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Graph models of λ-calculus at work, and variations

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Dedicated to Klaus Keimel on the occasion of his 65th birthday

This paper surveys what we have learned during the last ten years about the lattice $\lambda\mathscr{T}$ of all λ-theories (= equational extensions of untyped λ-calculus), via the sets $\lambda\mathscr{C}$ consisting of the λ-theories that are representable in a uniform class $\mathscr{C}$ of λ-models. This includes positive answers to several questions raised in Berline (2000), as well as several independent results, the state of the art on the long-standing open questions concerning the representability of $\lambda_\beta, \lambda_{\beta\eta}, H$ as theories of models, and 22 open problems.

We will focus on the class $\mathscr{G}$ of graph models, since almost all the existing semantic proofs on $\lambda\mathscr{T}$ have been, or could be, more easily, obtained via graph models, or slight variations of them. But in this paper we will also give some evidence that, for all uniform classes $\mathscr{C}, \mathscr{C}'$ of proper λ-models living in functional semantics, $\lambda\mathscr{C} - \lambda\mathscr{C}'$ should have cardinality 2^ω, provided $\mathscr{C}$ is not included in $\mathscr{C}'$.

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1. Introduction

In this paper *λ-calculus* means Church's untyped λ-calculus, and we assume a basic knowledge of its syntax, which can be found in Barendregt (1984). For this introduction *λ-model* just means 'model of λ-calculus'. In particular λ-models interpret λ-terms modulo $\alpha\beta$-conversion (λ_β), and they are *extensional* if the interpretation is modulo $\alpha\beta\eta$-conversion ($\lambda_{\beta\eta}$).

The problems

The main concern of this paper is the study of the complete lattice $\lambda\mathscr{T}$, which consists of all λ-theories, ordered by inclusion. A *λ-theory* is a congruence on the set Λ of all λ-terms (with respect to abstraction and application), which contains λ_β. Such a congruence usually arises either from syntactical and computational considerations or as the equational theory $Th(\mathscr{M})$ of a λ-model $\mathscr{M}$. The correspondence Th is far from being injective, and we will also see that the question of its 'surjectivity' is a key one. A λ-model $\mathscr{M}$ is called a *proper model* if it is not the term model of $Th(\mathscr{M})$. The best known computational theories, besides λ_β and $\lambda_{\beta\eta}$, are the theories H, H^*, BT, LT, NT, whose definitions are recalled in Section 3.2. It is here enough to say that BT equates two λ-terms iff they have the same Böhm tree, and that LT and NT are variants of BT, that H^* and NT have been proved to be equal, and, finally, that BT, H and H^* are sensible theories. Recall that a theory is *sensible* if it equates all the unsolvable terms.

We are interested in exploring the following generic questions.

Problem 1. What do models tell us about the structure of $\lambda\mathscr{T}$?

Given a class $\mathscr{C}$ of proper λ-models, a λ-theory T *is representable in* $\mathscr{C}$ if there is an $\mathscr{M}$ in $\mathscr{C}$ such that $Th(\mathscr{M}) = T$. We use $\lambda\mathscr{C}$ to denote the set of λ-theories representable in $\mathscr{C}$. The following problem is, of course, intimately linked to the study of Th.

Problem 2. What are the links between $\lambda\mathscr{T}$ and $\lambda\mathscr{C}$, where $\mathscr{C}$ is some class of proper models?

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What should be very clear is that, even though $\lambda\mathscr{T}$, as well as the $\lambda\mathscr{C}'s$ for all interesting $\mathscr{C}'s$, all have cardinality 2^{ω}, only a few explicitly definable theories can be proved to be the theory of a proper λ-model. Among these exceptions are BT, LT and NT. Whether $\lambda_{\beta}, \lambda_{\beta\eta}$ and H are representable are long-standing questions, due to Barendregt and Honsell, respectively, and will be discussed throughout the paper.

We have already surveyed the state of the art concerning the three generic problems presented in this introduction in Berline (2000, Section 6), where we listed a number of related open questions. In the past four years significant progress has been achieved in a series of papers, mainly by Antonino Salibra, either alone or in collaboration with Bucciarelli, Lusin or the current author. This makes it worthwhile presenting in this paper a revised view of the state of the art on $\lambda\mathscr{T}$, with a reorganisation of the material, an update to the list of problems and a presentation of new problems that we consider to be very natural in light of the new picture.

The conclusion of the paper will recapitulate the new results, and will, in particular, make precise the questions raised in the previous survey that have been answered, following the order in which these questions appear there.

To express the results we are interested in, it is convenient to introduce the notions of 2^{ω}-*high*, 2^{ω}-*wide* and 2^{ω}-*broad* (a notion stronger than 2^{ω}-wide), which will allow us to make precise three interesting possible ways in which $\lambda\mathscr{T}$ and the $\lambda\mathscr{C}'s$ can be of cardinality 2^{ω}.

Models and semantics

It is well known that it took nearly 40 years to construct a proper λ-model (Scott 1969), and for this Scott had to establish Scott's continuous semantics. This first model, named $\mathscr{D}_{\infty}$, was an extensional model, built as an inverse limit. Soon afterwards, two models admitting a much simpler construction, $\mathscr{P}_{\omega}$ and $\mathscr{E}$, were proposed by Scott and Plotkin (see, for example, Barendregt (1984, page 469)) and by Engeler and Plotkin (Engeler 1981; Plotkin 1993), respectively. More generally, the class $\mathscr{G}$ of *graph models*, which are called Engeler–Plotkin–Scott algebras in Longo (1983), was isolated ($\mathscr{E}$ and $\mathscr{P}_{\omega} \in \mathscr{G}$). Let us recall that $\mathscr{E}$ is the simplest of all proper λ-models, and that all the graph models, and, more generally, all the models we call 'webbed models' (see below) can be seen as more or less sophisticated variants of $\mathscr{E}$. Let us recall also that $Th(\mathscr{P}_{\omega}) = Th(\mathscr{E}) = BT$, and $Th(\mathscr{D}_{\infty}) = H^{*} = NT$ (Hyland 1976; Wadsworth 1976; Longo 1983) (cf Barendregt (1984, Chapter 19)).

A graph model is described by a pair (D, i), called its *web*, where D is an infinite set and i an injection from $D^{*} \times D$ into D, where D^{*} is the set of all finite subsets of D. Such a model interprets a λ-term as a subset of D, and abstractions as codes in $\mathscr{P}(D)$ of Scott's continuous functions on $\mathscr{P}(D)$, the code being computed using i. Hence, all the graph models belong to $\mathscr{C}_{cont}$, where $\mathscr{C}_{cont}$ is the class of all the λ-models that can be built inside Scott's semantics. We let $\mathscr{G}^{den}$ be the class of graph models with countable web; without loss of generality one can suppose in this case that $D = N$, nevertheless, $\lambda\mathscr{G}^{den}$ is very rich.

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More generally, *webbed models* can be defined, roughly speaking, as those λ-models that interpret λ-terms in some $\mathscr{R}(D) \subseteq \mathscr{P}(D)$ for some set D, and abstractions by codes in $\mathscr{R}(D)$ of functions on $\mathscr{R}(D)$ enjoying some specific properties, including continuity. Graph models form the simplest class of webbed models.

Section 5 presents graph models, the methods available for building them, and some of the equational and inequational properties that they all share. There are two known ways of building graph models: *free and forcing completions of a partial web*. *Free completion* mimics and generalises the construction of $\mathscr{E}$. It is, essentially, a recursive procedure, similar in essence to defining intersection type systems (Coppo *et al.* 1984), though much more general in some sense, but where redundancies are systematically cleaned. As noted in Bucciarelli and Salibra (2003), free completion also allows us to define a notion of product in $\mathscr{G}$. *Forcing completion* is a non-recursive procedure, which allows one, in particular, to prove the easiness of Ω (that is, the fact that $\Omega = t$ is consistent with λ_β for all closed term t) via graph models (Baeten and Boerboom 1979). With Salibra, we recently generalised this method and used this to prove that $\lambda\mathscr{G}$, and some related sets of λ-theories, are 2^ω-broad, and that, moreover, $\lambda\mathscr{T}$ contains an interval sublattice that is distributive, 2^ω-high and 2^ω-broad. Let us define a *positive graph model* as a model that can be presented as the free completion of a *positive partial pair*, in the sense of Berline (2000) or of Definition 42. Positive graph models are sensible (the converse is false). In contrast, models built by forcing are most of the time non-sensible.

In *Section 6* we will study the structure of $\lambda\mathscr{G}$ and of $\lambda\mathscr{G}^s$, which is the class of theories of sensible graph models. In particular, we will see that $\lambda\mathscr{G}$, $\lambda\mathscr{G}^{den}$ and $\lambda\mathscr{G}^s$ are 2^ω-broad; but whether $\lambda\mathscr{G}$ and $\lambda\mathscr{G}^{den}$ are equal is an open question. We will also see that $\lambda\mathscr{G}^s$ has a largest element (Bucciarelli and Salibra 2004), which is not true for $\lambda\mathscr{G}$, and this greatest element happens to be BT. On the other hand, we will see that several results that are true for $\lambda\mathscr{G}$ still hold for $\lambda\mathscr{G}^s$, but that transferring the results from $\lambda\mathscr{G}$ to $\lambda\mathscr{G}^s$ ranges, when possible, from trivial to highly difficult (case of the uncountability of $\lambda\mathscr{G}^s$). A key question, which we raised in Berline (2000) and which is still open, concerns the existence of a (strong) approximation theorem allowing us to manage uniformly all the positive graph models. Since we now know that $\lambda\mathscr{G}^s$ admits BT as a top element, this would imply that BT is the unique positive graph theory.

We focus on the class $\mathscr{G}$ of graph models for the following reasons. First, all the recent positive results on $\lambda\mathscr{T}$ that were proved semantically have been proved using $\mathscr{G}$, and second, most of the results ever obtained on $\lambda\mathscr{T}$ via other subclasses of $\mathscr{C}_{cont}$ could have been obtained using $\mathscr{G}$, and at a significantly lower cost. Of course, this approach has to be viewed with some caution, since, for example, $\mathscr{G}$ contains no extensional models. But when dealing with extensional theories, the same philosophy applies to Krivine's extension $\mathscr{K}$ of $\mathscr{G}$ (see below and Section 7.1). In particular, $\mathscr{D}_\infty$ and its variants can be presented as elements of $\mathscr{K}$, which is the simplest way of getting rid of the inverse limit construction, and the simplest way of studying these models (Krivine 1990). However, in Section 5.2 we will give an example of equations that are consistent with λ_β but cannot

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be satisfied by models of $\mathscr{K}$, and, in particular, of $\mathscr{G}$ (Corollary 37), with the consequence that the easiness of the λ-term $\Omega_3 I$ cannot be proved using models of $\mathscr{K}$; however, it was recently shown in Alessi *et al.* (2001) that this easiness could be proved semantically using filter models. Apart from this example, the interest of classes wider than $\mathscr{G}$ or $\mathscr{K}$ only shows up when one needs a model whose underlying domain cannot be a complete lattice, for example because one wishes to model some extension of λ-calculus involving added constants with special behaviours.

Incidentally, there are two other classes of problems where using graph models makes life easier. First, the simplest models of $\mathscr{G}$, beginning with $\mathscr{E}$, can be used for proving some operational or syntactic properties of lambda-calculus in a more economical and more fluid way than when using intersection type systems (*cf.* Berline (2000)). Second, $\mathscr{G}$ and $\mathscr{K}$ can even prove interesting for showing the consistency of extensions of λ-calculus involving new constants; the example of the *strong* surjective pairing (Krivine 1990) is sketched in Section 5.5. To summarise, most of the time, using $\mathscr{G}$ (or $\mathscr{K}$ when necessary) brings the same information as when using wider classes of models of $\mathscr{C}_{cont}$ for studying $\lambda\mathscr{T}$, but more elegantly and at a significantly lower cost.

Section 7 deals with $\mathscr{K}$, and with classes $\mathscr{C}$ belonging to other semantics than Scott's one. The main variants (or refinements) of Scott's semantics are Berry and Girard's *stable semantics*, Bucciarelli and Ehrard's *strongly stable semantics,* and various *weakly continuous semantics* (a survey of these semantics and the diverse classes of models, as well as a complete bibliography can be found in Berline (2000)). What makes these semantics workable is the existence, in each of them, of classes analogous to $\mathscr{G}, \mathscr{K}$ (see Section 7); moreover, some models of $\mathscr{C}_{cont}$, like $\mathscr{D}_\infty$, $\mathscr{E}$ and $\mathscr{P}_\omega$, have analogues in these classes. Hence, in general, all the positive results proved for $\lambda\mathscr{G}$ and $\lambda\mathscr{K}$ can be transferred to these other classes; in particular, all are 2^ω-large and 2^ω-incomplete. Thus, the new information we get on $\lambda\mathscr{T}$ using these classes arises from the following third generic question.

Problem 3. What are the relative positions of the diverse $\lambda\mathscr{C}$'s in $\lambda\mathscr{T}$?

The deepest results on Problem 3, which compare $\lambda\mathscr{C}_{cont}$ and its stable and strongly stable analogues, were obtained in the nineties and were included in the survey in Berline (2000). The information we have added here is, first, that these results still hold (and more easily) when one replaces $\mathscr{C}_{cont}$ by $\mathscr{G}$, and, second, that one can greatly amplify some of these results without cost by using generalised forcing instead of usual forcing, since this allows us to prove the 2^ω-broadness of some sets of the shape $\lambda\mathscr{C} - \lambda\mathscr{C}'$, instead of their mere non-emptyness. In order to discuss this, we also have to go deeper into the presentation of the earlier results than we did in Berline (2000). We will also see (at the end of Section 2) that *game semantics*, although it has attracted much interest these last ten years in the context of *typed* λ-calculi, is not really relevant here.

We conclude this introduction with a remark that also suggests some questions. As we will see, subsets of $\lambda\mathscr{T}$ can be proved to be 2^ω-wide or 2^ω-broad by either syntactic or

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semantic means, but at present it seems we cannot say anything non-trivial about the height of $\lambda\mathscr{G}$, and hence about the height of any interesting $\lambda\mathscr{C}$.

2. Preliminaries

λ-toys

λ-calculus. Λ and Λ° are the sets of λ-terms and closed λ-terms (also called *combinators*), respectively. For some specific λ-terms we set: $I \equiv \lambda x.x$, $\varepsilon \equiv \lambda x\lambda y.xy$, $V \equiv \lambda x\lambda y.x$, $F \equiv \lambda x.\lambda y.y$, $\Omega \equiv \delta\delta$ where $\delta \equiv \lambda x.xx$, $x(y)$, $\Omega_3 \equiv \delta_3\delta_3$ where $\delta_3 \equiv \lambda x.xxx$, and $Y \equiv \lambda f.(\lambda x.f(xx))\lambda x.f(xx)$; a more traditional notation for V is K (when not viewed as a boolean). We will denote $\alpha\beta$-conversion by λ_β and $\alpha\beta\eta$-conversion by $\lambda_{\beta\eta}$. Finally, we recall that the *order* of a λ-term u is the largest integer $n \geqslant 0$ such that u is $\alpha\beta$-convertible to $\lambda x_1 ... \lambda x_n.u'$ for some u', if there is one such n, and ∞ if this is true for all n.

λ-theories. A *λ-theory* is a congruence on Λ (with respect to the operators of abstraction and application) that contains λ_β; it can also be seen as a (specific) set of equations between λ-terms. The set of all λ-theories will be denoted by $\lambda\mathscr{T}$. We will often drop the 'λ' of 'λ-theory', since we will not use any other kind of theories here. A λ-theory is *extensional* if it contains the equation $I = \varepsilon$, and we let $\lambda\mathscr{T}_e$ be the set of extensional theories. It is clear that λ_β and $\lambda_{\beta\eta}$ are the least elements of $\lambda\mathscr{T}$ and $\lambda\mathscr{T}_e$, respectively.

The *λ-theory generated* (or *axiomatised*) *by a set* E of equations is the least λ-theory containing it, and is denoted by T_E. A theory T is *finitely axiomatisable* if $T = T_E$ for some finite E; then it is clear from the existence of a pair constructor $\lambda x\lambda y\lambda z.zxy$ in Λ, that $T = T_E$ for some singleton set E. We recall that $E \vdash r = s$ means that $r = s \in T_E$. A set of equations E is *inconsistent* if $T_E = Top \equiv \Lambda \times \Lambda$; in particular, Top is the unique inconsistent λ-theory. For example, $V = F$ is inconsistent, as well as $F = I$.

We do not ask for consistency in the definition of 'λ-theory', since it will be convenient to consider Top as a λ-theory (and it is obviously extensional).

Sensible λ-theories. A λ-theory is *sensible* if it is *consistent and* all the unsolvable terms are congruent, and is *semi-sensible* if it is *consistent and* no solvable term is congruent to an unsolvable term; obviously, λ_β and $\lambda_{\beta\eta}$ are semi-sensible and non-sensible. It is well known (and easy to prove) that sensible theories are semi-sensible.

Easiness. Given a λ-theory T, a closed term u is *T-easy* if for all other $t \in \Lambda^0$ we have that $T \cup \{u = t\}$ is consistent. It is easy to check that a T-easy term is necessarily unsolvable. On the other hand, the consistency of $T' = T \cup \{u = I\}$ implies that $u = \lambda x.u \notin T$ (otherwise $F = I$ is in T'); in particular, the existence of a T-easy term implies that T is non-sensible.

A closed term u is *easy* if it is λ_β-easy, namely, if $\{u = t\}$ is consistent for all closed terms t. The term Ω is the best known easy term, and it enjoys further properties that make it possible to prove its easiness semantically (via graph models), using a forcing technique developed in Baeten and Boerboom (1979).

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λ-models. In this paper the word *λ-model* is best understood as a generic expression covering 'any possible model of untyped λ-calculus'. Alternatively, it can be given the precise definition of *λ-model* in Barendregt (1984), or be defined as any reflexive object of a cartesian closed category with enough points, or just be understood as 'the union of all the concrete classes of models the reader has in mind'. Examples of such classes will be recalled in Sections 2 and 7.1. Each λ-model $\mathscr{M}$ induces a λ-theory, denoted here by $Th(\mathscr{M})$ or by $=_{\mathscr{M}}$. Thus: $t =_{\mathscr{M}} t'$ if and only if $t = t' \in Th(\mathscr{M})$ if and only if t, t' have the same interpretation in $\mathscr{M}$. We will call any λ-model $\mathscr{M}$ that is not a term model a *proper λ-model.* If $\mathscr{M}$ is a proper model, then $Th(\mathscr{M}) \neq Top$, since otherwise $\mathscr{M}$ would be a singleton model and hence the term model of Top. Finally, a λ-model is *sensible* when $Th(\mathscr{M})$ is.

Representability of theories in classes of models.

Definition 1. Given a λ-theory T:

(i) A λ-model $\mathscr{M}$ is *a model of* T if $T \subseteq Th(\mathscr{M})$.
(ii) A λ-model $\mathscr{M}$ *represents* T if $T = Th(\mathscr{M})$.
(iii) T *is representable* if it is representable by a proper λ-model.

Definition 2. Given a class $\mathscr{C}$ of λ-models and a theory T:

(i) $\mathscr{C}$ *represents* T if there is some $\mathscr{M} \in \mathscr{C}$ representing T.
(ii) $\mathscr{C}$ *omits* T if there is no $\mathscr{M} \in \mathscr{C}$ representing T.
(iii) $\mathscr{C}$ is *complete* for $\mathscr{S} \subseteq \lambda\mathscr{T}$ if $\mathscr{C}$ represents all the elements of $\mathscr{S}$.
(iv) $\mathscr{C}$ is 2^{ω}*-incomplete* if it omits 2^{ω} different λ-theories.

Notation 3.

— $\lambda\mathscr{C}$ is the set of λ-theories that are representable in $\mathscr{C}$.
— $\lambda\mathscr{C}^e$ is the set of extensional λ-theories that are representable in $\mathscr{C}$.
— $\lambda\mathscr{C}^s$ is the set of sensible λ-theories that are representable in $\mathscr{C}$.

$\lambda\mathscr{T}$ happens to be equal to the instance of $\lambda\mathscr{C}$ corresponding to the case where $\mathscr{C}$ is the class of all possible term models, but from now on $\mathscr{C}$ will always denote a class whose definition does not refer to the syntax of λ-calculus. We will also assume that $\mathscr{C}$ is *uniform,* in the sense that it should not be defined as a union (definition by case). Such a class has no reason to contain term models, and in practice none does. To be more concise, we will adopt the following convention.

Claim 4. From now on $\mathscr{C}$ denotes a uniform class of proper models.

In particular, $\mathscr{C}$ can be any of the concrete classes of models that we will introduce later on.

Sets

N denotes the set of positive integers. For every set S: $card(S)$ denotes the cardinality of S; S^* is the set of all finite subsets of S; $\mathscr{P}(S)$ is the powerset of S; and $S^{<\omega}$, S^{ω} and $S^{\leqslant\omega}$

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are, respectively, the sets of all finite, infinite and finite or infinite sequences of elements of S. We use $l(\bar{s})$ to denote the length of the sequence $\bar{s}$. When writing $g(\bar{x})$, where g is a function, we will always understand $l(\bar{x})$ to be the arity of g. For any function $f : S \to S'$ we define $f^+ : \mathscr{P}(S) \to \mathscr{P}(S')$ by $f^+(A) = \{f(x) : x \in A\}$ and $f^- : \mathscr{P}(S') \to \mathscr{P}(S)$ by $f^-(B) = \{x : f(x) \in B\}$.

Posets

Partially ordered sets are called *posets* for short. The least (or *bottom*) element of a poset, if any, is denoted by $\perp$. Given a set S, the *flat partial order* $S_\perp$ is the poset obtained from S, viewed as a discrete ordered set, by adding a bottom element, $\perp$. The interval notation will have the obvious meaning; for example, given a poset $\mathscr{S} \equiv (S, \sqsubseteq)$ and $s, s' \in S$, we let $[s, s'] \equiv \{ s'' \in S \,/\, s \sqsubseteq s'' \sqsubseteq s' \}$ and $[s, s'[= [s, s'] - \{s'\}$.We say that $A \subseteq S$ is *closed downwards* in $\mathscr{S}$ if $l \in A$ and $l' \sqsubseteq l$ imply $l' \in A$, and the definition of *closed upwards* is symmetric.

Given a poset $\mathscr{S} \equiv (S, \sqsubseteq)$ and $S' \subseteq S$, we recall that: S' is a *a chain* of $\mathscr{S}$ if it is totally ordered by $\sqsubseteq$, and S' is *discrete* when its elements are pairwise incomparable for $\sqsubseteq$, which means that, for all $s, s' \in S'$, $s \sqsubseteq s'$ implies $s = s'$. S' is *dense* in $\mathscr{S}$ if $card(S') \geqslant 2$ and for all distinct $s, s' \in S'$ we have that $]s, s'[\cap S'$ is non-empty, and $\mathscr{S}$ itself is a *dense* poset if S is dense in $\mathscr{S}$. Finally, S' is an *antichain of* $\mathscr{S}$ if, whenever we are given $s, s' \in S'$, $s \neq s'$, the top element $\top$, if there is one, is the only possible common upper bound of s, s' in $\mathscr{S}$.

Lattices

A *lattice* is a poset $(S, \leqslant)$ such that any two elements $s, s' \in S$ have a least upper bound $s \vee s'$ and a greatest lower bound $s \wedge s'$; then $\leqslant$ is definable from $\vee$ or $\wedge$. A lattice is *complete* if any $A \subseteq S$ has a least upper bound (then all A also have a greatest lower bound); in particular, a complete lattice has a top and a bottom element. An *interval sublattice* of the lattice $(S, \leqslant)$ is an interval that is closed under $\vee$ and $\wedge$, and, hence, is a sublattice; in particular, each closed interval $[s, s']$ of a lattice is a lattice interval. A *lattice identity* is an equation $P = Q$ where P, Q are terms in the language $\{\vee, \wedge\}$; a lattice identity is *trivial* if it holds in all lattices. A well-known and useful non-trivial lattice identity is *distributivity:* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$; a weaker condition is *modularity*, which expresses the fact that distributivity holds whenever $y \leqslant x$.

Describing the size of a poset

We introduce now the following definitions, which will be useful for expressing how large, in various senses, some subsets of $\lambda\mathscr{T}$ can be. First we recall that a poset $\mathscr{S} = (S, \sqsubseteq)$ *embeds* a poset $\mathscr{S}' = (S', \sqsubseteq')$ if there is an injection $f : S' \to S$ such that for all $x, y \in S'$ we have: $x \sqsubseteq' y$ if and only if $f(x) \sqsubseteq f(y)$.

Definition 5. A poset $\mathscr{S}$ is *c-high* (respectively, *c-wide* or *c-broad*), where c is a cardinal, if $\mathscr{S}$ has a chain (respectively, a discrete subset or an antichain) of cardinality c.

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Lemma 6. If a complete lattice embeds a dense poset, it also embeds the reals $(R, <)$ and hence it is 2^{ω}-high.

Lemma 7. If an ordered set embeds $(\mathscr{P}(N), \subseteq)$, it is 2^{ω}-high and 2^{ω}-wide.

Proof. The height follows from Lemma 6, once we notice that $\mathscr{P}(N)$ contains a dense set of infinite and co-infinite subsets of N. The width comes from the fact that it is easy to build 2^{ω} pairwise incomparable subsets of N, for example $\{A_f \,/\, f : P \to P\}$, where P is the set of prime numbers and $A_f \equiv \{\, p^{f(p)} \,/\, p \in P\}$. □

Scott's semantics

Cpos (complete partial orders) and (*Scott-) continuous functions* between cpos are defined in Barendregt (1984, Chapter I.2); all complete lattices are cpos. Two cpos $\mathscr{D}$ and $\mathscr{D}'$ are Scott-isomorphic if and only if they are isomorphic as posets, namely, if there is a bijection between them such that $f(x) < f(y)$ if and only if $x < y$, and we will denote this by $\mathscr{D} \simeq \mathscr{D}'$. If $\mathscr{D}, \mathscr{D}'$ are cpos, then $[\mathscr{D} \to \mathscr{D}']$ denotes the cpo of all the continuous functions from $\mathscr{D}$ into $\mathscr{D}'$. A *reflexive cpo* is a triple $(\mathscr{D}, A, \lambda)$ such that $\lambda \in [[\mathscr{D} \to \mathscr{D}] \to \mathscr{D}]$ and $A \in [\mathscr{D} \to [\mathscr{D} \to \mathscr{D}]]$ and $A \circ \lambda = id$. Reflexive cpos model λ-calculus as follows (for more details, see Barendregt (1984, Section V.5)).

Let $Env_{\mathscr{D}}$ be the set of environments ρ mapping the set of the variables of λ-calculus into $\mathscr{D}$. For $\rho \in Env$ and $d \in \mathscr{D}$ let $\rho[x : d]$ be the environment that takes value d on x and agrees with ρ on the other variables. The interpretation $|t| : Env_{\mathscr{D}} \to \mathscr{D}$ of a λ-term t that is relative to $(\mathscr{D}, A, \lambda)$ is defined by induction as follows:

(i) $|x|_{\rho} = \rho(x)$
(ii) $|tu|_{\rho} = A(|t|_{\rho})(|u|_{\rho})$
(iii) $|\lambda x.t|_{\rho} = \lambda(d \in \mathscr{D} \mapsto |t|_{\rho[x:d]})$.

Graph models are based on cpos of the form $(\mathscr{P}(D), \subseteq)$, for some infinite set D; such cpos are, of course, complete lattices. If D is countable, a function $g : \mathscr{P}(D) \to \mathscr{P}(D)$ is continuous if and only if it is monotone and commutes with all increasing unions.

Further conventions on sets. When dealing with a graph model based on $\mathscr{P}(D)$, greek letters $\alpha, \beta, \ldots$ will always represent elements of D, small Latin letters a, b, c will represent elements of D^*, that is, finite subsets of D, and $\bar{a}, \bar{b}, \bar{c} \ldots$ elements of $(D^*)^{<\omega}$. Also, (a, α) is the usual set-theoretic pair, and $(\bar{a}, \alpha)$ is defined by induction as follows: $(\bar{a}, \alpha) = \alpha$ if $l(\bar{a}) = 0$ and $(b\bar{c}, \alpha) =_{def} (b, (\bar{c}, \alpha))$.

Traces of continuous functions. Products of cpos are again cpos. By '*a continuous function g of arity n on* $\mathscr{P}(D)$', we mean $g \in [\mathscr{P}(D)^n \to \mathscr{P}(D)]$.

A continuous function g on $\mathscr{P}(D)$, of any arity, is completely determined by its *trace*, which is defined by

$$tr(g) =_{def} \{\, (\bar{a}, \alpha) \;\; : \;\; \alpha \in g(\bar{a}) \,\}. \tag{1}$$

The trace can be viewed as the relevant part of the graph $graph(g)$ of g. Note, however, that, when the arity is one, $tr(g) \subseteq D^* \times D \subseteq \mathscr{P}(D) \times D$, while $graph(g) \subseteq \mathscr{P}(D) \times \mathscr{P}(D)$.

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Graph models owe their name to the fact that continuous functions are encoded in them via (a sufficient fragment of) their graphs, namely their traces.

Other semantics of λ-calculus

Apart from *Scott's continuous semantics*, the key examples are *Berry's stable semantics* and the *Bucciarelli–Ehrhard strongly stable semantics*, which are refinements of Scott's continuous semantics capturing some aspects of the sequentiality of λ-calculus. When we refer to the '*three main semantics*' we will mean one of these and, for brevity, we will call the λ-models living inside them the *continuous, stable* and *strongly stable λ-models*, respectively. These semantics are *functional*, in the sense that λ-terms are interpreted by morphisms that are functions; this is also the case for a further family of semantics, namely the κ-continuous semantics, where κ is any regular cardinal (κ-continuity is a weakening of Scott's continuity). No further details of these semantics beyond what is stated in this subsection should be needed to read this paper, but, if necessary, Berline (2000) contains a more detailed presentation.

Notation 8. $\mathscr{C}_{cont}$, $\mathscr{C}_{st}$ and $\mathscr{C}_{sts}$ will denote the classes of continuous, stable and strongly stable λ-models, respectively.

Similarly, $\mathscr{C}_{\kappa-cont}$ and $\mathscr{C}_{games}$ will denote the classes of models corresponding to the κ-semantics and to the game semantics, respectively.

Our main concern is that of *representability problems*. In this context *game semantics* (which is not a functional semantics) is not relevant, since it was proved in Franco (2001) that $\lambda\mathscr{C}_{games}$ (at least in the case of Abramsky *et al.* games), only contains the theories BT, LT and H^* (see Section 3.2 for a definition of these theories) while all the functional semantics happen to be very rich and are also able to represent these theories. Since no systematic study of $\lambda\mathscr{C}_{\kappa-cont}$ has yet been undertaken, we will nearly always deal here with the three main semantics, with particular emphasis on $\mathscr{G}$ (the other subclasses of models will only be treated in Section 7).

3. The lattice of λ-theories

3.1. *The complete lattice* $\lambda\mathscr{T}$

$\lambda\mathscr{T}$, ordered by inclusion, is naturally equipped with the structure of a *complete* lattice, where the meet of a family of λ-theories is their intersection ∩, and the join, written + here, is the least equivalence relation containing their union. Since Λ is countable, $card(\lambda\mathscr{T}) \leqslant 2^{\omega}$, and we will see soon that $\lambda\mathscr{T}$ is in fact 2^{ω}- high and 2^{ω}-broad.

Two theories T, T' are *incompatible if* $T \cup T'$ is inconsistent, or, equivalently, if $T + T' = Top$. Hence an *antichain of* $\lambda\mathscr{T}$ is a set of theories that are pairwise incompatible.

Notation 9.

(i) $\lambda\mathscr{T}^{re}$ denotes the set of recursively enumerable λ-theories.
(ii) $\lambda\mathscr{T}^{s}$ is the set of all sensible λ-theories.
(iii) $\lambda\mathscr{T}^{s/2}$ is the set of all semi-sensible λ-theories.

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Notation 10. For $\mathscr{S} \subseteq \lambda\mathscr{T}$:

(i) $\mathscr{S}^{re} = \mathscr{S} \cap \lambda\mathscr{T}^{re}$.

(ii) $\mathscr{S}_e = \mathscr{S} \cap \lambda\mathscr{T}_e$ will be called *the extensional analogue* of $\mathscr{S} \subseteq \lambda\mathscr{T}$.

It follows immediately from their definitions that $\lambda\mathscr{T}^s$, $\lambda\mathscr{T}^{s/2}$ and their extensional analogues are closed under (finite or infinite) intersection, that $\lambda\mathscr{T}^s$, $\lambda\mathscr{T}_e$ and $\lambda\mathscr{T}_e^s$ are closed upwards (omitting Top in the sensible case) and that $\lambda\mathscr{T}^{s/2}$ is closed downwards. Obviously, $\lambda\mathscr{T} = [\lambda_\beta, Top]$ while $\lambda\mathscr{T}_e = [\lambda_{\beta\eta}, Top]$, and we will see below that $\lambda\mathscr{T}^s$, $\lambda\mathscr{T}^{s/2}$ and their extensional analogues are also closed intervals of $\lambda\mathscr{T}$, and hence complete sublattices. As for recursively enumerable theories, note that they are non-sensible (this can be inferred from Barendregt (1984, Chapter 17.1)), but that plenty of them are semi-sensible, starting with λ_β. Of course, $\lambda\mathscr{T}^{re}$ contains all the finitely axiomatisable theories.

In the next two subsections we briefly recall the 'historical' results on $\lambda\mathscr{T}$. Most of them (or their proofs) are purely syntactic, and all of them can be found in Barendregt (1984).

3.2. *The roles of* H, BT, H^*

Since $\lambda\mathscr{T}^s$ is closed under intersections, it has a smallest element, which is the intersection of all the sensible theories, and is traditionally denoted by H.

The theories H, BT, H^* were the first theories to be isolated, and they happen to play an important role with respect to the structure of $\lambda\mathscr{T}$. The theory BT contains $t = t'$ if and only if the λ-terms t, t' have the same Böhm tree; hence it is a sensible theory, which is strictly bigger than H, and BT is non-extensional; thus no theory below BT can be extensional; in particular, H is not. The theory H^* contains $t = t'$ if and only if for all context $C[-]$ we have that $C[t]$ is solvable if and only if $C[t']$ is solvable. It follows immediately from this definition that H^* is sensible and that every semi-sensible theory is included in H^*; hence H^* is the unique maximal sensible (and semi-sensible) theory. Since $\lambda\mathscr{T}^s$ is closed upwards, there is no theory between H^* and Top, since $\lambda\mathscr{T}^{s/2}$ is closed downwards, we get $\lambda\mathscr{T}^{s/2} = [\lambda_\beta, H^*]$. Finally,

$$\lambda_\beta \subsetneq H \subsetneq BT \subsetneq H^* \subsetneq Top \tag{2}$$

$$]H^*, Top[= \varnothing \tag{3}$$

$$\lambda\mathscr{T}^s = [H, H^*] = [H, Top[\tag{4}$$

$$\lambda\mathscr{T}^{s/2} = [\lambda_\beta, H^*], \tag{5}$$

while, of course,

$$\lambda\mathscr{T} = [\lambda_\beta, Top]. \tag{6}$$

It was proved in Hyland (1976) and Wadsworth (1976) that $H^* = Th(\mathscr{D}_\infty)$, where we recall that $\mathscr{D}_\infty$ is Scott's first model; in particular, H^* is extensional. It is interesting to note that there is another characterisation of H^*, in terms of trees: $H^* = NT$, where NT equals two terms if and only if they have the same Nakajima tree (*cf.* Barendregt (1984, Exercise 19.4.4)). Finally, it is worth introducing Levy and Longo's theory LT, which

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is also defined in terms of trees. LT is a 'lazy' version of BT, which equates two unsolvable terms u and v if and only if they have the same order. We have $LT \subseteq BT$ and $\Omega = \lambda x.\Omega \in BT - LT$, hence $LT \in \lambda\mathscr{T}^{s/2} - \lambda\mathscr{T}^{s}$.

The extensional case. Given a theory T, let T_η denote the smallest extensional theory containing T and $I = \varepsilon$. Since H^* is extensional, T_η is consistent for all semi-sensible theory; furthermore, H_η is the smallest extensional sensible theory, and $H_\eta \subseteq BT_\eta \subseteq H^*$. In fact, these inclusions are strict, so, finally, in the extensional case we get similar inclusions and equalities to the above, with $\lambda_{\beta\eta}$, H_η and BT_η replacing λ_β, H and BT.

3.3. *Size and shape of the key intervals*

The following results show, in particular, that all the intervals mentioned in the previous subsection are as high and wide as possible. The proofs of the theorems can be found in Barendregt (1984, Chapters 17.1 and 16.3) and the proofs of the corollaries are immediate. The proof of the first proposition is recalled and that of the second given to allow comparison with the semantic proofs that will be given later on.

Theorem 11 (Visser 1980). If $T \in \lambda\mathscr{T}^{re}$ and $r = s \notin T$, there exists u such that, for all t, $T \cup \{u = t\} \nvdash r = s$.

Corollary 12. If T is recursively enumerable, there is a T-easy term.

Corollary 13. If T is recursively enumerable, T is non-sensible.

Proposition 14. $\lambda\mathscr{T}^{re}$ is a dense subset of $\lambda\mathscr{T}$, and the same holds for the extensional analogues.

Proof. Let $T, T' \in \lambda\mathscr{T}^{re}$ be such that $T \subsetneq T'$, let $r = s \in T' - T$ and, finally, let $S = T \cup \{ur = us\}$, where u is given by Theorem 11. It is clear that $T \subseteq T_S \subseteq T'$, that $T_S \in \lambda\mathscr{T}^{re}$. Since $T_S \cup \{u = I\} \vdash r = s$, we have $T_S \neq T$ by Theorem 11. If $T' = T_S$, then $T \cup \{ur = us\} \vdash r = s$, but $T \cup \{u = \lambda x.I\} \vdash ur = us$, hence $T \cup \{u = \lambda x.I\} \vdash r = s$, which contradicts Theorem 11. Thus $T \subsetneq T_S \subsetneq T'$. □

Proposition 15. If $T \in \lambda\mathscr{T}^{re}$, then $[T, Top]$ is 2^ω-high and 2^ω-broad.

Sketch of proof. The fact that $[T, Top]$ is 2^ω-high follows immediately from Proposition 14 and Lemma 6. The fact that it is 2^ω-broad follows from Theorem 11, plus a compactness argument. Let u be a T-easy term, let $\hat{n}$ denote the n-th Church integer. It is enough to prove that for each sequence $s = (t_n)_{n\in\omega}$ of λ-terms, the set $E_s = \cup E_{s,n}$ is consistent with T, where $E_{s,n} = \{u\hat{1} = t_1, \ldots, u\hat{n} = t_n\}$. Indeed, if s is such that $t_n = t_m$ is consistent only if $n = m$, for example, if s is itself the sequence of the Church integers, then the sets $E_{\sigma s}$, where σs is obtained from s by action of the permutation σ of ω, will generate pairwise inconsistent theories. Now, to prove that E_s is consistent with T it is enough, by compactness, to show that each of the $T \cup E_{s,n}$ is consistent. But, using the usual tricks of λ-calculus, given the sequence s, there is for each n a λ-term $cond_n$ such

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that $cond_n\hat{j} =_\beta t_j$ for all $j \leqslant n$. Now, by Theorem 11, $u = cond_n$ is consistent with T, which implies the consistency of $E_{s,n} \cup T$. □

Corollary 16. $\lambda\mathscr{T}$ is 2^ω-high and 2^ω-broad, and similarly for $\lambda\mathscr{T}_e$.

Theorem 17 (Barendregt et al. 1978). $(\mathscr{P}(N), \subseteq)$ can be embedded in $]H_\eta, BT_\eta[$, and also in $]H, BT[$.

Corollary 18. $\lambda\mathscr{T}^s$,$\lambda\mathscr{T}^{s/2}$ are 2^ω-high and 2^ω-wide, and the same is true for their extensional analogues.

This corollary follows from Theorem 17 and Lemma 7, since the three 'extensional' classes contain $]H_\eta, BT_\eta[$, and the other three contain $]H, BT[$.

In the following we will see that the 2^ω-broadness of $\lambda\mathscr{T}$ and the 2^ω-width of $\lambda\mathscr{T}^s$ and $\lambda\mathscr{T}^{s/2}$ can also be given semantic proofs, via $\mathscr{G}$.

We conclude this section by quoting a recent very nice result due to Statman, which we will not use in this paper, but reminds us (as with the existence of easy terms) that surprising things can happen in $\lambda\mathscr{T}$.

Theorem 19 (Statman 2001). There is an equation $e \notin \lambda_\beta$ that is consistent with all consistent theories.

From Corollary 16 and Zorn's Lemma it follows that the set of maximal consistent λ-theories is 'maximally large', namely 2^ω-broad; and we also know that H^* is one of its elements. Now, Statman's Theorem is equivalent to saying that there is an equation $e \notin \lambda_\beta$ in the intersection of all these theories.

3.4. *Studying the lattice properties of* λT

At the end of the nineties, Antonino Salibra launched a research program to explore the lattice $\lambda\mathscr{T}$ using techniques of universal algebra.The first result was presented in Salibra (2000), where the lattice of λ-theories is shown to be isomorphic to the lattice of equational theories of a suitable class of algebras. Then the first important remark is that not every lattices can be a lattice of equational theories; examples of such constraints are the Zipper condition and the ET condition (*cf.* Lusin and Salibra (2004)), which are not identities. Salibra proposed the following conjecture, and recently proved with Lusin an approximation of it, which involves bounded versions of $+$, and can be found in Lusin and Salibra (2004).

Conjecture 4 (Salibra 2000). $\lambda\mathscr{T}$ satisfies no non-trivial lattice identity.

Theorem 20 (Salibra 2001a). $\lambda\mathscr{T}$ is not modular (and hence not distributive).

Sketch of proof. The modularity law (see Section 2) fails for $x = H_\eta$, $y = H$ and $z = T$, where T is generated by the equation $\Omega = I$. Since H contains the equation $\Omega = \lambda x.\Omega$, we have $T + H = Top$, and hence $H_\eta \cap (H + T) = H_\eta$. On the other hand, $(H_\eta \cap H) + (H_\eta \cap T) = H + (H_\eta \cap T)$. Obviously, $\varepsilon = I \in H_\eta$; on the other hand,

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$\varepsilon = I \notin H + (H_\eta \cap T)$ (Salibra 2001a), hence $H + (H_\eta \cap T) \neq H_\eta$. In fact, the proof works with any sensible extensional theory instead of H_η. □

Obviously, the above theorem is no longer true for sublattices, in general. For example the interval lattices $\{Top\}$ and $[H^*, Top]$, which have one and two elements, respectively, satisfy a lot of identities, including distributivity.

Problem 5. (Salibra) Are there large intervals of $\lambda\mathscr{T}$ that satisfy interesting lattice identities?

There are good reasons to be interested in intervals of the form $[T, Top]$; the first is that $[T, Top]$ is isomorphic to the lattice of congruences on the term-algebra Λ/T, which is a bridge to universal algebra, and a second is that we know from Proposition 15 that for all $T \in \mathscr{T}^{re}$, and, in particular, for all finitely axiomatisable T, the interval $[T, Top]$ is as large as it can be (as we can express it to be), namely, 2^ω-high and 2^ω-broad.

Theorem 21 (Berline and Salibra 2004). There is a finitely axiomatisable theory T such that $[T, Top]$ is distributive.

The proof, which is semantic, will be given in Section 5.4 as a direct application of generalised forcing over graph models. Moreover, we will have for free that $[T, Top] \cap \lambda\mathscr{G}$ is 2^ω-broad, and hence that $\lambda\mathscr{T}$, and also $[T, Top]$ for this T, are also 2^ω-broad.

3.5. *Representability problems*

We already know that $Th(\mathscr{D}_\infty) = H^*$ and that $BT = Th(\mathscr{E}) = Th(\mathscr{P}_\omega)$; in particular, Th is non-injective (where we take as an informal domain of definition for Th the union of all conceivable $\mathscr{C}'s$). In fact, as we will see, BT can even be represented by 2^ω models of $\mathscr{G}$, and BT and H^* can also be represented in each of the other main semantics! The question of the surjectivity of Th is much more difficult; Salibra's Theorem 27 below is a strong, although not yet definite, indication that the answer should be negative, and that, moreover, the range of Th is 2^ω-incomplete.

About $\lambda_\beta, \lambda_{\beta\eta}$ and H. As mentioned in the introduction, the problems of the representability of λ_β,$\lambda_{\beta\eta}$ and of H are due to Barendregt and Honsell, respectively. Both are long-standing questions (nearly as old as the existence of Scott's continuous semantics), even though they were only first discussed in print in Honsell and Ronchi (1992).

Problem 6 (Barendregt; Honsell). Are λ_β,$\lambda_{\beta\eta}$,H representable?

These questions are still open. The first partial answers only concern $\lambda_{\beta\eta}$ and date back as recently as 1995, and all the other partial results obtained so far are essentially negative. Nevertheless, exploring these questions has allowed us to gather interesting information about the diverse $\lambda\mathscr{C}'s$, as we will see in the next subsection. The only positive result is the following one. Note, however, that the model built in its proof does not really answer Barendregt's problem in spirit, since it is 'essentiality syntactic', in the sense that its construction is based on the syntax of λ-calculus.

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Theorem 22 (Di Gianantonio et al. 1995). $\lambda_{\beta\eta}$ is representable by (a kind of) weakly continuous model.

Hint of proof. Let us say that a function (between adequate cpos) is *DHP-continuous* if it commutes with sups of increasing sequences indexed by ω_1, and let us call $\mathscr{C}_{DHP\text{-}cont}$ the class of λ-models associated to the corresponding semantics. Starting from the term model $\Lambda_{\beta\eta}$ of $\lambda_{\beta\eta}$, the authors build an inverse limit $\mathscr{M}$ in an adequate category such that $\mathscr{M} \in \mathscr{C}_{DHP\text{-}cont}$ and there is a function $p : \mathscr{M} \to \Lambda_{\beta\eta}$ such that $p(|t|^{\mathscr{M}})$ is the class of t modulo $\lambda_{\beta\eta}$, for all $t \in \Lambda^0$. Then, obviously, $Th(\mathscr{M}) = \lambda_{\beta\eta}$. □

About the $\lambda\mathscr{C}'s$. The question of the representability of $\lambda_\beta, \lambda_{\beta\eta}$, and H generated a wealth of related questions. In the following, $\mathscr{C}$ denotes any uniform class of proper models we are interested in. Of course, the following three problems are not independent.

Problem 7. Is $\mathscr{C}$ complete for $\lambda\mathscr{T}$ (or some more adequate subset of $\lambda\mathscr{T}$)?

We will see in the next section that all known semantics are 2^ω-incomplete.

Problem 8. Are $\lambda_\beta, \lambda_{\beta\eta}, H$ representable in $\mathscr{C}$?

We will see that, for $\mathscr{C} = \mathscr{G}$, the answer is 'no' for $\lambda_\beta, \lambda_{\beta\eta}$, and is still open for H.

Problem 9. Is there a least element in $\lambda\mathscr{C}$? If yes, does it admit another (and preferably nice) characterisation?

Theorem 23 (Di Gianantonio et al. 1995). $\lambda\mathscr{C}^e_{cont}$ has a least element.

Hint. Using the axiom of choice, choose one continuous model $\mathscr{M}_T$ for each theory $T \in \lambda\mathscr{C}^e_{cont}$. Starting from the cartesian product Π of the $\mathscr{M}_T$ it is possible to build an inverse limit $\mathscr{M}$ in an adequate category, in such a way that $\mathscr{M} \in \mathscr{C}^e_{cont}$ and $Th(\mathscr{M}) \subseteq Th(\mathscr{M}_T)$ for all T. The proof of this last point in Di Gianantonio *et al.* (1995) uses 'logical relations' between Π and $\mathscr{M}$; since logical relations do not distinguish terms having the same applicative behaviour, this proof can only work with extensional models.

Problem 10. Compare the shape of $\lambda\mathscr{C}$ with that of $\lambda\mathscr{T}$ or of a more appropriate sublattice.

These problems were presented in Berline (2000), though sometimes with a different formulation. Since then, several results have been obtained, and we will discuss them from Section 5 on. In particular, the case of $\mathscr{C} = \mathscr{G}$ has been thoroughly studied; but some key questions remain open.

4. Theories of ordered models

4.1. *Omitting λ_β and $\lambda_{\beta\eta}$*

Definition 24. We define a *p.o λ-model* to be any λ-model $\mathscr{M}$ such that application is monotone for some *non-trivial* partial order $\leqslant$ on the carrier set of $\mathscr{M}$, and a *p.o$_\perp$ λ-model*

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to be any *p.o* λ-model having a bottom element. We will write $\leqslant_{\mathscr{M}}$ for the preorder induced by $\leqslant$ on Λ^0; thus $t \leqslant_{\mathscr{M}} u$ if and only if $|t|_{\mathscr{M}} \leqslant |u|_{\mathscr{M}}$.

Graph models, and more generally all the proper models that we meet in λ-calculus, are $p.o_{\perp}$ λ-models. Not only are there well-known computational motivations for considering *p.o* λ-models, but it takes much energy to find λ-models that are not *p.o* λ-models. The first one was built in 1995 (Plotkin 1996) and soon afterwards it could be proved that the term models of λ_β and $\lambda_{\beta\eta}$ were not orderable. This is a corollary of the following beautiful theorem due to Peter Selinger, which also gives a partial answer to Problem 8.

Theorem 25 (Selinger 1996; 2003). Let $\mathscr{M}$ be a *p.o* λ-model. If $Th(\mathscr{M}) = \lambda_\beta$ or $\lambda_{\beta\eta}$, then the order is trivial on the interpretations of closed terms.

Sketch of the proof in Selinger (1996). There is a term $A \in \Lambda^0$ such that $Axxxy =_{\lambda\beta} Axyyy$ while $Axxxy \neq_{\lambda\beta\eta} Axxyy$, for variables x, y. Hence, for all closed terms t, u and variable s, we have that $t \leqslant_{\mathscr{M}} u$ implies $A(st)(st)(st)(su) =_{\mathscr{M}} A(st)(st)(su)(su)$. It is then enough to prove that $A(st)(st)(st)(su) \neq_T A(st)(st)(su)(su)$ if $t \neq_T u$, where $T = \lambda_\beta$ or $\lambda_{\beta\eta}$. This follows from a non-trivial lemma, which states that if $t \neq_T u$, then st and su behave like distinct variables: for all $B, C \in \Lambda^0$, if $B(st)(su) =_T C(st)(su)$, then $Bxy =_T Cxy$. The idea for finding A is the following. Let f be any closed term such that $(fx)^3 =_{\lambda\beta} fx$, where $v^3 = v \circ v \circ v$ and $v \circ u = \lambda x.v(ux)$. Take $A = \lambda x_1\lambda x_2\lambda x_3\lambda x_4.(fx_1) \circ (fx_2) \circ (fx_3) \circ (fx_4)$; then, obviously, $Axxxy =_{\lambda\beta} Axyyy$. The next step is to show that for $f = Y\lambda g.\lambda z.(gz)^3$ the equation $Axxxy = Axxyy$ can be refuted in some extensional model. A further clever insight of P. Selinger is to use a (3 element!) partial model of $\beta\eta$-reduction in order to achieve this refutation. □

Recall that the one, and only, example we have of an $\mathscr{M}$ satisfying the hypothesis of the above theorem is the DHP-continuous model built in Di Gianantonio *et al.* (1995) to prove Theorem 22, which, furthermore, is essentially syntactic.

Problem 11. Does the above statement hold for H?

Remark 26. It follows from the proof of Theorem 25 that, for the above term A, we have $Th(\mathscr{M}) \supseteq \{A(st)(st)(st)(su) = A(st)(st)(su)(su) \,/\, t \leqslant_{\mathscr{M}} u\}$ for all *p.o.* λ-models $\mathscr{M}$.

4.2. *Omitting dense sets of theories*

Salibra (2001b) gives a proof that (except maybe for very exotic orders) all uniform classes of ordered models are 2^ω-incomplete; the proof mixes tricky arguments from topology and universal algebra. The following consequence of this result already covers all the semantics we have met so far, and admits a simpler proof, due to Salibra and Plotkin (Salibra 2003), which is composed of the next two lemmas. In particular, Theorem 27 gives a positive answer to Question 2 of Berline (2000, Section 6.1, page 149) (*cf.* Section 8).

Theorem 27 (Salibra 2001b). The class of all $p.o_{\perp}$ λ-models omits 2^ω-high intervals of $\lambda\mathscr{T}$.

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In the following, (V, F) could be replaced by any pair (t, r) of terms such that $t = r$ is inconsistent.

Let Π and Π' be the λ-theories axiomatised by $\{\Omega xx = \Omega\}$ and $\{\Omega xx = \Omega\ ,\ \Omega = \Omega(\Omega VF)\Omega\}$, respectively. Clearly, $\Pi \subseteq \Pi' \subseteq H$; in particular, Π and Π' are consistent.

Lemma 28. The interval $[\Pi, \Pi']$ is 2^{ω}-high.

Sketch of proof. From Theorem 14, it is enough to prove that $\Pi \subsetneq \Pi'$. It is proved in Salibra (2003) that $(u = v) \in \Pi \iff (\Omega uv = \Omega) \in \Pi$; applying this result twice to the equation $\Omega = \Omega(\Omega VF)\Omega$, we see that it is not in Π, since otherwise Π would contain $V = F$. □

Lemma 29. The interval $[\Pi, \Pi'[$ is omitted by all $p.o_{\perp}$ λ-models.

Proof. Any p.o. model $\mathscr{M}$ of Π satisfies $\Omega = \Omega\perp\perp \leqslant \Omega VF$, hence it also satisfies $\Omega = \Omega\Omega\Omega \leqslant \Omega(\Omega VF)\Omega \leqslant \Omega(\Omega VF)(\Omega VF) = \Omega$, hence $\Omega = \Omega(\Omega VF)\Omega$. Thus $Th(\mathscr{M}) \supseteq \Pi$ implies $Th(\mathscr{M}) \supseteq \Pi'$. □

Corollary 30. All the known classes $\mathscr{C}$ of λ-models are 2^{ω}-incomplete.

Note that, given a specific class $\mathscr{C}$, it is in general possible to find more natural intervals (for example, $[\lambda_{\beta\eta}, Top]$ for $\mathscr{G}$, since no graph model is extensional). We will also see below that for each of the usual classes $\mathscr{C}$, $\lambda\mathscr{T} - \lambda\mathscr{C}$ is 2^{ω}-broad. While Theorem 27 and its proof produce 2^{ω}-high intervals, which are, furthermore, independent of $\mathscr{C}$, and hence bring uniformity.

5. Graph models

5.1. *Definition*

For brevity we will confuse graph models and their webs, hence we have the following definition.

Definition 31. A *graph model* is a pair (D, i), where D is a non-empty set and $i : D^* \times D \to D$ is a total injective function.

Such a pair will also be called a *total pair*. It follows from the definition that D is infinite. A total pair (D, i) generates a reflexive cpo $(\mathscr{P}(D), \lambda_i, A_i)$, where λ_i and A_i are defined as follows:

$$\lambda_i(g) = \{\, i(a, \alpha) \ :\ \alpha \in g(a) \,\}.$$

The left inverse $A_i \in [\mathscr{P}(D) \to [\mathscr{P}(D) \to \mathscr{P}(D)]]$ of λ_i (which allows the interpretation of application in the model) is defined by

$$A_i(d)(d') = \{\alpha \in D : (\exists a \subseteq d')\ i(a, \alpha) \in d\}$$

where d, d' are arbitrary subsets of D. When no ambiguity occurs we write $d\,d'$ instead of $A_i(d)(d')$. The interpretation $|t|^i : Env_{\mathscr{P}(D)} \to \mathscr{P}(D)$ of a λ-term t with respect to (D, i) is

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hence defined by induction as follows:

— $|x|^i_\rho = \rho(x)$
— $|tu|^i_\rho = \{\alpha : (\exists a \subseteq |u|^i_\rho)\ \ i(a,\alpha) \in |t|^i_\rho\}$
— $|\lambda x.t|^i_\rho = \{\, i(a,\alpha) : \alpha \in |t|^i_{\rho[x:a]}\}$.

Since $|t|^i_\rho$ only depends on the value of ρ on the free variables of t, we just write $|t|^i$ if t is closed, and just $|t|$ if i is clear from the context.

Example 32.

$$\begin{aligned} |I|^i \equiv |\lambda x.x|^i &= \{\, i(a,\alpha) \,/\, \alpha \in a\} \\ |K|^i \equiv |\lambda x.\lambda y.x|^i &= \{\, i(a,i(b,\alpha)) \,/\, \alpha \in a\,\} \\ |\varepsilon|^i \equiv |\lambda x\lambda y.xy|^i &= \{\, i(a,i(b,\alpha)) \,/\, \exists b' \subseteq b\ (b',\alpha) \in a\,\} \\ |\delta|^i \equiv |\lambda x.xx|^i &= \{i(a,\alpha) \,/\, \alpha \in aa\,\}. \end{aligned}$$

Remark 33. It is easy to check that for all $t \in \Lambda^0$ and all (D,i) we have:

$$|t|^i \cap range(i) \subseteq |\varepsilon t|^i \subseteq range(i).$$

Recall that $\mathscr{G}$ denotes the class of graph models and $\mathscr{G}^{den}$ denotes the class of graph models whose web is countable; thus, all the models in $\mathscr{G}^{den}$ have cardinality 2^ω. The elements of $\lambda\mathscr{G}$ will be called *graph theories* for short.

5.2. *First equational and inequational properties*

Lemma 34. No graph model is extensional.

Proof. In all graph models (D,i) we have $i(\{i(\varnothing,\alpha)\}, i(\{\alpha\},\alpha)) \in |\varepsilon| - |I|$. □

This first, and now old, observation happens to be the simplest instance of a general, and much less trivial, recent result, which we will state in Section 6.1.

Lemma 35. In all (D,i) and for all $t \in \Lambda^0$ we have:

(i) $|I| \subset |\varepsilon|$ if and only if i is onto.
(ii) $|t| \subseteq |\varepsilon t|$ if and only if $|t| \subseteq range(i)$.

Proof.

(i) It is easy to check this part, and the proof is left as an exercise.
(ii) This part follows easily from Remark 33. □

Lemma 36 (Kerth 1995). In all graph models (D,i) we have:

(i) $|\delta_3| \neq D$ and,
(ii) if $\alpha \in |\Omega_3|$, then $\alpha = i(\bar{a},\alpha)$ for some $n > 0$ and $a_1, \ldots, a_n \subseteq |\delta_3|$.

Proof.

(i) It is easy to see that $|\delta_3|$ contains no element of the form $i(\varnothing,\gamma)$.
(ii) The proof of this part, which can be found in Kerth (1995, Example 5.3.7), is a non-straightforward, but not difficult, exercise. □

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Corollary 37 (Kerth 1995). No graph model satisfies $\Omega_3 u = I$ (for any $u \in \Lambda^0$).

Proof. Let (D, i) be a graph model, and let $\gamma \in D - |\delta_3|$. We have $i(\{\gamma\}, \gamma) \in |I|$. If $i(\{\gamma\}, \gamma) \in |\Omega_3 u|$, there is $a \subseteq |u|$ such that $\gamma' = i(a, i(\{\gamma\}, \gamma)) \in |\Omega_3|$. By (ii) of the above lemma and using the injectivity of i, we get $\gamma \in |\delta_3|$, which is a contradiction. □

This corollary, which remains true for $\mathscr{K}$ (and beyond) shows a concrete limitation of $\mathscr{G}$ and $\mathscr{K}$. Indeed, $\Omega_3 I$ is an easy term; this was proved syntactically in Jiacopini and Venturi Zelli (1985), but was only recently given a semantic proof in Alessi *et al.* (2001), where the authors build, for each closed t, a filter model of $\Omega_3 I = t$. Now, the above corollary shows that there can exist no semantic proof via graph models or models of $\mathscr{K}$, in contrast to the case Ω, since $\Omega_3 I = I$ is satisfied in no such model. This essentially answers Question 5 of Berline (2000, Section 6.3, page 152).

From Lemma 36, Bucciarelli and Salibra also derived the following interesting consequences; the first is immediate, also using Lemma 35.

Corollary 38. $|\Omega_3| \subseteq |\varepsilon\Omega_3|$ is true in all graph models.

Combining this property with Theorem 25 we get the following corollary.

Corollary 39 (Bucciarelli and Salibra 2004). $\mathscr{G}$ omits λ_β.

Proposition 40 (Bucciarelli and Salibra 2004). Each sensible graph model interprets all the closed unsolvable terms by the empty set.

Proof. It is enough to show that if a graph model (D, i) satisfies $v = \lambda x.v$ and $v \subseteq \Omega_3$ for some closed term v, then (D, i) interprets v by $\varnothing$. Suppose $\alpha \in |v|$, then also $i(a, \alpha) \in |v| \subseteq |\Omega_3|$, for all $a \in D^*$. By Lemma 36, $i(a, \alpha) = i(b, \beta)$ for some $b \subseteq |\delta_3|$ and some β; hence $a = b \subseteq |\delta_3|$; since a is arbitrary, we would have $D \subseteq |\delta_3|$, which is a contradiction. □

The class $\mathscr{G}_{lazy}$ of *lazy graph models* differs from $\mathscr{G}$ by a small variation in the definition of λ_i, which works only if i is non-surjective: fix $\gamma \in D - range(i)$, then define $\lambda'_i(f) \equiv \lambda_i(f) \cup \{\gamma\}$. It is easy to check that $(\mathscr{P}(D), \lambda'_i, A_i)$ is still a reflexive cpo. The first and simplest example of a lazy graph model is the lazy variation $\mathscr{E}_{lazy}$ of $\mathscr{E}$ introduced in Longo (1983); as we have already noted, $Th(\mathscr{E}_{lazy}) = LT$, hence LT is representable in $\mathscr{G}_{lazy}$.

5.3. *Building graph models*

There are only two methods for building graph models, and both consist of *completing a partial pair into a total one*, either *freely* or *by forcing*. When applied to total pairs, both methods leave them unchanged. There is also a product construction, called $\mathscr{G}$-product below, which is a particular case of free completion.

All these methods can be extended to the other classes of webbed models, with more or less ease (see Section 7). The systematic extension of the free completion method to $\mathscr{K}$ was written down in Berline (2000), but some particular cases had been previously worked out by several authors, beginning with Krivine (1990) (*cf.* Berline (2000)).

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5.3.1. *Partial pairs.*

Definition 41. A *partial pair* is a pair (P, j) where P is a non-empty set and j is a partial (possibly total) injection from $P^* \times P$ to P, written $j : P^* \times P \rightharpoonup P$.

The simplest example of a partial pair is $(P, \varnothing)$, and, of course, any total pair is a partial pair. In the examples, j will be described by its graph (in the usual sense). A crucial dichotomy occurs between those pairs that we call positive, and the others. A definition of a positive pair, in full generality, was proposed in Berline (2000, page 125), and it was left to the reader to write down alternative formalisations. Such a (more intuitive, but possibly less general) formalisation is the following.

Definition 42. (P, j) is *positive* if there exists a function $v : P \to \{+, -\}$ such that $((a, \alpha), \beta) \in graph(j)$ implies $v(\beta) = v(\alpha)$ and $v(\beta) = -v(\gamma)$ for all $\gamma \in a$.

5.3.2. *The free completion method.* This method, which is just called 'completion' in Berline (2000), and 'canonical completion' elsewhere, generalises the construction of $\mathscr{E}$. It was introduced in Longo (1983), who also proved that the graph model P_ω is isomorphic to the free completion of the pair$(\{0\}, \{(\varnothing, 0), 0\})$, in a sense, we will not make precise here. It was then used on a larger scale by Kerth (see Sections 6.2 and 6.3), who also transferred the method to other semantics (Kerth 1995; Kerth 1998b; Kerth 2001), and it was also used recently in Bucciarelli and Salibra (2003; 2004). *Free* refers here to the fact that the graph model (D, i) is built in an inductive and canonical way from the partial pair (P, j) that we start with, as freely as possible.

Definition 43. The *free completion of the partial pair* (P, j) is the total pair (D, i) where D is the smallest set such that

$$D = P \cup ((D^* \times D) - dom(j)),$$

and i is defined by

$$\begin{aligned} i(a, \alpha) &= j(a, \alpha) \text{ if } (a, \alpha) \in dom(j) \\ i(a, \alpha) &= (a, \alpha) \text{ otherwise.} \end{aligned}$$

Hence D is countable and could be defined as the union of an increasing sequence of sets D_n, while i is (globally) defined at the end. We now recall the key examples (others can be found in Berline (2000)).

Example 44 (Engeler's model). $\mathscr{E}_P$ is the free completion of the (positive) pair $(P, \varnothing)$; in this case i is just inclusion. Note that each element of $\mathscr{E}_P$ can be written uniquely as $(\bar{a}, \alpha)$ for some $\alpha \in P$ and $\bar{a} \in D^{*<\omega}$.

Example 45. *The model* $\mathscr{P}_P$ is the free completion of the (positive) partial pair (P, j_P) where $dom(j_P) = \{\varnothing\} \times P$ and $j_P(\varnothing, p) = p$ for all $p \in P$.

Example 46. *The model* $\mathscr{P}'_P$ is defined as above except that $dom(j_P) = \{(\{p\}, p) : p \in P\}$ and $j_P(\{p\}, p) = p$ for all $p \in P$.

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Definition 47. We define the *positive graph theories* to be the theories of graph models that can be obtained by freely completing a positive pair.

Formally, the definitions of $\mathscr{P}_P$ and $\mathscr{P}'_P$ are similar, and the web of $\mathscr{P}'_P$ is as simple as for $\mathscr{P}_P$. However, the two families behave quite differently, as we will see soon; in particular, all the models $\mathscr{P}_P$ are sensible, while in $\mathscr{P}'_P$ we have $|\Omega| = P$ and $|\Omega t|_\rho = P \cap |t|_\rho$ (Honsell and Ronchi 1992). The key structural difference between the two families is that $\mathscr{P}_P$ is generated by a positive pair, which is not the case for $\mathscr{P}'_P$.

Indeed, if a model happens to be the free completion of a positive pair, we have a (uniform) control on its theory. First it can be proved that positive graph theories are *sensible* (*cf.* Berline (2000, page 125)); the more direct way to prove it is to use a reducibility method (in the spirit of Tait) directly in the model, as is done there. Second, if, furthermore, one can apply the *strong* approximation theorem[†] in the spirit of Hyland and Wadsworth (Hyland 1976; Wadsworth 1976), which is the case for the $\mathscr{E}_P$'s and the $\mathscr{P}_P$'s (in particular for P_ω), then $Th(D, i)$ is completely known, and equal to BT. An open problem, which we raised in Berline (2000), and for which we have only partial positive answers yet, is whether this is always true for models generated by positive pairs. We will return to this point later on.

Remark 48. The theory of $\mathscr{E}_P$ (respectively, $\mathscr{P}_P$ and $\mathscr{P}'_P$) is independent of P.

Remark 49. Viewing P_ω as $\mathscr{P}_{\{0\}}$ makes it easier to study $Th(P_\omega)$.

The reason P_ω and $\mathscr{P}_{\{0\}}$ can be proved to have the same theory is that we have a good notion of isomorphism between webs. The definition is as follows: let us say that $\theta : D \to D'$ is a *morphism* between (D, i) and (D', i') if for all a, α we have $\theta(i(a, \alpha)) = i'(\theta^+(a), \theta(\alpha))$; by definition, θ is an *isomorphism* if it is also a bijection (then its inverse is also an isomorphism). This notion of isomorphism is good in the sense that isomorphic webs generate models with the same theory; but the mere notion of morphism happens to be disappointing (one would of course like to induce an inclusion of theories).

5.3.3. *The graph-product construction.*

Definition 50 (Bucciarelli and Salibra 2003). We define the *$\mathscr{G}$-product of the family* $(D_e, i_e)_{e \in E}$ of graph models, where the D_e are supposed to be disjoint, to be the free completion (D_E, i_E) of $(\cup D_e, \cup i_e)$, and hence a graph model.

Note that $\mathscr{G}^{den}$ is only closed under countable products (that is, E should be countable). In the following it will be clear that the families we have chosen can be supposed to consist of disjoint sets without loss of generality. Finally, in the rest of this subsection we freely use the notation of the above definition.

In order to make clearer the presentation of the two following propositions, which express the key technical properties of the $\mathscr{G}$-product, and that of their consequences, we will state them separately. However, they are not independent, in the sense that their only

[†] The more general, but weaker, Approximation Theorem in Honsell and Ronchi (1992), though helpful, is not enough.

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known proof is global and by mutual induction on t, once the family $(f_e)_{e\in E}$ has been exhibited.

Proposition 51 (Bucciarelli and Salibra 2003). For all $t \in \Lambda^\circ$ we have:
$|t|^{i_e} = |t|^{i_E} \cap D_e$.

Corollary 52. $Th(D_E, i_E) \subseteq \bigcap_{e\in E} Th(D_e, i_e)$.

Corollary 53. Any $\mathscr{G}$-product of semi-sensible graph models is semi-sensible.

Proposition 54 (Bucciarelli and Salibra 2003). There is a family $(f_e)_{e\in E}$ of functions $f_e : D_E \to D_E$ such that:

(i) D_E is the (disjoint) union of the $f_e^{-1}(D_e)$, $e \in E$.
(ii) $f_e(|t|^{i_E}) \subseteq |t|^{i_E}$.

Corollary 55 (Bucciarelli and Salibra 2004). Any $\mathscr{G}$-product of sensible graph models is sensible.

Proof. By Proposition 40, we have to prove that all unsolvables v are interpreted by $\varnothing$ in the product. Suppose $\alpha \in |v|^{i_E}$; by (i) of Proposition 54, there is an $e \in E$ such that $f_e(\alpha) \in D_e$, and by (ii) of the same proposition, $f_e(\alpha) \in |v|^{i_E}$. Thus $f_e(\alpha) \in |v|^{i_e}$ by Proposition 51, which contradicts the sensibility of (D_e, i_e), using Proposition 40 once more. □

5.3.4. *The forcing completion method.* The origin of this method is in Baeten and Boerboom (1979), where it was used to prove the 'easiness' of Ω. In the simpler presentation proposed in Zylberajch (1991) (see, for example, Berline and Salibra (2004) for a published proof), one starts from a partial pair $(D, \varnothing)$, where D is an infinite countable set, and builds by induction a total $i : D^* \times D \to D$, hence a graph model (D, i). Thus, here, D is fixed during the construction. Another difference is that the inductive construction itself depends on the consistency problem we are interested in, and, moreover, it heavily exploits the fact that the interpretation of Ω can be quite freely constrained. The method was generalised in Zylberajch (1991) to families of terms having a similar behaviour to Ω, and then transferred by other authors to other classes of models and other semantics (*cf.* Section 7.2). One may also have to start from pairs (D, p_0), with small constraints on p_0.

5.3.5. *Comparing free and forcing completions.* In addition to the differences mentioned above, it is worth noting the following points where the two methods behave differently.

Control or non-control over $Th(\mathscr{M})$? Forcing *never* gives us control on the whole of $Th(\mathscr{M})$, even if we start from a positive partial web. A first consequence is that, even though $(D, \varnothing)$ is a positive web, it is likely that no model built by completing $(D, \varnothing)$ by forcing will be sensible, and most of them are, furthermore, clearly non-semi-sensible; thus forcing cannot be used to produce sensible theories or to study $\lambda\mathscr{G}^s$ and $\lambda\mathscr{T}^s$. A

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second consequence is that forcing cannot be used to study the height of $\lambda\mathscr{G}$ since, given two models $\mathscr{M}$, $\mathscr{M}'$ built by forcing, we will never be able to prove that $Th(\mathscr{M}) \subseteq Th(\mathscr{M}')$.

Preserving recursivity or not. From a recursive partial web, free completion builds a recursive total web (hence a graph model that could be viewed as a reasonable intersection type system – *cf.* Berline (2000)), while non-trivial forcing always creates a non-recursive web.

Possible cardinalities of the webs. Forcing produces models with countable webs, while free completion can be used for building webs of any infinite cardinality.

Mass production of models. Free completion allows for the mass production of *non-isomorphic* graph models (and probably might allow for 2^ω non-isomorphic *sensible* graph models), while the usual forcing does not. However, and as we will see in the next subsection, it is possible to extend the method so that it becomes very easy to create 2^ω graph models with pairwise inconsistent theories. Such theories are neither semi-sensible nor recursively enumerable. As already mentioned, this generalisation also has other interesting applications to $\lambda\mathscr{T}$.

5.3.6. *Generalised forcing.* In Berline and Salibra (2004) the authors generalise the forcing method in three directions. They first note that the method works with other 'operators' besides forcing (but this is inessential for our current discussion), second, that it can be applied to terms with parameters in $\cup[P(D)^n \to P(D)]$, where the union is taken over $n \geqslant 0$, which opens the way to a lot of potential applications and, third, that it allows us to treat any (finite and) infinite sequence of closed terms instead of a single term t as above. Given a set D we define the set Λ_D of *D-generalised λ-terms* with the same inductive definition as for Λ, except that Λ_D is also closed under all continuous functions on $P(D)$ of arbitrary arity $n \geqslant 0$. In other words, we add the clauses $P(D) \subseteq \Lambda_D$ and, if $f \in [P(D)^n \to P(D)]$, then $n \geqslant 1$, and if $t_1, \ldots, t_n \in \Lambda_D$, then $f(t_1, \ldots, t_n) \in \Lambda_D$.

As a corollary of the main results proved in the paper, we get the following theorem.

Theorem 56 (Berline and Salibra 2004). There is a sequence of easy terms $(v_k)_{k<\omega}$ such that, for all countable sets D and all sequences $(r_k)_{k\in\omega} \in {\Lambda_D}^\circ$, there is $i : D^* \times D \to D$ such that the graph model (D, i) satisfies $v_k = r_k$ for all k.

The simplest infinite such sequence is $v_k = \Omega\pi_{k,k}$, where $\pi_{k,k} \equiv \lambda x_1 \ldots \lambda x_k . x_k$; the result is, however, far from straightforward.

5.4. *Applications to $\lambda\mathscr{T}$*

We now present two direct applications of generalised forcing to $\lambda\mathscr{T}$, which, furthermore, give evidence that the method is very promising, and should produce many other interesting consequences in the future. First we prove, semantically, that $\lambda\mathscr{T}$ is 2^ω-broad.

Proposition 57 (Berline and Salibra 2004). $\lambda\mathscr{G}^{den}$ is 2^ω-broad, hence $\lambda\mathscr{T}$ is also.

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Proof. Let $s = (t_k)_{k\in\omega}$ be an infinite sequence of closed normal terms such that $t_n = t_m$ is consistent only if $m = n$, for example, s is the sequence of Church integers. Because of Theorem 56, for each permutation σ of ω, there is an i_σ such that the graph model $G_\sigma \equiv (D, i_\sigma)$ satisfies $v_k = t_{\sigma(k)}$ for all k. Now, by Böhm's theorem, $t_m = t_n$ is inconsistent if $m \neq n$, thus $Th(G_\sigma)$ and $Th(G_{\sigma'})$ are inconsistent if $\sigma \neq \sigma'$. □

Proposition 58. There is a finitely axiomatisable theory T such that $[T, Top]$ is a distributive lattice.

Proof. Given D, intersection $\cap$ and union $\cup$ are two binary continuous functions on $\mathscr{P}(D)$. Let $r_1, r_2 \in \Lambda_D$ be defined by $r_1 \equiv \lambda x\lambda y.(x \cap y)$ and $r_2 \equiv \lambda x\lambda y.(x \cup y)$. Let $G \equiv (D, i)$ be any graph model satisfying $v_1 = r_1$ and $v_2 = r_2$, where v_1, v_2 are as in Theorem 56. Since $\mathscr{P}(D)$ is a (distributive) lattice, G also satisfies the equations between terms of Λ that express the fact that we have a (distributive) lattice when v_1, v_2 play the role of join and meet (for example distributivity itself is expressed by the equation $v_1x(v_2yz) = v_2(v_1xy)(v_1xz)$). Let T be the λ-theory generated by these equations. We have proved that T is consistent and that there are two terms that make its term model Λ/T a (distributive) lattice. Thus, the lattice of congruences of Λ/T is the lattice of congruences of a lattice, and hence a distributive lattice, by McKenzie *et al.* (1987, Theorem 2.50, and the remark following the proof of this theorem). Since $[T, Top]$ is isomorphic to this lattice, it is also a distributive lattice. □

By slightly modifying the proof, we have automatically that $[T, Top]$ is 2^ω-broad: just work as in the proof of Proposition 57, but with sequences $r_1, r_2, t_1, \ldots, t_n, \ldots$ where r_1 and r_2 are as above (and untouched by the permutations σ). This also proves point (i) of the following remark; to prove point (ii), one just has to use sequences $r_1, r_2, \hat{n}$.

Remark 59.

(i) There exist 2^ω pairwise inconsistent theories T such that $[T, Top]$ is a distributive lattice.

(ii) There exist ω pairwise inconsistent and finitely axiomatisable theories T such that $[T, Top]$ is a distributive lattice.

5.5. *Graph models and strong surjective pairing*

A λ-model $\mathscr{M}$ *models surjective pairing (SP)* if there are $c, p_1, p_2 \in \mathscr{M}$ such that the equations $p_1(cxy) = x$, $p_2(cxy) = y$ and $c(p_1z)(p_2z) = z$ are satisfied in the model. It is well known that the term models of λ_β and $\lambda_{\beta\eta}$ do not model surjective pairing (see Klop (1980), *cf.* Barendregt (1984, Example 15.4.4)). On the other hand, it is easy to see that any proper model $\mathscr{M} = (\mathscr{D}, A, \lambda)$ living, say, in $\mathscr{C}_{cont}$, and such that $\mathscr{D} \simeq \mathscr{D} \times \mathscr{D}$ models surjective pairing, since then one can indeed take $c = \lambda(f)$, $p_1 = \lambda(\pi_1)$ and $p_2 = \lambda(\pi_2)$, where $f : \mathscr{D} \times \mathscr{D} \to \mathscr{D}$ and $(\pi_1, \pi_2) : \mathscr{D} \to \mathscr{D} \times \mathscr{D}$ are any pair of inverse isomorphisms. Note that all graph models have this property, since for all infinite sets D we have $\mathscr{P}(D) \times \mathscr{P}(D) \simeq \mathscr{P}(D)$; to see this, note that clearly $\mathscr{P}(D) \simeq \mathscr{P}(D_1) \times \mathscr{P}(D_2)$ for any *good* partition $D = D_1 \cup D_2$ of D, where *good* means into two subsets of the same cardinality as D.

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$\mathscr{M}$ models strong surjective pairing (Strong SP) if $\mathscr{M}$ models SP in such a way that it also satisfies $p_1xy = p_1(xy)$ and $p_2xy = p_2(xy)$. It was proved in Krivine (1990) (and rediscovered independently in Durfee (1997)) that $\mathscr{D}_\infty$ models Strong SP. Then Jiang proved in her thesis (Jiang 1993; Jiang 1995) that one could build, by forcing, extensional models of $\mathscr{K}$ that also satisfied much stronger constraints.

We wish to point out here that (a lot of) graph models can also model Strong SP, and that the simplest model of Strong SP is Engeler's model $\mathscr{E}_P$, P infinite (the simplest extensional models being $\mathscr{D}_\infty$ or $\mathscr{P}_\infty$, when presented as $\mathscr{K}$-models, and also relative to an infinite P). In fact, the following proof is just the relevant simplification of Krivine's proof for $\mathscr{D}_\infty$. Jiang's variants of strong pairing could also have been modelled more simply within graph models, at the price of loosing extensionality.

Proposition 60. If P is infinite, $\mathscr{E}_P$ models strong surjective pairing.

Proof. Let P be an infinite set, let $(D, \subseteq)$ be the web of $\mathscr{E}_P$, let $P = P_1 \cup P_2$ be a good partition of P, and, finally, let $D_i = \{(\bar{a}, \alpha) : \alpha \in P_i\}$ for $i = 1, 2$ (thus $D = D_1 \cup D_2$ is also a good partition of D). Now let $\varphi_i : P \to P_i$ be two bijections, let $\bar{\varphi}_i : D \to D$ be defined by $\bar{\varphi}_i(\bar{a}, \alpha) = (\bar{a}, \varphi_i(\alpha))$ if $\alpha \in P$, let $\pi_i = \bar{\varphi}_i^- : \mathscr{P}(D) \to \mathscr{P}(D)$ and let $f : \mathscr{P}(D) \times \mathscr{P}(D) \to \mathscr{P}(D)$ be defined by $f(d_1, d_2) = \bar{\varphi}_1^+(d_1) \cup \bar{\varphi}_2^+(d_2)$. Then it is clear that f and (π_1, π_2) are inverse isomorphisms, and it is also easy to check that $p_i = \lambda(\pi_i)$ satisfies, for $i = 1, 2$, the further condition needed for Strong SP. □

6. The structure of $\lambda\mathscr{G}$

In order to have a clearer picture of $\lambda\mathscr{G}$, we continue to feel free of any respect for historical chronology. As we said earlier, we will see that extending some of the results obtained for $\mathscr{G}$ to $\mathscr{G}^s$, and hence from $\lambda\mathscr{T}$ to $\lambda\mathscr{T}^s$, ranges, when possible, from straightforward to highly difficult.

The links between $\lambda\mathscr{G}$ and $\lambda\mathscr{G}^{den}$ are simpler, in a sense, since all the results proved in this paper for $\lambda\mathscr{G}$ also hold for $\lambda\mathscr{G}^{den}$, and with the same proofs, because one only uses models built by forcing or free completion of finite or countable partial webs, and hence in $\mathscr{G}^{den}$. However, the following natural question is, as far as we know, still open.

Problem 12. Does $\lambda\mathscr{G} = \lambda\mathscr{G}^{den}$?

6.1. *Least and largest elements of $\lambda\mathscr{G}$ and $\lambda\mathscr{G}^s$*

The following lemma is a corollary of the proof of Proposition 57, which provides 2^ω pairwise inconsistent graph theories, but it can also be given a direct and shorter proof, as below.

Lemma 61. $\lambda\mathscr{G}$ and $\lambda\mathscr{G}^{den}$ have no greatest element.

Proof. Let $\mathscr{M} \in \mathscr{G}^{den}$ satisfying $\Omega = I$ (by forcing). Then $Th(\mathscr{E})$ and $Th(\mathscr{M})$ are incompatible, since $\Omega = \lambda x.\Omega$ is in $BT = Th(\mathscr{E})$, and the set $\{\Omega = I\,, \Omega = \lambda x\Omega\}$ is inconsistent. □

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The situation happens to be different for $\mathscr{G}^s$, as discovered by Bucciarelli and Salibra. The proof of the following theorem, which gives the key to a positive answer, and also generalises the fact that no graph model is extensional, is quite technical and can be found in Bucciarelli and Salibra (2004).

Theorem 62 (Bucciarelli and Salibra 2004). All the equations of $H^* - BT$ are false in all graph models.

Corollary 63 (Bucciarelli and Salibra 2004). $\lambda\mathscr{G}^s$ and $\lambda\mathscr{G}^{s/2}$ have a greatest element, which is BT in both cases.

Proof. Let $T \in \lambda\mathscr{G}^s$. Since T is sensible, we have $T \subseteq H^*$; then, since T omits each of the equations of $H^* - BT$, we have $T \subseteq BT$. □

Theorem 64 (Bucciarelli and Salibra 2003). If $\mathscr{C} \subseteq \mathscr{G}$ is closed under countable $\mathscr{G}$-products, $\lambda\mathscr{C}$ has a least element.

First proof. Let E be the (countable) set consisting of the equations that fail to hold in some graph model. For each $e \in E$ let (D_e, i_e) be a model where e is false, let T_E be the theory of the $\mathscr{G}$-product (D_E, i_E) of this family, and, finally, let T_e be the theory of (D_e, i_e). By Theorem 52, $T_E \subseteq \cap T_e$. Furthermore, if an equation does not belong to T_E, that is, fails in (D_E, i_E), which belongs to $\mathscr{C}$ by hypothesis, then it is one of the $e's$ and it fails in the corresponding (D_e, i_e), hence it is not in T_e. Thus $T_E = \cap T_e$. □

Second proof. This proof is simpler but assumes that $\mathscr{C}$ is closed under arbitrary products.

For each $T \in \lambda\mathscr{C}$ take a model (D_T, i_T) whose theory is T. The product (D_E, i_E) has theory $T_E \subseteq \cap\lambda\mathscr{C}$. Since $T_E \in \lambda\mathscr{C}$ we have $T_E = \cap\lambda\mathscr{C}$. □

Since $\mathscr{G}$,$\mathscr{G}^s$ and $\mathscr{G}^{s/2}$ are closed under arbitrary products (by Corollary 53 and Proposition 55), and $\mathscr{G}^{den}$ under countable products, we get the following corollary.

Corollary 65. $\lambda\mathscr{G}, \lambda\mathscr{G}^s, \lambda\mathscr{G}^{s/2}$ and $\lambda\mathscr{G}^{den}$ have least elements.

Let $T_{\min}$ and $T^s_{\min}$ be the least elements of $\lambda\mathscr{G}$ and $\lambda\mathscr{G}^s$. Since $T_{\min} \subseteq H$, $T_{\min}$ is semi-sensible, and since $\lambda_\beta \notin \lambda\mathscr{G}$ (by Corollary 39) we have $\lambda_\beta \subsetneq T_{\min}$. Bucciarelli and Salibra made the following conjecture concerning $\lambda\mathscr{G}^s$.

Conjecture 13 (Bucciarelli and Salibra 2004). $T^s_{\min} = H$.

6.2. *Width and height of $\lambda\mathscr{G}$ and $\lambda\mathscr{G}^{den}$*

We have already proved via forcing that $\lambda\mathscr{G}$ is 2^ω-broad. We now present some earlier results and proofs obtained via free completion, which are mainly due to Kerth (Kerth 1994; 1995). These proofs bring further information on $\lambda\mathscr{G}$ and $\lambda\mathscr{T}$, and also raise other kinds of questions. Everywhere $\mathscr{G}$ could be replaced by $\mathscr{G}^{den}$.

Theorem 66 (Kerth 1994). $\lambda\mathscr{G}$ is 2^ω-wide.

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Sketch of proof. Kerth produced a family of non-positive pairs $(A_W, j_W)_{W\in\mathscr{P}(N)}$, and sets of equations $R_W = \{YF_{n+1} = YF_2 : n \in W\}$, where $F_n = \lambda x.\lambda x_1 \dots \lambda x_n.x$, such that (A_W, j_W) satisfies all the equations of R_W and no equation of $R_N - R_W$. Since $R_W \subseteq R_{W'}$ if and only if $W \subseteq W'$, and since $(\mathscr{P}(N), \subseteq)$ contains 2^ω pairwise incomparable sets (see the proof of Lemma 7), we deduce immediately that there are 2^ω pairwise incomparable graph theories. □

These models are non-sensible, since all the YF_n are clearly unsolvable. Later, Kerth produced in Kerth (1995), on the same principles, another family of theories, for which he had the hope (finally fulfilled) that they could be proved to be sensible. In both cases, producing the pairs, the equations R_W, and checking that the models had distinct theories, even if not difficult, required non-trivial observations and some computations. Let us observe that this is no longer the case for our proof of Proposition 57.

Problem 14. Can we say something about the height of $\lambda\mathscr{G}$?

6.3. *Width of $\lambda\mathscr{G}^s$*

6.3.1. *Producing 2 sensible graph theories is difficult.* From the remarks in Section 5.3.5 it follows that the only method available for producing sensible graph models is the free completion method. Completing freely positive pairs brings sensibility for sure; but how many sensible theories can we obtain in this way? In Berline (2000, Question 1 of page 153) we asked whether webbed models generated by positive partial webs necessarily satisfy the (strong) Approximation Theorem of Hyland and Wadsworth. If this is the case, all positive graph theories contain BT. Positive partial results, which we obtained with Salibra (article in preparation), argue for a positive answer, but we met hard resistance to going further. Anyway, in view of Theorem 62 and its corollary, we can now rephrase our question as follows (in the case of graph models).

Conjecture 15. BT is the only positive graph theory.

If this can be proved, the same arguments should work for proving that LT is the only positive lazy graph theory, and it will also have analogues in other classes of webbed models.

The state of the art. To date, Salibra and the author have proved the following partial (unpublished) results, which rather argue in favour of the conjecture. First, all positive graph models give the same interpretation to terms t, t', provided they have the same Böhm tree if this tree is *almost hereditarily head closed (ahhc)*, which means that all but a finite number of its nodes are equal to $\perp$ or have the form $\lambda\bar{x}.y$ with $y \in \bar{x}$. Second, all positive graph models (D, i) interpret the Curry and Turing fixed point combinators Y and Θ by the least fixed point operator $\mathscr{Y}$ of the complete lattice $\mathscr{P}(D)$, and hence satisfy $Y = \Theta$. Note that all fixed point combinators have the same Böhm tree, which consists of one infinite branch where all nodes, except the root, are labelled by the same variable y; the root itself being labelled by $\lambda y.y$. This tree is thus (nearly) the simplest example of a

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non-ahhc tree. Nevertheless, we are as yet unable to prove that all positive graph models interpret all the fixed point combinators by $\mathscr{Y}$, even if we can, of course, treat other fixed point combinators besides Y and Θ; moreover, our proofs are different for the Y-like and Θ-like tree cases.

Thus, to produce sensible graph theories different from BT, we are left with the free completion of non-positive pairs, which explains retroactively why Kerth's task of producing 2^ω-sensible graph theories was so complex. Indeed, proving the sensibility of models generated by non-positive webs happens to be very difficult.

More generally, producing a non-extensional sensible proper model whose theory is different from BT is difficult. This problem concerns *all* the classes of λ-models, since forcing completion and free completion are the only systematic ways we know for building models concretely. Even the inverse limit construction can, in practice, always be presented as a free completion.

The conceptual interest of $\mathscr{G}$ here is that, because its exploration is easier, the problem could be met, isolated, stated, and hence understood, more easily.

6.3.2. *But producing 2^ω sensible graph theories is possible.*

Theorem 67 (Kerth 1995; David 1999). $\lambda\mathscr{G}^s$ is 2^ω-wide.

Sketch of proof. There are two very difficult steps. First, Kerth built 2^ω graph models G_W as the free completions of adequate non-positive pairs (A_W, i_W), $W \in \mathscr{P}(N)$ (Kerth 1995; Kerth 1998b). Then he reduced the proof that the G_W were sensible (although the pair was not positive) to a syntactic conjecture on the head-normalisation of non-solvable terms, for which he gave much evidence. The second step was the proof of this conjecture, four years later, by R. David (David 1999; David 2001). □

We can be more precise: Kerth's created pairs (A_W, i_W) such that G_W satisfies $YF_n = \Theta F_n$ if and only if $n \in W$, where Y and Θ are the Turing and Curry fixed point operators, and here $F_n \equiv \lambda x.\lambda y_1 \ldots \lambda y_n.\lambda z.(z)x$. In particular, $G_W \vDash Y \neq \Theta$ for all $W \neq \varnothing$.

Corollary 68. $\lambda\mathscr{G} \cap [H, BT]$ is 2^ω-wide.

Proof. The proof is by Theorem 67 and Corollary 63. □

Corollary 69. There are 2^ω theories of proper models included in $[H, BT]$.

This corollary largely answers positively Question 2 in Berline (2000, page 151).

However, the following questions remain open, since forcing essentially builds non-sensible models, and they should be difficult.

Problem 16. Is $\lambda\mathscr{G}^s$ 2^ω-broad?

Problem 17. Is there a (large) distributive lattice of sensible theories?

Problem 18. Can we get information about the height of $\lambda\mathscr{G}^s$?

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7. The other $\lambda\mathscr{C}'s$

7.1. *The other key classes of models.*

All the classes of models presented below live within one of the three main semantics (*cf.* Section 7), and are classes of *webbed models* (see Berline (2000) for a more detailed presentation of each of them). The methods used for building graph models or for proving positive results about $\mathscr{G}$ can be adapted to these other classes $\mathscr{C}$, at a cost that depends on $\mathscr{C}$.

— *The simplest subclass of* $\mathscr{C}_{cont}$, *strictly including* $\mathscr{G}$, is the class $\mathscr{K}$ of *K-models*, which was isolated in Krivine (1990; 1993)[†]. The interest of $\mathscr{K}$ over $\mathscr{G}$ is that it contains (plenty of) extensional models. In particular, Scott's $\mathscr{D}_\infty$ and Park's $\mathscr{P}_\infty$ live in $\mathscr{K}$, and their description as K-models is much simpler than their previous descriptions, for example, as inverse limits, and considerably simplifies their study. Finally, it is worth noting that $\mathscr{D}_\infty$ and $\mathscr{P}_\infty$ are, respectively, the *extensional completion* of the graph models $\mathscr{P}_P$ and $\mathscr{P}'_P \in \mathscr{G}$, a process that is presented in Berline (2000, page 130). To be accurate, $\mathscr{D}_\infty$ and $\mathscr{P}_\infty$ are also built starting from a set P, and hence should also be indexed by P. This notation is somewhat misleading, but respects the traditional notation for Scott's and Park's models.
More general, and hence more complex, classes of webbed models living in $\mathscr{C}_{cont}$ have been introduced and used in the literature (for example, filter-models, information systems, and pcs-models), which are also surveyed and compared in Berline (2000). As already noted, even though $\lambda\mathscr{C}_{cont} \supsetneq \lambda\mathscr{K}$, it is exceptional that we have to go beyond $\mathscr{K}$ to prove things about $\lambda\mathscr{C}^{cont}$.

— *The key subclass of* $\mathscr{C}_{st}$ is the class $\mathscr{G}_{coh}$ of reflexive coherent spaces, or *G*-models ('*G*' for 'Girard').

— *The key subclass of* $\mathscr{C}_{sts}$ is the class $\mathscr{G}_{hycoh}$ of Ehrard's reflexive hypercoherences, or *H*-models ('*H*' for 'hypercoherence').

7.1.1. *Comparison with* $\mathscr{G}$ *and* $\mathscr{K}$. The classes $\mathscr{G}_{coh}$ and $\mathscr{G}_{hycoh}$ are the analogues of $\mathscr{G}$ in their respective semantics, in the following sense. First, $\mathscr{G}$, $\mathscr{G}_{coh}$ and $\mathscr{G}_{hycoh}$ are the simplest classes of webbed models one can find in $\mathscr{C}_{cont}$, $\mathscr{C}_{st}$ and $\mathscr{C}_{sts}$, respectively. Second, the definitions of $\mathscr{G}_{coh}$ and $\mathscr{G}_{hycoh}$ can be seen as *the* variants of the definition of $\mathscr{G}$ adapted to $\mathscr{C}_{st}$ and $\mathscr{C}_{sts}$. On the other hand, $\mathscr{G}_{coh}$ and $\mathscr{G}_{hycoh}$ can also be viewed as the analogues of $\mathscr{K}$, since they are rich in extensional models, while $\mathscr{G}$ contains none. A final remark is that the definition and study of $\mathscr{G}_{hycoh}$ is significantly more complicated than the other two.

7.1.2. *Analogues of the key models of* $\mathscr{G}$ *and* $\mathscr{K}$ *in* $\mathscr{G}_{coh}$ *and* $\mathscr{G}_{hycoh}$. The constructions of $\mathscr{E}, \mathscr{P}_P, \mathscr{P}'_P \in \mathscr{G}$ and of $\mathscr{D}_\infty, \mathscr{P}_\infty \in \mathscr{K}$ can be mimicked in both $\mathscr{G}_{coh}$ and $\mathscr{G}_{hycoh}$.

[†] Let us just recall here that *K*-models are generated from webs of the form $(D, \preceq, i)$, (D, i) in the same way as for graph models, and $\preceq$ is a preorder on D; furthermore, i has to be compatible with $\preceq$ in some sense, which has to be made precise.

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Notation 70. When the construction of a model $\mathscr{M}$ of $\mathscr{G}$ or $\mathscr{K}$ can be mimicked in $\mathscr{G}_{coh}$ (respectively, $\mathscr{G}_{hycoh}$) we use $\mathscr{M}^{st}$ (respectively, $\mathscr{M}^{sts}$) to denote its analogue.

Note that $\mathscr{M}^{st}$ and $\mathscr{M}^{sts}$ need not exist, and when they do, they need not have the same equational theory as $\mathscr{M}$, although this may happen. Examples of both situations will be given below.

7.2. *Transferring the completion methods.*

Free and forcing completions can be extended to $\mathscr{K}, \mathscr{G}_{coh}$, and $\mathscr{G}_{hycoh}$. This often allows one to extend the positive results that have been proved for $\lambda\mathscr{G}$ to the three other classes, and hence to get new information on $\lambda\mathscr{T}$. However, the notions of a web and a partial web are more complex than for graph models, for which partial webs are just partial pairs, as defined earlier. For example, the web of a K-model has the shape $(D, \leqslant, i)$, where $\leqslant$ is a preorder on D, and i and $\leqslant$ have to satisfy some compatibility condition; the notion of partial web is then defined accordingly (and the cpo is now the complete lattice $\mathscr{S}(D)$ of all downward closed subsets of $(D, \leqslant)$, still ordered by inclusion, instead of $\mathscr{P}(D)$).

The free completion method in $\mathscr{K}, \mathscr{G}_{coh}$, and $\mathscr{G}_{hycoh}$. Scott's model $\mathscr{D}_\infty$ and Park's model $\mathscr{P}_\infty$ are the extensional completions of the graph models $\mathscr{P}$ and $\mathscr{P}'$. In particular, $\mathscr{D}_\infty$ is generated by a positive partial web, while $\mathscr{P}_\infty$ is not; hence we can expect different theories, and we will see below that the difference between the behaviour of $\mathscr{D}_\infty$ and $\mathscr{P}_\infty$ is still deeper. It was shown in Honsell and Ronchi (1992) that $\mathscr{P}_\infty$ is non-sensible, and this is also the case of its graph analogue $\mathscr{P}'$, for similar reasons.

The free completion method was systematically adapted to $\mathscr{G}_{coh}$ by Kerth (Kerth 1995; Kerth 2001) and to $\mathscr{G}_{hycoh}$ by Bastonero (Bastonero 1996).

The forcing method in $\mathscr{K}, \mathscr{G}_{coh}$, and $\mathscr{G}_{hycoh}$. Y. Jiang was the first to build extensional models (of $\mathscr{K}$) by forcing (Jiang 1993; Jiang 1995) (the aim was to produce models enjoying a strong notion of surjective pairing, combined with other properties). Later, Bastonero used forcing to build an extensional model $\mathscr{M} \in \mathscr{C}_{cont}$, such that $Th(\mathscr{M}) \notin \lambda\mathscr{C}_{st} \cup \lambda\mathscr{G}_{hycoh}$ (Bastonero 1996).

The forcing method was systematically adapted to $\mathscr{G}_{coh}$ (that is, to the stable semantics) in Kerth (1995; 1999), and to G_{hycoh} (that is, to the strongly stable semantics) by Bastonero and Kerth (unpublished).

7.3. *Transferring results from $\mathscr{G}$ to the $\mathscr{C}'s$*

Of course, only the positive results are transferable, in the good cases.

The first result shows that BT and H^* belong to $\lambda\mathscr{C}_{cont} \cap \lambda\mathscr{C}_{st} \cap \lambda\mathscr{C}_{sts}$, which shows, in particular, that the three main semantics are non-exclusive. This result has to be contrasted with the situation for Plotkin's PCF (Plotkin 1977), which is one of the best known typed λ-calculi. Indeed, the three 'standard models' of PCF in the main semantics have different, and even incomparable, equational theories (Jim and Meyer 1991; Bucciarelli 1993).

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Theorem 71 (Bastonero and Gouy 1996; 1999).

(i) $BT \in \lambda\mathscr{G} \cap \lambda\mathscr{G}_{coh} \cap \lambda\mathscr{G}_{hycoh}$ and
(ii) $H^* \in \lambda\mathscr{K} \cap \lambda\mathscr{G}_{coh} \cap \lambda\mathscr{G}_{hycoh}$.

Sketch of proofs.

(i) Not only can the construction of $\mathscr{E}$ be mimicked in $\mathscr{G}_{coh} \subseteq \mathscr{C}_{st}$ and $\mathscr{G}_{hycoh} \subseteq \mathscr{C}_{sts}$, *but also* the proof that $Th(\mathscr{E}) = BT$, and hence $BT \in \lambda\mathscr{G} \cap \lambda\mathscr{G}_{coh} \cap \lambda\mathscr{G}_{hycoh}$ (refer to Bastonero (1996) for $\mathscr{C}_{st}$ and Bastonero (1998) for $\mathscr{C}_{sts}$).
(ii) The same is true for $\mathscr{D}_\infty$ (refer to Gouy (1995; 1996) for $\mathscr{C}_{st}$ and Bastonero and Gouy (1996; 1999) for $\mathscr{C}_{sts}$), hence $H^* \in \lambda\mathscr{K} \cap \lambda\mathscr{G}_{coh} \cap \lambda\mathscr{G}_{hycoh}$. Only the results on $\mathscr{D}_\infty$ have been published, since at the beginning we were focusing on extensional theories; the fact that the same worked for $\mathscr{E}$, but, of course, with simpler proofs, was only emphasised later (Bastonero 1996)). □

Digression. What about LT, which is the third theory represented in $\mathscr{C}_{games}$? We have not checked the details, but have a strong feeling that the construction of $\mathscr{E}_{lazy}$ in $\mathscr{G}_{lazy}$ can also be mimicked in (possibly slight variations of) $\mathscr{G}_{coh}$ and $\mathscr{G}_{hycoh}$, as well as the proof that $Th(\mathscr{E}_{lazy}) = LT$, and hence also that $LT \in \lambda\mathscr{C}_{cont} \cap \lambda\mathscr{C}_{st} \cap \lambda\mathscr{C}_{sts}$.

The next theorem and remark show that $\lambda\mathscr{C}_{st}$ and $\lambda\mathscr{C}_{sts}$ are as 'rich' as $\lambda\mathscr{C}_{cont}$. The proof uses forcing relative to $\mathscr{G}_{coh}$ (and $\mathscr{G}_{hycoh}$).

Theorem 72 (Kerth 1995; 2001). $\mathscr{G}_{coh}$ is 2^ω-wide, and hence also $\lambda\mathscr{C}_{st}$.

Remark 73. Bastonero noticed later that forcing, and hence the proof of this theorem, could be adapted to $\mathscr{G}_{hycoh}$, but this remained unpublished. It is also likely that generalised forcing can be developed, at least for $\mathscr{G}_{coh}$, and hence that we can replace 2^ω-wide by 2^ω-broad.

7.4. *Relative positions of the $\lambda\mathscr{C}'s$*

We already know that $\lambda\mathscr{C}_{cont}$, $\lambda\mathscr{C}_{st}$ and $\lambda\mathscr{C}_{sts}$ are 2^ω-wide and 2^ω-incomplete, and that their intersection contains BT, H^* and probably LT; moreover, we know that $\lambda\mathscr{T} - \lambda\mathscr{C}$ is 2^ω-high, for each of these $\mathscr{C}$. We show now that they are far from representing the same theories. First we state the problem we started from when studying these questions.

Conjecture 19. $\lambda\mathscr{C}_{cont}$, $\lambda\mathscr{C}_{st}$ and $\lambda\mathscr{C}_{sts}$ are pairwise incomparable (for inclusion).

Xavier Gouy and Olivier Bastonero got very close to proving this, since they succeeded in proving it for the triple $\mathscr{C}_{cont}$, $\mathscr{C}_{st}$ and $\mathscr{G}_{hycoh}$. We will see below how one can improve this result using generalised forcing.

Theorem 74 (Bastonero and Gouy 1996; 1999).

(i) $Th(\mathscr{P}^{st}_\infty) \notin \lambda\mathscr{C}_{cont} \cup \lambda\mathscr{G}_{hycoh}$
(ii) $Th(\mathscr{P}^{sts}_\infty) \notin \lambda\mathscr{C}_{cont} \cup \lambda\mathscr{C}_{st}$.

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Theorem 75 (Bastonero 1996; 1998). One can build (by forcing) a λ-model $\mathscr{M} \in \mathscr{K}^e$ such that $Th(\mathscr{M}) \notin \lambda\mathscr{C}_{st} \cup \lambda\mathscr{G}_{hycoh}$.

Hint for proof. Let $u = \Omega\,\lambda x.\Omega$ and $v = \Omega\,\lambda x\lambda y.\Omega$. There is a set F consisting of seven equations and one inequation, relating Ω, u and v, and there is an equation $t_1 = t_2$, such that no model of $\mathscr{C}_{st} \cup \mathscr{G}_{hycoh}$ can satisfy $F' = F \cup \{t_1 = t_2\}$, but one can build, by forcing, a model $\mathscr{M} \in \mathscr{K}^e$ satisfying F'. Since Bastonero was only concerned with extensionality, F contains $I = \varepsilon$, and he had to use the extension of forcing that is adequate for $\mathscr{K}^e$. However, the proof can be easily modified/simplified such that $\varepsilon = I$ is removed from F, and $\mathscr{M}$ is built by the usual forcing in $\mathscr{G}$. We now give some information on the construction of $\mathscr{M}$ in this simpler setting. A triple (p_0, h, D) is exhibited, in this order, such that p_0 and h are infinite, $D \supseteq h$, and (D, p_0) is a partial pair satisfying the two easy constraints that make forcing possible from (D, p_0). Then a pair (D, i), $i \supseteq p_0$, is built by forcing such that $\mathscr{M} = (D, i)$ satisfies $\Omega = h$; but in fact p_0 was also chosen in such a way that $\Omega = h$ implies F' in $\mathscr{M}$, which ends the (sketch of) the proof. □

From the above theorem and the observation that $\mathscr{M}$ could be built in $\mathscr{G}$, we immediately derive the following statements (note that none of them is a consequence of the others).

Corollary 76.

(i) $\mathscr{C}_{cont}$, $\mathscr{C}_{st}$ and $\mathscr{G}_{hycoh}$ are pairwise incomparable.
(ii) $\mathscr{K}$, $\mathscr{C}_{st}$ and $\mathscr{G}_{hycoh}$ are pairwise incomparable.
(iii) $\mathscr{K}$, $\mathscr{G}_{coh}$ and $\mathscr{G}_{hycoh}$ are pairwise incomparable.
(iv) $\mathscr{G}$, $\mathscr{G}_{coh}$ and $\mathscr{G}_{hycoh}$ are pairwise incomparable.

Thus, to answer positively the initial conjecture, it only remains to prove the following conjecture.

Conjecture 20.

(i) $\lambda\mathscr{C}_{cont} - \lambda\mathscr{C}_{sts} \neq \varnothing$.

Moreover, it should be the case that:

(ii) $Th(\mathscr{M}) \in \lambda\mathscr{C}_{cont} - \lambda\mathscr{C}_{sts}$, with $\mathscr{M}$ as in the proof of Theorem 75.

The reason Bastonero could not conclude in his thesis that $Th(\mathscr{M}) \notin \lambda\mathscr{C}_{sts}$ is that he was unable to prove that a model of $\mathscr{C}_{sts}$ satisfying F cannot satisfy $t_1 = t_2$, with t_1, t_2 as in the proof of Theorem 75. Indeed, in the case of $\mathscr{G}_{hycoh}$ (respectively, $\mathscr{C}_{st}$), he could exhibit a strongly stable function (respectively, a stable function) g separating t_1 and t_2 in all models of F; but his definition of g relied, in the strongly stable case, on the fact that we were dealing with webbed models, which is not the case for all the models of $\mathscr{C}_{sts}$ (while in the stable case the definition of g was not problematic).

A wealth of incompleteness proofs for the three main semantics. Salibra's Theorem 27 was a definite and uniform argument proving that each of the main semantics is incomplete (and even 2^{ω}-incomplete). But, historically, the incompleteness of $\mathscr{C}_{cont}$ was first proved by Honsell and Ronchi, by producing an operational λ-theory that could not be the theory

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of a continuous model (Honsell and Ronchi 1992). The proof, which was already quite technical, was adapted to the stable case by Gouy (Gouy 1995), but the new proof was so complicated that it became clear that one had to find a different idea for $\mathscr{C}_{sts}$. Such other approaches were provided by Bastonero and Gouy (*cf.* Theorems 74 and 75 above), which hence provided two new incompleteness proofs for each of $\mathscr{C}_{cont}, \mathscr{C}_{st}$ and $\mathscr{G}_{hycoh.}$ To summarise, we have four different proofs for the incompleteness of $\mathscr{C}_{cont}$, four for $\mathscr{C}_{st}$, three for the subclass $\mathscr{G}_{hycoh}$ of $\mathscr{C}_{sts}$, and only Salibra's proof for $\mathscr{C}_{sts}$.

The final semantic problem. To gain a more accurate picture of the relative positions of $\lambda\mathscr{C}_{cont}, \lambda\mathscr{C}_{st}$ and $\lambda\mathscr{C}_{sts}$, one should have an idea of the width and height of $\lambda\mathscr{T} - \lambda\mathscr{C}$, and of the width of $\lambda\mathscr{C} - \lambda\mathscr{C}'$, for $\mathscr{C}, \mathscr{C}'$ any distinct uniform classes of proper models. By Salibra's incompleteness theorem, we know that for each ordered semantics we have that $\lambda\mathscr{T} - \lambda\mathscr{C}$ is 2^{ω}-high, but the question of the value of the height of $\lambda\mathscr{C} - \lambda\mathscr{C}'$ for any other $\mathscr{C}'$ is completely open, since we still know nothing about that of $\lambda\mathscr{C}$. On the other hand, it is likely that the following conjecture (which only concerns uniform classes living in functional semantics) is true, and that all its instances will be accessible via generalised forcing.

Conjecture 21. For all $\mathscr{C}, \mathscr{C}'$, either $\mathscr{C} \subseteq \mathscr{C}'$ or $\lambda\mathscr{C} - \lambda\mathscr{C}'$ is 2^{ω}-broad.

We can already give a positive answer to two instances of this conjecture. First we observe that generalised forcing is extendible to the extensional models of $\mathscr{K}$, in the same way that the usual forcing is (one just has to mix, in a straightforward way, the development of Jiang (1995) or Bastonero (1996) with the one in Berline and Salibra (2004)). Hence $\lambda\mathscr{K}^e$ is 2^{ω}-broad. Thus we have the following proposition.

Proposition 77. $\lambda\mathscr{K} - \lambda\mathscr{G}$ is 2^{ω}-broad.

Next we observe that, using Theorem 56 (generalised forcing), one can get for free 2^{ω} versions of the model $\mathscr{M}$ of Theorem 75 in $\lambda\mathscr{G} - (\lambda\mathscr{C}_{st} \cup \lambda\mathscr{G}_{hycoh})$ with pairwise inconsistent theories ($\lambda\mathscr{K}^e$ could also replace $\lambda\mathscr{G}$). Hence we have the following proposition.

Proposition 78. $\lambda\mathscr{G} - (\lambda\mathscr{C}_{st} \cup \lambda\mathscr{G}_{hycoh})$ is 2^{ω}-broad.

Proving the conjecture for other classes of the form $\lambda\mathscr{C}$-$\lambda\mathscr{C}'$ requires that we are able to prove by a forcing technique that this set is non-empty, and that we can extend generalised forcing to $\lambda\mathscr{C}$.

8. Conclusions

It is now time to sum up the results that have been achieved over the past four years.

Answers to questions raised or quoted in Berline (2000, Section 6).

1 Salibra's Theorem 27 states that all the known ordered semantics are 2^{ω}-incomplete, which proves, in particular, that $\mathscr{C}_{sts}$ is incomplete. This answers, very generously, [*Question 2 of Section 6.1, page 149*], which recalled the conjecture we had with Bastonero that $\mathscr{C}_{sts}$ was incomplete (recall that Bastonero could only prove in his

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thesis that $\mathscr{G}_{hycoh}$ was incomplete). Salibra's proof was original, and initially inspired by universal algebra and general topology.

2 Bucciarelli and Salibra proved that λ_β could not be the theory of a graph model, which answers the instance of [*Question 1, Section 6.2, page 149*] corresponding to $\mathscr{G}$.

3 Bucciarelli and Salibra's definition and study of the $\mathscr{G}$-product (*cf.* Section 5.3) allowed them to prove that there was a minimal graph-theory and a minimal sensible graph theory, and hence gave a positive answer to the instance of [*Question 2 of Section 6.2, page 149*] corresponding to $\mathscr{G}$ (and it also shows the way for larger classes of webbed models). But they leave open the question of whether the minimal sensible graph theory is H.

4 Bucciarelli and Salibra proved that the set of sensible graph theories has a greatest element, which, furthermore, is BT (Theorem 62). This has the following two consequences. First, our [*Question 1 of Section 6.3, page 151*] can now be reformulated as follows: 'Is BT the only positive graph theory?'. Second, combined with Kerth and David's theorem on the existence of 2^ω sensible graph theories, it shows that there exists at least one, and even 2^ω, theories of sensible proper models below BT, which answers positively the two items of [*Question 2 of Section 6.3, page 151*]. As for the first question, we conjecture in fact that the answer is positive, and have proved some partial unpublished results with Salibra.

5 Starting from an example of Kerth (see Section 5.2), Alessi, Dezani and Honsell exhibited a result (the easiness of $\Omega_3 I$) that admits a semantic proof using filter models, but for which there exists no semantic proof using $\mathscr{G}$ or $\mathscr{K}$ or, more generally, using any class of models whose underlying cpo is a prime algebraic domain. This also answers (part of) [*Question 5 of Section 6.3, page 152*] since it gives examples of theories represented in the continuous semantics, and via algebraic domains, but not via prime algebraic domains.

The four other recent results. The first to be mentioned was Statman's Theorem on the existence of a non-trivial equation consistent with any consistent theory. The following two results answer questions raised by Antonino Salibra. In particular, Salibra conjectured that the complete lattice $\lambda\mathscr{T}$ of all λ-theories does not satisfy any non-trivial lattice identity, and proved that it is not modular, and hence not distributive. Then he proved with the author, using graph models, that there was a finitely axiomatisable λ-theory T such that $[T, Top]$ was a (very large) distributive lattice (see Section 5.3.6). To prove this, we generalised the Baeten–Boerboom method of forcing in a way that also allowed us to prove that $\lambda\mathscr{G}$ was 2^ω-broad, as well as various smaller sets of graph theories. Also, we are now in a position to conjecture that 'all' the $\lambda\mathscr{C}' - \lambda\mathscr{C}$ are 2^ω-broad, and prove some particular cases.

Partial results concerning the distributivity question had previously been obtained by Lusin and Salibra (Lusin and Salibra 2004) (the existence of a lattice with weaker properties) and Alessi, Dezani and Lusin (Alessi *et al.* 2004) (the existence of a filter model where Ω acts as union), but with different methods.

Some long-standing open questions still persist, such as the representability of H, and λ_β and $\lambda_{\beta\eta}$ by reasonable proper models, as well as the more recent question concerning the existence of a positive graph theory different from BT.

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We would like to conclude this survey with a final conjecture, whose proof, if it is true, should force us to develop new tools for studying proper models.

Conjecture 22. The class of graph models omits all recursively enumerable λ-theories.

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