) where

- M is a compact Spin^c manifold possibly with non-empty boundary ∂M,
- φ is a continuous map from M to X, $\varphi : M \longrightarrow X$ with $\varphi(\partial M) \subset Y$,
- $\sigma \in K_0(\Gamma(M,\varphi^*\mathcal{A}^{op}))$.

Keeping $X,Y,\mathcal{A}$ fixed, denote by $\{(M,\varphi,\sigma)\}$ the collection of all twisted-by-$\mathcal{A}$ K-cycles on (X,Y). On this collection impose the equivalence relation $\sim$ generated by the two elementary steps:

- bordism,
- vector bundle modification.

We remark that the bordism between two twisted-by-$\mathcal{A}$ K-cycles (M_0,φ_0,σ_0) and (M_1,φ_1,σ_1) on (X,Y) is given by (W,φ,σ) as in the absolute case except that $M_0 \sqcup M_1$ is a regularly embedded submanifold of codimension 0 in ∂W and $\varphi(\partial W - \{M_0 \sqcup M_1\}) \subset Y$.

The topological twisted-by-$\mathcal{A}$ K-homology of (X,Y), denoted by $K_*^{top}(X,Y;\mathcal{A})$, is the set of equivalence classes of twisted-by-$\mathcal{A}$ K-cycles:

$$K_*^{top}(X,Y;\mathcal{A}) := \{(M,\varphi,\sigma)\}/\sim .$$

Addition in $K_*^{top}(X,Y;\mathcal{A})$ is disjoint union of twisted-by-$\mathcal{A}$ K-cycles :

$$(M,\varphi,\sigma) + (M',\varphi',\sigma') = (M \sqcup M',\varphi \sqcup \varphi',\sigma \oplus \sigma').$$

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The additive inverse of (M,φ,σ) is $(-M,\varphi,\sigma)$. The equivalence relation $\sim$ on twisted-by-$\mathcal{A}$ K-cycles (M,φ,σ) preserves the dimension of M modulo 2. Therefore, as an abelian group, $K_*^{top}(X,Y;\mathcal{A})$ is the direct sum

$$K_*^{top}(X,Y;\mathcal{A}) = K_0^{top}(X,Y;\mathcal{A}) \oplus K_1^{top}(X,Y;\mathcal{A})$$

where $K_j^{top}(X,Y;\mathcal{A})$ is the subgroup of $K_*^{top}(X,Y;\mathcal{A})$ generated by those twisted-by-$\mathcal{A}$ K-cycles (M,φ,σ) on (X,Y) such that every connected component of M has dimension $\equiv j \mod 2$, $j = 0,1$.

Lemma 6 *(***Excision***) Let $U \subset Y \subset X$ be a subspace such that the closure of U lies in the interior of Y. Then the inclusion*

$$i : (X-U, Y-U) \longrightarrow (X,Y)$$

induces isomorphisms $i_* : K_*^{top}(X-U,Y-U;\mathcal{A}) \to K_*^{top}(X,Y;\mathcal{A})$.

Proof: We will define an inverse map to i_*. Let (M,φ,σ) be a twisted-by-$\mathcal{A}$ K-cycle for $(X,\mathcal{A})$. Choose a Morse function $f : M \to \mathbb{R}$ separating $f^{-1}(\overline{U})$ and $M - f^{-1}(Y)$, such that

$$a = \sup_{y \in M - f^{-1}(Y)} \{f(y)\} < b = \inf_{y \in f^{-1}(\overline{U})} \{f(y)\}.$$

Take a regular value $c \in [a,b]$. Define

$$M_1 = f^{-1}((-\infty,c])$$

which can be given a smooth structure by smoothing the corner as in [9]. Note that M_1 is then a smooth sub-manifold of M with its boundary component

$$f^{-1}(c) \cup \left(f^{-1}((-\infty,c]) \cap M\right)$$

mapped to $X-U$ under φ. Then by restricting (E,φ,σ) to M_1, we obtain a twisted-by-$\mathcal{A}$ K-cycle (M_1,φ_1,σ_1) for $(X-U,Y-U;\mathcal{A})$. It can be checked that this construction defines a homomorphism

$$j_* : K_*^{top}(X,Y;\mathcal{A}) \longrightarrow K_*^{geo}(X-U,Y-U;\mathcal{A}).$$

To see that $i_* \circ j_* = Id$ we can apply the bordant manifold

$$\bigcup_{t \in [c,d]} f^{-1}((-\infty,t]) \subset M \times \mathbb{R}$$

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for some $d > \sum_{y \in M}\{f(y)\}$, which is equipped with the induced twisting datum from (M,φ,σ), to obtain a bordism between (M,φ,σ) and (M_1,φ_1,σ_1) as twisted-by-$\mathcal{A}$ K-cycles over $(X,Y;\mathcal{A})$. For a twisted-by-$\mathcal{A}$ K-cycle over $(X,Y;\mathcal{A})$ arising from a twisted-by-$\mathcal{A}$ K-cycle over $(X-U,Y-U;\mathcal{A})$, choose an appropriate regular value c in the above construction. Then we obtain the same twisted-by-$\mathcal{A}$ K-cycle $(X-U,Y-U;\mathcal{A})$. This implies that $j_* \circ i_* = Id$. □

Now introduce the six-term sequence

$$\begin{array}{ccccc} K_0^{top}(Y,\mathcal{A}|_Y) & \longrightarrow & K_0^{top}(X,\mathcal{A}) & \longrightarrow & K_0^{top}(X,Y;\mathcal{A}) \\ \uparrow & & & & \downarrow \\ K_1^{top}(X,Y;\mathcal{A}) & \longleftarrow & K_1^{top}(X,\mathcal{A}) & \longleftarrow & K_1^{top}(Y,\mathcal{A}|_Y) \end{array}$$

where the maps $K_j^{top}(Y,\mathcal{A}) \to K_j^{top}(X,\mathcal{A})$ are

$$(M,\varphi,\sigma) \mapsto (M,i \circ \varphi,\sigma)$$

with $i : Y \to X$ is the inclusion of Y in X. The maps $K_j^{top}(X,\mathcal{A}) \to K_j^{top}(X,Y,\mathcal{A})$ are

$$(M,\varphi,\sigma) \mapsto (M,\varphi,\sigma).$$

That is, the twisted-by-$\mathcal{A}$ K-cycle (M,φ,σ) on X is mapped to the twisted-by-$\mathcal{A}$ K-cycle (M,φ,σ) on (X,Y). Finally, the maps $K_0^{top}(X,Y;\mathcal{A}) \to K_1^{top}(Y,\mathcal{A})$ and $K_1^{top}(X,Y;\mathcal{A}) \to K_0^{top}(Y,\mathcal{A})$ are

$$(M,\varphi,\sigma) \mapsto (\partial M\ ,\ \varphi|\partial M\ ,\ \sigma|\partial M).$$

We will establish exactness of this six-term sequence. For example, consider

$$K_0^{top}(X,\mathcal{A}) \to K_0^{top}(X,Y;\mathcal{A}) \to K_1^{top}(Y,\mathcal{A}).$$

It is immediate that the composition of these two maps is zero. The remaining exactness results follow by applying Lemma 5 and the corollary to Lemma 5. Thus let (M,φ,σ) be a twisted-by-$\mathcal{A}$ K-cycle on (X,Y) such that $(\partial M\ ,\ \varphi|\partial M\ ,\ \sigma|\partial M)$ is zero in $K_1^{top}(Y,\mathcal{A})$. According to the corollary to Lemma 5, $(\partial M\ ,\ \varphi|\partial M\ ,\ \sigma|\partial M)$ is normally bordant (as a twisted-by-$\mathcal{A}$ K-cycle on Y) to a twisted-by-$\mathcal{A}$ K-cycle on Y which bounds. Hence a normal bundle ν with even dimensional fibers can be chosen for ∂M such that $\nu\#(\partial M\ ,\ \varphi|\partial M\ ,\ \sigma|\partial M)$ bounds. That is, there exists a $(W,\widetilde{\varphi},\widetilde{\sigma})$ such that:

1. W is a compact Spin^c manifold with boundary,

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2. $\widetilde{\varphi}\colon W \to Y$ is a continuous map,

3. $\widetilde{\sigma} \in K_0(\Gamma(W,\widetilde{\varphi}^*\mathcal{A}))$,

4. $(\partial W,\widetilde{\varphi}|_{\partial W},\widetilde{\sigma}|_{\partial W}) \cong \nu\#(\partial M\,,\varphi|\partial M\,,\sigma|\partial M)$.

By choosing ν to have a sufficiently large fiber dimension, it can be assumed that ν extends from ∂M to M. Let $(\widetilde{W},\Psi,\kappa)$ be the twisted-by-$\mathcal{A}$ K-cycle on X obtained by joining together — along ∂W — $\nu\#(M,\varphi,\sigma)$ and $(-W,\widetilde{\varphi},\widetilde{\sigma})$. Note that $\kappa \in K_0(\Gamma(W,\Psi^*\mathcal{A}^{op}))$ is obtained from $s_*\sigma$ and $\widetilde{\sigma}$ using the Mayer-Vietoris argument in twisted K-theory. Then (W,Ψ,κ) is an element in $K_0^{top}(X,\mathcal{A})$ which is mapped to (M,φ,σ) by the map

$$K_0^{top}(X,\mathcal{A}) \to K_0^{top}(X,Y;\mathcal{A}).$$

This last fact establishes the exactness of the six-term sequence.

There is a natural map $\eta_{X,Y} : K_*^{top}(X,Y;\mathcal{A}) \to KK^*(\Gamma(X-Y,\mathcal{A}),\mathbb{C})$ defined in a similar way as in the absolute case. Moreover the above six-term exact sequence is compatible with the six-term exact sequence for Kasparov K-homology

$$\begin{array}{ccccccc}
\longrightarrow K_j^{top}(X;\mathcal{A}) & \longrightarrow & K_j^{top}(X,Y;\mathcal{A}) & \longrightarrow & K_{j-1}^{top}(Y,\mathcal{A}|_Y) & \longrightarrow \\
\downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_{X,Y}} & & \downarrow{\scriptstyle \eta_Y} & \\
\longrightarrow KK^j(\Gamma(X,\mathcal{A})\,,\mathbb{C}) & \longrightarrow & KK^j(I\,,\mathbb{C}) & \longrightarrow & KK^{j-1}(\Gamma(Y,\mathcal{A}|_Y)\,,\mathbb{C}) & \longrightarrow .
\end{array}$$

Now the proof of Theorem 3 is completed by a standard argument on the number of cells in X.

(1) When the number of cells in X is 1, then X consists of a single point. In this case the twisting datum is necessarily trivial, so the twisted index map is just the usual index map which is an isomorphism [5].

(2) When X is a finite CW-complex obtained by attaching a k-cell to a CW subcomplex Y with an attaching map f, then the excision lemma implies that

$$\begin{array}{ccccc}
K_j^{top}(X,Y;\mathcal{A}) & \xrightarrow{\cong} & K_*^{top}(D^k,S^{k-1},f^*\mathcal{A}) & \xrightarrow{\cong} & K_*^{top}(D^k,S^{k-1}) \\
\downarrow{\scriptstyle \eta_{X,Y}} & & \downarrow{\scriptstyle \eta_{D^k,S^{k-1}}} & & \downarrow{\scriptstyle \eta} \\
KK^j(\Gamma(X-Y,\mathcal{A}),\mathbb{C}) & \xrightarrow{\cong} & KK^j(\Gamma(D^k-S^{k-1},\mathcal{A}),\mathbb{C}) & \xrightarrow{\cong} & KK^j(\Gamma(D^k-S^{k-1},\mathcal{A}),\mathbb{C}).
\end{array}$$

Again the isomorphism η for the untwisted case in [5] gives the isomorphism $\eta_{X,Y}$. The inductive step is accomplished by applying the five lemma to the two 6-term exact sequences connected by the η-maps.

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7. The group of D-cycles

In this section, we introduce another notion of K-cycles for $(X,\mathcal{A})$, called D-cycles, which is closely related to the notion of D-branes in string theory. We begin with the following:

Definition A D-cycle for $(X,\mathcal{A})$ is a 4-tuple $(M,E,\varphi,\mathcal{S})$ such that:

1. M is a closed oriented C^∞ Riemannian manifold,
2. E is a complex vector bundle on M,
3. φ is a continuous map from M to X,
4. $\mathcal{S}$ is a spinor bundle for $\mathrm{Cliff}^+_{\mathbb{C}}(TM)\otimes\varphi^*\mathcal{A}^{op}$.

Remark (1) If M is even dimensional, then $\mathrm{Cliff}^+_{\mathbb{C}}(TM)$ is the twisting datum on M whose fiber at $p\in M$ is the complexified Clifford algebra $\mathbb{C}\otimes_{\mathbb{R}}\mathrm{Cliff}(T_pM)$. As usual, T_pM is the tangent space to M at p. On the other hand, if M is odd dimensional, given $p\in M$ choose a positively oriented orthonormal basis $e_1,e_2,\ldots,e_n$ for T_pM. Set $n=2r+1$ and define $\omega(p)\in\mathbb{C}\otimes_{\mathbb{R}}\mathrm{Cliff}(T_pM)$ by:

$$\omega(p)=i^{r+1}e_1e_2\ldots e_n.$$

Then $\omega(p)$ does not depend on the choice of positively oriented orthonormal basis for T_pM. Also $\omega(p)$ is in the center of $\mathrm{Cliff}_{\mathbb{C}}(T_pM)=\mathbb{C}\otimes_{\mathbb{R}}\mathrm{Cliff}(T_pM)$ and $\omega(p)^2=1$.

Now set

$$\mathrm{Cliff}^+_{\mathbb{C}}(T_pM)=\{a\in\mathrm{Cliff}_{\mathbb{C}}(T_pM)\,|\,\omega(p)a=a\}.$$

Then $\mathrm{Cliff}^+_{\mathbb{C}}(TM)$ is the twisting datum on M whose fiber at $p\in M$ is $\mathrm{Cliff}^+_{\mathbb{C}}(T_pM)$.

(2) The existence of a spinor bundle $\mathcal{S}$ for $\mathrm{Cliff}^+_{\mathbb{C}}(TM)\otimes\varphi^*\mathcal{A}^{op}$ implies

$$DD(\mathrm{Cliff}^+_{\mathbb{C}}(TM))=\varphi^*(DD(\mathcal{A})).$$

By standard algebraic topology, $DD(\mathrm{Cliff}^+_{\mathbb{C}}(TM))$ is the third (integral) Stiefel-Whitney class of M, so the existence of $\mathcal{S}$ implies

$$W_3(M)=\varphi^*(DD(\mathcal{A}))$$

which is the Freed-Witten anomaly cancellation condition for Type IIB D-branes as explained in [24].

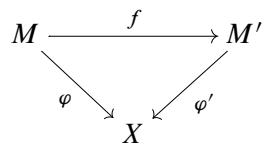
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Definition. Keeping $(X,\mathcal{A})$ fixed, denote by $\{(M,E,\varphi,\mathcal{S})\}$ the collection of all D-cycles for $(X,\mathcal{A})$. On this collection impose the equivalence relation $\sim$ generated by the three elementary steps:

- bordism,
- direct sum - disjoint union,
- vector bundle modification.

These three elementary moves can be precisely defined as follows.

Isomorphism. Two D-cycles $(M,E,\varphi,\mathcal{S})$, $(M',E',\varphi',\mathcal{S}')$ for $(X,\mathcal{A})$ are *isomorphic* if there is an orientation preserving isometric diffeomorphism $f : M \to M'$ such that the diagram



commutes, and $f^*E' \cong E$, $f^*\mathcal{S}' \cong \mathcal{S}$.

Bordism. Two D-cycles $(M_0,E_0,\varphi_0,\mathcal{S}_0)$, $(M_1,E_1,\varphi_1,\mathcal{S}_1)$ for $(X,\mathcal{A})$ are *bordant* if there exists a 4-tuple $(W,E,\Phi,\mathcal{S})$ such that W is a compact oriented Riemannian manifold with boundary, E is a complex vector bundle on W, Φ is a continuous map from $W \to X$ and

$$(\partial W, E|\partial W, \Phi|\partial W, \mathcal{S}^{(+)}|\partial W) \cong (M_0,E_0,\varphi_0,\mathcal{S}_0) \sqcup (-M_1,E_1,\varphi_1,\mathcal{S}_1).$$

When W is of odd dimension,

$$\mathrm{Cliff}_{\mathbb{C}}^{(+)}(TW)|\partial W \cong \mathrm{Cliff}_{\mathbb{C}}(T(\partial W).$$

Then $\mathcal{S}^{(+)} = \mathcal{S}$. When W is of even dimension, then $\mathcal{S}^{(+)}$ is the positive part of $\mathcal{S}$, the (+1)-eigenbundle of the chirality section of $\mathrm{Cliff}_{\mathbb{C}}^{(+)}(TW)$.

Direct sum-disjoint union. Let $(M,E,\varphi,\mathcal{S})$ be a D-cycle for $(X,\mathcal{A})$ and let E' be a complex vector bundle on M, then

$$(M,E,\varphi,\mathcal{S}) \sqcup (M,E',\varphi,\mathcal{S}) \sim (M,E \oplus E',\varphi,\mathcal{S}).$$

Vector bundle modification. Let $(M,E,\varphi,\mathcal{S})$ be a D-cycle for $(X,\mathcal{A})$ and let F be a Spin^c vector bundle on M with even dimensional fibers. Then, with $\mathcal{S}_F$ being the Spinor bundle for the Spin^c vector bundle F,

$$(M,E,\varphi,\mathcal{S}) \sim (S(F \oplus 1_{\mathbb{R}}),\beta \otimes \pi^*E,\varphi \circ \pi,\pi^*\mathcal{S}_F \otimes \pi^*\mathcal{S}),$$

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with the other notation in this equivalence being the same as that introduced previously. The group of D-cycles for $(X,\mathcal{A})$, denoted $K_*^{geo}(X,\mathcal{A})$, is the set of equivalence classes of these D-cycles under the equivalence defined above in terms of the three relations. Addition in $K_*^{geo}(X,\mathcal{A})$ is disjoint union of D-cycles:

$$(M,E,\varphi,\mathcal{S}) + (M',E',\varphi',\mathcal{S}') = (M \sqcup M', E \sqcup E', \varphi \sqcup \varphi', \mathcal{S} \sqcup \mathcal{S}').$$

The equivalence relation $\sim$ on D-cycles $(M,E,\varphi,\mathcal{S})$ preserves the dimension of M modulo 2 so that we have

$$K_*^{geo}(X,\mathcal{A}) = K_0^{geo}(X,\mathcal{A}) \oplus K_1^{geo}(X,\mathcal{A})$$

where $K_j^{geo}(X,\mathcal{A})$ is the subgroup of $K_*^{geo}(X,\mathcal{A})$ generated by those $(X,\mathcal{A})$ D-cycles $(M,E,\varphi,\mathcal{S})$ such that every connected component of M has dimension $\equiv j$ modulo 2, $j = 0,1$.

Let $(M,E,\varphi,\mathcal{S})$ be a D-cycle on $(X,\mathcal{A})$ such that every connected component of M has its dimension congruent to j modulo 2. Let D_E denote the Dirac element of M tensored with E. It can be described in terms of the $1/2$-signature operator on M. Then as in [16], D_E yields an element in the Kasparov K-homology group

$$[D_E] \in KK^j(C(M, \mathrm{Cliff}_{\mathbb{C}}^{(+)}(M)),\mathbb{C}).$$

The isomorphism $\Psi_{\mathcal{S}}$ defined in Lemma 2 in Section 3 reads as

$$\Psi : KK^j(C(M, \mathrm{Cliff}_{\mathbb{C}}^{(+)}(M)),\mathbb{C}) \longrightarrow KK^j(\Gamma(M,\varphi^*\mathcal{A}),\mathbb{C}).$$

The map of C^*-algebras $\Gamma(X,\mathcal{A}) \to \Gamma(M,\varphi^*\mathcal{A})$ induces a homomorphism of abelian groups

$$\varphi_* : KK^j(\Gamma(M,\varphi^*\mathcal{A}),\mathbb{C}) \longrightarrow KK^j(\Gamma(X,\mathcal{A}),\mathbb{C}).$$

Then $(M,E,\varphi,\mathcal{S}) \mapsto \varphi_*[D_E]$ yields a homomorphism of abelian groups, denoted by

$$\mu : K_*^{geo}(X,\mathcal{A}) \longrightarrow KK^*(\Gamma(X,\mathcal{A}),\mathbb{C}).$$

We call this the twisted index map in twisted K-homology of $(X,\mathcal{A})$.

8. The charge map $K_*^{geo}(X,\mathcal{A}) \longrightarrow K_*^{top}(X,\mathcal{A})$

In this Section normal bundle modification is used to define a homomorphism of abelian groups $h\colon K_*^{geo}(X,\mathcal{A}) \longrightarrow K_*^{top}(X,\mathcal{A})$. Here h should be viewed as the map which sends a D-cycle $(M,E,\varphi,\mathcal{S})$ to its charge.

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Definition Given a D-cycle $(M,E,\varphi,\mathcal{S})$ choose a normal bundle ν for M, with even dimensional fibers. Then:

$$h(M,E,\varphi,\mathcal{S}) := \nu\#(M,E,\varphi,\mathcal{S}) = (S(\nu \oplus 1_{\mathbb{R}}), \varphi \circ \pi, \sigma).$$

Note that $S(\nu \oplus 1_{\mathbb{R}})$ is a Spin^c manifold because its tangent bundle is stably trivialized. Here σ is the element in $K_0(\Gamma(S(\nu \oplus \mathbf{1}_{\mathbb{R}}, (\varphi \circ \pi)^* \mathcal{A}^{op}))$ obtained from E as follows:

1. Denote by $s_*\colon K^0(M) \to K_0(\Gamma(S(\nu \oplus \mathbf{1}_{\mathbb{R}}), \pi^*\mathrm{Cliff}_{\mathbb{C}}(\nu))$ the Gysin homomorphism associated to the canonical section $s\colon M \to S(\nu \oplus \mathbf{1}_{\mathbb{R}})$.

2. Observe that, as twisting data on M, a trivialisation of $TM \oplus \nu$ gives an equivalence

$$\mathrm{Cliff}^+_{\mathbb{C}}(TM) \otimes \varphi^*\mathcal{A}^{op} \sim \mathrm{Cliff}_{\mathbb{C}}(\nu)^{op} \otimes \varphi^*\mathcal{A}^{op},$$

   a stable isomorphism of bundles of elementary C^*-algebras. Hence the given spinor bundle $\mathcal{S}$ for $\mathrm{Cliff}^+_{\mathbb{C}}(TM) \otimes \varphi^*\mathcal{A}^{op}$ determines a spinor bundle $\widetilde{\mathcal{S}}$ for $\mathrm{Cliff}_{\mathbb{C}}(\nu)^{op} \otimes \varphi^*\mathcal{A}^{op}$. Then $\pi^*\widetilde{\mathcal{S}}$ is a spinor bundle for $\pi^*\mathrm{Cliff}_{\mathbb{C}}(\nu)^{op} \otimes (\varphi \circ \pi)^*\mathcal{A}^{op}$ and therefore yields an isomorphism of abelian groups

$$\chi\colon K_0(\Gamma(S(\nu \oplus \mathbf{1}_{\mathbb{R}}), \pi^*\mathrm{Cliff}_{\mathbb{C}}(\nu))) \longrightarrow K_0(\Gamma(S(\nu \oplus \mathbf{1}_{\mathbb{R}}), (\varphi \circ \pi)^*\mathcal{A}^{op})).$$

3. $\sigma := \chi(s_*[E])$.

Question: Is the map $h : K_j^{geo}(X,\mathcal{A}) \to K_j^{top}(X,\mathcal{A})$ an isomorphism for any locally finite CW-complex?

Given a positive answer to this question then we would have the following corollary which extends the result in [24] from the situation where X is a manifold to that where X can be any locally finite CW-complex X.

Consequence: Let X be a locally finite CW-complex with a twisting datum $\mathcal{A}$ defined by a principal $PU(H)$-bundle $\mathcal{P}$, and $K_*^{geo}(X,\mathcal{P})$ be the twisted geometric K-homology of $(X,\mathcal{P})$ in [24]. Then the twisted index map

$$\mu : K_*^{geo}(X,\mathcal{P}) \longrightarrow KK_c^*(\Gamma(X,\mathcal{A}),\mathbb{C})$$

is an isomorphism.

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9. Properly supported K-cycles and D-cycles

Let X be a CW-complex which is locally finite, countable, and finite dimensional. Assume that $\mathcal{A}$ is a twisting datum on X. If X is not compact (i.e., X is not a finite CW-complex), then Kasparov twisted K-homology $KK^*(\Gamma_0(X,\mathcal{A}),\mathbb{C})$ is not compactly supported. Hence in this case the analogous K-cycle theory cannot use compactly supported K-cycles.

Definition. A *properly supported* K-cycle for $(X,\mathcal{A})$ is a triple (M,φ,σ) such that:

1. M is a Spin^c manifold without boundary.
2. φ is a continuous proper map from M to X.
3. σ is an element in the representable (i.e., not compactly supported) K_0 -theory of $\Gamma_0(M,\varphi^*\mathcal{A}^{op})$.

Keeping $(X,\mathcal{A})$ fixed, denote by $\{(M,\varphi,\sigma)\}_\Lambda$ the collection of all properly supported K-cycles for $(X,\mathcal{A})$. On this collection impose the equivalence relation $\sim$ generated by the two elementary steps:

- bordism,
- vector bundle modification.

The properly supported K-cycle (or topological) twisted-by-$\mathcal{A}$ K-homology of X, denoted $K_*^{top}(X,\mathcal{A})_\Lambda$, is the set of equivalence classes of these K-cycles. Addition in $K_*^{top}(X,\mathcal{A})_\Lambda$ is disjoint union of properly supported K-cycles:

$$(M,\varphi,\sigma)+(M',\varphi',\sigma')=(M\sqcup M',\varphi\sqcup\varphi',\sigma\oplus\sigma').$$

As explained in the previous cases $K_*^{top}(X,\mathcal{A})_\Lambda = K_0^{top}(X,\mathcal{A})_\Lambda \oplus K_1^{top}(X,\mathcal{A})_\Lambda$ where $K_j^{top}(X,\mathcal{A})_\Lambda$ is the subgroup of $K_*^{top}(X,\mathcal{A})_\Lambda$ generated by those properly supported $(X,\mathcal{A})$ K-cycles (M,φ,σ) such that every connected component of M has dimension $\equiv j$ modulo 2, $j=0,1$.

The natural map $\eta : K_j^{top}(X,\mathcal{A})_\Lambda \longrightarrow KK^j(\Gamma_0(X,\mathcal{A}),\mathbb{C})$ is defined as in the compactly-supported case above.

Theorem 7 *Let X be CW-complex which is locally finite, countable, and finite dimensional. Let $\mathcal{A}$ be a twisting datum on X. Then the natural map*

$$\eta : K_j^{top}(X,\mathcal{A})_\Lambda \longrightarrow KK^j(\Gamma_0(X,\mathcal{A}),\mathbb{C}) \qquad j=0,1$$

is an isomorphism of abelian groups.

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The proof of this Theorem follows step by step the proof of Theorem 3.

Properly Supported Index Problem. With $(X,\mathcal{A})$ as in Theorem 7, the properly supported index problem is:
Given an element $\xi \in KK^*(\Gamma_0(X,\mathcal{A}),\mathbb{C})$ *explicitly construct a properly supported* K*-cycle* $(M,\varphi,\sigma))$ *for* $(X,\mathcal{A})$ *with* $\eta(M,\varphi,\sigma) = \xi$.

Definition. A *properly supported* D-cycle for $(X,\mathcal{A})$ is a 4-tuple $(M,\Sigma,\varphi,\mathcal{S})$ such that :

1. M is an oriented C^∞ Riemannian manifold without boundary,
2. Σ is an element in the representable (i.e., not compactly supported) even K-theory of M,
3. φ is a continuous proper map from M to X,
4. $\mathcal{S}$ is a spinor bundle for $\mathrm{Cliff}^+_{\mathbb{C}}(TM) \otimes \varphi^*\mathcal{A}^{op}$.

Remarks Keeping $(X,\mathcal{A})$ fixed, denote by $\{(M,\Sigma,\varphi,\mathcal{S})\}_\Lambda$ the collection of all properly supported D-cycles for $(X,\mathcal{A})$. On this collection impose the equivalence relation $\sim$ generated by the two elementary steps:

- bordism,
- vector bundle modification.

The group of properly supported D-cycles for $(X,\mathcal{A})$, denoted $K_*^{geo}(X,\mathcal{A})_\Lambda$, is the set of equivalence classes of these D-cycles.

Addition in $K_*^{geo}(X,\mathcal{A})_\Lambda$ is disjoint union of properly supported K-cycles :

$$(M,\Sigma,\varphi,\mathcal{S}) + (M',\Sigma',\varphi',\mathcal{S}') = (M \sqcup M',\Sigma \oplus \Sigma',\varphi \sqcup \varphi',\mathcal{S} \oplus \mathcal{S}').$$

The equivalence relation $\sim$ on properly supported D-cycles $(M,\Sigma,\varphi,\mathcal{S})_\Lambda$ preserves the dimension of M modulo 2. Therefore, as we had previously $K_*^{geo}(X,\mathcal{A})_\Lambda = K_0^{geo}(X,\mathcal{A})_\Lambda \oplus K_1^{geo}(X,\mathcal{A})_\Lambda$ where $K_j^{geo}(X,\mathcal{A})_\Lambda$ is the subgroup of $K_*^{geo}(X,\mathcal{A})_\Lambda$ generated by those properly supported $(X,\mathcal{A})$ D-cycles $(M,\Sigma,\varphi,\mathcal{S})$ such that every connected component of M has dimension $\equiv j$ modulo 2, $j = 0,1$.

As in the compactly supported case modification by an even rank normal bundle ν gives a homomorphism of abelian groups

$$h_\Lambda \colon K_*^{geo}(X,\mathcal{A})_\Lambda \longrightarrow K_*^{top}(X,\mathcal{A})_\Lambda,$$

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$$h_\Lambda(M,\Sigma,\varphi,\mathcal{S}) := \nu\#(M,\Sigma,\varphi,\mathcal{S}) = (S(\nu \oplus 1_{\mathbb{R}})\,,\varphi\circ\pi\,,\sigma).$$

Question: Is the map $h_\Lambda : K_*^{geo}(X,\mathcal{A})_\Lambda \to K_*^{top}(X,\mathcal{A})_\Lambda$ an isomorphism?

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https://doi.org/10.1017/is013004029jkt226 Published online by Cambridge University Press

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Received: December 22, 2011

https://doi.org/10.1017/is013004029jkt226 Published online by Cambridge University Press