region except at the singularities—unbounded in number. Such an expansion is more compact than analytic continuation by Taylor's series; it cannot be valid if the singularities are not enumerable.

We take this opportunity to mention that in (1), p. 549: (*a*) in line 1, 'in and on C' should be 'for ζ in C'; and (*b*) in lines 4, 3 from the bottom, m should be defined as the lower bound of M for all paths connecting the z_s. But for the referee's comments both this paper and (1) would be less clear, less correct.

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Integrals involving Bessel functions and Whittaker functions

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The object of the present note is to evaluate some integrals involving Bessel functions and Whittaker functions in terms of Appell's function F_4 with the help of operational calculus.

If we use the operational pairs ((3), p. 283)

$$\left.\begin{aligned}\phi(p) &= 2a^{\frac{1}{2}}p^{\lambda+\frac{1}{2}}K_{2\mu}(2a^{\frac{1}{2}}p^{\frac{1}{2}}),\\ &\doteqdot t^{-\lambda}e^{-a/2t}W_{\lambda,\mu}(a/t) = h(t),\end{aligned}\right\} \tag{1}$$

where $\Re(a) > 0$ and ((3), p. 281, equation 23)

$$\begin{aligned}\psi(p) &= \exp\left\{-\frac{b^2+c^2}{p}\right\}I_\nu\left(\frac{2bc}{p}\right),\\ &\doteqdot J_\nu(2bt^{\frac{1}{2}})\,J_\nu(2ct^{\frac{1}{2}}) = g(t),\end{aligned}$$

where $\Re(\nu+1) > 0$, in Goldstein's form of Parseval's relation ((4), p. 105) we find that

$$\begin{aligned}\int_0^\infty t^{-\lambda-1}\exp\left[-\frac{a}{2t}-\frac{b^2+c^2}{t}\right]W_{\lambda,\mu}(a/t)\,I_\nu\left(\frac{2bc}{t}\right)dt\\ = 2a^{\frac{1}{2}}\int_0^\infty t^{\lambda-\frac{1}{2}}K_{2\mu}(2a^{\frac{1}{2}}t^{\frac{1}{2}})\,J_\nu(2bt^{\frac{1}{2}})\,J_\nu(2ct^{\frac{1}{2}})\,dt.\end{aligned}$$

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On making slight changes in the variables we see that this becomes

$$\int_0^\infty t^{\lambda-1}\exp\left[-(\tfrac{1}{2}a+b^2+c^2)\,t\right]\mathrm{W}_{\lambda,\mu}(at)\,\mathrm{I}_\nu(2bct)\,dt = 4a^{\frac{1}{2}}\int_0^\infty t^{2\lambda}\,\mathrm{K}_{2\mu}(2a^{\frac{1}{2}}t)\,\mathrm{J}_\nu(2bt)\,\mathrm{J}_\nu(2ct)\,dt.$$

The integral on the right can now be evaluated using Bailey's result ((2), p. 38) to give

$$\begin{aligned}\int_0^\infty t^{\lambda-1}\exp\left[-(\tfrac{1}{2}a+b^2+c^2)\,t\right]\mathrm{W}_{\lambda,\mu}(at)\,\mathrm{I}_\nu(2bct)\,dt \\ = \frac{(bc)^\nu\,\Gamma(\frac{1}{2}+\lambda-\mu+\nu)\,\Gamma(\frac{1}{2}+\lambda+\mu+\nu)}{a^{\lambda+\nu}[\Gamma(\nu+1)]^2} \\ \times \mathrm{F}_4\left[\tfrac{1}{2}+\lambda-\mu+\nu,\tfrac{1}{2}+\lambda+\mu+\nu;\,\nu+1,\nu+1;\,-\frac{b^2}{a},-\frac{c^2}{a}\right], \qquad (2)\end{aligned}$$

where $\Re(\lambda\pm\mu+\nu+\frac{1}{2})>0$ and $\Re[a+(b\pm c)^2]>0$.

In the same way the formula

$$\begin{aligned}\int_0^\infty t^{\lambda-1}\exp\left[-(\tfrac{1}{2}a-b^2-c^2)\,t\right]\mathrm{W}_{\lambda,\mu}(at)\,\mathrm{I}_\nu(2bct)\,dt \\ = \frac{(bc)^\nu\,\Gamma(\frac{1}{2}+\lambda-\mu+\nu)\,\Gamma(\frac{1}{2}+\lambda+\mu+\nu)}{a^{\lambda+\nu}[\Gamma(\nu+1)]^2} \\ \times \mathrm{F}_4\left[\tfrac{1}{2}+\lambda-\mu+\nu,\tfrac{1}{2}+\lambda+\mu+\nu;\,\nu+1,\nu+1;\frac{b^2}{a},\frac{c^2}{a}\right], \qquad (3)\end{aligned}$$

where $\Re(\frac{1}{2}+\lambda\pm\mu+\nu)>0$ and $\Re[a-(b\pm c)^2]>0$ can be established on using the relation given in (1) and ((3), p. 281, equation 22),

$$\begin{aligned}\psi(p) &= \exp\left(\frac{b^2+c^2}{p}\right)\mathrm{I}_\nu\left(\frac{2bc}{p}\right), \\ &\doteqdot \mathrm{I}_\nu(2bt^{\frac{1}{2}})\,\mathrm{I}_\nu(2ct^{\frac{1}{2}}) = g(t),\end{aligned}$$

where $\Re(\nu+1)>0$ and applying Bailey's formula ((1), p. 17).

Similarly, if we take the correspondences ((3), p. 284)

$$\left.\begin{aligned}\phi(p) &= 2p\mathrm{I}_\nu(2bp^{\frac{1}{2}})\,\mathrm{K}_\nu(2cp^{\frac{1}{2}}), \\ &\doteqdot \frac{1}{t}\exp\left\{-\frac{b^2+c^2}{t}\right\}\mathrm{I}_\nu\left(\frac{2bc}{t}\right) = h(t),\end{aligned}\right\} \qquad (4)$$

where $\Re[(b\pm c)^2]>0$ and ((3), p. 293)

$$\begin{aligned}\psi(p) &= \frac{p^{\lambda+1}\,\Gamma(\frac{1}{2}-\lambda+\mu)}{a^{\frac{1}{2}}\Gamma(1+2\mu)}\,e^{a/2p}\,\mathrm{M}_{\lambda,\mu}\left(\frac{a}{p}\right), \\ &\doteqdot t^{-\lambda-\frac{1}{2}}\mathrm{I}_{2\mu}(2a^{\frac{1}{2}}t^{\frac{1}{2}}) = g(t),\end{aligned}$$

where $\Re(\frac{1}{2}-\lambda+\mu)>0$ and make use of Bailey's result, we obtain

$$\begin{aligned}\int_0^\infty t^{-\lambda-1}\exp\left[(\tfrac{1}{2}a-b^2-c^2)\,t\right]\mathrm{M}_{\lambda,\mu}(at)\,\mathrm{I}_\nu(2bct)\,dt \\ = \frac{a^{\mu+\frac{1}{2}}b^\nu\,\Gamma(\frac{1}{2}-\lambda+\mu+\nu)}{c^{1+\nu+2\mu-2\lambda}\,\Gamma(\nu+1)}\,\mathrm{F}_4\left[\tfrac{1}{2}-\lambda+\mu,\tfrac{1}{2}-\lambda+\mu+\nu;\,2\mu+1,\nu+1;\,\frac{a}{c^2},\frac{b^2}{c^2}\right], \qquad (5)\end{aligned}$$

where $\Re(\frac{1}{2}-\lambda+\mu+\nu)>0$ and $\Re[(b\pm c)^2]>0$, $\Re(a)$.

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Finally, on taking the relation given in (4) and ((3), p. 282, equation 32)

$$\Psi(p) = p^{1-\lambda}K_{\mu}(a/p),$$
$$\doteqdot 2^{-\lambda}t^{\lambda-1}\pi^{\frac{1}{2}}S_2\left(\frac{\mu-1}{2}, -\frac{\mu+1}{2}, \frac{1-\lambda}{2}, -\frac{\lambda}{2}; \frac{at}{4}\right) = g(t)$$

and making use of Bailey's formula, it follows that

$$\int_0^{\infty} t^{\lambda-1}S_2\left(\frac{\mu-1}{2}, -\frac{\mu+1}{2}, \frac{1-\lambda}{2}, -\frac{\lambda}{2}; \frac{at}{4}\right) K_{\nu}(bt^{\frac{1}{2}})\, I_{\nu}(ct^{\frac{1}{2}})\, dt$$
$$= \sum_{\mu, -\mu} \frac{a^{\mu}(bc)^{\nu}\,\Gamma(-\mu)\,\Gamma(\lambda+\mu+\nu)}{\pi^{\frac{1}{2}}\,2^{2-3\lambda-\mu}(b^2+c^2)^{\lambda+\mu+\nu}\,\Gamma(\nu+1)}$$
$$\times F_4\left[\frac{\lambda+\mu+\nu}{2}, \frac{\lambda+\mu+\nu+1}{2}; \mu+1, \nu+1; \frac{4a^2}{(b^2+c^2)^2}, \frac{4b^2c^2}{(b^2+c^2)^2}\right], \quad (6)$$

where $\Re(\lambda \pm \mu + \nu) > 0$, $\Re(b \pm c) > 0$ and $a > 0$.

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On the number of particles in a steady-state universe

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In a letter to *Nature*, R. Schlegel (1) raised a question about the consistency of steady-state cosmology. In a response to this letter Ursell (2) pointed out that Schlegel's argument was open to question. Schlegel (2) maintained his position. I propose to demonstrate conclusively that Schlegel's position is mathematically untenable.

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