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MODEL COMPLETENESS FOR THE REAL FIELD WITH THE WEIERSTRASS $\wp$ FUNCTION

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Abstract We prove model completeness for the expansion of the real field by the Weierstrass $\wp$ function as a function of the variable z and the parameter (or period) τ. We need to existentially define the partial derivatives of the $\wp$ function with respect to the variable z and the parameter τ. To obtain this result, it is necessary to include in the structure function symbols for the unrestricted exponential function and restricted sine function, the Weierstrass ζ function and the quasi-modular form E_2 (we conjecture that these functions are not existentially definable from the functions $\wp$ alone or even if we use the exponential and restricted sine functions). We prove some auxiliary model-completeness results with the same functions composed with appropriate change of variables. In the conclusion, we make some remarks about the non-effectiveness of our proof and the difficulties to be overcome to obtain an effective model-completeness result, and how to extend these results to appropriate expansion of the real field by automorphic forms.

Keywords: model completeness; Weierstrass systems; elliptic functions; $\wp$ function; Weierstrass ζ function; modular forms; o-minimality; definable

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Secondary 03C64; 03C98; 14H52; 33E05

1. Introduction

O-minimal techniques have been used to prove important results in algebraic geometry. Pila [**16**] proved in 2011 the André–Oort conjecture for $\mathbb{C}^n$, where he makes essential use of his work with Wilkie [**17**] on rational points on definable sets in an o-minimal structure, and work by Peterzil and Starchenko [**14**] on the definability of Weierstrass's $\wp$ function in the o-minimal structure $\mathbb{R}_{an,\exp}$ (the expansion of the real field by real analytic functions restricted to $[-1,1]^n$, with the unrestricted exponential function). This structure is model complete, so the $\wp$ function is existentially definable. This suggests the question of which 'minimal' reduct of $\mathbb{R}_{an,\exp}$ is still model complete and defines $\wp$.

We prove here model completeness of expansions of the real fields by $\wp$ function as a function of the variable z and of the parameter τ. The structure obtained is interpretable

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in $\mathbb{R}_{an,\exp}$ and so it is o-minimal. We present a model-complete reduct of $\mathbb{R}_{an,\exp}$ (with finitely many function symbols and with the unrestricted exponential function) in which the function symbols of that structure are existentially interpretable (and conversely). This gives the desired result. Because of the techniques we use, we need the exponential function, restricted sine function and modular forms.

We conjecture that the exponential function and the restricted sine function are necessary for such a definition; that is, we conjecture that these functions are not *existentially* definable from $\wp$ and $\wp'$ alone (although they are definable from $\wp$; see [14, Theorem 5.7, p. 545] for the exponential function and [19, § 20.222, pp. 438–439] for the sine function).

This work is organized as follows. In the following section, we present the general techniques to prove model completeness for reducts of $\mathbb{R}_{an,\exp}$. The main results are stated in § 3. The proofs are given in § 4. In the concluding remarks, in § 5, we deal with the problem of finding an effective proof of these model-completeness results.

Notation. $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{N}$ the set of non-negative integers, $\mathbb{R}$ the field of real numbers and $\mathbb{C}$ the field of complex numbers; $\mathfrak{Re}(z)$ and $\mathfrak{Im}(z)$ denote the real and imaginary parts of z, the letter i denotes $\sqrt{-1}$ and $\bar{x}$ denotes the tuple $(x_1, \ldots, x_n)$ for some unspecified n; and $\mathbb{H} = \{z \in \mathbb{C} : \mathfrak{Im}(z) > 0\}$. Other notation is explained in the text.

2. Model-completeness criteria

We use the following criteria for model completeness. They are based on the reduction to the so-called Weierstrass systems of the proof of model completeness of the structure $\mathbb{R}_{an}$ by Denef and van den Dries in [4], as suggested by van den Dries in [5] and presented in [2]. The versions below are based on these papers and in [6] (extending the result to $\mathbb{R}_{an,\exp}$), as presented in [3].

We stress here the point that these results rely on the (topological) compactness of closed and bounded polyintervals in $\mathbb{R}^n$ and polydiscs in $\mathbb{C}^n$, which gives a non-effective feature to the proofs. We discuss this in the concluding remarks.

Recall that the theory T of a structure M is called *strongly model complete* if for each formula $\varphi(\bar{x})$ there is a quantifier-free formula $\psi(\bar{x}, \bar{y})$ such that

$$T \vdash \forall \bar{x}\, [\varphi(\bar{x}) \leftrightarrow \exists \bar{y}\, \psi(\bar{x}, \bar{y})]$$

and $T \vdash \forall \bar{x} \forall \bar{y} \forall \bar{z} (\psi(\bar{x}, \bar{y}) \wedge \psi(\bar{x}, \bar{z}) \rightarrow \bar{y} = \bar{z})$. We say that such a formula is a *strong (existential) formula* and that the set it defines is *strongly definable*.

The first criterion deals with functions restricted to compact polyintervals and is the main step to obtain the more general result.

Theorem 2.1 (Bianconi [3, Theorem 2]). *Let* $\hat{R} = \langle \mathbb{R}, \textit{constants}, +, -, \cdot, <, (F_\lambda)_{\lambda \in \Lambda} \rangle$ *be an expansion of the field of real numbers, where for each* $\lambda \in \Lambda$, F_λ *is the restriction to a compact polyinterval* $D_\lambda \subseteq \mathbb{R}^{n_\lambda}$ *of a real analytic function whose domain contains* D_λ, *and defined as zero outside* D_λ, *such that there exists a complex analytic function* g_λ *defined in a neighbourhood of a polydisc* $\Delta_\lambda \supseteq D_\lambda$ *and such that:*

(1) g_λ *is strongly definable in* $\hat{R}$ *and the restriction of* g_λ *to* D_λ *coincides with* F_λ *restricted to the same set;*

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(2) *for each* $\boldsymbol{a} \in \Delta_\lambda$ *there exists a compact polydisc* Δ *centred at* $\boldsymbol{a}$ *and contained in the domain of* g_λ*, such that all the partial derivatives of the restriction of* g_λ *to* Δ *are strongly definable in* $\hat{R}$*.*

Under these hypotheses, the theory of $\hat{R}$ *is strongly model complete.*

Now we introduce the unrestricted exponential function. This result is analogous to the one for $\mathbb{R}_{an,\exp}$ proved in [**6**].

Theorem 2.2 (Bianconi [3, Theorem 4]). *Let* $\hat{R}$ *be the structure described in Theorem 2.1. We assume that the functions*

$$\exp\restriction_{[0,1]}(x) = \begin{cases} \exp x & \text{if } 0 \leqslant x \leqslant 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$\sin\restriction_{[0,\pi]}(x) = \begin{cases} \sin x & \text{if } 0 \leqslant x \leqslant \pi, \\ 0 & \text{otherwise,} \end{cases}$$

have representing function symbols in its language. The expansion $\hat{R}_{\exp}$ *of* $\hat{R}$ *by the inclusion of the (unrestricted) exponential function* '$\exp$' *is strongly model complete.*

We use this theorem to prove model completeness of the expansion of the field of real numbers by functions defined in unbounded sets of the real numbers using the exponential function, and the restricted sine function to existentially interpret that structure into another that satisfies its hypotheses.

3. The main results

To state the main theorems, we need to introduce some notation and definitions. We follow closely the work [**14**] by Peterzil and Starchenko on the definability of the $\wp$ function (as a function of the variable z and the parameter τ) on $\mathbb{R}_{an,\exp}$ and use their notation for easy comparison with their work.

We define $\mathfrak{F}_d = \{\tau \in \mathbb{H} : -1/2 \leqslant \mathfrak{Re}(\tau) < 1/2, \text{ and } \mathfrak{Im}(\tau) \geqslant d\}$, where $d = \sqrt{3}/2$. Observe that this set contains the usual fundamental domain of the action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$, the set $\mathfrak{F} = \{\tau \in \mathbb{H} : -1/2 \leqslant \mathfrak{Re}(\tau) < 1/2, |\tau| \geqslant 1\}$. For each $\tau \in \mathbb{H}$, we define the set $E_\tau = \{z \in \mathbb{C} : z = a\tau + b, \text{ for } a, b \in \mathbb{R}, a, b \geqslant 0 \text{ and } a, b < 1\}$ and the set $E^{\mathfrak{F}_d} = \{(\tau, z) \in \mathbb{H} \times \mathbb{C} : \tau \in \mathfrak{F}_d \text{ and } z \in E_\tau\}$. The latter set is semi-algebraic and contains a fundamental region for the actions of $SL(2,\mathbb{Z})$ on $\mathbb{H}$ and the action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{C}$, generated by the translations $z \mapsto z + 1$ and $z \mapsto z + \tau$.

We write $z = x + \mathrm{i}y$ and $\tau = \mu + \mathrm{i}\nu$, with x, y, μ, ν real variables, and define the functions

$$RP(\mu, \nu; x, y) = \begin{cases} \mathfrak{Re}(\wp)(\tau; z) & \text{if } (\tau, z) \in E^{\mathfrak{F}_d}, \\ 0 & \text{otherwise,} \end{cases}$$

$$IP(\mu, \nu; x, y) = \begin{cases} \mathfrak{Im}(\wp)(\tau; z) & \text{if } (\tau, z) \in E^{\mathfrak{F}_d}, \\ 0 & \text{otherwise,} \end{cases}$$

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the real and imaginary parts of Weierstrass's $\wp$ function,

$$RZ(\mu,\nu;x,y)=\begin{cases}\mathfrak{Re}(\zeta)(\tau;z) & \text{if } (\tau,z)\in E^{\mathfrak{F}_d},\\ 0 & \text{otherwise,}\end{cases}$$

$$IZ(\mu,\nu;x,y)=\begin{cases}\mathfrak{Im}(\zeta)(\tau;z) & \text{if } (\tau,z)\in E^{\mathfrak{F}_d},\\ 0 & \text{otherwise,}\end{cases}$$

the real and imaginary parts of Weierstrass's zeta function, which can be defined by

$$\frac{\partial\zeta}{\partial z}(\tau;z)=-\wp(\tau;z)\quad\text{and}\quad\lim_{z\to 0}\left(\zeta(\tau;z)-\frac{1}{z}\right)=0;$$

and also

$$RE_2(\mu,\nu)=\begin{cases}\mathfrak{Re}(E_2)(\tau) & \text{if } \tau\in\mathfrak{F}_d,\\ 0 & \text{otherwise,}\end{cases}$$

$$IE_2(\mu,\nu)=\begin{cases}\mathfrak{Im}(E_2)(\tau) & \text{if } \tau\in\mathfrak{F}_d,\\ 0 & \text{otherwise,}\end{cases}$$

the real and imaginary parts of the quasi-modular form $E_2(\tau)$

$$E_2(\tau)=1+\frac{3}{\pi^2}\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2},$$

which appears in the derivatives of the Weierstrass $\wp$ and ζ functions with respect to the parameter τ (see the proof of Lemma 4.4 for more details).

We can now state the main result, where exp is the unrestricted exponential function and $\sin\restriction_{[0,\pi]}$ is the sine function restricted to the interval $[0,\pi]$ and defined as 0 outside this interval.

Theorem 3.1. *The structure* $\mathcal{R}_\wp=\langle\mathbb{R}$, *constants*, $+$, $-$, $\cdot$, $<$, RP, IP, RZ, IZ, RE_2, IE_2, exp, $\sin\restriction_{[0,\pi]}\rangle$ *is model complete.*

The idea of the proof is to use the change of variables $q=\exp(-2\pi\mathrm{i}\tau)$, $u=\exp(-2\pi\mathrm{i}z)$, to interpret the structure $\mathcal{R}_\wp$ into an auxiliary structure that satisfies the hypotheses of Theorem 2.2. This is done in the following section.

4. Proof of Theorem 3.1

We first prove an auxiliary result, which uses the model-completeness criteria from Theorem 2.1. These criteria apply to the restriction to compact polyintervals of the real and imaginary parts of complex analytic functions. The functions in the structure $\mathcal{R}_\wp$ are defined in unbounded (hence non-compact) sets, but admit a definable change of coordinates that maps their domains onto relatively compact ones, transforming the original question into an amenable setting. The fact that the functions are invariant

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under translation $z \mapsto z+1$ and $\tau \mapsto \tau+1$ allows us to use the Fourier transformation $u = \exp(2\pi \mathrm{i} z)$ and $q = \exp(2\pi \mathrm{i}\tau)$.

We define the sets $\tilde{E}_\tau = \{z \in \mathbb{C} : \text{there are } 0 \leqslant a, b \leqslant 1, \text{ with } a+b \leqslant 1, \text{ such that } z = 1/8 + a/4 + \tau/8 + b\tau/4\}$, the convex closure of the parallelogram with vertices $(1+\tau)/8$, $(3+\tau)/8$, $(1+3\tau)/8$ and $(3+3\tau)/8$. Define $\tilde{E}^{\mathfrak{F}_d} = \{(\tau, z) \in \mathbb{H} \times \mathbb{C} : \tau \in \mathfrak{F}_d, \text{ and } z \in \tilde{E}_\tau\}$.

The image of the set $\tilde{E}^{\mathfrak{F}_d}$ under the mapping $(\tau; z) \mapsto (q; u)$ is the set $M_\delta = \{(q, u) \in \mathbb{C}^2 : 0 < |q| \leqslant \delta \text{ and } (\log u)/(2\pi\mathrm{i}) \in \tilde{E}_{(\log q)/2\pi\mathrm{i}}\}$, where $\delta = \exp(-2\pi d) < 1$.

We apply this change of variables to the functions $\wp$, ζ and E_2.

For the $\wp$-function, we have [**18**, Theorem 6.2, p. 50]

$$\wp_0(q; u) = (2\pi\mathrm{i})^2 \left[\sum_{m \in \mathbb{Z}} \frac{u q^m}{(1-uq^m)^2} + \frac{1}{12} - 2 \sum_{m \geqslant 1} \frac{q^m}{(1-q^m)^2} \right].$$

For the zeta function, we have [**18**, Theorem 6.3, pp. 52–53]

$$\frac{1}{2\pi\mathrm{i}} \zeta_0(q; u) = \sum_{n \geqslant 0} \frac{-q^n u}{1-q^n u} + \sum_{n \geqslant 1} \frac{q^n u^{-1}}{1-q^n u^{-1}} + \frac{1}{2\pi\mathrm{i}} \eta(1) z - \frac{1}{2},$$

where we have a logarithmic term $z = (\log u)/2\pi\mathrm{i}$ and

$$\frac{1}{(2\pi\mathrm{i})^2} \eta(1) = \frac{1}{12} \left[-1 + 24 \sum_{n \geqslant 1} \frac{q^n}{(1-q^n)^2} \right].$$

We see that these series converge in an open neighbourhood of the image M_δ and also converge if $q = 0$. Observe that because, for all $\tau \in \mathfrak{F}_d$ and $z \in \tilde{E}_\tau$, z is bounded away from zero and so the logarithmic term in the series of ζ_0 has no singularity in some open neighbourhood of M_δ.

Write $q = q_0 + \mathrm{i} q_1$ and $u = u_0 + \mathrm{i} u_1$. We put $\delta = \exp(-2\pi d)$, with $d = \sqrt{3}/2$, and define the functions

$$RP_0(q_0, q_1; u_0, u_1) = \begin{cases} \mathfrak{Re}(\wp_0)(q; u) & \text{if } (q_0, q_1; u_0, u_1) \in M_\delta, \\ 0 & \text{otherwise,} \end{cases}$$

$$IP_0(q_0, q_1; u_0, u_1) = \begin{cases} \mathfrak{Im}(\wp_0)(q; u) & \text{if } (q_0, q_1; u_0, u_1) \in M_\delta, \\ 0 & \text{otherwise,} \end{cases}$$

and the real and imaginary parts of Weierstrass's $\wp$ function,

$$RZ_0(q_0, q_1; u_0, u_1) = \begin{cases} \mathfrak{Re}(\zeta_0)(q; u) & \text{if } (q_0, q_1; u_0, u_1) \in M_\delta, \\ 0 & \text{otherwise,} \end{cases}$$

$$IZ_0(q_0, q_1; u_0, u_1) = \begin{cases} \mathfrak{Im}(\zeta_0)(q; u) & \text{if } (q_0, q_1; u_0, u_1) \in M_\delta, \\ 0 & \text{otherwise.} \end{cases}$$

The quasi-modular form $E_2(\tau) = 1 + (3/\pi^2) \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} (m\tau + n)^{-2}$ is invariant under the translation $\tau \mapsto \tau + 1$; it admits an expansion as a q-series ($q = \exp(2\pi\mathrm{i}\tau)$).

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We define the function $\tilde{E}_2(q)$ as the Fourier transform of the function $E_2(\tau)$, which is a power series in the variable q converging in the unit disc centred at the origin, which is an open neighbourhood of the closed disc Δ_δ ($\delta = \exp(-\pi\sqrt{3}) < 1$) centred at $q = 0$ (see [**20**, p. 19]), and define

$$RE_2^*(q_0, q_1) = \begin{cases} \mathfrak{Re}(\tilde{E}_2)(q) & \text{if } (q_0, q_1) \in \mathfrak{F}_d, \\ 0 & \text{otherwise,} \end{cases}$$

$$IE_2^*(q_0, q_1) = \begin{cases} \mathfrak{Im}(\tilde{E}_2)(q) & \text{if } (q_0, q_1) \in \mathfrak{F}_d, \\ 0 & \text{otherwise.} \end{cases}$$

With this notation, we prove the following auxiliary result.

Theorem 4.1. *The structure* $\mathcal{R}_{\text{aux}} = \langle \mathbb{R}, \text{constants}, +, -, \cdot, <, RP_0, IP_0, RZ_0, IZ_0, RE_2^*, IE_2^*, \exp\restriction_{[0,1]}, \sin\restriction_{[0,\pi]} \rangle$ *is model complete.*

We divide the proof into three steps, focusing on each item of the model-completeness criteria from Theorem 2.1, where in Step 1 we deal with Item 1, and in Steps 2 and 3 we deal with Item 2.

Step 1. The model-completeness criteria require that:

(...) for each $\lambda \in \Lambda$, F_λ is the restriction to a compact polyinterval $D_\lambda \subseteq \mathbb{R}^{n_\lambda}$ of a real analytic function whose domain contains D_λ, and defined as zero outside D_λ, such that there exists a complex analytic function g_λ defined in a neighbourhood of a polydisc $\Delta_\lambda \supseteq D_\lambda$ and such that

1. g_λ is strongly definable in $\hat{R}$ and the restriction of g_λ to D_λ coincides with F_λ restricted to the same set;

(...)

This comes true for the structure $\mathcal{R}_{aux}$, as we now prove. The strategy is to define extensions of the real analytic part of the functions of the structure to polyintervals with twice the size of the original ones. We transform back to the variables (τ, z), and use the addition formulas in the variable z for the elliptic functions and modularity equations in the variable τ, and then transform again to the variables (q, τ).

We first state a lemma presenting known formulas for the complex analytical continuation of the real and imaginary parts of a given complex analytic function. We write a proof for the sake of completeness.

Lemma 4.2. *The real and imaginary parts,* $F_R(x, y)$ *and* $F_I(x, y)$*, of the complex analytic function* $F(z)$*,* $z = x + \mathrm{i}y$*, defined in a polydisc* $\Delta_\rho = \{z \in \mathbb{C}^n : |z_i| < \rho, 1 \leqslant i \leqslant n\}$*, are real analytic functions and admit complex analytic continuations*

$$\tilde{F}_R(z, w) = \frac{F(z + \mathrm{i}w) + \overline{F(\bar{z} + \mathrm{i}\bar{w})}}{2}, \qquad \tilde{F}_I(z, w) = (-\mathrm{i})\frac{F(z + \mathrm{i}w) - \overline{F(\bar{z} + \mathrm{i}\bar{w})}}{2},$$

with $z, w \in \Delta_{\rho/2}$.

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Proof. This is a consequence of the formulas $\mathfrak{Im}(F) = \mathfrak{Re}(-\mathrm{i}F)$ and

$$\mathfrak{Re}(F)(x+\mathrm{i}y) = \frac{F(x+\mathrm{i}y) + \overline{F(\overline{x+\mathrm{i}y})}}{2}.$$

If $F(x+\mathrm{i}y) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha (x+\mathrm{i}y)^\alpha$, $\overline{F(\overline{x+\mathrm{i}y})} = \sum_{\alpha\in\mathbb{N}^n} \bar{c}_\alpha (\bar{x}+\mathrm{i}\bar{y})^\alpha$. If $z, w \in \Delta_{\rho/2}$, then $z+\mathrm{i}w, \bar{z}+\mathrm{i}\bar{w} \in \Delta_\rho$. □

Thus, to prove the strong definability of the complex analytic continuation of the functions RP_0, IP_0, RZ_0, IZ_0, RE_2^*, IE_2^*, we only need to prove the strong definability of the extensions of their real analytic part to a bigger domain M_δ^*, such that if $z, w \in M_\delta$, then $z+\mathrm{i}w, \bar{z}+\mathrm{i}\bar{w} \in M_\delta^* = \{(q,u) : (q/2, u/2) \in M_\delta\}$.

We recall here the definition of the set M_δ in a form appropriate to make explicit the definition of M_δ^*, where $\delta = \exp(-\pi\sqrt{3})$:

$$M_\delta = \left\{ (q,u) \in \mathbb{C}^2 : 0 < |q| \leqslant \delta,\ \exists a, b \in [0,1]\,(a+b \leqslant 1), \right.$$
$$\left. \frac{\log u}{2\pi\mathrm{i}} = \frac{1+2a}{8} + \frac{1+2b}{8}\frac{\log q}{2\pi\mathrm{i}} \right\}.$$

If $(q,u) \in M_\delta$, then $(p,v) = (2q, 2u) \in M_\delta^*$, and thus,

$$M_\delta^* = \left\{ (p,v) \in \mathbb{C}^2 : 0 < |p| \leqslant 2\delta,\ \exists a, b \in [0,1]\,(a+b \leqslant 1), \right.$$
$$\left. \frac{\log v}{2\pi\mathrm{i}} = \frac{1+2a}{8} + \frac{1+2b}{8}\frac{\log p}{2\pi\mathrm{i}} \right\}.$$

Lemma 4.3. *The functions RP_0, IP_0, RZ_0, IZ_0, RE_2^*, IE_2^*, restricted to M_δ admit strongly definable real analytic extensions to M_δ^*.*

Proof. This is a consequence of the addition formulas in the variable z for the functions $\wp$ and ζ, and the modularity formulas related to the parameter τ.

If $|p| \leqslant 2\delta = 2\exp(-\pi\sqrt{3}) = \exp(\log -2\pi(-\log 2/2\pi + \sqrt{3}/2))$, then

$$\mathfrak{Im}(\log p/2\pi\mathrm{i}) \geqslant \sqrt{3}/2 - \log 2/2\pi > 0.75 = 3/4.$$

The union of the set $\{\tau \in \mathbb{H} : -1/2 \leqslant \mathfrak{Re}(\tau) \leqslant 1/2,\ |\tau| \geqslant 1\}$ with its images under the maps ST^{-1} and ST, where $S(\tau) = -1/\tau$ and $T(\tau) = \tau + 1$, contains the set $\mathfrak{F}_{3/4}$.

The formula in [**20**, Proposition 6, p. 19],

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) - \frac{6}{\pi}\mathrm{i}c(c\tau+d),$$

provides the desired extensions of RE_2^*, IE_2^*.

From the series of $\wp(\tau;z)$ and $\zeta(\tau;z)$, we obtain $\wp(\tau+1;z) = \wp(\tau;z)$, $\wp(-1/\tau;z) = \tau\wp(\tau;\tau z)$, $\zeta(\tau+1;z) = \zeta(\tau;z)$ and $\zeta(-1/\tau;z) = \tau\zeta(\tau;\tau z)$. These formulas, together with the summation formulas for $\wp$ and ζ in the variable z, and the mapping of these formulas

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under $(\tau; z) \mapsto (q, u) = (\exp(-2\pi i\tau); \exp(-2\pi i z))$, give the expected extensions of RP_0, IP_0, RZ_0, IZ_0. □

Step 2. The model-completeness criteria require that:

2. for each $\boldsymbol{a} \in \Delta_\lambda$ there exists a compact polydisc Δ centred at $\boldsymbol{a}$ and contained in the domain of g_λ, such that all the partial derivatives of the restriction of g_λ to Δ are strongly definable in $\hat{R}$.

The strategy here is to show that the required partial derivatives are definable. We again go back to the variable (τ, z), and return to the variables (q, u) using the chain rule. In this step, we deal with the differential equations that allow us to strongly define the partial derivatives, and Step 3 contains the end of the proof of this item.

The following two lemmas show this.

Recall the differential equation satisfied by the $\wp$ function, where the prime indicates a derivative with respect to the variable z:

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

where g_2 and g_3 are functions of the variable τ,

$$g_2(\tau) = 60{\sum}'(m\tau + n)^{-4}, \quad g_3(\tau) = 140{\sum}'(m\tau + n)^{-6},$$

and $\sum'$ denotes the summation of the series for all pairs $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$.

Lemma 4.4. *The field $K = \mathbb{C}(z, \tau, \wp, \wp', g_2, g_3, \zeta, E_2)$ is closed under differentiation with respect to the variables z and τ.*

Proof. We recall the differential equation satisfied by $\wp$, where $\wp'$ indicates the (partial) derivative with respect to z:

$$\wp'^2(\tau; z) = 4\wp(\tau; z)^3 - g_2(\tau)\wp(\tau; z) - g_3(\tau),$$

from which we can obtain all partial derivatives with respect to z as rational functions of $\wp$, $\wp'$, g_2 and g_3.

By definition, $\partial\zeta(\tau; z)/\partial z = -\wp(\tau; z)$, from which we can obtain all partial derivatives of ζ with respect to the variable z.

There remains the need to differentiate with respect to the variable (or parameter) τ.

We start with the functions g_2 and g_3, which are modular forms of degree 4 and 6, respectively. Here we follow [**20**, §§ 2.1, 2.3 and 5.1].

Write $g_2 = 60E_4$ and $g_3 = 140E_6$, where

$$E_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m + n\tau)^{2k}},$$

for $k = 2, 3$.

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For $k = 1$, the series does not converge absolutely, but we can define

$$E_2(\tau) = 1 + \frac{3}{\pi^2} \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^2},$$

where the sums cannot be interchanged (the series converges conditionally). This defines a *quasi-modular form of weight 2*, which has the transformation formula

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{6}{\pi} \mathrm{i} c(c\tau + d).$$

Although E_2 is not a modular form, its transformation formula is simple. Its importance is that the ring $\mathbb{C}[E_2, E_4, E_6]$ contains all the modular forms and is closed under differentiation because of Ramanujan's formulas

$$E_2' = \frac{E_2^2 - E_4}{12}, \quad E_4' = \frac{E_2 E_4 - E_6}{3}, \quad E_6' = \frac{E_2 E_6 - E_4^2}{2},$$

where $E_k'(\tau)$ denotes $(1/2\pi\mathrm{i})\mathrm{d}E_k/\mathrm{d}\tau$ (see [**20**, Proposition 15, p. 49]).

For the functions $\wp$, $\wp'$ and ζ, we use the following formulas (see [**1**, Formulas 18.6.19–22, p. 641], or [**9**, Chapter IX, pp. 291–299] for their deduction). We omit showing the variables of the functions $\wp$, $\wp'$ and ζ, and consider g_2 and g_3 as variables and denote $\Delta = g_2^3 - 27g_3^2$:

$$\Delta \frac{\partial \wp}{\partial g_3} = \left(3g_2\zeta - \frac{9}{2} g_3 z\right) \wp' + 6g_2\wp^2 - 9g_3\wp - g_2^2,$$

$$\Delta \frac{\partial \wp}{\partial g_2} = \left(-\frac{9}{2} g_3\zeta + \frac{g_2^2 z}{4}\right) \wp' - 9g_3\wp^2 + \frac{g_2^2}{2}\wp + \frac{3}{2} g_2 g_3,$$

$$\Delta \frac{\partial \zeta}{\partial g_3} = -3\zeta\left(g_2\wp + \frac{3g_3}{2}\right) + \frac{z}{2}\left(9g_3\wp + \frac{g_2^2}{2}\right) - \frac{3}{2} g_2\wp',$$

$$\Delta \frac{\partial \zeta}{\partial g_2} = \frac{\zeta}{2}\left(9g_3\wp + \frac{g_2^2}{2}\right) - \frac{g_2 z}{2}\left(\frac{g_2\wp}{2} + \frac{3g_3}{4}\right) + \frac{9}{4} g_3\wp'.$$

From these, we can obtain the derivatives of $\wp'$ with respect to g_2 and g_3. Because of these formulas, we see the need for the ζ function in this field, and we conjecture that this function is not existentially definable from the other functions of the structure.

The chain rule gives the derivatives of $\wp$, $\wp'$ and ζ with respect to the variable τ, as required. □

As an immediate consequence, we have the following.

Lemma 4.5. *The partial derivatives of all orders of the real and imaginary parts of the functions $\wp$ and ζ, and form E_2, are strongly definable from their real and imaginary parts.*

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Proof. The only missing formulas are $g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3)$ and $g_3 = 4e_1e_2e_3$, where $e_1 = \wp(\tau; 1/2)$, $e_2 = \wp(\tau; \tau/2)$ and $e_3 = \wp(\tau; (1+\tau)/2)$ are the roots of the polynomial $4\wp^3 - g_2\wp - g_3$. The correct choice of the branch of the square root of this polynomial provides the definability of $\wp'$.

If we write $z = x + \mathrm{i}y$ and $\tau = \alpha + \mathrm{i}\beta$, we have $\wp(\alpha + \mathrm{i}\beta; x + \mathrm{i}y) = RP(\alpha, \beta; x, y) + \mathrm{i}IP(\alpha, \beta; x, y)$, $\zeta = RZ(\alpha, \beta; x, y) + \mathrm{i}IZ(\alpha, \beta; x, y)$ and $E_2(\tau) = RE_2(\alpha, \beta) + \mathrm{i}IE_2(\alpha, \beta)$. From these formulas, we can obtain the strong definability of all partial derivatives of $\wp$, ζ and E_2 with respect to both variables z and τ. □

Step 3. We finish the verification of the Item 2 of Theorem 2.1 showing that the functions are definably extended to an open neighbourhood of the original domains. This was partially done in Step 1. This implies that *for each* $\boldsymbol{a} \in \Delta_\lambda$ *there exists a compact polydisc* Δ *centred at* $\boldsymbol{a}$ *and contained in the domain of* g_λ*, such that all the partial derivatives of the restriction of* g_λ *to* Δ *are strongly definable in* $\hat{R}$.

Lemma 4.6. *Extensions of the function* E_2 *can be strongly defined from* E_2*. Extensions of the functions* $\wp$ *and* ζ *in both variables can be strongly defined from these functions.*

Proof. Recall that $E_2(\tau \pm 1) = E_2(\tau)$ and, in general,

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{6\mathrm{i}c(c\tau + d)}{\pi^2},$$

which allows the definition of extensions of E_2 beyond the fundamental domain.

The function $\wp$ satisfies the addition formulas

$$\wp(\tau; u + v) = \frac{1}{4}\left(\frac{\wp'(\tau; u) - \wp'(\tau; v)}{\wp(\tau; u) - \wp(\tau; v)}\right)^2 - \wp(\tau; u) - \wp(\tau; v)$$

and, for the case where $u = v$,

$$\wp(\tau; 2u) = -2\wp(\tau; u) - \left(\frac{\wp''(\tau; u)}{2\wp'(\tau; u)}\right)^2.$$

It also satisfies $\wp(\tau + 1; z) = \wp(\tau; z)$ and $\wp(-1/\tau; z) = \tau\wp(\tau; \tau z)$. We recall that the Möbius transformations $S(\tau) = -1/\tau$ and $T(\tau) = \tau + 1$ generate the group $SL(2, \mathbb{Z})$. With these equations, we can strongly define the desired extensions of $\wp$.

The function ζ satisfies $\zeta(\tau; z + \omega) = \zeta(\tau; z) + \eta(\omega)$, where $\eta : \Lambda_\tau \to \mathbb{C}$ is a group homomorphism from the period lattice $\Lambda_\tau = \{m + n\tau : m, n \in \mathbb{Z}\}$ into $\mathbb{C}$. For $\omega \in \Lambda_\tau \setminus 2\Lambda_\tau$, we have the formula $\eta(\omega) = 2\zeta(\tau; \omega/2)$ (see [**18**, Proposition 5.2, pp. 40–41]). The zeta function also satisfies a kind of summation formula [**19**, § 20.41, Example 1, p. 446],

$$[\zeta(\tau; x) + \zeta(\tau; y) + \zeta(\tau; z)]^2 + \wp(\tau; x) + \wp(\tau; y) + \wp(\tau; z) = 0,$$

if $x + y + z = 0$. From the series defining the zeta function, we obtain $\zeta(\tau + 1; z) = \zeta(\tau; z)$ and $\zeta(-1/\tau; z) = \tau\zeta(\tau; \tau z)$.

This proves the lemma and finishes the proof of Theorem 4.1. □

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Proof of Theorem 3.1. By the model-completeness Theorem 2.2, the expansion of the structure $\mathcal{R}_{aux}$ including the full exponential function is strongly model complete. With this result at hand, it is easy to see that the structure $\mathcal{R}_{\wp}$ is also strongly model complete.

Indeed, a definable set in the structure $\mathcal{R}_{\wp}$ can be mapped onto a set definable in $\mathcal{R}_{aux,\exp}$, and so it can be (strongly) existentially definable. This can be mapped back onto the original set and preserving the strong definability, because the map and its inverse are both strongly definable. □

Remark. If we take the limit of $\wp(\tau; z)$ for $\tau \to \mathrm{i}\infty$, with $|\mathfrak{Re}(\tau)| \leqslant 1/2$, we obtain the series

$$\sum_{n=-\infty}^{+\infty} \frac{1}{(z-n)^2} - 2\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{\sin^2(\pi z)} - \frac{\pi^2}{3}$$

(see [**19**, § 20.222, pp. 438–439] and [**1**, Formula 4.3.92, p. 75]). Therefore, unless the restricted sine function were (strongly, or just existentially) definable from the functions $\wp$, ζ, E_2 and exp, the reduction of the structure $\mathcal{R}_{\wp}$ without the restricted sine function would not be model complete. Actually, we conjecture that without either the sine or the exponential functions the (reduced) structure $\mathcal{R}_{\wp}$ may not be model complete. Observe that both functions are definable from the function $\wp$ ([**14**, Theorem 5.7, p. 545] shows that exp is definable).

5. Concluding remarks

Our method is based on the fact that the interval $[-1, 1]$ is a compact set and so the proofs bear a non-effective feature.

Recently, Macintyre [**11**] has shown how to prove an effective version of the model completeness of the expansion of the field of the real numbers by the $\wp$ function with parameter $\tau = \mathrm{i}$. His proof can be modified to prove effective model completeness at least for the cases where $\wp$ admits complex multiplication and, perhaps, for all cases. He uses the inverse function of $\wp$, which is an elliptic integral and thus is a Pfaffian function (see the details in [**12**]), so it is amenable to the treatment used in his work with Alex Wilkie on the decidability of the real exponential field [**13**]. In the latter paper, they prove an effective model-completeness result for the expansion of the real field with the restricted exponential function, from which we can pinpoint three main ideas.

(1) Pfaffian functions are amenable to good induction proofs on their complexity.

(2) There exist computable bounds on the number of connected components of zero sets of Pfaffian functions.

(3) The structure is polynomially bounded, and they show that there is a computable estimate on the behaviour at infinity of the one-variable definable functions.

Their methods do not seem to apply to our case, because the derivative of $\wp$ with respect to the parameter τ involves the modular forms E_4 and E_6, which are not Pfaffian functions (but are *Noetherian* functions) and satisfy a non-triangular system of differential equations involving E_2, E_4 and E_6.

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Of the three ideas stated above, only the third one (behaviour at infinity) seems to be more easily achievable. Some work towards the first and second ideas (computable bounds on the number of connected components) is surveyed in [7].

Two open questions on existential definability have arisen in this paper.

(1) Are the real and imaginary parts of the Weierstrass ζ function (existentially) definable from $\wp$ and exp?

(2) Peterzil and Starchenko proved in [14, Theorem 5.7, p. 545] that the real exponential function is definable from the $\wp$ function. We ask if it is existentially definable.

Finally, as the referee has pointed out, the techniques of this paper should be applied to try to prove analogous results for covering maps of $\mathcal{A}_g$ and to general and mixed Shimura varieties; the appropriate functions are definable in $\mathbb{R}_{an,\exp}$, by the results of Peterzil and Starchenko [15], Klingler *et al.* [10] and Gao [8], respectively.

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