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Entropy and its variational principle for non-compact metric spaces

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Abstract. In the present paper, we introduce a natural extension of Adler, Konheim and MacAndrew topological entropy for proper maps of locally compact separable metrizable spaces and prove a variational principle which states that this topological entropy, the supremum of the Kolmogorov–Sinai entropies and the minimum of the Bowen entropies always coincide. We apply this variational principle to show that the topological entropy of automorphisms of simply connected nilpotent Lie groups always vanishes. This shows that the Bowen formula for the Bowen entropy of an automorphism of a non-compact Lie group with respect to some invariant metric is just an upper bound for its topological entropy.

1. *Introduction*

Topological entropy was introduced in [**1**] by Adler, Konheim and MacAndrew (AKM) to study the dynamics of a continuous map $T : X \to X$ defined on a compact space X. They also conjectured the well-known variational principle which states that

$$h(T) = \sup_{\mu} h_{\mu}(T),$$

where $h(T)$ is the AKM topological entropy of T, $h_{\mu}(T)$ is the Kolmogorov–Sinai (KS) entropy of T with respect to μ and the supremum ranges over the set $\mathcal{P}_T(X)$ of all T-invariant Borel probability measures μ on X.

Bowen [**2**] extended the concept of topological entropy for non-compact metric spaces. This approach uses the concept of spanning sets, which are defined by means of a given metric. For compact spaces this definition of entropy is, in fact, independent of the metric and, indeed, coincides with the AKM topological entropy.

However, for non-compact spaces the Bowen entropy depends on the metric. Handel and Kitchens obtained [**5**, Lemma 1.5] the following variational principle for locally compact separable metric spaces:

$$\sup_{\mu} h_{\mu}(T) = \inf_{d} h_{d}(T),$$

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where $h_d(T)$ is the Bowen entropy of T with respect to d and the infimum ranges over all metrics d on X.

In the present paper, we improve this result showing that the above infimum is always attained at d, whenever d is a metric satisfying some special conditions. These metrics are called admissible metrics and always exist when X is a locally compact separable metric space. Moreover, when T is a proper map, the supremum and the minimum coincide with a natural extension of AKM topological entropy, introduced in §2. This natural extension and the admissible metrics are closed related to the one-point compactification of X. In §3 this variational principle is proved.

The idea of obtaining a variational principle for a non-compact space by passing to the one-point compactification is not new. It was first used by Gurevich in his calculation of the supremum of the KS entropies for topological Markov shifts with countably many states [**4**]. This result does not imply our results because they are confined to topological Markov shifts. On the other hand, it does not follow from our results because topological Markov shifts with infinite set of states are not always locally compact and not always proper.

In §4 we apply the variational principle to determine the topological entropy of automorphisms of Lie groups. First we consider linear isomorphisms of a finite-dimensional vector space and characterize their recurrent sets in terms of their multiplicative Jordan decompositions. We show that the topological entropy of these linear isomorphisms always vanishes. This shows that the Bowen formula for its Bowen entropy with respect to the Euclidian metric d is just an upper bound for its null topological entropy. Remember that the Bowen formula is given by

$$h_d(T) = \sum_{|\lambda|>1} \log |\lambda|,$$

where λ runs through the eigenvalues of the linear isomorphism T listed with multiplicity (cf. [**9**, Theorem 8.14]). Thus it might be an interesting problem to calculate the topological entropy of a given automorphism ϕ of a general non-compact Lie group G, since the classical formula for its Bowen entropy with respect to some invariant metric is just an upper bound for its topological entropy. We conclude this paper by providing an answer for this problem when G is a simply connected nilpotent Lie group. We show that the topological entropy of its automorphisms also vanishes.

2. *Topological entropy and admissible metrics*

We start this section by presenting an extension of the AKM topological entropy introduced in [**1**]. Let X be a topological space and $T : X \to X$ be a *proper* map, i.e., T is a continuous map such that the pre-image by T of any compact set is compact. An *admissible* cover of X is an open and finite cover α of X such that, for each $A \in \alpha$, the closure or the complement of A is compact. Given an admissible cover α of X, for every $n \in \mathbb{N}$, we have that the set given by

$$\alpha^n = \{A_0 \cap T^{-1}(A_1) \cap \cdots \cap T^{-n}(A_n) \mid A_i \in \alpha\}$$

is also an admissible cover of X, since T is a proper map. Given an admissible cover α of X, we denote by $N(\alpha^n)$ the smallest cardinality of all subcovers of α^n. Exactly as in the

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compact case, it can be shown that the sequence $\log N(\alpha^n)$ is subadditive, which implies the existence of the limit

$$h(T, \alpha) = \lim_{n\to\infty} \frac{1}{n} \log N(\alpha^n).$$

The *topological entropy of the map* T is thus defined as

$$h(T) = \sup_{\alpha} h(T, \alpha),$$

where the supremum is taken over all admissible covers α of X. We note that, when X is compact, the above definition coincides with the AKM definition, since in this case every continuous map is proper and every open and finite cover of X is admissible.

The following result generalizes, for non-compact spaces, the relation between the topological entropies of two semi-conjugated maps.

PROPOSITION 2.1. *Let $T : X \to X$ and $U : Y \to Y$ be two proper maps, where X and Y are topological spaces. If $f : Y \to X$ is a proper surjective map such that $f \circ U = T \circ f$, then we have that $h(T) \leq h(U)$.*

Proof. Let α be an admissible cover of X. Since f is a proper map, it follows that

$$f^{-1}(\alpha) = \{f^{-1}(A) \mid A \in \alpha\}$$

is an admissible cover of Y. We claim that if β is a subset of α^n, then $f^{-1}(\beta)$ is a subset of $f^{-1}(\alpha)^n$. In fact, $B \in \beta$ if and only if

$$B = A_0 \cap T^{-1}(A_1) \cap \cdots \cap T^{-n}(A_n)$$

where $A_i \in \alpha$, for each $i \in \{0, \ldots, n\}$. Thus we have that

$$\begin{aligned} f^{-1}(B) &= f^{-1}(A_0) \cap f^{-1}(T^{-1}(A_1)) \cap \cdots \cap f^{-1}(T^{-n}(A_n)) \\ &= f^{-1}(A_0) \cap U^{-1}(f^{-1}(A_1)) \cap \cdots \cap U^{-n}(f^{-1}(A_n)), \end{aligned}$$

where we have used the fact that $f^{-1} \circ T^{-i} = U^{-i} \circ T^{-1}$, since $f \circ U = T \circ f$. Thus it follows that $f^{-1}(\beta) \subset f^{-1}(\alpha)^n$. Conversely, we claim that if γ is a subset of $f^{-1}(\alpha)^n$, then $\gamma = f^{-1}(\beta)$, where β is some subset of α^n. Proceeding analogously, if $C \in \gamma$, then

$$\begin{aligned} C &= f^{-1}(A_0) \cap U^{-1}(f^{-1}(A_1)) \cap \cdots \cap U^{-n}(f^{-1}(A_n)) \\ &= f^{-1}(A_0 \cap T^{-1}(A_1) \cap \cdots \cap T^{-n}(A_n)), \end{aligned} \tag{1}$$

where $A_i \in \alpha$, for each $i \in \{0, \ldots, n\}$. Thus it follows that $C = f^{-1}(B)$, for some $B \in \alpha^n$, which implies that $\gamma = f^{-1}(\beta)$, where β is some subset of α^n.

If β is a subcover of α^n, then $f^{-1}(\beta)$ is a subcover of $f^{-1}(\alpha)^n$. Conversely, if γ is a subcover of $f^{-1}(\alpha)^n$, then $\gamma = f^{-1}(\beta)$, where β is some subcover of α^n. In fact, we have already shown that $\gamma = f^{-1}(\beta)$, for some subset β of α^n. It remains to show that β is a cover of X. But this is immediate, since $\gamma = f^{-1}(\beta)$ is a cover of Y, f is surjective and $B = f(f^{-1}(B))$.

Hence we have that $N(\alpha^n) = N(f^{-1}(\alpha)^n)$ and, taking logarithms, dividing by n and taking limits, it follows that

$$h(T, \alpha) \leq h(U, f^{-1}(\alpha)) \leq h(U).$$

Since α is an arbitrary admissible cover of X, we get that $h(T) \leq h(U)$. □

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Now we introduce two different concepts of entropy with respect to a given metric. First we remember the Bowen definition, introduced in [**2**]. Given a metric space (X, d) and a continuous map $T : X \to X$, we consider the metric given by

$$d_n(x, y) = \max\{d(T^i(x), T^i(y)) \mid 0 \leq i \leq n\},$$

which is equivalent to the original metric d. Given a subset $Y \subset X$, a subset S is an (n, ε)-spanning set of Y if and only if, for every $y \in Y$, there exists $x \in S$ such that $d_n(x, y) < \varepsilon$. In other words, the collection of all ε-balls of d_n centered at points of S is a cover of Y. We denote by $S_n(\varepsilon, Y)$ the smallest cardinality of all (n, ε)-spanning sets of Y and define

$$s(\varepsilon, Y) = \lim_{n\to\infty} \sup \frac{1}{n} \log S_n(\varepsilon, Y).$$

It can be proved that $s(\varepsilon, Y)$ is monotone with respect to ε, so we can define

$$h_d(T, Y) = \lim_{\varepsilon\downarrow 0} s(\varepsilon, Y).$$

The *Bowen entropy of the map T with respect to the metric d* is thus defined as

$$h_d(T) = \sup_K h_d(T, K),$$

where the supremum is taken over all compact subsets K of X, where $h_d(T)$ can be infinite.

The second definition, which we call *d-entropy*, is given by

$$h^d(T) = \sup_Y h_d(T, Y),$$

where now the supremum is taken over all subsets Y of X instead of just the compact ones. Note that here the metric d is denoted in the superscript. Since we have that $h_d(T, Y)$ is monotone with respect to Y, it follows that

$$h^d(T) = h_d(T, X).$$

We immediately notice that $h_d(T) \leq h^d(T)$ always holds and, when X is compact, we get the equality. Denoting $S_n(\varepsilon, X)$ and $s(\varepsilon, X)$, respectively, by $S_n(\varepsilon)$ and $s(\varepsilon)$, we get that

$$h^d(T) = \lim_{\varepsilon\downarrow 0} s(\varepsilon)$$

and that

$$s(\varepsilon) = \lim_{n\to\infty} \sup \frac{1}{n} \log S_n(\varepsilon).$$

Our first result is a generalization of the well-known result which states that, if (X, d) is compact, then the topological and the Bowen entropies coincide. In the non-compact case, this remains true if we work with the d-entropy and require some regularity in the metric d. Let (X, d) be a metric space and denote by $B(x, \delta)$ the open ball centered at x with radius $\delta > 0$. The metric d is *admissible* if the following conditions are satisfied:

(1) If $\alpha_\delta = \{B(x_1, \delta), \ldots, B(x_k, \delta)\}$ is a cover of X, for every $\delta \in (a, b)$, where $0 < a < b$, then there exists $\delta_\varepsilon \in (a, b)$ such that $\alpha_{\delta_\varepsilon}$ is admissible.

(2) Every admissible cover of X has a Lebesgue number.

We observe that, if (X, d) is compact, then d is automatically admissible.

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PROPOSITION 2.2. *Let $T : X \to X$ be a proper map, where (X, d) is a metric space. If d is an admissible metric, then $h(T) = h^d(T)$.*

Proof. First we claim that, for any admissible cover α of X, it follows that $s(|\alpha|) \leq h(T, \alpha)$, where $|\alpha|$ is the maximum of the diameters of $A \in \alpha$. In fact, the elements of α^n are given by $A_0 \cap T^{-1}(A_1) \cap \cdots \cap T^{-n}(A_n)$, where $A_i \in \alpha$. Let β be a subcover of α^n. For each $B \in \beta$, take an $x \in B$ and consider S the set of all such points. We claim that S is an $(n, |\alpha|)$-spanning set of X. In fact, let $y \in X$ and take some $A_0 \cap T^{-1}(A_1) \cap \cdots \cap T^{-n}(A_n) \in \beta$ containing y. Taking $x \in S$ such that $x \in A_0 \cap T^{-1}(A_1) \cap \cdots \cap T^{-n}(A_n)$, we have that $d(T^i(x), T^i(y)) < |\alpha|$, for every $i \in \{0, 1, \ldots, n\}$, since $T^i(x), T^i(y) \in A_i$. Hence, for each subcover β of α^n, there exists an $(n, |\alpha|)$-spanning set of X such that $S_n(|\alpha|) \leq \#S \leq \#\beta$. Thus we get that $S_n(|\alpha|) \leq N(\alpha^n)$. Taking logarithms, dividing by n and taking limits, it follows as claimed that $s(|\alpha|) \leq h(T, \alpha)$.

Now we claim that, for all $\varepsilon > 0$, there exists $\delta_\varepsilon \in (\varepsilon/2, \varepsilon)$ such that $h(T, \alpha_{\delta_\varepsilon}) \leq s(\varepsilon/2)$, where $\alpha_{\delta_\varepsilon}$ is an admissible cover of balls with radius equals to δ_ε. In fact, if $S = \{x_1, \ldots, x_k\}$ is an $(n, \varepsilon/2)$-spanning set of X, for every $\delta \in (\varepsilon/2, \varepsilon)$, we have that

$$\beta_\delta = \{B(x_i, \delta) \cap \cdots \cap T^{-n}(B(T^n(x_i), \delta)) \mid x_i \in S\}$$

is a cover of X. To see this, given $x \in X$, there exists $x_i \in S$ such that $d_n(x, x_i) < \varepsilon/2 < \delta$ and thus we have that

$$x \in B(x_i, \delta) \cap \cdots \cap T^{-n}(B(T^n(x_i), \delta)).$$

This implies that

$$\alpha_\delta = \{B(T^l(x_i), \delta) \mid x_i \in S, 0 \leq l \leq n\}$$

is a finite cover of X, for every $\delta \in (\varepsilon/2, \varepsilon)$. Since d is an admissible metric, there exists $\delta_\varepsilon \in (\varepsilon/2, \varepsilon)$ such that $\alpha_{\delta_\varepsilon}$ is an admissible cover of balls with radius equals to δ_ε. Furthermore, we have that $\beta_{\delta_\varepsilon}$ is a subcover of $\alpha^n_{\delta_\varepsilon}$. Hence, for each $(n, \varepsilon/2)$-spanning set S of X, there exist $\delta_\varepsilon \in (\varepsilon/2, \varepsilon)$ and a subcover $\beta_{\delta_\varepsilon}$ of $\alpha^n_{\delta_\varepsilon}$ such that $N(\alpha_{\delta_\varepsilon}) \leq \#\beta_{\delta_\varepsilon} \leq \#S$. Thus it follows that $N(\alpha_{\delta_\varepsilon}) \leq S_n(\varepsilon/2)$. Taking logarithms, dividing by n and taking limits, we get as claimed that there exists $\delta_\varepsilon \in (\varepsilon/2, \varepsilon)$ such that $h(T, \alpha_{\delta_\varepsilon}) \leq s(\varepsilon/2)$. Since $|\alpha_{\delta_\varepsilon}| \leq 2\delta_\varepsilon \leq 2\varepsilon$, we have that

$$s(2\varepsilon) \leq s(|\alpha_{\delta_\varepsilon}|) \leq h(T, \alpha_{\delta_\varepsilon}) \leq s(\varepsilon/2).$$

Taking limits with $\varepsilon \downarrow 0$, it follows that

$$h^d(T) = \lim_{\varepsilon \downarrow 0} h(T, \alpha_{\delta_\varepsilon}) = \sup_{\varepsilon > 0} h(T, \alpha_{\delta_\varepsilon}).$$

In order to complete the proof, it remains to show that the above supremum is equal to $h(T)$. For any admissible cover α of X, take ε to be a Lebesgue number of this cover. We claim that $N(\alpha^n) \leq N(\alpha^n_{\delta_\varepsilon})$, where $\alpha_{\delta_\varepsilon}$ is given above. In fact, since every element of $\alpha^n_{\delta_\varepsilon}$ is given by $B_0 \cap \cdots \cap T^{-n}(B_n)$, where $\{B_0, \ldots, B_n\}$ are balls of radius $\delta_\varepsilon < \varepsilon$, there exist $\{A_0, \ldots, A_n\} \subset \alpha$ such that $B_i \subset A_i$, for all $i \in \{0, \ldots, n\}$. Thus we have that

$$B_0 \cap \cdots \cap T^{-n}(B_n) \subset A_0 \cap \cdots \cap T^{-n}(A_n),$$

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showing that, for each subcover β of $\alpha^n_{\delta_\varepsilon}$, there exists a subcover γ of α^n such that $N(\alpha^n) \leq \#\gamma \leq \#\beta$. Hence we get as claimed that $N(\alpha^n) \leq N(\alpha^n_{\delta_\varepsilon})$. Taking logarithms, dividing by n and taking limits, it follows that

$$h(T, \alpha) \leq h(T, \alpha_{\delta_\varepsilon}) \leq \sup_{\varepsilon>0} h(T, \alpha_{\delta_\varepsilon}),$$

which shows that

$$h(T) = \sup_{\varepsilon>0} h(T, \alpha_{\delta_\varepsilon}),$$

completing the proof. □

From now on we assume that X is a locally compact separable metric space. Thus we have associated with X its one-point compactification, which we denote by $\widetilde{X}$. We have that $\widetilde{X}$ is defined as the disjoint union of X with $\{\infty\}$, where ∞ is some point not in X called the *point at infinity*. The topology in $\widetilde{X}$ consists of the former open sets in X and the sets $A \cup \{\infty\}$, where the complement of A in X is compact. Let $T : X \to X$ be a proper map. Defining $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ by

$$\widetilde{T}(\widetilde{x}) = \begin{cases} T(\widetilde{x}), & \widetilde{x} \neq \infty, \\ \infty, & \widetilde{x} = \infty, \end{cases}$$

we have that $\widetilde{T}$ is also a proper map, called the *extension of T to $\widetilde{X}$*. To see this, we only need to verify that $\widetilde{T}$ is continuous at ∞. If $A \cup \{\infty\}$ is a neighborhood of ∞, then the complement of A is compact and we have that

$$\widetilde{T}^{-1}(A \cup \{\infty\}) = T^{-1}(A) \cup \{\infty\}$$

is also a neighborhood of ∞, since T is proper and thus the complement of $T^{-1}(A)$ is also compact. We note that the separability of X is equivalent to the metrizability of $\widetilde{X}$.

The following result shows that the restriction to X of any metric on $\widetilde{X}$ is always admissible and that their respective entropies coincide.

PROPOSITION 2.3. *Let $T : X \to X$ be a proper map, where X is a locally compact separable space. Let d be the metric given by the restriction to X of some metric $\widetilde{d}$ on $\widetilde{X}$, the one-point compactification of X. Then it follows that d is an admissible metric and that*

$$h^d(T) = h^{\widetilde{d}}(\widetilde{T}),$$

where $\widetilde{T}$ is the extension of T to $\widetilde{X}$. In particular, we get that $h(T) = h(\widetilde{T})$.

Proof. First we show that $S_n(\varepsilon)$ is finite for every $\varepsilon > 0$. Let $\widetilde{S} = \{\widetilde{x}_1, \ldots, \widetilde{x}_k\} \subset \widetilde{X}$ be an $(n, \varepsilon/2)$-spanning set of $\widetilde{X}$. By the density of X in $\widetilde{X}$, it follows that there exist $\{x_1, \ldots, x_k\} \subset X$, such that $\widetilde{d}_n(x_i, \widetilde{x}_i) < \varepsilon/2$. If $x \in X \subset \widetilde{X}$, we have that $\widetilde{d}_n(x, \widetilde{x}_i) < \varepsilon/2$, for some $\widetilde{x}_i \in \widetilde{S}$. Hence it follows that

$$d_n(x, x_i) \leq \widetilde{d}_n(x, \widetilde{x}_i) + \widetilde{d}_n(x_i, \widetilde{x}_i) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

showing that $S = \{x_1, \ldots, x_k\}$ is an (n, ε)-spanning set of X. If we choose $\widetilde{S}$ such that $\#\widetilde{S} = \widetilde{S}_n(\varepsilon/2)$, then we get as claimed that

$$S_n(\varepsilon) \leq \widetilde{S}_n(\varepsilon/2) < \infty.$$

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Now assume that $\alpha_\delta = \{B(x_1, \delta), \ldots, B(x_k, \delta)\}$ is a cover of X, for every $\delta \in (a, b)$, where $0 < a < b$. Since, for each fixed δ, the number of balls is finite, it follows that there exists $\delta_\varepsilon \in (a, b)$ such that $\infty \notin \widetilde{S}(x_i, \delta_\varepsilon)$, for every $i \in \{1, \ldots, k\}$, where $\widetilde{S}(x, r)$ is the sphere in $\widetilde{X}$ of radius r centered at x. Denoting by $\widetilde{B}(x, r)$ the open ball in $\widetilde{X}$ of radius r centered in x, there are two alternatives: (1) the point ∞ is inside $\widetilde{B}(x_i, \delta_\varepsilon)$; or (2) the point ∞ is not in the closure of $\widetilde{B}(x_i, \delta_\varepsilon)$. In the first case, the complement of $B(x_i, \delta_\varepsilon)$ in X is equal to the complement of $\widetilde{B}(x_i, \delta_\varepsilon)$ in X, which is compact. In the second alternative, there exist an open neighborhood U of ∞ which has empty intersection with $\widetilde{B}(x_i, \delta_\varepsilon)$. Thus $B(x_i, \delta_\varepsilon) = \widetilde{B}(x_i, \delta_\varepsilon)$ is in the complement of U in X, which is compact. This shows that the closure or the complement of $B(x_i, \delta_\varepsilon)$ in X is compact, showing that $\alpha_{\delta_\varepsilon}$ is already admissible.

We now show that every admissible cover of X has a Lebesgue number. If X is compact, the existence of a Lebesgue number is a classical result. Thus assume that X is not compact. If α is an admissible cover of X, there exists an open cover $\widetilde{\alpha}$ of $\widetilde{X}$ such that $A \in \alpha$ if and only if there exists $\widetilde{A} \in \widetilde{\alpha}$, with $A = \widetilde{A} \cap X$. In fact, if α is an admissible cover of X, there exists at least one $A \in \alpha$ with compact complement in X and such that its closure in X is not compact. If we define $\widetilde{A} = A \cup \{\infty\}$, we have that $\widetilde{A}$ is an open neighborhood of $\{\infty\}$ in $\widetilde{X}$. Thus defining

$$\widetilde{\alpha} = \{\widetilde{A}\} \cup \{B \in \alpha \mid B \neq A\},$$

it follows as claimed that $\widetilde{\alpha}$ is an open cover of $\widetilde{X}$ such that $A \in \alpha$ if and only if there exists $\widetilde{A} \in \widetilde{\alpha}$, with $A = \widetilde{A} \cap X$. If ε is a Lebesgue number for $\widetilde{\alpha}$, then we claim that ε is also a Lebesgue number for α. To see this, if $B(x, \varepsilon)$ is a ball in X, there exists $\widetilde{A} \in \widetilde{\alpha}$ such that $B(x, \varepsilon) \subset \widetilde{A}$. Thus it follows that $B(x, \varepsilon) \subset A$, where $A = \widetilde{A} \cap X \in \alpha$, completing the proof that d is an admissible metric.

Finally, we show that $h^d(T) = h^{\widetilde{d}}(\widetilde{T})$. Let $S = \{x_1, \ldots, x_k\}$ be an $(n, \varepsilon/2)$-spanning set of X. By the density of X in $\widetilde{X}$, if $\widetilde{x} \in \widetilde{X}$, there exists $x \in X$ such that $\widetilde{d}_n(\widetilde{x}, x) < \varepsilon/2$, since d_n is topologically equivalent to d. Thus we have that $d_n(x, x_i) < \varepsilon/2$, for some $x_i \in S$. Hence it follows that

$$\widetilde{d}_n(\widetilde{x}, x_i) \leq \widetilde{d}_n(\widetilde{x}, x) + d_n(x, x_i) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

showing that $S = \{x_1, \ldots, x_k\}$ is an (n, ε)-spanning set of $\widetilde{X}$. Choosing S such that $\#S = S_n(\varepsilon/2)$, we get that $\widetilde{S}_n(\varepsilon) \leq S_n(\varepsilon/2)$. Since we have also shown above that $S_n(\varepsilon) \leq \widetilde{S}_n(\varepsilon/2)$, taking logarithms, dividing by n and taking limits, we get that

$$s(\varepsilon) \leq \widetilde{s}(\varepsilon/2) \quad \text{and} \quad \widetilde{s}(\varepsilon) \leq s(\varepsilon/2).$$

Therefore it follows that

$$h^d(T) = \lim_{\varepsilon \downarrow 0} s(4\varepsilon) \leq \lim_{\varepsilon \downarrow 0} \widetilde{s}(2\varepsilon) = h^{\widetilde{d}}(\widetilde{T}) \leq \lim_{\varepsilon \downarrow 0} s(\varepsilon) = h^d(T),$$

showing that $h^d(T) = h^{\widetilde{d}}(\widetilde{T})$. The last statement of the proposition now follows by applying Proposition 2.2. □

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3. *The variational principle*

In this section we present a full extension for non-compact sets of the well-known variational principle involving entropies. We start with a proper map $T : X \to X$ and remember the concept of entropy associated with some given T-invariant probability measure on X. A *T-invariant probability measure on X* is a Borel measure μ such that $\mu(X) = 1$ and $\mu(T^{-1}(A)) = \mu(A)$, for all Borel subsets $A \subset X$. We recall that $\mathcal{P}_T(X)$ denotes the set of all T-invariant probability measures on X. A Borel partition $\mathcal{A}$ of X is a partition of X into Borel sets. Given a finite Borel partition $\mathcal{A}$ of X, for every $n \in \mathbb{N}$, we have that the set given by

$$\mathcal{A}^n = \{A_0 \cap T^{-1}(A_1) \cap \cdots \cap T^{-n}(A_n) \mid A_i \in \mathcal{A}\}$$

is also a finite Borel partition of X, since T is continuous. For a given finite Borel partition $\mathcal{A}$ of X, we define its associated n-entropy as

$$H(\mathcal{A}^n) = \sum_{B \in \mathcal{A}^n} \phi(\mu(B)),$$

where $\phi : [0, 1] \to \mathbb{R}$ is the continuous function given by

$$\phi(x) = \begin{cases} -x \log(x), & x \in (0, 1], \\ 0, & x = 0. \end{cases}$$

It can be shown that the sequence $H(\mathcal{A}^n)$ is subadditive, which implies the existence of the limit

$$h(T, \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} H(\mathcal{A}^n).$$

The *μ-entropy of the map T* is thus defined as

$$h_\mu(T) = \sup_{\mathcal{A}} h(T, \mathcal{A}),$$

where the supremum is taken over all finite Borel partitions $\mathcal{A}$ of X. The next result was first proved by Handel and Kitchens (see [**5**, Proof of Lemma 1.5]). They used the fact that the supremum of the KS entropies taken over all invariant probability measures is equal to the supremum taken over all ergodic invariant probability measures. We present here an elementary proof of the quoted result which does not use this fact.

LEMMA 3.1. *Let $T : X \to X$ be a proper map such that $\mathcal{P}_T(X) \neq \emptyset$ and $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ be its extension in the one-point compactification $\widetilde{X}$ of the locally compact separable space X. Then it follows that*

$$\sup_{\mu} h_\mu(T) = \sup_{\widetilde{\mu}} h_{\widetilde{\mu}}(\widetilde{T}),$$

where the suprema are taken, respectively, over $\mathcal{P}_T(X)$ and over $\mathcal{P}_{\widetilde{T}}(\widetilde{X})$.

Proof. If $\mu \in \mathcal{P}_T(X)$, defining $\widetilde{\mu}(\widetilde{A}) = \mu(\widetilde{A} \cap X)$, it is immediate that $\widetilde{\mu} \in \mathcal{P}_{\widetilde{T}}(\widetilde{X})$, since X and $\{\infty\}$ are $\widetilde{T}$-invariant sets. It is also immediate that $h_\mu(T) = h_{\widetilde{\mu}}(\widetilde{T})$, showing that

$$\sup_{\mu} h_\mu(T) \leq \sup_{\widetilde{\mu}} h_{\widetilde{\mu}}(\widetilde{T}).$$

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Now let $\widetilde{\mu} \in \mathcal{P}_{\widetilde{T}}(\widetilde{X})$. If $\widetilde{\mu}(\infty) = 1$, it is immediate that $h_{\widetilde{\mu}}(\widetilde{T}) = 0$, since X and $\{\infty\}$ are $\widetilde{T}$-invariant sets. Thus we can assume that $\widetilde{\mu}(\infty) = c < 1$. It is also immediate that

$$\mu = \left(\frac{1}{1-c}\right)\widetilde{\mu}|_{\mathcal{B}(X)} \in \mathcal{P}_T(X).$$

We claim that $h_{\widetilde{\mu}}(\widetilde{T}) \leq h_\mu(T)$. If $\widetilde{\mathcal{A}} = \{\widetilde{A}_1, \ldots, \widetilde{A}_l\}$ is a measurable partition of $\widetilde{X}$, defining $A_i = \widetilde{A}_i \cap X$, it follows that $\mathcal{A} = \{A_1, \ldots, A_l\}$ is a measurable partition of X. We have that $B \in \mathcal{A}^n$ if and only if there exists $\widetilde{B} \in \widetilde{\mathcal{A}}^n$, with $B = \widetilde{B} \cap X$. This follows because

$$\widetilde{T}^{-j}(\widetilde{A}_i) \cap X = T^{-j}(A_i)$$

for any $j \in \{0, \ldots, n\}$ and any $i \in \{1, \ldots, l\}$, since X and $\{\infty\}$ are $\widetilde{T}$-invariant sets. It follows that

$$H(\widetilde{\mathcal{A}}^n) = \sum_{\widetilde{B} \in \widetilde{\mathcal{A}}^n} \phi(\widetilde{\mu}(\widetilde{B})) = \phi(\widetilde{\mu}(\widetilde{B}_\infty)) + \sum_{\widetilde{B} \neq \widetilde{B}_\infty} \phi(\widetilde{\mu}(\widetilde{B})),$$

where $\infty \in \widetilde{B}_\infty$. Since $\widetilde{\mu}(\widetilde{B}) = (1-c)\mu(B)$, for each $\widetilde{B} \neq \widetilde{B}_\infty$, it follows that

$$H(\widetilde{\mathcal{A}}^n) = b + \sum_{B \in \mathcal{A}^n} \phi(a\mu(B)),$$

where $a = 1 - c$ and $b = \phi(\widetilde{\mu}(\widetilde{B}_\infty)) - \phi(a\mu(B))$. Hence

$$\begin{aligned} H(\widetilde{\mathcal{A}}^n) &= b - \sum_{B \in \mathcal{A}^n} a\mu(B)\log(a\mu(B)) \\ &= b - a\left(\sum_{B \in \mathcal{A}^n} \mu(B)(\log(a) + \log(\mu(B)))\right) \\ &= b - a\log(a)\left(\sum_{B \in \mathcal{A}^n} \mu(B)\right) + a\left(\sum_{B \in \mathcal{A}^n} \phi(\mu(B))\right) \\ &= b + \phi(a) + aH(\mathcal{A}^n). \end{aligned}$$

It follows that $H(\widetilde{\mathcal{A}}^n) \leq d + H(\mathcal{A}^n)$, since $a = 1 - c \leq 1$ and

$$b + \phi(a) \leq d = 2\max\{\phi(x) \mid x \in [0, 1]\}.$$

Dividing by n and taking limits, we get that

$$h_{\widetilde{\mu}}(\widetilde{T}, \widetilde{\mathcal{A}}) \leq h_\mu(T, \mathcal{A}) \leq h_\mu(T).$$

Since $\widetilde{\mathcal{A}}$ is arbitrary, we have that

$$h_{\widetilde{\mu}}(\widetilde{T}) \leq h_\mu(T) \leq \sup_\mu h_\mu(T).$$

Since $\widetilde{\mu} \in \mathcal{P}_{\widetilde{T}}(\widetilde{X})$ is also arbitrary, it follows that

$$\sup_{\widetilde{\mu}} h_{\widetilde{\mu}}(\widetilde{T}) \leq \sup_\mu h_\mu(T),$$

completing the proof. □

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We now prove the variational principle for entropies in a locally compact separable space. Handel and Kitchens have proved that [**5**, Lemma 1.5]

$$\sup_{\mu} h_{\mu}(T) = \inf_{d} h_{d}(T).$$

Here we improve this result by showing that the above infimum is always attained at d, whenever d is an admissible metric. Moreover, the supremum and the minimum coincide with the topological entropy introduced in §2.

THEOREM 3.2. *Let $T : X \to X$ be a continuous map, where X is a locally compact separable space. Then it follows that*

$$\sup_{\mu} h_{\mu}(T) = h(T) = \min_{d} h_{d}(T),$$

where the minimum is attained whenever d is an admissible metric.

Proof. The variational principle for compact spaces states that

$$h(\widetilde{T}) = \sup_{\widetilde{\mu}} h_{\widetilde{\mu}}(\widetilde{T}).$$

By [**5**, Proposition 1.4], we have that

$$\sup_{\mu} h_{\mu}(T) \leq \inf_{d} h_{d}(T).$$

Applying Lemma 3.1, it follows that

$$h(\widetilde{T}) = \sup_{\mu} h_{\mu}(T) \leq \inf_{d} h_{d}(T).$$

Applying Propositions 2.2 and 2.3, we get that

$$h_{d}(T) \leq h^{d}(T) = h(T) = \sup_{\mu} h_{\mu}(T) \leq \inf_{d} h_{d}(T),$$

where, in the first two terms, d is any admissible metric. □

4. *Topological entropy of automorphisms*

In this section we compute the topological entropy for automorphisms of some Lie groups. Recall that the recurrent set of a continuous map $T : X \to X$ is given by the T-invariant set

$$R(T) = \{x \in X \mid T^{n_k} x \to x \text{ for some } n_k \to \infty\}.$$

If X is a separable metric space, a consequence of the Poincaré recurrence theorem is that the support of every $\mu \in \mathcal{P}_T(X)$ is contained in $\overline{R(T)}$, the closure of $R(T)$. It follows that

$$h_{\mu}(T) = h_{\mu|_{\overline{R(T)}}}(T|_{\overline{R(T)}}).$$

As an immediate consequence of the variational principle, we obtain the following result.

PROPOSITION 4.1. *We have that*

$$h(T) = h(T|_{\overline{R(T)}}).$$

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We now consider linear isomorphisms of a finite-dimensional vector space. First we need to determine the recurrent set of a linear isomorphism in terms of its multiplicative Jordan decomposition (see [**6**, Lemma 7.1, p. 430]). If $T : V \to V$ is a linear isomorphism, where V is a finite-dimensional vector space, then we can write $T = T_H T_E T_U$, where $T_H : V \to V$ is diagonalizable in V with positive eigenvalues, $T_E : V \to V$ is an isometry relative to some appropriate norm and $T_U : V \to V$ is a linear isomorphism which can be decomposed into the sum of the identity map plus some nilpotent linear map. The linear maps T_H, T_E and T_U commute, are unique and called, respectively, the *hyperbolic*, the *elliptic* and the *unipotent* components of the multiplicative Jordan decomposition of T. In the next result, we prove that the recurrent set $R(T)$ of T is given as the intersection of the fixed points of the hyperbolic and unipotent components. Using this characterization and the variational principle proved in the previous section, we also show that the topological entropy of T always vanishes. Note that the Dirac measure at the origin of V is always in $\mathcal{P}_T(V)$.

PROPOSITION 4.2. *Let $T : V \to V$ be a linear isomorphism, where V is a finite-dimensional vector space. Then the recurrent set of T is given by*

$$R(T) = \mathrm{fix}(T_H) \cap \mathrm{fix}(T_U).$$

Furthermore, we have that $h(T) = 0$.

Proof. Let $\mathbb{P}T : \mathbb{P}V \to \mathbb{P}V$ be the map induced by T in the projective space of V and, for a subspace $W \subset V$, denote by $\mathbb{P}W$ its projection in $\mathbb{P}V$. By [**3**, Theorem 4.9], we have that

$$R(\mathbb{P}T) = \mathrm{fix}(\mathbb{P}T_H) \cap \mathrm{fix}(\mathbb{P}T_U).$$

By linearity, it is immediate that $\mathbb{P}\langle R(T)\rangle \subset R(\mathbb{P}T)$, where $\langle A\rangle$ denotes the linear subspace generated by $A \subset V$. Thus it follows that

$$R(T) \subset \mathrm{eig}(T_H) \cap \mathrm{eig}(T_U),$$

where $\mathrm{eig}(S)$ denotes the union of the eigenspaces of a linear map $S : V \to V$. Since T_U is unipotent, all of its eigenvalues are equals to one, implying that $\mathrm{eig}(T_U) = \mathrm{fix}(T_U)$. Hence $R(T) \subset \mathrm{eig}(T_H) \cap \mathrm{fix}(T_U)$. Now let $v \in R(T)$ be such that $T_H v = \lambda v$, for some $\lambda > 0$. Since the multiplicative Jordan decomposition is commutative, it follows that

$$|T^n v| = |T_E^n T_H^n T_U^n v| = |T_H^n v| = \lambda^n |v|,$$

where we have used the fact that T_E is an isometry relative to some appropriate norm $|\cdot|$. This shows that λ is equals to one and thus that $R(T)$ is contained in $\mathrm{fix}(T_H) \cap \mathrm{fix}(T_U)$. Since $\mathrm{fix}(T_H) \cap \mathrm{fix}(T_U)$ is invariant by T_E, we can consider the restriction of T to this set, which is just the restriction of T_E. Thus we get that $R(T) = \mathrm{fix}(T_H) \cap \mathrm{fix}(T_U)$, since the restriction of T_E to $\mathrm{fix}(T_H) \cap \mathrm{fix}(T_U)$ is an isometry whose orbits have compact closure.

Now to show that $h(T) = 0$, using Proposition 4.1 and the fact that $R(T)$ is a closed set, we first note that

$$h(T) = h(T|_{R(T)}).$$

By the variational principle, it follows that

$$h(T) \le h_d(T|_{R(T)}),$$

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for every metric d. Since $T|_{R(T)} = T_E|_{R(T)}$ is an isometry relative to some appropriate metric d, we get that $h_d(T|_{R(T)}) = 0$, completing the proof. □

We note that the Bowen formula for the Bowen entropy of a linear isomorphism with respect to the Euclidian metric is just an upper bound for its null topological entropy. Thus it might be an interesting problem to calculate the topological entropy of a given automorphism ϕ of a general non-compact Lie group G, since the classical formula for its Bowen entropy with respect to some invariant metric is just an upper bound for its topological entropy. Remember that the classical formula is given by

$$h_d(\phi) = \sum_{|\lambda|>1} \log|\lambda|,$$

where λ runs through the eigenvalues of $\mathrm{d}_1\phi : \mathfrak{g} \to \mathfrak{g}$, the differential of the automorphism ϕ at the identity element of G. Note that the Dirac measure at the identity of G is always in $\mathcal{P}_\phi(G)$. This allows us to apply the variational principle obtained in the previous section to study the topological entropy of an automorphism of a Lie group.

The following theorem gives an answer to the above problem when G is a nilpotent Lie group with finite fundamental group. First we extent the multiplicative Jordan for automorphisms of a simply connected Lie group.

LEMMA 4.3. *Let $\phi : G \to G$ be an automorphism, where G is a connected and simply connected Lie group. Then $\phi = \phi_H\phi_E\phi_U$, where ϕ_H, ϕ_E and ϕ_U are unique automorphisms of G such that $\mathrm{d}_1\phi_H\mathrm{d}_1\phi_E\mathrm{d}_1\phi_U$ is the multiplicative Jordan decomposition of $\mathrm{d}_1\phi$. The automorphisms ϕ_H, ϕ_E and ϕ_U commute.*

Proof. If $\theta_H\theta_E\theta_U$ is the multiplicative Jordan decomposition of $\mathrm{d}_1\phi$, we have that θ_H, θ_E and θ_U are automorphisms of the Lie algebra $\mathfrak{g}$ of G (see [**8**, §3]). Since G is connected and simply connected, it follows (see [**7**, p. 74]) that there exist unique automorphisms ϕ_H, ϕ_E, ϕ_U such that

$$\mathrm{d}_1\phi_H = \theta_H, \quad \mathrm{d}_1\phi_E = \theta_E, \quad \mathrm{d}_1\phi_U = \theta_U.$$

Since $\psi(\exp(X)) = \exp(\mathrm{d}_1\psi X)$, whenever ψ is an automorphism, we get that

$$\phi(\exp(X)) = \phi_H(\phi_E(\phi_U(\exp(X)))).$$

The connectedness of G implies that $\phi = \phi_H\phi_E\phi_U$. Now the commutation property of this decomposition follows from the commutation property of the multiplicative Jordan decomposition. □

As in the linear case, we call the maps ϕ_H, ϕ_E and ϕ_U, respectively, the *hyperbolic*, the *elliptic* and the *unipotent* components of the multiplicative Jordan decomposition of ϕ.

THEOREM 4.4. *Let $\phi : G \to G$ be an automorphism, where G is a connected nilpotent Lie group. If G is simply connected, then the recurrent set of ϕ is given by*

$$R(\phi) = \mathrm{fix}(\phi_H) \cap \mathrm{fix}(\phi_U).$$

Furthermore, we have that $h(\phi) = 0$.

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Proof. If G is a connected and simply connected nilpotent Lie group, we have that the exponential map is a diffeomorphism between $\mathfrak{g}$ and G (see [**7**, Theorem 1.127, p. 107]). Since $\phi(\exp(X)) = \exp(\mathrm{d}_1\phi X)$, it follows that ϕ and $\mathrm{d}_1\phi$ are conjugated maps and thus that $R(\phi) = \exp(R(\mathrm{d}_1\phi))$. By Proposition 4.2 and Lemma 4.3, we get that $h(\phi) = 0$ and that

$$\begin{aligned} R(\phi) &= \exp(\mathrm{fix}(\mathrm{d}_1\phi_H) \cap \mathrm{fix}(\mathrm{d}_1\phi_U)) \\ &= \exp(\mathrm{fix}(\mathrm{d}_1\phi_H)) \cap \exp(\mathrm{fix}(\mathrm{d}_1\phi_U)) \\ &= \mathrm{fix}(\phi_H) \cap \mathrm{fix}(\phi_U). \end{aligned}$$

□

Note that for a nilpotent Lie group G either its fundamental group $\pi_1(G)$ is trivial or it is infinite. In the latter case, there exist automorphisms with positive topological entropy, even when G is abelian. In fact, the matrix

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

induces an automorphism ϕ on the compact abelian Lie group $G = \mathbb{R}^2/\mathbb{Z}^2$. Since $\mathrm{d}_1\phi = T$, by the Bowen formula, we have that

$$h(\phi) = \log\left(\frac{1+\sqrt{5}}{2}\right) > 0.$$

The application above deals with homeomorphisms (in fact automorphisms); however, the results presented in this paper can be applied to general proper maps such as the following endomorphism of $\mathbb{C}^* = \mathbb{C} - 0$ given by

$$\phi^* : \mathbb{C}^* \to \mathbb{C}^*, \quad z \mapsto z^2.$$

If d is some invariant metric of C^*, by the Bowen formula, the Bowen entropy of ϕ^* is given by $h_d(\phi^*) = 2\log(2)$, since

$$\mathrm{d}_1\phi^* = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

We conclude this paper by computing the topological entropy of ϕ^*.

PROPOSITION 4.5. *We have that*

$$h(\phi^*) = \log(2).$$

Proof. The map $\phi = \phi^*|_{S^1}$ is an endomorphism of S^1. By the Bowen formula (which still remains true for endomorphisms), we have that $h(\phi) = \log(2)$. It is easy to show that $R(\phi^*) \subset S^1$. Applying Proposition 4.1, we get that

$$h(\phi^*) = h(\phi^*|_{\overline{R(\phi^*)}}) = h(\phi|_{\overline{R(\phi)}}) = h(\phi) = \log(2).$$

□

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