COMPOSITIO MATHEMATICA

On the linear AFL: the non-basic case

Qirui Li and Andreas Mihatsch

Compositio Math. **161** (2025), 385–425.

doi: 10.1112/S0010437X24007577

FOUNDATION
COMPOSITIO
MATHEMATICA



LONDON
MATHEMATICAL
SOCIETY
EST. 1865



https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Compositio Math. 161 (2025) 385–425
doi:10.1112/S0010437X24007577

On the linear AFL: the non-basic case

Qirui Li and Andreas Mihatsch

Abstract

The linear arithmetic fundamental lemma (AFL) is a conjectural identity of intersection numbers on Lubin–Tate deformation spaces and derivatives of orbital integrals. It was introduced for elliptic orbits by Li, and Howard and Li. For elliptic orbits, the relevant intersection problem is formulated for the basic isogeny class. In the present article, we extend the conjecture to all orbits and all isogeny classes. Our main result is a reduction of the non-basic cases of the AFL to the basic ones, which relies on an analysis of the connected-étale sequence. Our results will be relevant in the global setting, where also locally non-elliptic orbits may contribute in a non-trivial way.

Contents

1 Introduction **386**
2 The FL **388**
2.1 Invariants . . . 388
2.2 Matching . . . 390
2.3 Orbital integrals for (E_1, E_2) . . . 390
2.4 The η-twisted orbital integrals for (E_0, E_3) . . . 391
2.5 Fundamental lemma . . . 392
2.6 Reduction formula . . . 393
2.7 Vanishing order . . . 394
3 The AFL **395**
3.1 Quadratic cycles . . . 395
3.2 Hecke correspondences . . . 397
3.3 Intersection numbers . . . 399
3.4 The AFL . . . 402
4 Reduction to elliptic invariant **404**
4.1 Fibration of lattice sets . . . 404
4.2 Application to orbital integrals . . . 405
4.3 Fibration of RZ spaces . . . 406
4.4 Degree computation . . . 408
Appendix A. Local intersection theory **414**
A.1 Cycle intersection . . . 415
A.2 Correspondences . . . 416

Received 20 March 2023, accepted in final form 18 July 2024, published online 23 June 2025.
2020 Mathematics Subject Classification 14G35, 11G18 (primary).
Keywords: arithmetic fundamental lemma; intersection number; Lubin–Tate space; orbital integral; p-divisible group.
© The Author(s), 2025. The publishing rights in this article are licensed to Foundation Compositio Mathematica under an exclusive licence.

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Appendix B. Partial Satake transformation for GL_n **419**
B.1 Partial Satake transformation 419
B.2 Satake transformation in terms of orbital integrals 421
B.3 Satake transformation of minuscule Hecke functions 421
B.4 Generating symmetric polynomials 422
B.5 Satake transformation of $\mathbf{T}_m$ 422
B.6 Partial Satake transformation of $\mathbf{T}_m$ 423
Acknowledgements **424**
References **424**

1. Introduction

The purpose of this article is to formulate a linear arithmetic fundamental lemma (AFL) conjecture also for non-basic isogeny classes. Our main result is a reduction of these non-basic cases to the basic one. The context and relevance of these results are as follows.

In general, AFLs are certain identities of intersection numbers on moduli spaces of p-divisible groups (Rapoport–Zink spaces or RZ spaces) and derivatives of orbital integrals. They are, in particular, local statements, formulated over a p-adic local field. Their motivation is global, however; namely they arise from a relative trace formula comparison approach to certain intersection problems of special cycles on Shimura varieties. We refer to the original paper of W. Zhang [Zha12b] or his surveys [Zha12a, Zha18] for an explanation of this global-to-local formalism.

There are currently two families of such conjectures. The first comes from the Gan–Gross–Prasad setting of a diagonal embedding of unitary groups $U(V) \to U(V) \times U(V \oplus 1)$. The original conjecture in this context [Zha12b] has been proved in work of W. Zhang [Zha21b], of W. Zhang and the second author [Mih22, MZ24] and of Z. Zhang [Zha21a]. Several variants have been formulated by Rapoport, Smithling and Zhang [RSZ20, RSZ18, RSZ17], Liu [Liu21], W. Zhang [Zha22], and Z. Zhang [Zha21a]. In all these cases, the global situation is such that intersection takes place in the basic locus. Correspondingly, all the cited works are concerned with basic RZ spaces.

The second family arises from the unitary variant of the Guo–Jacquet restriction problem [Guo96]. That is, the special cycles in question come from an embedding $U_E(V) \to U_F(V)$, where V is a hermitian space for some field extension F/F_0 with an additional action of a quadratic extension E/F. At inert places of F/F_0, this leads to an AFL for basic unitary RZ spaces that is the content of a forthcoming work by Leslie, Xiao and Zhang. At a split place v, however, it leads to intersection problems on RZ spaces for all isogeny classes of dimension 1 and height $2n$ strict p-divisible O_{F_v}-modules with O_{E_v}-action. These are the non-basic linear AFLs from the title of the paper. We mention that the phenomenon of non-basic intersection at split places also occurs for unitary groups over central simple algebras and refer to [HM23] and [LM23] for intersection number identities in such situations.

We now describe our results in some detail. They also pertain to the function field setting. We do not exclude the prime 2.

Let E/F be an unramified quadratic extension of a non-archimedean local field F with uniformizer $\pi \in F$. Denote by $\breve{F}$ the completion of a maximal unramified extension of F and fix an embedding $E \to \breve{F}$. Let $\mathbb{X}$ be a 1-dimensional, height $2n$, not necessarily formal, strict π-divisible O_F-module over the residue field $\mathbb{F}$ of $\breve{F}$ (cf. Definition 3.1). In addition, consider a pair of embeddings

$$\beta = (\beta_1, \beta_2), \quad \beta_i : E \longrightarrow \mathrm{End}^0(\mathbb{X}).$$

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

The π-divisible O_F-module $\mathbb{X}$ decomposes in an essentially unique way as $\mathbb{X} = \mathbb{X}^0 \times \mathbb{X}^1$ with $\mathbb{X}^0$ connected and $\mathbb{X}^1$ étale. Since $\mathrm{End}^0(\mathbb{X}) = \mathrm{End}^0(\mathbb{X}^0) \times \mathrm{End}^0(\mathbb{X}^1)$, giving a pair β is the same as giving analogous pairs β^0, β^1 for the two factors.

Consider now the RZ space M of strict π-divisible O_F-modules with quasi-isogeny to $\mathbb{X}$. It is a formal scheme; abstractly,

$$M \cong \coprod_{\mathbb{Z} \times (GL_{2n^1}(F)/GL_{2n^1}(O_F))} \mathrm{Spf}\, O_{\breve{F}}[\![t_1, \ldots, t_{2n-1}]\!], \tag{1.1}$$

where $2n^1$ is the height of the étale factor $\mathbb{X}^1$. There are two analogous RZ spaces Z_1, Z_2 that parametrize strict O_E-modules together with an O_E-linear quasi-isogeny to $(\mathbb{X}, \beta_i)$. Forgetting about the O_E-action defines closed immersions $Z_1, Z_2 \to M$ that allow them to be viewed as cycles in middle dimension. Assuming that (β_1, β_2) is regular semi-simple (cf. Definition 2.1), the centralizer of the image $\beta_1(E) \cup \beta_2(E)$ is an étale F-algebra $L \subseteq \mathrm{End}^0(\mathbb{X})$ of degree n. Write $L = \prod_{j \in J} L^j$ as product of fields, pick uniformizers $\pi_j \in L^j$ and define $\Gamma = \prod_{j \in J} \pi_j^{\mathbb{Z}} \subset L^\times$. Then $L^\times$ acts compatibly on Z_1, Z_2 and M, and the quotient $\Gamma \backslash (Z_1 \cap Z_2)$ is artinian. We define

$$\mathrm{Int}(\beta) := \ell_{O_{\breve{F}}}(\mathcal{O}_{\Gamma \backslash (Z_1 \cap Z_2)}). \tag{1.2}$$

To β one may also associate a matching pair (cf. Definition 2.6)

$$\alpha = (\alpha_0, \alpha_3), \quad \alpha_i : F \times F \longrightarrow M_{2n}(F),$$

which is unique up to conjugation. Given an element $f \in \mathcal{H} = \mathbb{C}[GL_{2n}(O_F) \backslash GL_{2n}(F) / GL_{2n}(O_F)]$ of the spherical Hecke algebra, one may define an orbital integral for the action by left and right multiplication on $GL_n(F \times F) \times GL_n(F \times F)$ on $GL_{2n}(F)$,

$$O(\alpha, f, s), \quad s \in \mathbb{C}. \tag{1.3}$$

This is an element of $\mathbb{C}[q_F^s, q_F^{-s}]$, where q_F is the residue cardinality of F. Under our assumption that α matches β, the central value $O(\alpha, f, 0)$ vanishes.

Conjecture 1.1 (Linear AFL). Assume α and β to be as above. Then there is an equality,

$$\frac{1}{\log(q_F)} \frac{d}{ds}\Big|_{s=0} O(\alpha, 1_{GL_{2n}(O_F)}, s) = \mathrm{Int}(\beta).$$

The basic case is precisely when $\mathbb{X} = \mathbb{X}^0$ is connected. In this situation, the above conjecture has (up to sign) already been formulated in work of the first author [Li22b] and in his joint work with Howard [HL20]. In fact, [Li22b] gives a definition of $\mathrm{Int}(\beta, f)$ for every $f \in \mathcal{H}$ and formulates Conjecture 1.1 in this generality, whereas [HL20] allows the cycles Z_1, Z_2 to be defined for two different quadratic extensions $E_1, E_2/F$ (biquadratic setting). We work in the combined generality throughout the paper and refer to Conjecture 3.10 for the general version.

Conjecture 1.1 is known when $n = 1$ or 2. The more general version, Conjecture 3.10, is known in all cases when $n = 1$, cf. [Li22b, HL20]. It is currently only known for the unit function $f = 1_{GL_{2n}(O_F)}$ (but possibly $E_1 \ncong E_2$) when $n = 2$, cf. [Li22a, Li25].

The following are our two main results.

Theorem 1.2. *Write $\beta = (\beta^0, \beta^1)$ for the two components of β with respect to $\mathbb{X} = \mathbb{X}^0 \times \mathbb{X}^1$. Then Conjecture 1.1 holds for β if and only if it holds for the connected component β^0.*

Corollary 1.3. *The AFL (Conjecture 1.1) holds for all isogeny classes $\mathbb{X}$ whose connected component $\mathbb{X}^0$ has height ≤ 4.*

We formulate and prove these two results also in the biquadratic case and for general spherical Hecke functions, cf. Theorem 3.12 and Corollary 3.13. In this generality, however, we have to

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

assume the Guo–Jacquet fundamental lemma (FL), respectively, its biquadratic variant for the whole spherical Hecke algebra; it is currently only known for the unit Hecke function and $E_1 = E_2$, see [Guo96].

The proof of Theorem 1.2 relies on the connected-étale sequence $0 \to X^0 \to X \to X^1 \to 0$ of the universal π-divisible O_F-module X over M. It provides a fibration map

$$M \longrightarrow M^0 \times_{\operatorname{Spf} O_{\breve{F}}} M^1 \tag{1.4}$$

to the product of the RZ spaces for $\mathbb{X}^0$ and $\mathbb{X}^1$. If Y^0 and Y^1 denote their universal π-divisible O_F-modules, then M can be identified with the space of torsion extensions $\underline{\operatorname{Ext}}^1(Y^1, Y^0)_{\mathrm{tors}}$. This space is in bijection with the Hom-functor $\underline{\operatorname{Hom}}(T(Y^1), Y^0)$, where $T(Y^1)$ denotes the Tate module. In this way, M is described as the total space of a π-divisible O_F-module over $M^0 \times_{\operatorname{Spf} O_{\breve{F}}} M^1$. This description allows to relate intersections in the three spaces and we prove the identity

$$\operatorname{Int}(\beta) = |\operatorname{Res}(\operatorname{Inv}(\beta^0), \operatorname{Inv}(\beta^1))|_F^{-1} O(\beta^1, 1_{GL_{2n^1}(O_F)}) \operatorname{Int}(\beta^0). \tag{1.5}$$

Here, $\operatorname{Inv}(\beta^0)$, $\operatorname{Inv}(\beta^1) \in F[T]$ denote the invariants of β^0 and β^1 as in Remark 2.2, $\operatorname{Res}(\cdot,\cdot)$ denotes the resultant of two polynomials, and $|\cdot|_F$ denotes the normalized absolute value on F. A similar identity can be shown for $O(\alpha, f, s)$, leading to the proof of Theorem 1.2. Again, we carry out all these ideas in the biquadratic setting and for arbitrary f.

The structure of this article is as follows. In § 2, we recall necessary background on matching, orbital integrals and the Guo–Jacquet FL. This is essentially taken from [HL20], our own addition being a formulation of the analytic side of Theorem 1.2. In § 3, we define our intersection numbers and formulate the AFL. § 4 is devoted to the proof of Theorem 1.2, its heart being the fibration argument for (1.4) in § 4.3 and a linear algebra computation that produces the resultant factor in (1.5). Finally, our Appendix A and Appendix B provide background material on correspondences and the partial Satake transform.

2. The FL

This section recalls necessary background on matching, orbital integrals and the FL from [Guo96] and [HL20].

2.1 Invariants

Fix a field F and an integer $n \geq 1$. We consider two quadratic étale extensions E_1/F and E_2/F and write σ_1 and σ_2, respectively, for their non-trivial involutions over F. Let $B = \prod_{j \in J} B^j$ be the product of a finite family $(B^j)_{j \in J}$ of central simple F-algebras. We assume that the degree $[B^j : F] = \dim_F(B^j)^{1/2}$ of each factor is even and that the total degree $[B : F] = \sum_{j \in J} [B^j : F]$ is $2n$.

We write $\beta : (E_1, E_2) \to B$ for pairs $(\beta_1 : E_1 \to B,\ \beta_2 : E_2 \to B)$ of embeddings as F-algebras into B. Giving a map $\beta_i : E_i \to B$ is the same as giving a tuple of maps $\beta_i^j : E_i \to B^j$. Without further mention, we only consider maps β_i^j that make B^j into a free E_i-module. This additional specification only matters if $E_i \cong F \times F$. In particular, pairs $\beta : (E_1, E_2) \to B$ are the same as tuples of pairs $\beta^j : (E_1, E_2) \to B^j$.

Let $E_3 \subset E_1 \otimes_F E_2$ be the fixed ring under $\sigma_1 \otimes \sigma_2$. It is a quadratic étale F-algebra and we denote by $\sigma_3 : E_3 \to E_3$ its non-trivial F-involution. Given β as above, Howard and Li [HL20, § 2] define a canonical element $\mathbf{s}_\beta \in E_3 \otimes_F B$ that centralizes $\beta_1(E_1)$ and $\beta_2(E_2)$. Their definition, which we do not need here, extends verbatim from central simple algebras to products of

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

such; in fact, simply $\mathbf{s}_\beta = (\mathbf{s}_{\beta^j})_{j\in J}$. They showed in [HL20, Proposition 2.2.3] that the reduced characteristic polynomial

$$\operatorname{charred}_{E_3\otimes_F B/E_3}(\mathbf{s}_\beta; T) = \prod_{j\in J} \operatorname{charred}_{E_3\otimes_F B/E_3}(\mathbf{s}_{\beta^j}; T) \in E_3[T]$$

is a square and make the following definition.

Definition 2.1.

(1) The set of invariants of degree n is the set of monic degree n polynomials $\delta \in E_3[T]$ that satisfy the symmetry

$$(-1)^n \delta(1-T) = \sigma_3(\delta(T)). \tag{2.1}$$

An invariant δ is called regular semi-simple if for every map $\rho : E_3 \to \overline{F}$, the polynomial $\rho(\delta)$ has n distinct non-zero roots.

(2) The invariant $\operatorname{Inv}(\beta) \in E_3[T]$ of a pair β is the unique monic square root of $\operatorname{charred}_{E_3\otimes_F B/E_3}(\mathbf{s}_\beta; T)$. It satisfies (2.1) by [HL20, Proposition 2.3.3]. A pair β is called regular semi-simple if its invariant is regular semi-simple.

Remark 2.2. If $E_1 = E_2$, then $E_3 \cong F \times F$. Fixing such an isomorphism, one may renormalize the invariant by considering only the first component of $\operatorname{Inv}(\beta; T) \in (F\times F)[T]$. This was done for the formulation of (1.5).

Note that $\operatorname{Inv}(\beta) = \prod_{j\in J} \operatorname{Inv}(\beta^j)$ and that $\operatorname{Inv}(\beta)$ only depends on the $B^\times$-conjugation orbit of β. For regular semi-simple β and simple B, a converse holds as follows.

Proposition 2.3 [HL20, Corollary 2.5.7]. *Assume that B is a central simple F-algebra. Two regular semi-simple pairs $\beta, \beta' : (E_1, E_2) \to B$ are $B^\times$-conjugate if and only if $\operatorname{Inv}(\beta) = \operatorname{Inv}(\beta')$.*

Remark 2.4. The converse direction of Proposition 2.3 does not hold for non-simple B. For example, take $E_1 = E_2 = F \times F$ and consider two regular semi-simple pairs $\beta^0, \beta^1 : (E_1, E_2) \to M_2(F)$ with $\operatorname{Inv}(\beta^0) \neq \operatorname{Inv}(\beta^1)$. Then

$$(\beta^0, \beta^1),\ (\beta^1, \beta^0) : (E_1, E_2) \longrightarrow B := M_2(F) \times M_2(F)$$

are both regular semi-simple with invariant $\operatorname{Inv}(\beta^0)\operatorname{Inv}(\beta^1)$. They are not $B^\times$-conjugate, however, because β^0 and β^1 are not $GL_2(F)$-conjugate.

An immediate question is, given E_1, E_2, B and a regular semi-simple invariant δ as above, whether or not there exists a pair $\beta : (E_1, E_2) \to B$ with $\operatorname{Inv}(\beta) = \delta$. This is partially answered by the following universal construction.

Proposition 2.5 [HL20, Proposition 2.5.6]. *Given E_1, E_2 and a regular semi-simple invariant $\delta \in E_3[T]$ of degree n, there exists an étale F-algebra L_δ of degree n and a quaternion algebra B_δ/L_δ, together with a pair of embeddings $\beta_\delta : (E_1, E_2) \to B_\delta$, that has the following properties.*

(1) *The image $\beta_{\delta,1}(E_1) \cup \beta_{\delta,2}(E_2)$ generates B_δ as F-algebra.*

(2) *The natural maps $\beta_{\delta,i}(E_i) \otimes_F L_\delta \to B_\delta$ are injective and make $\beta_{\delta,i}(E_i) \otimes_F L_\delta$ into a maximal commutative F-subalgebra of B_δ.*

(3) *For every product B of central simple F-algebras of total degree $[B : F] = 2n$, and for every pair $\beta : (E_1, E_2) \to B$ of invariant δ, there is a unique embedding of F-algebras $\iota : B_\delta \to B$ such that $\beta = \iota \circ \beta_\delta$. Conversely, if $\beta = \iota \circ \beta_\delta$ for an embedding $\iota : B_\delta \to B$ of F-algebras, then $\operatorname{Inv}(\beta) = \delta$.*

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Proof. Starting from (E_1, E_2, δ), the datum $\beta_\delta : (E_1, E_2) \to B_\delta$ is constructed during the proof of [HL20, Proposition 2.5.6]. (The algebra B_δ is denoted by $\mathcal{F}(\varphi_1, \varphi_2)$ there.) It has properties (1) and (2) by construction. Property (3) is stated in [HL20, Proposition 2.5.6] for simple B. The statement for general B is obtained by the same proof. □

Thus, pairs β of invariant δ for B are the same as F-algebra maps $B_\delta \to B$. Implicit in Proposition 2.5 is moreover the statement that the centralizer of a regular semi-simple pair β is an étale F-algebra of degree n. Namely, it is isomorphic to the center $L_\delta \subset B_\delta$. By the construction of B_δ in [HL20], it may be described (up to isomorphism) by

$$L_\delta \cong (E_3[T]/(\delta(T)))^{\sigma_3 = \mathrm{id}}, \tag{2.2}$$

where σ_3 is extended to $E_3[T]$ as $\sigma_3(T) = 1 - T$.

2.2 Matching

From now on, we let F be a non-archimedean local field with normalized absolute value $|\cdot|$ and assume E_1, E_2 to be étale field extensions. Put $E_0 = F \times F$. Then the invariant $\mathrm{Inv}(\alpha)$ of a pair $\alpha = (\alpha_0, \alpha_3) : (E_0, E_3) \to M_{2n}(F)$ lies in $E_3[T]$ as well.

Definition 2.6. Two regular semi-simple pairs $\alpha : (E_0, E_3) \to M_{2n}(F)$ and $\beta : (E_1, E_2) \to B$ are said to match if $\mathrm{Inv}(\alpha) = \mathrm{Inv}(\beta)$.

Let $\delta \in E_3[T]$ be regular semi-simple and of degree n. Let $\alpha_\delta : (E_0, E_3) \to A_\delta$ be the universal datum for (E_0, E_3, δ) from Proposition 2.5 and let $L_\delta = \mathrm{Cent}(A_\delta)$. By part (2) of Proposition 2.5, the L_δ-algebra $E_0 \otimes_F L_\delta \cong L_\delta \times L_\delta$ embeds into A_δ which implies $A_\delta \cong M_2(L_\delta)$. In particular, there exists an embedding $\iota : A_\delta \to M_{2n}(F)$. By Proposition 2.5(3), $\mathrm{Inv}(\iota \circ \alpha_\delta) = \delta$. It follows that for every regular semi-simple invariant δ of degree n, there exists a pair $\alpha : (E_0, E_3) \to M_{2n}(F)$ with $\mathrm{Inv}(\alpha) = \delta$. On the other hand, given E_1, E_2, B and δ, there need not exist a pair $\beta : (E_1, E_2) \to B$ with $\mathrm{Inv}(\beta) = \delta$. Definition 2.6 is asymmetric in this sense.

2.3 Orbital integrals for (E_1, E_2)

The FL compares η-twisted orbital integrals of pairs $(E_0, E_3) \to M_{2n}(F)$ with orbital integrals of matching pairs $(E_1, E_2) \to M_{2n}(F)$. In this section, we define the second kind. Throughout, we denote by $\Lambda_{\mathrm{std}} := O_F^{2n}$ the standard lattice and by $\mathcal{H} = \mathbb{C}[GL_{2n}(O_F) \backslash GL_{2n}(F) / GL_{2n}(O_F)]$ the spherical Hecke algebra.

Definition 2.7. Let $\beta : (E_1, E_2) \to M_{2n}(F)$ be regular semi-simple and write $H_i := \mathrm{Cent}(E_i)^\times$. Choose $g_i \in GL_{2n}(F)$ such that $g_i \Lambda_{\mathrm{std}}$ is O_{E_i}-stable. For $f \in \mathcal{H}$, define

$$O(\beta, f) := \int_{(H_1 \cap H_2) \backslash (H_1 \times H_2)} f(g_1^{-1} h_1^{-1} h_2 g_2)\, dh_1\, dh_2. \tag{2.3}$$

Here we endow $H_i \cong GL_n(E_i)$ with the Haar measure such that $\mathrm{Vol}(GL_n(O_{E_i})) = 1$. The stabilizer $H_1 \cap H_2$ is the group of units $L^\times$, where $L = \mathrm{Cent}(\beta(E_1) \cup \beta(E_2)) \cong L_\delta$ is an étale F-algebra of degree n by Proposition 2.5. We endow it with the measure such that $\mathrm{Vol}(O_L^\times) = 1$. Convergence of the integral follows from Lemma 2.8. The definition is then independent of the chosen g_i: Any second choice would come from $H_i g_i GL_{2n}(O_F)$ and f is bi-$GL_{2n}(O_F)$-invariant.

Lemma 2.8. *The subset $(H_1 \cap H_2) \backslash (H_1 \times H_2) \subset GL_{2n}(F)$ is closed. In particular, the integrand in (2.3) has compact support.*

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Proof. Set $G = GL_{2n}(F)$. There is an isomorphism

$$G\backslash(G/H_1 \times G/H_2) \xrightarrow{\sim} H_1\backslash G/H_2, \quad (g_1, g_2) \longmapsto g_1^{-1}g_2.$$

By Skolem–Noether, via $(g_1, g_2) \mapsto (g_1\beta_1 g_1^{-1}, g_2\beta_2 g_2^{-1})$, the source is in bijection with the G-orbits of pairs $(E_1, E_2) \to M_{2n}(F)$. By Proposition 2.3, the orbit of β agrees with the locus where the invariant takes value $\mathrm{Inv}(\beta)$. It is, in particular, closed. Since the orbit of β maps to H_1H_2 in the target, it follows that $(H_1 \cap H_2)\backslash(H_1 \times H_2) \subset GL_{2n}(F)$ is a closed subset as claimed. □

The orbital integral $O(\beta, f)$ can also be understood as an f-weighted lattice count. Let $\mathcal{L}(\beta_i)$ denote the O_{E_i}-stable lattices in F^{2n}. This set is stable under the $L^\times$-action because $L^\times$ centralizes $\beta_i(E_i)$. Let $g_i \in GL_{2n}(F)$ be fixed as in Definition 2.7 and set $K_i = \mathrm{Stab}(g_i\Lambda_{\mathrm{std}}) \cap H_i$. Note that H_i is isomorphic to $GL_n(E_i)$ which acts transitively on the set of O_{E_i}-lattices in E_i^n. Thus, there is a bijection

$$H_i/K_i \xrightarrow{\sim} \mathcal{L}(\beta_i), \quad h_i \longmapsto h_i g_i \Lambda_{\mathrm{std}}. \tag{2.4}$$

As f lies in the spherical Hecke algebra, the restriction $f|_{g_1^{-1}K_1h_1^{-1}h_2K_2g_2}$ is constant for every element $(h_1, h_2) \in H_1 \times H_2$. Moreover, the volume of $L^\times\backslash(L^\times \cdot (h_1K_1 \times h_2K_2))$ is equal to

$$\mathrm{Vol}(L^\times \cap h_1K_1h_1^{-1} \cap h_2K_2h_2^{-1})^{-1} = \mathrm{Vol}(\mathrm{Stab}_{L^\times}(h_1g_1\Lambda_{\mathrm{std}}, h_2g_2\Lambda_{\mathrm{std}}))^{-1}. \tag{2.5}$$

We introduce the following notation. Given two O_F-lattices $\Lambda_1, \Lambda_2 \subset F^{2n}$, we write $f(\Lambda_1, \Lambda_2)$ for the value of f at the relative position of Λ_1 and Λ_2. That is, $f(\Lambda_1, \Lambda_2) := f(k_1^{-1}k_2)$ where the $k_i \in GL_{2n}(F)$ are chosen with $\Lambda_i = k_i\Lambda_{\mathrm{std}}$. In addition, choose a cocompact, free, discrete subgroup $\Gamma \subseteq L^\times$. Combining (2.4) and (2.5) with Definition 2.7, we obtain

$$\begin{aligned} O(\beta, f) &= \sum_{(\Lambda_1,\Lambda_2)\in L^\times\backslash(\mathcal{L}(\beta_1)\times\mathcal{L}(\beta_2))} \mathrm{Vol}(\mathrm{Stab}_{L^\times}(\Lambda_1, \Lambda_2))^{-1} f(\Lambda_1, \Lambda_2) \\ &= \mathrm{Vol}(\Gamma\backslash L^\times)^{-1} \sum_{(\Lambda_1,\Lambda_2)\in\Gamma\backslash(\mathcal{L}(\beta_1)\times\mathcal{L}(\beta_2))} f(\Lambda_1, \Lambda_2). \end{aligned} \tag{2.6}$$

2.4 The η-twisted orbital integrals for (E_0, E_3)

Let $\eta : E_3^\times \to \{\pm 1\}$ be the character from local class field theory for the extension $E_1 \otimes_F E_2/E_3$. Our aim is to define certain η-twisted orbital integrals for regular semi-simple pairs $\alpha : (E_0, E_3) \to M_{2n}(F)$. We only consider spherical Hecke functions in this article. For these, the definition of such integrals is only interesting when η restricts to the trivial character on $O_{E_3}^\times$; cf. [HL20, Proposition 3.2.7]. Thus, we assume from now on that $E_1 \otimes_F E_2/E_3$ is unramified. Equivalently, we assume that one out of E_1, E_2 is unramified over F. This leaves us with the following two cases.

(1) If E_1, E_2 are both unramified over F, then $E_3 \cong F \times F$ and $E_1 \otimes_F E_2 \cong E_1 \times E_1$ is unramified over E_3.

(2) If precisely one out of E_1, E_2 is ramified over F, then $E_1 \otimes_F E_2$ is a biquadratic field extension of F such that $\sigma_1 \otimes \sigma_2$ acts non-trivially on the residue field. It follows that E_3/F is ramified and $E_1 \otimes_F E_2/E_3$ unramified.

In particular, we see that in both cases η can be defined by $\eta(x) = (-1)^{[O_{E_3} : xO_{E_3}]}$, where the index[1] is meant as O_F-lattices in E_3. We define η on $GL_n(E_3)$ by $\eta(h) := \eta(\det h)$. Denote by

[1] If $\Lambda' \subset \Lambda \subset V$ are O_F-lattices in a finite-dimensional F-vector space V, then we set $[\Lambda : \Lambda'] = \log_q |\Lambda/\Lambda'|$ where $q = q_F$. For general $\Lambda', \Lambda \subset V$, we put $[\Lambda : \Lambda'] = [\Lambda : \pi^n\Lambda'] - n\dim(V)$ for $n \gg 0$.

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

$|\cdot|_F$ the normalized absolute value on F. Recall that $E_0 = F \times F$. We also define the character

$$|\cdot| : GL_n(E_0) \longrightarrow \mathbb{R}^\times, \quad (a,b) \longmapsto \left|\frac{\det(a)}{\det(b)}\right|. \tag{2.7}$$

The definition of $O(\beta, f)$ in (2.3) required the choice of auxiliary elements $g_1, g_2 \in GL_{2n}(F)$. The situation for η-twisted orbital integrals is similar, but we need to be more careful to ensure independence of such an auxiliary choice. Consider a regular semi-simple pair $\alpha : (E_0, E_3) \to M_{2n}(F)$. We write $\mathcal{L}(\alpha_i)$ for the set of $\alpha_i(O_{E_i})$-lattices in F^{2n}. Any lattice $\Lambda_0 \in \mathcal{L}(\alpha_0)$ is of the form $\Lambda_0^+ \times \Lambda_0^-$, where $\Lambda_0^+ \subset \alpha_0(1,0)F^{2n}$ is an $\alpha_0(O_F \times 0)$-lattice and $\Lambda_0^- \subset \alpha_0(0,1)F^{2n}$ an $\alpha_0(0 \times O_F)$-lattice. Since α is regular semi-simple, one can show (omitted) that the two natural maps obtained from α_3,

$$E_3 \otimes_F (\alpha_0(1,0) \cdot F^{2n}) \longrightarrow F^{2n}, \quad E_3 \otimes_F (\alpha_0(0,1) \cdot F^{2n}) \longrightarrow F^{2n}$$

are bijective. In particular, given $(\Lambda_0, \Lambda_3) \in \mathcal{L}(\alpha_0) \times \mathcal{L}(\alpha_3)$, there are well-defined lattice indices

$$[\Lambda_3 : O_{E_3}\Lambda_0^+], \quad [\Lambda_3 : O_{E_3}\Lambda_0^-] \in \mathbb{Z}. \tag{2.8}$$

Definition 2.9. Let $\alpha : (E_0, E_3) \to M_{2n}(F)$ be regular semi-simple. Given $(\Lambda_0, \Lambda_3) \in \mathcal{L}(\alpha_0) \times \mathcal{L}(\alpha_3)$ and $s \in \mathbb{C}$, the transfer factor of (Λ_0, Λ_3) is defined by

$$\Omega(\Lambda_0, \Lambda_3, s) = (-1)^{[\Lambda_3 : O_{E_3}\Lambda_0^+]} \cdot (q^{[\Lambda_3 : O_{E_3}\Lambda_0^-] - [\Lambda_3 : O_{E_3}\Lambda_0^+]})^{s/2}. \tag{2.9}$$

This function has the transformation property

$$\Omega(h_0\Lambda_0, h_3\Lambda_3, s) = |h_0|^s \eta(h_3)\Omega(\Lambda_0, \Lambda_3, s), \quad h_0 \in H_0,\ h_3 \in H_3. \tag{2.10}$$

Choose $g_0, g_3 \in GL_{2n}(F)$ such that $g_i\Lambda_{\mathrm{std}} \in \mathcal{L}(\alpha_i)$. For $f \in \mathcal{H}$, define

$$O(\alpha, f, s) := \Omega(g_0\Lambda_{\mathrm{std}}, g_3\Lambda_{\mathrm{std}}) \cdot \int_{(H_0 \cap H_3)\backslash(H_0 \times H_3)} f(g_0^{-1}h_0^{-1}h_3g_3)|h_0|^s \eta(h_3)\, dh_0\, dh_3. \tag{2.11}$$

The integrand here is well-defined by [HL20, Lemma 3.2.3] which states that η and $|\cdot|$ are trivial on $H_0 \cap H_3$. Moreover, the normalization of Haar measures in (2.11) is as in Definition 2.7 and the same remarks on convergence apply. Identity (2.10) allows (2.11) to be rewritten as

$$O(\alpha, f, s) = \int_{(H_0 \cap H_3)\backslash(H_0 \times H_3)} f(g_0^{-1}h_0^{-1}h_3g_3)\Omega(h_0g_0\Lambda_{\mathrm{std}}, h_3g_3\Lambda_{\mathrm{std}})\, dh_0\, dh_3. \tag{2.12}$$

As f is spherical, $f(g_0^{-1}h_0^{-1}h_3g_3)$ only depends on the lattices $g_0\Lambda_{\mathrm{std}}$ and $\Lambda_3\Lambda_{\mathrm{std}}$ and not on the precise choices of g_0, g_3. Using that H_i acts transitively on $\mathcal{L}(\alpha_i)$, we see that $O(\alpha, f, s)$ is actually independent of g_0 and g_3.

There is a lattice count formula for $O(\alpha, f, s)$ that is analogous to (2.6): let L denote the centralizer of α and pick a cocompact subgroup $\Gamma \subset L^\times$ as before. Then (2.12) gives

$$\begin{aligned} O(\alpha, f, s) &= \sum_{(\Lambda_0,\Lambda_3) \in L^\times\backslash(\mathcal{L}(\alpha_0)\times\mathcal{L}(\alpha_3))} \mathrm{Vol}(\mathrm{Stab}_{L^\times}(\Lambda_0, \Lambda_3))^{-1}\, f(\Lambda_0, \Lambda_3)\Omega(\Lambda_0, \Lambda_3, s) \\ &= \mathrm{Vol}(\Gamma\backslash L^\times)^{-1} \sum_{(\Lambda_0,\Lambda_3) \in \Gamma\backslash(\mathcal{L}(\alpha_0)\times\mathcal{L}(\alpha_3))} f(\Lambda_0, \Lambda_3)\Omega(\Lambda_0, \Lambda_3, s). \end{aligned} \tag{2.13}$$

2.5 Fundamental lemma

Recall that we assumed from § 2.4 that at least one out of E_1, E_2 is unramified over F.

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Conjecture 2.10 (Guo–Jacquet FL). Assume that $\alpha : (E_0, E_3) \to M_{2n}(F)$ and $\beta : (E_1, E_2) \to M_{2n}(F)$ are regular semi-simple and matching. Let $f \in \mathcal{H}$ be a spherical Hecke function. Then

$$O(\alpha, f, 0) = O(\beta, f). \tag{2.14}$$

This has been conjectured by Guo and Jacquet [Guo96] for $E_1 = E_2$. The biquadratic case ($E_1 \neq E_2$) has been conjectured by Howard and Li [HL20]. The following cases are known.

(1) For $E_1 = E_2$ both unramified and the unit Hecke function $f = 1_{GL_{2n}(O_F)}$, the FL is the main result of [Guo96].

(2) For general E_1, E_2 and $n = 1$, the conjecture is known for the full spherical Hecke algebra, cf. [HL20].

(3) For general E_1, E_2 and $n = 2$, the conjecture is known for the unit element $1_{GL_{2n}(O_F)}$, which is due to the first author [Li25].

Remark 2.11. FL-type identities for the spherical Hecke algebra may sometimes be deduced from the FL for the unit function, using global arguments, which goes back to Clozel [Clo90] and Labesse [Lab90]. This idea was already mentioned in [Guo96] for the above FL conjecture, but does not seem to have been brought to fruition yet.

2.6 Reduction formula

Assume that $n = n^0 + n^1$ and that the two regular semi-simple pairs

$$\alpha : (E_0, E_3) \longrightarrow M_{2n}(F), \quad \beta : (E_1, E_2) \longrightarrow M_{2n}(F)$$

factor through $M_{2n^0}(F) \times M_{2n^1}(F)$. Denote their components by α^0, α^1 and β^0, β^1, respectively. Our aim is to relate the orbital integrals $O(\alpha, f, s)$ and $O(\beta, f)$ with those for the Levi factors, $O(\alpha^j, f, s)$ and $O(\beta^j, f)$, respectively, $j = 0, 1$. This relation is due to Guo [Guo96] when $E_1 = E_2$.

We work with the following generators (as $\mathbb{C}$-vector space) of $\mathcal{H}$. For $m \in \mathbb{Z}_{\geq 0}$, write

$$f(m) = 1_{\{g \in M_{2n}(O_F),\ v(\det(g)) = m\}}. \tag{2.15}$$

Given a tuple $m = (m_1, \ldots, m_r) \in \mathbb{Z}^r_{\geq 0}$, put

$$f(m) = f(m_1) * f(m_2) * \cdots * f(m_r). \tag{2.16}$$

Concretely, $f(m)(g)$ is the number of tuples of O_F-lattices $(\Lambda_r, \ldots, \Lambda_0)$ in F^{2n} such that

$$gO_F^{2n} = \Lambda_0 \subseteq \Lambda_1 \subseteq \cdots \subseteq \Lambda_r = O_F^{2n}, \quad [\Lambda_i : \Lambda_{i-1}] = m_i \text{ for all } i. \tag{2.17}$$

Also define $[\pi] = 1_{\pi GL_{2n}(O_F)}$ which is invertible with inverse $[\pi]^{-1} = 1_{\pi^{-1} GL_{2n}(O_F)}$. By Theorem B.3,

$$\mathcal{H} = \sum_{k \in \mathbb{Z},\ r \geq 0,\ m = (m_1, \ldots, m_r) \in \mathbb{Z}^r_{\geq 0}} \mathbb{C} \cdot [\pi]^k * f(m). \tag{2.18}$$

We remark that this is not a direct sum, meaning there are non-trivial relations among the generators $[\pi]^k * f(m)$. We also mention that

$$O(\alpha, [\pi]^k * f, s) = O(\alpha, f, s), \quad O(\beta, [\pi]^k * f) = O(\beta, f) \tag{2.19}$$

for every $f \in \mathcal{H}$ and $k \in \mathbb{Z}$. These identities are obtained from $([\pi]^k * f)(g) = f(\pi^{-k} g)$ and a variable substitution in the definitions of $O(\alpha, f, s)$ and $O(\beta, f)$. Let $\mathcal{H}^j = \mathbb{C}[GL_{2n^j}(O_F) \backslash GL_{2n^j}(F) / GL_{2n^j}(O_F)]$, for $j = 0, 1$, denote the spherical Hecke algebra of $GL_{2n^j}(F)$. There is a partial Satake transformation

$$\mathcal{S} : \mathcal{H} \longrightarrow \mathcal{H}^0 \otimes_{\mathbb{C}} \mathcal{H}^1$$

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

which is discussed in detail in (B.2) and Proposition B.7. It is described on the above generators[2] by

$$\mathcal{S}([\pi]^k * f(m)) = \sum_{m^0,m^1\in\mathbb{Z}^r_{\geq 0},\ m=m^0+m^1} q^{n^1\cdot|m^0|+n^0\cdot|m^1|}([\pi]^k * f(m^0)) \otimes ([\pi]^k * f(m^1)). \tag{2.20}$$

Here, $|m| := \sum_{i=1}^r m_i$ denotes the sum of all coefficients. Taking the product of the orbital integrals for α^0 and α^1 and for β^0 and β^1, respectively, defines orbital integrals

$$O^{\mathrm{Levi}}((\alpha^0,\alpha^1),f,s), \quad O^{\mathrm{Levi}}((\beta^0,\beta^1),f), \quad f\in\mathcal{H}^0\otimes_{\mathbb{C}}\mathcal{H}^1.$$

Theorem 2.12. *Given α and β as above, there are the following relations of orbital integrals on $GL_{2n}(F)$ and its Levi $GL_{2n^0}(F)\times GL_{2n^1}(F)$:*

$$\begin{aligned} O(\alpha,f,s) &= |\mathrm{Disc}_{E_3/F}|_F^{-\frac{n^0\cdot n^1}{2}}|\mathrm{Res}(\mathrm{Inv}(\alpha^0),\mathrm{Inv}(\alpha^1))|_F^{-1}\cdot O^{\mathrm{Levi}}((\alpha^0,\alpha^1),\ \mathcal{S}(f),s),\\ O(\beta,f) &= |\mathrm{Disc}_{E_1/F}\cdot\mathrm{Disc}_{E_2/F}|_F^{-\frac{n^0\cdot n^1}{2}}|\mathrm{Res}(\mathrm{Inv}(\beta^0),\mathrm{Inv}(\beta^1))|_F^{-1}\cdot O^{\mathrm{Levi}}((\beta^0,\beta^1),\ \mathcal{S}(f)). \end{aligned} \tag{2.21}$$

Here, we have used the notation $|x|_F := |\mathrm{Nm}_{E_3/F}(x)|_F^{1/2}$ for elements $x\in E_3$.

Remark 2.13. The resultant $\mathrm{Res}(\mathrm{Inv}(\alpha^0),\mathrm{Inv}(\alpha^1))$ lies in $E_3^\times$. Namely, α being regular semi-simple in particular means that $\mathrm{Inv}(\alpha)=\mathrm{Inv}(\alpha^0)\mathrm{Inv}(\alpha^1)$ is separable, which implies that $\mathrm{Inv}(\alpha^0)$ and $\mathrm{Inv}(\alpha^1)$ have no common zeros. The same applies to all other analogous resultant terms in this article.

Proof of Theorem 2.12. These are [Guo96, Propositions 2.1 and 2.2] when $E_1=E_2$. We give a proof for the general case in § 4. □

2.7 Vanishing order

We now combine the reduction formula with the following vanishing statement.

Proposition 2.14 [HL20, Proposition 3.3.3]. *Assume that $\alpha:(E_0,E_3)\to M_{2n}(F)$ is regular semi-simple and matches to a pair $\beta:(E_1,E_2)\to B$, where B is a central division algebra of degree $2n$ over F. Then*

$$O(\alpha,f,0)=0.$$

This relies on a functional equation for the orbital integral that relates $O(\alpha,f,s)$ with $O(\alpha,f,-s)$, we refer to [HL20].

Definition 2.15. Let δ be a regular semi-simple invariant and let $\beta_\delta:(E_1,E_2)\to B_\delta$ be the universal datum from Proposition 2.5. Denote the center by $L_\delta\subseteq B_\delta$, write $L_\delta=\prod_{j\in J}L^j$ for its factorization into fields and let B^j/L^j be the corresponding factor of B_δ. We define the order of δ as

$$\mathrm{ord}(\delta) := |\{j\in J\mid B^j \text{ is a division algebra}\}|. \tag{2.22}$$

Lemma 2.16. *Let L/F be a field extension of degree d and let C/L be a quaternion algebra. There exists a central division algebra D/F of degree $[D:F]=2d$ together with an embedding $C\to D$ if and only if C is non-split.*

Proof. Clearly, C is non-split if there exists an embedding $C\to D$ as stated. The converse direction is obtained as follows. We denote the Hasse invariant of a central simple algebra D/F by

[2] In Appendix B, the notation for $f(m)$ is $\mathbf{T}_m$ or $\mathbf{T}_{\mathrm{GL}_{2n},m}$, see (B.4).

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

$\mathrm{Ha}(D) \in \mathbb{Q}/\mathbb{Z}$. Recall that $D \mapsto \mathrm{Ha}(D)$ defines an isomorphism $\mathrm{Br}(F) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$. Also recall that for every field extension L/F, the identity $\mathrm{Ha}(L \otimes_F D) = [L:F] \cdot \mathrm{Ha}(D)$ holds. In particular, [Dra83, Theorem 9.12 (c)] implies that for every central division algebra D/F and for every field extension L/F such that $[L:F] \mid [D:F]$, there exists an embedding $L \to D$ as F-algebras. In this situation, [Dra83, Corollary 9.1] shows that $\mathrm{Ha}(\mathrm{Cent}_D(L)) = [L:F] \cdot \mathrm{Ha}(D)$. Thus, if $[D:F] = 2[L:F]$, then there exists an embedding $L \to D$ and $\mathrm{Cent}_D(L)$ is a quaternion division algebra over L which is isomorphic to C if C is non-split. □

Corollary 2.17. *Let $\alpha : (E_0, E_3) \to M_{2n}(F)$ be a regular semi-simple pair. Then*

$$\mathrm{ord}_{s=0} O(\alpha, f, s) \geq \mathrm{ord}(\mathrm{Inv}(\alpha)).$$

Proof. Let $\delta = \mathrm{Inv}(\alpha)$ and let $\beta_\delta : (E_1, E_2) \to B_\delta$ be the universal datum for (E_1, E_2, δ). Write $L_\delta = \prod_{j \in J} L^j$ and $B_\delta = \prod_{j \in J} B^j$ as before. Let $\delta = \prod_{j \in J} \delta^j$ be the corresponding factorization of δ, see (2.2), and note that $B^j = B_{\delta^j}$. By Lemma 2.16, B^j embeds into a central division algebra of degree $2[L^j : F]$ if it is a division algebra. Then Proposition 2.14 states that $O(\alpha^j, f^j) = 0$ for every spherical Hecke function f^j on $GL_{2[L^j:F]}(F)$ and every pair $\alpha^j : (E_0, E_3) \to GL_{2[L^j:F]}(F)$ of invariant δ^j. Combining this observation with the product formula Theorem 2.12 and performing induction on $|J|$, we obtain the corollary. □

Let q_F be the residue cardinality of F. We next consider the normalized first derivative

$$\partial\mathrm{Orb}(\alpha, f) := \frac{1}{\log(q_F)} \frac{d}{ds}\bigg|_{s=0} O(\alpha, f, s). \tag{2.23}$$

Corollary 2.18. *Assume that $\alpha : (E_0, E_3) \to M_{2n}(F)$ is regular semi-simple and that $f \in \mathcal{H}$ is such that*

$$O(\alpha, f, 0) = 0, \quad \partial\mathrm{Orb}(\alpha, f) \neq 0. \tag{2.24}$$

Then $\mathrm{ord}(\mathrm{Inv}(\alpha)) = 1$. In particular, there are uniquely determined n^0, n^1 with $n = n^0 + n^1$ such that α matches a pair $\beta : (E_1, E_2) \to D_{1/2n^0} \times M_{2n^1}(F)$. Here, D_λ denotes the central division algebra of Hasse invariant λ over F.

Proof. Set $\mathrm{ord}(\alpha) = \mathrm{ord}(\mathrm{Inv}(\alpha))$. Corollary 2.17 provides $\mathrm{ord}(\alpha) \leq 1$ and we are required to exclude the case $\mathrm{ord}(\alpha) = 0$. This relies on the functional equation [HL20, Proposition 3.2.8], saying

$$O(\alpha, f, s) = \varepsilon(\alpha, s) O(\alpha, f, -s) \tag{2.25}$$

for a factor $\varepsilon(\alpha, s) \in \pm |F^\times|_F^s$ that only depends on α. The proof of [HL20, Proposition 3.3.3], combined with Theorem 2.12 and Lemma 2.16, shows that the sign of $\varepsilon(\alpha, s)$ is $(-1)^{\mathrm{ord}(\alpha)}$. Assume it were positive, say $\varepsilon(\alpha, s) = q_F^{rs}$. Taking the derivative $(d/ds)|_{s=0}$ of (2.25), we obtain that

$$\log(q_F)\, \partial\mathrm{Orb}(\alpha, f) = r \log(q_F) O(\alpha, f) - \log(q_F)\, \partial\mathrm{Orb}(\alpha, f).$$

This is impossible under our assumptions (2.24). Thus, necessarily $\mathrm{ord}(\alpha) = 1$. The existence of (n^0, n^1) and of the pair β is now evident from Lemma 2.16: take $n^0 = [L^{j_0} : F]$ and $n^1 = n - n^0$, where $j_0 \in J$ is the unique index in (2.22) such that B^{j_0} is division. □

3. The AFL

3.1 Quadratic cycles

Let F, E_1 and E_2 as well as $n \geq 1$ be as before. In particular, E_1 and E_2 are quadratic étale field extensions of F of which at least one is unramified. Denote by $\breve{F}/F$ the completion of a

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

maximal unramified extension and by $\mathbb{F}$ its residue field. Let $\breve{E}$ be a composite of $\breve{F}$, E_1 and E_2, and fix embeddings $E_1, E_2 \to \breve{E}$.

Definition 3.1. Let S be an $O_{\breve{F}}$-scheme such that $\pi \in \mathcal{O}_S$ is locally nilpotent.

(1) Assume F is p-adic. A strict π-divisible O_F-module over S is a pair (X, ι) consisting of a p-divisible group X over S and a strict action $\iota : O_F \to \mathrm{End}(X)$. The latter means $\iota(a) = a$ on $\mathrm{Lie}(X)$.

(2) Assume $F \cong \mathbb{F}_q(\!(\pi)\!)$ is a local function field. A strict π-divisible O_F-module over S is a π-divisible group[3] (X, ι) over S (for F) in the sense of [HS19, Definition 7.1] that is strict, meaning that $\iota(a) = a$ on $\mathrm{Lie}(X)$.

The theories of these two types of objects are completely parallel, at least to the extent required by this article. We also remark that the objects in part (2) are equivalent to minuscule local O_F-shtuka, cf. [HS19, Theorem 8.3].

Let $\mathbb{X}/\mathbb{F}$ be a strict π-divisible O_F-module of height $2n$ and dimension 1 and consider a pair of embeddings

$$\beta : (E_1, E_2) \longrightarrow \mathrm{End}^0(\mathbb{X}).$$

Considering its identity component and maximal étale quotient, $\mathbb{X}$ decomposes in a unique way as a product $\mathbb{X} = \mathbb{X}^0 \times \mathbb{X}^1$ with $\mathbb{X}^0$ connected and $\mathbb{X}^1$ étale. The heights of $\mathbb{X}^0$ and $\mathbb{X}^1$ are even because of the existence of β and we write $\mathrm{ht}(\mathbb{X}^0) = 2n^0$ and $\mathrm{ht}(\mathbb{X}^1) = 2n^1$. Then

$$\mathrm{End}(\mathbb{X}) \cong O_D \times M_{2n^1}(O_F),$$

where D/F denotes a central division algebra of Hasse invariant $1/2n^0$. The datum of β is, thus, equivalent to that of two pairs

$$\beta^0 : (E_1, E_2) \to \mathrm{End}^0(\mathbb{X}^0), \quad \beta^1 : (E_1, E_2) \to \mathrm{End}^0(\mathbb{X}^1).$$

Lemma 3.2. *Assume that β is regular semi-simple. Then* $\mathrm{ord}(\mathrm{Inv}(\beta)) = 1$. *Conversely, for every regular semi-simple invariant δ with* $\mathrm{ord}(\delta) = 1$, *there is a unique pair $(\mathbb{X}, \beta)$ as above up to isogeny such that* $\mathrm{Inv}(\beta) = \delta$.

Proof. Write $L_\delta = \prod L^j$ and $B_\delta = \prod B^j$ as in Definition 2.15. As we showed during the proof of Corollary 2.18, there exists an F-algebra embedding $B_\delta \to D_{1/2n^0} \times M_{2n^1}(F)$ if and only if there is a unique index j_0 with B^{j_0} a division algebra and $[L^{j_0} : F] = n^0$. The pairs $(2n^0, 2n^1)$ with $n^0 + n^1 = n$ on the other hand are in bijection with the possible isogeny classes of $\mathbb{X}$ (Dieudonné–Manin classification). Finally, any embedding of B_δ into $D_{1/2n^0} \times M_{2n^1}(F)$ has to be a product of two embeddings

$$B^{j_0} \longrightarrow D_{1/2n^0}, \quad \prod_{j \neq j_0} B^j \longrightarrow M_{2n^1}(F)$$

and, hence, all such embeddings are conjugate by Skolem and Noether. This proves the uniqueness up to isogeny. □

Remark 3.3. An analogous uniqueness statement holds for local Shimura data of EL type for inner forms of $GL_{2n}(F)$, see [LM23, Proposition 4.5]. It would be interesting to know if there is a more group-theoretic explanation for this phenomenon.

[3]These are called π-divisible local Anderson modules in [HS19]. Our terminology is chosen for uniformity with the p-adic case.

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Let $M \to \operatorname{Spf} O_{\breve{E}}$ denote the RZ space of $\mathbb{X}$, that is

$$M(S) = \left\{ (X, \rho) \;\middle|\; \begin{array}{l} X/S \text{ a strict } \pi\text{-divisible } O_F\text{-module} \\ \rho : \overline{S} \times_S X \to \overline{S} \times_{\operatorname{Spec} \mathbb{F}} \mathbb{X} \text{ a quasi-isogeny} \end{array} \right\}. \tag{3.1}$$

Here, $\overline{S} := \mathbb{F} \otimes_{O_{\breve{E}}} S$ denotes the special fiber of S. Analogously define, for $i = 1, 2$,

$$Z(\beta_i)(S) = \left\{ (X, \rho) \;\middle|\; \begin{array}{l} X/S \text{ a strict } \pi\text{-divisible } O_{E_i}\text{-module} \\ \rho : \overline{S} \times_S X \to \overline{S} \times_{\operatorname{Spec} \mathbb{F}} (\mathbb{X}, \beta_i) \text{ an } O_{E_i}\text{-linear quasi-isogeny} \end{array} \right\}. \tag{3.2}$$

The above spaces are well-known to be formally of finite type and formally smooth over $\operatorname{Spf} O_{\breve{E}}$. The relative dimension of M is $2n-1$, that of the $Z(\beta_i)$ is $n-1$. Moreover, the reduced subschemes M_{red} and $Z(\beta_i)_{\mathrm{red}}$ are 0-dimensional. The forgetful maps $Z(\beta_1), Z(\beta_2) \to M$ are closed immersions and allow the $Z(\beta_i)$ to be viewed as cycles on M.

3.2 Hecke correspondences

Throughout this article, we work with a non-standard definition of integral models of Hecke operators that is derived from our non-standard presentation (2.18) of the Hecke algebra. Therefore, we begin with some general remarks on Hecke correspondences that justify this approach and provide the comparison with the definitions in [Li22b] and [HL20].

Write $Z^c(M)$ for the group of cycles on M of codimension c, similarly for $M \times_{\operatorname{Spf} O_{\breve{E}}} M$. By correspondence on M we mean an element of

$$\operatorname{Corr}(M) := \{ z \in Z^{2n-1}(M \times_{\operatorname{Spf} O_{\breve{E}}} M) | p_1, p_2 : \operatorname{Supp}(z) \to M \text{ both finite} \}. \tag{3.3}$$

Correspondences form a ring through convolution and act on $Z^\bullet(M)$; we refer to Appendix A for these definitions and their properties.

Recall that $\mathcal{H}$ denotes the spherical Hecke algebra of $GL_{2n}(F)$ with respect to $GL_{2n}(O_F)$. Given an indicator function $1_\mu \in \mathcal{H}$, there is a definition of Hecke correspondence $[R(1_\mu)] \in \operatorname{Corr}(M)$. It is uniquely characterized by the requirement that its generic fiber parametrizes pairs $((X_1, \rho_1), (X_2, \rho_2))$ such that $\rho_2^{-1}\rho_1 : X_1 \to X_2$ is of type μ. The general definition of $[R(f)]$, $f \in \mathcal{H}$, is obtained by linear extension. Given cycles z_1, z_2 on M that are of complementary dimension and such that $\operatorname{Supp}[R(f)] \cap (\operatorname{Supp} z_1 \times_{\operatorname{Spf} O_{\breve{E}}} \operatorname{Supp} z_2)$ is artinian, there is an intersection number $(z_1, [R(f)] * z_2)$. In this way, the Hecke action on cycles on M and the resulting intersection numbers are defined unambiguously.

An explicit construction of $[R(f)]$ in terms of formal scheme models $R(f) \to M \times M$ is given in [Li22b, § 6.2] or [RSZ21, § 4.3], it relies on the notion of Drinfeld level structure. By Proposition A.8, the Hecke action $[R(f)] : Z^\bullet(M) \to Z^\bullet(M)$ is then expressed as

$$[R(f)] * z = p_{1,*}[R(f) \times_{M \times M} p_2^{-1}(z)].$$

(The bracket notation indicates passage to the cycle.) Assume, furthermore, that $Z_1, Z_2 \subset M$ are closed formal subschemes of pure complementary dimension, both defined by regular sequences, and such that $R(f) \cap (Z_1 \times_{\operatorname{Spf} O_{\breve{E}}} Z_2)$ is artinian. The mentioned models $R(f)$ are even regular, so Corollary A.10 implies

$$([Z_1], [R(f)] * [Z_2]) = \ell_{O_{\breve{E}}}(\mathcal{O}_{R(f) \times_{M \times M} (Z_1 \times Z_2)}). \tag{3.4}$$

This provides the promised link of the intersection numbers from [Li22b, Li25] (given by the right-hand side) with the canonical ones on the left-hand side. We now come to our definition of Hecke correspondences, which, by the above arguments, lead to the same intersection numbers.

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Definition 3.4.

(1) Given $m \geq 0$, denote by $(p_1, p_2) : R(m) \to M \times_{\operatorname{Spf} O_{\breve{E}}} M$ the functor

$$R(m)(S) = \left\{ (X_0 \xrightarrow{\varphi} X_1, \rho) \;\middle|\; \begin{array}{l} (X_1, \rho) \in M(S), \\ \varphi \text{ an } O_F\text{-linear isogeny of height } m \end{array} \right\}. \tag{3.5}$$

The map to $M \times_{\operatorname{Spf} O_{\breve{E}}} M$ is given by

$$(X_0 \xrightarrow{\varphi} X_1, \rho) \longmapsto ((X_0, \rho \circ \varphi), (X_1, \rho)).$$

It is a closed immersion that identifies $R(m)$ with the locus of those pairs $((X_0, \rho_0), (X_1, \rho_1))$ such that the quasi-isogeny $\rho_1^{-1}\rho_0 : X_0 \to X_1$ is an isogeny of degree m.

(2) For a tuple $m = (m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}$, we define

$$R(m) := R(m_1) \underset{p_2, M, p_1}{\times} R(m_2) \underset{p_2, M, p_1}{\times} \cdots \underset{p_2, M, p_1}{\times} R(m_r)$$

by composition. In other words, $R(m)$ represents the functor

$$R(m)(S) = \left\{ (X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_r} X_r, \rho) \;\middle|\; \begin{array}{l} \text{Each } X_i \text{ a } \pi\text{-divisible strict } O_F\text{-module}, \\ (X_r, \rho) \in M(S), \\ \varphi_i \text{ an } O_F\text{-linear isogeny of height } m_i \end{array} \right\}. \tag{3.6}$$

We often write $(X_\bullet, \rho)$ for such chains of isogenies. The map to $M \times_{\operatorname{Spf} O_{\breve{E}}} M$ takes the chain $(X_\bullet, \rho)$ to $((X_0, \rho \circ \varphi_r \circ \cdots \circ \varphi_1), (X_r, \rho))$.

(3) For $k \in \mathbb{Z}$, multiplication in the framing defines an automorphism

$$[\pi]^k : M \longrightarrow M, \quad (X, \rho) \longmapsto (X, \pi^k \rho).$$

Taking its graph, we view $[\pi]^k : M \to M \times_{\operatorname{Spf} O_{\breve{E}}} M$ as correspondence.

Proposition 3.5. *For any $m = (m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}$, the formal scheme $R(m)$ is Cohen–Macaulay and the two projection maps $p_1, p_2 : R(m) \to M$ finite flat. In particular, Definition 3.4 provides a model for the Hecke correspondence:*

$$[[\pi]^k \underset{p_2, M, p_1}{\times} R(m)] = [R([\pi]^k * f(m))] \quad \text{in } \operatorname{Corr}(M).$$

Proof. First consider the case $r = 1$ and the first projection $p_1 : R(m) \to M$. Let $(X, \rho) \in M(S)$. The fiber of p_1 over (X, ρ) is the set of O_F-linear isogenies $X \to Y$ of height m to strict π-divisible O_F-modules Y, up to isomorphism in Y. Considering the occurring kernels, this set is in bijection with certain finite locally free subgroups $H \subset X$ that we now describe.

In the p-adic setting, H has to be O_F-stable, of order q_F^m, and such that O_F acts on X/H strictly, meaning $\iota(a) = a$ in $\operatorname{End}(\operatorname{Lie}(X/H))$. Varying S and (X, ρ), this description shows that p_1 is relatively representable in projective schemes.

In the function field setting, we still consider O_F-stable H of order q_F^m. We also need to make sure that X/H is again a π-divisible O_F-module, however. This is the case if and only if each truncation $(X/H)[\pi^r]$, $r \geq 0$, is balanced as $\mathbb{F}_{q_F}$-module scheme. (Being balanced is a condition on the primitive elements in the Hopf algebra of the group scheme, cf. [Pog17, Definition 5.13]. This property is equivalent to being a strict $\mathbb{F}_q$-module scheme, cf. [HS19, Remark 5.3]. It is required for the truncations of a π-divisible O_F-module by definition, cf. [HS19, Definition 7.1].) We are given that all $X[\pi^r]$ are balanced. Then all $(X/H)[\pi^r]$ are balanced if and only if H is balanced, because this property is well-behaved in exact sequences. The condition for a finite locally free group scheme to be balanced is open and closed on the base, cf. [Pog17, Remark 5.17].

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Finally, we impose that the O_F-action on X/H is strict. In this way, we have again shown that p_1 is relatively representable in projective schemes.

Now consider the special fiber $\operatorname{Spec}\mathbb{F}\times_M R(m)\to\operatorname{Spec}\mathbb{F}$. It is of dimension 0 because a 1-dimensional π-divisible O_F-module over $\mathbb{F}$ has at most finitely many subgroups of order q_F^i. Thus, p_1 is finite and we obtain $\dim R(m)\le\dim M$.

The map $R(m)\to M\times_{\operatorname{Spf}O_{\breve{E}}}M$ is a closed immersion and $M\times_{\operatorname{Spf}O_{\breve{E}}}M$ is regular. We claim that $R(m)$ is locally defined by at most $2n-1$ equations. Let $X_i=p_i^*(X)$ be the pullback of the universal π-divisible O_F-module from the ith component, $i=1,2$. Write $R(m)=V(I)$ for an ideal sheaf I and consider the Hodge filtrations of $X_1|_{V(I^2)}$ and $X_2|_{V(I^2)}$ as well as the following map on (O_F-linear) Grothendieck–Messing crystals coming from φ.[4]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \longrightarrow & D_{X_1}(V(I^2)) & \longrightarrow & \operatorname{Lie}(X_1|_{V(I^2)}) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle D_\varphi(V(I^2))} & & & & \\ 0 & \longrightarrow & F_2 & \longrightarrow & D_{X_2}(V(I^2)) & \longrightarrow & \operatorname{Lie}(X_2|_{V(I^2)}) & \longrightarrow & 0 \end{array} \tag{3.7}$$

Note that $I/I^2\subset\mathcal{O}_{V(I^2)}$ is defined by the condition that it is the minimal ideal $J\subset\mathcal{O}_{V(I^2)}$ such that the universal quasi-isogeny $\rho_2^{-1}\rho_1: X_1|_{V(I^2)}\to X_2|_{V(I^2)}$ is an isogeny mod J. (Here, (X_1,ρ_1,X_2,ρ_2) denotes the universal point over $M\times_{\operatorname{Spf}O_{\breve{E}}}M$.) By Grothendieck–Messing theory, I/I^2 equals the ideal defined by the condition that the composite map $F_1\to\operatorname{Lie}(X_2|_{V(I^2)})$ in (3.7) vanishes. The source is a vector bundle of rank $2n-1$, the target a line bundle. Thus, $R(m)$ is locally defined by $2n-1$ equations as claimed. In particular, also $\dim R(m)\ge\dim M$ everywhere, so $R(m)$ is pure of dimension $2n$. It follows from the smoothness of M that $R(m)$ is Cohen–Macaulay.

The flatness of both p_1 and p_2 is now deduced from 'miracle flatness': a local homomorphism $R\to S$ of noetherian local rings with R regular and S Cohen–Macaulay is flat if and only if its fiber dimension is constant, see [Sta18, 00R4].

The obtained properties now extend to general r: the composition of finite flat correspondences is finite flat, so both $p_1,p_2: R(m_1,\dots,m_r)\to M$ are finite flat. Being finite flat over a regular local ring implies Cohen–Macaulay.

Finally, the identity $[[\pi]^k\times_{p_2,M,p_1}R(m)]=[R([\pi]^k*f(m))]$ is clear for generic fibers. Since $R(m)$ is flat over M, it is also flat over $O_{\breve{E}}$, so the identity extends to cycles on M. □

3.3 Intersection numbers

Definition 3.6. For $m=(m_1,\dots,m_r)\in\mathbb{Z}_{\ge0}$ and β as before, we define $I(\beta,m)$ by the following Cartesian diagram.

$$\begin{array}{ccc} I(\beta,m) & \longrightarrow & Z(\beta_2)\times_{\operatorname{Spf}O_{\breve{E}}}Z(\beta_1) \\ \downarrow & \square & \downarrow \\ R(m) & \longrightarrow & M\times_{\operatorname{Spf}O_{\breve{E}}}M \end{array} \tag{3.8}$$

Lemma 3.7. *Assume that β is regular semi-simple. Then $I(\beta,m)$ is a (possibly infinite) union of artinian schemes.*

Proof. This follows *a posteriori* from the length computations in [Li22b, HL20]. We give an alternative *a priori* argument. Let $\operatorname{Spf}A\subset I(\beta,f)$ be an affine closed formal subscheme, given by

[4] We refer to [HS19, Theorem 9.8] for the fact that the deformation theory of π-divisible O_F-modules in equal characteristic is the same as that in the p-adic setting.

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

some chain $(X_\bullet, \rho) \in R(m)(\operatorname{Spf} A)$. Then $X := X_r$ algebraizes to a strict π-divisible O_F-module X over $\operatorname{Spec} A$ that comes with a pair of embeddings

$$(\rho^{-1}\beta_1\rho,\ (\rho\varphi_r \cdots \varphi_1)^{-1}\beta_2(\rho\varphi_r \cdots \varphi_1)) : (O_{E_1}, O_{E_2}) \longrightarrow \operatorname{End}^0(X). \tag{3.9}$$

The intersection of its image with $\operatorname{End}(X)$ generates an order in a quaternion algebra B over an étale F-algebra L of degree n because β is regular semi-simple, see Proposition 2.5. Moreover, $B \ncong M_2(L)$ because B embeds into $\operatorname{End}^0(\mathbb{X})$. Our task is to show that ρ algebraizes, i.e. that $(X_\bullet, \rho) \in R(m)(\operatorname{Spec} A)$.

It is well-known that p-divisible groups in characteristic 0 are étale. The analogous statement holds in the function field setting. Namely let X/S be a strict π-divisible O_F-module. The three properties X being π^∞-torsion, $\pi \in \mathcal{O}_S^\times$ and $\iota(\pi) = \pi$ in $\operatorname{End}(\operatorname{Lie}(X))$ imply $\operatorname{Lie}(X) = 0$. The only strict π-divisible O_F-module of height $2n$ over a geometric point with $\pi \neq 0$ is, thus, $(F/O_F)^{2n}$ (up to isomorphism). Its endomorphism ring is $M_{2n}(O_F)$ and there is no possibility for a pair of invariant $\operatorname{Inv}(\beta)$ as in (3.9). Thus, A has to be π^k-torsion, $k \gg 0$.

Concerning O_F-fields with $\pi = 0$, the geometric isogeny class of $\mathbb{X}$ is the only one of height $2n$ and dimension 1 that admits a pair of embeddings of invariant $\operatorname{Inv}(\beta)$, see Lemma 3.2. Thus, all fibers of X are of the same geometric isogeny class. In particular, the connected component of identity X^0 is again a strict π-divisible O_F-module, cf. [Mes72, Proposition II.4.9] (p-adic case) and [HS19, Proposition 10.16] (function field case).

Let $A \to R$ be a local homomorphism to the ring of integers in a complete, algebraically closed, non-archimedean field K. Denote its residue field by k. We use index notation X_R, X_K, X_k, etc. to denote base changes to R, K, k, etc. The map

$$\operatorname{End}(\mathbb{X}_R) \longrightarrow \operatorname{End}(\mathbb{X}_K) = O_{D_{1/2n^0}} \times M_{2n^1}(O_F)$$

is injective, so the reduction map

$$\operatorname{End}(\mathbb{X}_R) \longrightarrow \operatorname{End}(\mathbb{X}_k)$$

is a bijection. If we manage to find an isogeny $\tau : X_R \to \mathbb{X}_R$, then the composition $(\rho\tau^{-1})\tau : X_R \to \mathbb{X}_R$ is a $(\operatorname{Spec} R)$-valued point of M that agrees with the given $\operatorname{Spf} R$-valued one. But any morphism from $\operatorname{Spec} R$ to M factors through one of the closed points of M because M is a disjoint union of formal spectra of complete local rings. Since R was arbitrary, this implies the statement of the lemma.

Since R is perfect, we may write $X_R = X_R^0 \times X_R^1$ and construct τ component-wise

$$\tau^0 : X_R^0 \longrightarrow \mathbb{X}_R^0, \quad \tau^1 : X_R^1 \longrightarrow \mathbb{X}_R^1.$$

Both X_R^1 and $\mathbb{X}_R^1$ are isomorphic to $(F/O_F)^{2n^1}$ and we take τ^1 as any isomorphism. For τ^0, we consider $X_R^0 \cong \operatorname{Spf} R[\![t]\!]$ as a formal group. (See [HS19, Corollary 10.12] for the equivalence of formal O_F-modules and connected strict π-divisible O_F-modules.) Since special and generic fiber of X_R^0 are both of height $2n^0$, and since K is algebraically closed, we may normalize the coordinate such that multiplication by $\pi \in O_F$ is expressed by

$$[\pi](t) = t^{q^{2n^0}}.$$

The power series describing addition and the O_F-action all commute with $[\pi](t)$ and now have coefficients in the finite field $\mathbb{F}_{q^{2n^0}}$. In other words, X_R^0 comes via base change from $\mathbb{F}$ and is, thus, isomorphic to $\mathbb{X}_R^0$. □

Let $L := L(\beta) \subset \operatorname{End}^0(\mathbb{X})$ denote the centralizer of $\beta_1(E_1) \cup \beta_2(E_2)$. Its units $L^\times$ act on all objects in (3.8) by transport of structure along ρ.

Lemma 3.8. *Assume that β is regular semi-simple. Then the quotient $L^\times \backslash \pi_0(I(\beta, m))$ is finite.*

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Proof. Since $I(\beta,m)(\mathbb{F})=\pi_0(I(\beta,m))$, this is purely a statement about $\mathbb{F}$-points. Given a π-divisible group X over $\mathbb{F}$, we write $X=X^0\times X^1$ for its decomposition into connected and étale part. Then every $\mathbb{F}$-point $(X,\rho)\in M(\mathbb{F})$ is (in a unique way) a product $(X^0\times X^1,\rho^0\times\rho^1)$, where $\rho^0:X^0\to\mathbb{X}^0$ and $\rho^1:X^1\to\mathbb{X}^1$ are two quasi-isogenies. Moreover, giving an isogeny $\varphi_i:X_{i-1}\to X_i$ of height m_i is the same as giving a pair of isogenies

$$\varphi_i^0:X_{i-1}^0\longrightarrow X_i^0,\quad \varphi_i^1:X_{i-1}^1\longrightarrow X_i^1$$

with $\mathrm{ht}(\varphi_i^0)+\mathrm{ht}(\varphi_i^1)=m_i$. In this way, we obtain

$$\pi_0(I(\beta,m))=\coprod_{m^0+m^1=m}\pi_0(I(\beta^0,m^0))\times\pi_0(I(\beta^1,m^1)) \tag{3.10}$$

where $m^0,m^1\in\mathbb{Z}^r_{\geq 0}$. Moreover, the centralizer is a product $L=L^0\times L^1$ and its units act diagonally on the right of (3.10). This reduces us to the two cases of $\mathbb{X}$ being connected or étale.

If $\mathbb{X}$ is connected, then $\pi_0(M)\cong\mathbb{Z}$ via $(X,\rho)\mapsto\mathrm{ht}(\rho)$. The element $\pi\in F^\times$ acts through translation by $2n$, proving the statement in this case. If $\mathbb{X}$ is étale, then $\pi_0(M)\cong GL_{2n}(F)/GL_{2n}(O_F)$ and the argument is the same as for the convergence of (2.3). □

In fact, $[R(f)]$ is invariant under the diagonal action of $\mathrm{Aut}^0(\mathbb{X})$ on $M\times_{\mathrm{Spf}\,O_{\breve{E}}}M$ for every f. Lemmas 3.7 and 3.8 imply that $\mathrm{Supp}[R(f)]\times_{M\times M}(Z(\beta_2)\times Z(\beta_1))$ is artinian and finite modulo $L^\times$ whenever β is regular semi-simple.

Definition 3.9. Let β be regular semi-simple. Endow $L=L(\beta)^\times$ with the measure such that $\mu(O_L^\times)=1$. Let $\Gamma\subseteq L^\times$ be a cocompact subgroup that acts freely on $\pi_0(M)$. For $f\in\mathcal{H}$, define the intersection number

$$\begin{aligned}\mathrm{Int}(\beta,f)&:=\mu(\Gamma\backslash L^\times)^{-1}(\Gamma\backslash[R(f)],\ \Gamma\backslash(Z(\beta_2)\times Z(\beta_1)))\\&=\sum_{x\in L^\times\backslash\pi_0(\mathrm{Supp}[R(f)]\times(Z(\beta_2)\times Z(\beta_1)))}\mu(\mathrm{Stab}(x))^{-1}([R(f)]_x,\ (Z(\beta_2)\times Z(\beta_1))_x).\end{aligned} \tag{3.11}$$

The intersection of the first expression is taken on $\Gamma\backslash(M\times M)$. The intersections of the second happen on the connected component x. More concretely, write $f=\sum_{i\in I}\lambda_i\cdot[\pi]^{k_i}*f(m^{(i)})$ for some finite index set I. (Recall that the $f(m^{(i)})$ were defined in (2.16).) Then, by Proposition 3.5 and Corollary A.10,

$$\mathrm{Int}(\beta,f)=\sum_{i\in I}\lambda_i\cdot\mathrm{Int}(\beta,m^{(i)}),$$

where the summands on the right-hand side are defined by

$$\begin{aligned}\mathrm{Int}(\beta,m)&:=\mu(\Gamma\backslash L^\times)^{-1}\ell_{O_{\breve{E}}}(\mathcal{O}_{\Gamma\backslash I(\beta,m)})\\&=\sum_{x\in L^\times\backslash\pi_0(I(\beta,m))}\mu(\mathrm{Stab}(x))^{-1}\ell_{O_{\breve{E}}}(\mathcal{O}_{I(\beta,m),x}).\end{aligned} \tag{3.12}$$

Here, we have also used the following analog of (2.19):

$$\mathrm{Int}(\beta,f)=\mathrm{Int}(\beta,[\pi]^k*f),\quad f\in\mathcal{H},\ k\in\mathbb{Z}. \tag{3.13}$$

Namely, $[\pi]^k*R(f)$ is defined by composing $R(f)$ with the graph of the automorphism $\pi^{-k}:M\to M$, $(X,\rho)\mapsto(X,\pi^{-k}\rho)$, and this automorphism maps $Z(\beta_1)$ to itself.

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

3.4 The AFL

Recall that $E_0 = F \times F$, and that $E_3 \subset E_1 \otimes_F E_2$ is the fixed ring under $\sigma_1 \otimes \sigma_2$. Recall further that

$$\partial\mathrm{Orb}(\alpha, f) = \frac{1}{\log(q_F)} \frac{d}{ds}\bigg|_{s=0} O(\alpha, f, s) \tag{3.14}$$

denotes the normalized first derivative. By Corollary 2.18 and Lemma 3.2, the following conjecture concerns precisely those invariants δ with $\mathrm{ord}(\delta) = 1$.

Conjecture 3.10. Let $\alpha : (E_0, E_3) \to M_{2n}(F)$ be a regular semi-simple pair of embeddings that matches to a pair $\beta : (E_1, E_2) \to \mathrm{End}^0(\mathbb{X})$. Then, for any $f \in \mathcal{H}$, there is an equality

$$\partial\mathrm{Orb}(\alpha, f) = \mathrm{Int}(\beta, f).$$

Remark 3.11.

(1) In case $\mathbb{X} = \mathbb{X}^0$ is connected, this is the conjecture of Howard and the first author [HL20] (respectively, the first author [Li22b] if $E_1 = E_2$). Note that the original conjecture in [Li22b] is slightly different, however, because it does not take into account connected components. This is remedied with the definition in (3.12). The difference is not yet visible for $n = 1$ or $n = 2$ because in these cases $L^\times$ acts transitively on $\pi_0(I(\beta, f))$. It is, unfortunately, quite challenging to verify an explicit example for $n = 3$ that shows that $\mathrm{Int}(\beta, f)$ is the correct definition. We offer a slightly different identity in Example 3.14 that provides some evidence, however. We also remark that our definition of $\mathrm{Int}(\beta, f)$ is the one that comes naturally from a global intersection problem.

(2) The case $n = 1$ of Conjecture 3.10 is proved in [HL20, Li22b]. The case of $n = 2$ and $f = 1_{GL_4(O_F)}$ is shown in [Li22a] (for $E_1 = E_2$) and the forthcoming work [Li25] (for $E_1 \neq E_2$).

(3) Assume $E_1 = E_2 = E$ is unramified over F. Given any pair β, one may write $\beta_2 = \mu\beta_1\mu^{-1}$ for some $\mu \in D^\times$. Assume that $|m| \not\equiv v_D(\gamma)$ modulo 2. Then $R(m) \cap (Z(\beta_2) \times Z(\beta_1)) = \emptyset$. One can also show (omitted) that the orbital integral then vanishes identically, $O(\alpha, f, s) = 0$. In particular, Conjecture 3.10 holds in that case.

It is enough to know Conjecture 3.10 for a $\mathbb{C}$-basis of $\mathcal{H}$. Our main result to this end is the following analogue of the analytic reduction formula Theorem 2.12.

Theorem 3.12. *For every $m \in \mathbb{Z}^r_{\geq 0}$, the following identity holds:*

$$\begin{aligned}\mathrm{Int}(\beta, m) = {} & |\mathrm{Disc}_{E_1/F} \cdot \mathrm{Disc}_{E_2/F}|_F^{-\frac{n^0 \cdot n^1}{2}} |\mathrm{Res}(\mathrm{Inv}(\beta^0), \mathrm{Inv}(\beta^1))|_F^{-1} \\ & \cdot \sum_{m=m^0+m^1} q^{n^1 \cdot |m^0| + n^0 \cdot |m^1|} \mathrm{Int}(\beta^0, m^0) O(\beta^1, f(m^1)).\end{aligned} \tag{3.15}$$

Here, we have used the notation $|x|_F := |\mathrm{Nm}_{E_3/F}(x)|_F^{1/2}$ for elements $x \in E_3$.

Comparing this with (2.21), we obtain the following corollary.

Corollary 3.13. *Assume that the FL (Conjecture 2.10) holds. Then the AFL (Conjecture 3.10) holds in all cases if it holds in all basic cases, meaning in all cases where $\mathbb{X} = \mathbb{X}^0$ is connected.*

More precisely, the AFL then holds for some $\mathbb{X}$, some $f(m)$ and some β if it holds for β^0 and every $f(m^0)$ with $0 \leq m^0 \leq m$.

In particular, the AFL holds whenever $\mathrm{ht}(\mathbb{X}^0) = 2$. It holds for the unit function $f = 1_{GL_{2n}(O_F)}$ if $\mathrm{ht}(\mathbb{X}^0) = 4$.

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Proof. We show the more precise, second statement. Assume that $\mathbb{X} = \mathbb{X}^0 \times \mathbb{X}^1$ is of height $2n = 2n^0 + 2n^1$. Consider a pair $\beta = (\beta^0, \beta^1) : (E_1, E_2) \to D_{2n^0} \times M_{2n^1}(F)$, and some $m \in \mathbb{Z}^r_{\geq 0}$. Let $\alpha^0 : (E_0, E_3) \to M_{2n^0}(F)$ match β^0 and let $\alpha^1 : (E_0, E_3) \to M_{2n^1}(F)$ match β^1. In particular, their direct sum

$$\alpha = (\alpha^0, \alpha^1) : (E_0, E_3) \to M_{2n}(F)$$

matches β. Set $\delta^j := \mathrm{Inv}(\alpha^j) = \mathrm{Inv}(\beta^j)$ in the following. In addition, define

$$\Theta = |\mathrm{Disc}_{E_1/F} \cdot \mathrm{Disc}_{E_2/F}|_F^{-\frac{n^0 \cdot n^1}{2}} |\mathrm{Res}(\delta^0, \delta^1)|_F^{-1}$$

and note that this expression equals

$$|\mathrm{Disc}_{E_3/F}|_F^{-\frac{n^0 \cdot n^1}{2}} |\mathrm{Res}(\delta^0, \delta^1)|_F^{-1}.$$

Namely, taking E_1 to be unramified, $\mathrm{Disc}_{E_1/F}$ is trivial and

$$\mathrm{Disc}_{E_2/F} = \mathrm{Disc}_{E_1 \otimes_F E_2/E_1} = \mathrm{Disc}_{E_1 \otimes_F E_3/E_1} = \mathrm{Disc}_{E_3/F},$$

where the outer identities follow from the unramifiedness of E_1/F and where the middle one follows from $E_1 \otimes_F E_3 \cong E_1 \otimes_F E_2$. Now we apply Theorem 2.12. Substituting the concrete expression for the partial Satake transformation $\mathcal{S}(f(m))$ from (2.20), taking the derivative at $s = 0$, and applying the vanishing of $O(\alpha^0, f(m^0))$ for every $m^0 \in \mathbb{Z}^r_{\geq 0}$ (see Proposition 2.14), this theorem states that

$$\partial\mathrm{Orb}(\alpha, f(m)) = \Theta \cdot \sum_{m = m^0 + m^1} q^{n^1 \cdot |m^0| + n^0 \cdot |m^1|} \, \partial\mathrm{Orb}(\alpha^0, f(m^0)) O(\alpha^1, f(m^1)). \tag{3.16}$$

On the other hand, Theorem 3.12 states that

$$\mathrm{Int}(\beta, m) = \Theta \cdot \sum_{m = m^0 + m^1} q^{n^1 \cdot |m^0| + n^0 \cdot |m^1|} \, \mathrm{Int}(\beta^0, m^0) O(\beta^1, f(m^1)). \tag{3.17}$$

The FL (Conjecture 2.10), which we assumed to hold, states that $O(\alpha^1, f(m^1))$ equals $O(\beta^1, f(m^1))$ for all m^1. Furthermore, our assumption also is that $\partial\mathrm{Orb}(\alpha^0, f(m^0)) = \mathrm{Int}(\beta^0, m^0)$ for all occurring m^0. Comparing (3.16) and (3.17) then shows that $\partial\mathrm{Orb}(\alpha, f(m)) = \mathrm{Int}(\beta, m)$ as was to be shown. The last statement of the corollary follows from the known cases of the AFL in Remark 3.11. □

The proof of Theorem 3.12 is given in the next section. Here, we conclude with the promised example on connected component counts.

Example 3.14. Assume that $\mathbb{X} = \mathbb{X}^0$ is connected and that $E = E_1 = E_2$ is unramified over F. Let L/F be an unramified field extension of degree n and choose an embedding $L \to D = \mathrm{End}^0(\mathbb{X})$. Write $B = \mathrm{Cent}_D(L)$ for its centralizer and let $\beta : (E, E) \to B$ be a pair of embeddings of F-algebras whose image generates B over F. This situation can only exist if n is odd because $E \otimes_F L$ then embeds into B by Proposition 2.5(2) and, hence, has to be a field. Denote by $Z = Z(O_L, \beta) \subseteq M$ the closed formal subscheme of (X, ρ) such that

$$\rho^{-1} O_L \rho, \quad \rho^{-1} \beta_1(O_E) \rho, \quad \rho^{-1} \beta_2(O_E) \rho \text{ all } \subseteq \mathrm{End}(X).$$

Let $\Gamma = \pi^{\mathbb{Z}}$. (Note that π is also a uniformizer for L.) Then $\Gamma \backslash Z(O_L, \beta)$ has $2n$ connected components that correspond to the $2n$ possible characters of $O_{\beta_1(E)L}$, acting on the Lie algebra. Imposing compatible Kottwitz conditions for the two O_E-actions leaves us with only n connected components. Each is isomorphic to the intersection $\Gamma \backslash I(\beta \otimes \mathrm{id}_L, 1_{GL_2(O_L)})$ that occurs in the formulation of the linear AFL for $GL_{2,L}$. Thus,

$$\ell_{O_E}(\mathcal{O}_{\Gamma \backslash Z}) = n \, \mathrm{Int}(\beta \otimes \mathrm{id}_L, 1_{GL_2(O_L)}).$$

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Let $\alpha : (E_0, E_3) \to M_{2n}(F)$ match β. The centralizer of α is isomorphic to L. In (2.13), we may sum only over the sets

$$\mathcal{L}(O_L[\alpha_0]) \subseteq \mathcal{L}(\alpha_0), \quad \mathcal{L}(O_L[\alpha_3]) \subseteq \mathcal{L}(\alpha_3)$$

of lattices Λ that are also O_L-stable to obtain a rational function in q^s,

$$\begin{aligned} &O(O_L[\alpha], 1_{GL_{2n}(O_F)}, s) \\ &\quad = \sum_{(\Lambda_0,\Lambda_3)\in L^\times\backslash(\mathcal{L}(O_L[\alpha_0])\times\mathcal{L}(O_L[\alpha_3]))} \mathrm{Vol}(\mathrm{Stab}_{L^\times}(\Lambda_0,\Lambda_3))^{-1}\delta(\Lambda_0,\Lambda_3)\Omega(\Lambda_0,\Lambda_3,s), \end{aligned} \tag{3.18}$$

with $\delta(\Lambda_0, \Lambda_3) = 1$ if $\Lambda_0 = \Lambda_3$ and 0 otherwise. Then

$$\begin{aligned} O(O_L[\alpha], 1_{GL_{2n}(O_F)}, s) &= O(\alpha \otimes \mathrm{id}_L, 1_{GL_2(O_L)}, s), \\ \partial\mathrm{Orb}(O_L[\alpha], 1_{GL_{2n}(O_F)}) &= n\,\partial\mathrm{Orb}(\alpha \otimes \mathrm{id}_L, 1_{GL_2(O_L)}), \end{aligned} \tag{3.19}$$

where the factor n stems from the difference in normalization by q_F or $q_L = q_F^n$. In this way, we have recovered the AFL conjecture over L for $n = 2$ and the unit Hecke function, but only with the right convention for connected component counting.

4. Reduction to elliptic invariant

The purpose of this section is to prove the reduction formulas Theorems 2.12 and 3.12. The next three sections reduce this to a degree computation that is accomplished in § 4.4.

4.1 Fibration of lattice sets

Let $V = V^0 \oplus V^1$ be a direct sum decomposition of $V = F^{2n}$ with $V^0 \cong F^{2n^0}$ and $V^1 \cong F^{2n^1}$. Denote by $p^j : V \to V^j$ the two projection maps. The fundamental observation is that there is a bijection of O_F-lattices $X \subseteq V$ and triples

$$\left\{ (X^0, X^1, s) \;\middle|\; \begin{array}{l} X^0 \subset V^0,\ X^1 \subset V^1 \text{ both } O_F\text{-lattices}, \\ s : X^1 \to V^0/X^0 \text{ any } O_F\text{-linear map} \end{array} \right\}.$$

It is given by

$$X \longmapsto (X^0 := X \cap V^0,\ X^1 := p^1(X),\ s_X := [X^1 \xrightarrow{t} X \xrightarrow{p^0} V^0/X^0]), \tag{4.1}$$

where $t : X^1 \to X$ is any choice of splitting for p^1. The inverse is given by

$$(X^0, X^1, s) \longmapsto X^0 + \{(\widetilde{s}(x^1), x^1),\ x^1 \in X^1\},$$

where $\widetilde{s} : X^1 \to V^0$ is any lift of s. The bijection moreover satisfies the following two functoriality properties.

(1) Assume $X \subseteq Y$. Then clearly $X^0 \subseteq Y^0$ and $X^1 \subseteq Y^1$. We claim that also the following diagram commutes.

$$\begin{array}{ccc} X^1 & \longrightarrow & Y^1 \\ {\scriptstyle s_X}\downarrow & & \downarrow{\scriptstyle s_Y} \\ V^0/X^0 & \longrightarrow & V^0/Y^0 \end{array} \tag{4.2}$$

Namely $(\widetilde{s}_X(x^1), x^1) \in X$ lying in Y means that there is $y^0 \in Y^0$ such that $\widetilde{s}_X(x^1) = \widetilde{s}_Y(x^1) + y_0$. Conversely, if for two lattices X and Y one has $X^j \subseteq Y^j$ and also that (4.2) commutes, then $X \subseteq Y$.

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

(2) Let $\zeta = (\zeta^0, \zeta^1) \in \mathrm{End}(V^0) \times \mathrm{End}(V^1) \subset \mathrm{End}(V)$ be an endomorphism. Then X is ζ-stable if and only if each X^j is ζ^j-stable and $\zeta^0 \circ s = s \circ \zeta^1$.

4.2 Application to orbital integrals

Consider a regular semi-simple pair $\alpha : (E_0, E_3) \to M_{2n}(F)$. Let B be the quaternion algebra generated by the image of α, denote by $L = \mathrm{Cent}(B)$ its center. Assume that $L = L^0 \times L^1$ is a product with eigenspace decomposition $V = V^0 \oplus V^1$. Given a tuple $m = (m_1, \dots, m_r) \in \mathbb{Z}^r_{\geq 0}$, define

$$\mathcal{L}(\alpha, m) = \left\{ (X_0, X_1, \dots, X_r) \;\middle|\; \begin{array}{l} X_0 \subset V \text{ an } O_{E_3}\text{-lattice, } X_r \subset V \text{ an } O_{E_0}\text{-lattice,} \\ X_i \subset V \text{ an } O_F\text{-lattice for } i = 1, \dots, r-1, \\ X_{i-1} \subseteq X_i \text{ of index } m_i \end{array} \right\}. \tag{4.3}$$

Note that this definition matches (2.17). The above discussion provides a map

$$\begin{aligned} \mathcal{L}(\alpha, m) &\longrightarrow \coprod_{m^0, m^1 \in \mathbb{Z}^r_{\geq 0},\ m = m^0 + m^1} \mathcal{L}(\alpha^0, m^0) \times \mathcal{L}(\alpha^1, m^1) \\ (X_0, \dots, X_r) &\longmapsto ((X^0_0, \dots, X^0_r), (X^1_0, \dots, X^1_r)). \end{aligned} \tag{4.4}$$

We use the shorter notation $X_\bullet$, $X^0_\bullet$ and $X^1_\bullet$ in the following. Let $\Gamma^j \subset L^{j,\times}$ be a subgroup with $\Gamma^j \times \mathcal{O}^\times_{L^j} \cong L^{j,\times}$ and put $\Gamma = \Gamma^0 \times \Gamma^1 \subseteq L^\times$. Then (4.4) is Γ-equivariant. Using (2.13) and the description of $f(m)$ in (2.17), we obtain

$$O(\alpha, [\pi]^k * f(m), s) = \sum_{X_\bullet \in \Gamma \backslash \mathcal{L}(\alpha, m)} \Omega(X_r, X_0, s). \tag{4.5}$$

The same results give

$$\begin{aligned} &O(\alpha^0, [\pi]^k * f(m^0), s) O(\alpha^1, [\pi]^k * f(m^1), s) \\ &\quad = \sum_{(X^0_\bullet, X^1_\bullet) \in \Gamma^0 \backslash \mathcal{L}(\alpha^0, m^0) \times \Gamma^1 \backslash \mathcal{L}(\alpha^1, m^1)} \Omega(X^0_r, X^0_0, s) \Omega(X^1_r, X^1_0, s). \end{aligned} \tag{4.6}$$

Observe now that the transfer factor from (2.9) is multiplicative: if $X_\bullet \mapsto (X^0_\bullet, X^1_\bullet)$ in (4.4), then

$$\Omega(X_r, X_0, s) = \Omega(X^0_r, X^0_0, s)\, \Omega(X^1_r, X^1_0, s). \tag{4.7}$$

By the explicit description of the partial Satake transform (2.20), the reduction formula Theorem 2.12 (for E_0 and E_3) is now reduced to the following proposition.

Proposition 4.1. *Each fiber of (4.4) over $\mathcal{L}(\alpha^0, m^0) \times \mathcal{L}(\alpha^1, m^1)$ has*

$$|\mathrm{Disc}_{E_3/F}|_F^{-\frac{n^0 \cdot n^1}{2}} |\mathrm{Res}\,(\mathrm{Inv}(\alpha^0), \mathrm{Inv}(\alpha^1))|_F^{-1} \cdot q^{n^1 |m^0| + n^0 \cdot |m^1|} \tag{4.8}$$

many elements. In other words, given any pair $(X^0_\bullet, X^1_\bullet) \in \mathcal{L}(\alpha^0, m^0) \times \mathcal{L}(\alpha^1, m^1)$, there are (4.8) many tuples $(\gamma_0, \dots, \gamma_r)$ of O_F-linear maps $\gamma_i : X^1_i \to V^0/X^0_i$ that fit into a commutative diagram

$$\begin{array}{ccccccc} X^1_0 & \longrightarrow & X^1_1 & \longrightarrow & \cdots & \longrightarrow & X^1_r \\ \downarrow{\scriptstyle \gamma_0} & & \downarrow{\scriptstyle \gamma_1} & & & & \downarrow{\scriptstyle \gamma_r} \\ V^0/X^0_0 & \longrightarrow & V^0/X^0_1 & \longrightarrow & \cdots & \longrightarrow & V^0/X^0_r \end{array} \tag{4.9}$$

and such that γ_0 is O_{E_3}-linear, and such that γ_r is O_{E_0}-linear.

Given a regular semi-simple pair $\beta : (E_1, E_2) \to M_{2n}(F)$ there is a completely analogous definition of a lattice set $\mathcal{L}(\beta, m)$. If β factors through $M_{2n^0}(F) \times M_{2n^1}(F)$, all the above considerations

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

apply to provide a fibration

$$\mathcal{L}(\beta, m) \longrightarrow \coprod_{m=m^0+m^1} \mathcal{L}(\beta^0, m^0) \times \mathcal{L}(\beta^1, m^1). \tag{4.10}$$

In this way, Theorem 2.12 (for E_1 and E_2) reduces to the following statement.

Proposition 4.2. *Each fiber of* (4.10) *over* $\mathcal{L}(\beta^0, m^0) \times \mathcal{L}(\beta^1, m^1)$ *has*

$$|\mathrm{Disc}_{E_1/F} \cdot \mathrm{Disc}_{E_2/F}|_F^{-\frac{n^0 \cdot n^1}{2}} |\operatorname{Res}(\mathrm{Inv}(\beta^0), \mathrm{Inv}(\beta^1))|_F^{-1} \cdot q^{n^1|m^0|+n^0\cdot|m^1|} \tag{4.11}$$

many elements. In other words, given any pair $(X_\bullet^0, X_\bullet^1) \in \mathcal{L}(\beta^0, m^0) \times \mathcal{L}(\beta^1, m^1)$, *there are* (4.11) *many tuples* $(\gamma_0, \dots, \gamma_r)$ *of* O_F*-linear maps* $\gamma_i : X_i^1 \to V^0/X_i^0$ *that fit into a commutative diagram*

$$\begin{array}{ccccccc} X_0^1 & \longrightarrow & X_1^1 & \longrightarrow & \cdots & \longrightarrow & X_r^1 \\ \downarrow{\scriptstyle \gamma_0} & & \downarrow{\scriptstyle \gamma_1} & & & & \downarrow{\scriptstyle \gamma_r} \\ V^0/X_0^0 & \longrightarrow & V^0/X_1^0 & \longrightarrow & \cdots & \longrightarrow & V^0/X_r^0 \end{array} \tag{4.12}$$

and such that γ_0 *is* O_{E_2}*-linear, and such that* γ_r *is* O_{E_1}*-linear.*

4.3 Fibration of RZ spaces

Our aim is to adapt the previous arguments to strict π-divisible O_F-modules. We begin with some general observations about the connected-étale sequence. Let S be a scheme such that $\pi \in \mathcal{O}_S$ is locally nilpotent. Assume that X/S has fiberwise constant étale rank. Then, by [Mes72, Proposition II.4.9] and [HS19, Proposition 10.16], respectively, it is an extension

$$0 \longrightarrow X^0 \longrightarrow X \longrightarrow X^1 \longrightarrow 0$$

of its maximal étale quotient by its identity component. We view π-divisible O_F-modules as sheaves for the fpqc-topology in the following. Then the maximal étale quotient X^1 has the canonical presentation

$$0 \longrightarrow T \longrightarrow V \longrightarrow X^1 \longrightarrow 0, \tag{4.13}$$

where

$$T := TX^1 := \varprojlim_{n \geq 0} X^1[\pi^n] \quad \text{and} \quad V := VX^1 := TX^1[\pi^{-1}]$$

denote the Tate module and rational Tate module of X^1, respectively. The natural map $V \to X^1$ in (4.13) is defined by

$$\pi^{-n}(\dots, x_2, x_1, x_0) \longmapsto x_n, \quad \text{where } (\dots, x_2, x_1, x_0) \in TX^1.$$

Applying $\mathrm{Hom}(-, X^0)$ and taking into account that multiplication by π is an automorphism of V while $\mathrm{Hom}(T, X^0)$ is torsion, we obtain an exact sequence

$$0 \longrightarrow \mathrm{Hom}(T, X^0) \longrightarrow \mathrm{Ext}^1(X^1, X^0) \longrightarrow \mathrm{Ext}^1(V, X^0). \tag{4.14}$$

The leftmost term here is torsion, the rightmost term torsion-free, so

$$\mathrm{Hom}(T, X^0) = \mathrm{Ext}^1(X^1, X^0)_{\mathrm{torsion}}.$$

By definition, an extension

$$0 \longrightarrow X^0 \longrightarrow X \longrightarrow X^1 \longrightarrow 0$$

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

here is torsion if and only if there exists an integer $N \geq 1$ and the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^0 & \longrightarrow & X^0 \oplus X^1 & \longrightarrow & X^1 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \pi^N} & & \\ 0 & \longrightarrow & X^0 & \longrightarrow & X & \longrightarrow & X^1 & \longrightarrow & 0 \end{array} \tag{4.15}$$

Proposition 4.3.

(1) *An extension $0 \to X^0 \to X \to X^1 \to 0$ is torsion if and only if there exists an isogeny $\varphi : X \to X^0 \oplus X^1$.*

(2) *Assume $\varphi^0 : X^0 \to Y^0$ and $\varphi^1 : X^1 \to Y^1$ are two homomorphisms with Y^1 étale. Assume furthermore that we are given two torsion extensions*

$$0 \longrightarrow X^0 \longrightarrow X \longrightarrow X^1 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow Y^0 \longrightarrow Y \longrightarrow Y^1 \longrightarrow 0.$$

Then there exists a (necessarily unique) map $\varphi : X \to Y$ that restricts to φ_0 and extends φ_1 if and only if the following diagram commutes.

$$\begin{array}{ccc} TX^1 & \xrightarrow{T\varphi^1} & TY^1 \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle Y} \\ X^0 & \xrightarrow{\varphi^0} & Y^0 \end{array}$$

Proof.

(1) The two π-divisible O_F-modules in the middle column of (4.15) have the same height and the kernel of the middle vertical arrow of (4.15) identifies with $X^1[\pi^N]$, is hence finite locally free. This shows the 'only if' assertion.
For the converse, we observe that any homomorphism $\varphi : X \to X^0 \oplus X^1$ provides two homomorphisms $\varphi^0 = \varphi|_{X^0} : X^0 \to X^0$ and $\varphi^1 = (\varphi \mod X^0) : X^1 \to X^1$. If $\ker(\varphi)$ is finite locally free, then also the intersection $\ker(\varphi^0) = \ker(\varphi) \cap X^0$ has this property because it is open and closed in $\ker(\varphi)$. Thus, φ^0 is an isogeny. Then $\ker(\varphi^1) = \ker(\varphi)/\ker(\varphi^0)$ is also finite locally free, so φ^1 is an isogeny. Let ψ^0 and ψ^1 be such that $\varphi^1\psi^1 = \pi^N$ and $\psi^0\varphi^0 = \pi^N$ for some $N \geq 0$. Then the class of X in Ext^1 is annihilated by π^{2N}.

(2) The connecting homomorphism in (4.14) has the description that X is constructed from a map $s_X : TX^1 \to X^0$ as $X = (X^0 \oplus VX^1)/TX^1$. The claim is then obvious. □

Apply these considerations now to (X, ρ), the universal pair over M. Let

$$0 \longrightarrow X^0 \longrightarrow X \longrightarrow X^1 \longrightarrow 0 \tag{4.16}$$

be its connected-étale sequence and let $S \to M$ be a point. Both X^0 and X^1 inherit a strict O_F-action and ρ provides a pair of quasi-isogenies

$$\rho^0 : \overline{S} \times_S X^0 \longrightarrow \overline{S} \times_{\mathrm{Spec}\,\mathbb{F}} \mathbb{X}^0, \quad \rho^1 : \overline{S} \times_S X^1 \longrightarrow \overline{S} \times_{\mathrm{Spec}\,\mathbb{F}} \mathbb{X}^1 \tag{4.17}$$

over the special fiber $\overline{S} = \mathbb{F} \otimes_{O_{\breve{E}}} S$. It follows that $\overline{S} \times_S X$ is isogeneous to $\overline{S} \times_S (X^0 \times X^1)$ and hence X is isogeneous to $X^0 \times X^1$. The extension class $S \times_M X$ is thus torsion by Proposition 4.3(1) and comes from a map $TX^1 \to X^0$.

Let $M_{\mathbb{X}^0}$ and $M_{\mathbb{X}^1}$ denote the RZ spaces for the two indicated strict π-divisible O_F-modules. Denote by (Y^0, ρ^0) and (Y^1, ρ^1) the universal points of $M_{\mathbb{X}^0}$ and $M_{\mathbb{X}^1}$, respectively. By abuse of notation, we continue to write (Y^j, ρ^j) in place of $p_j^*(Y^j, \rho^j)$ for their pullbacks to $M_{\mathbb{X}^0} \times M_{\mathbb{X}^1}$. The Hom-functor $\underline{\mathrm{Hom}}(TY^1, Y^0)$ is then itself a strict π-divisible O_F-module. (In fact, since $M_{\mathbb{X}^0} \times M_{\mathbb{X}^1}$ is a union of formal spectra of strict complete local rings, it is isomorphic to $(Y^0)^{2n^1}$.)

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Thus, we obtain a map

$$\begin{aligned} M_{\mathbb{X}} &\longrightarrow \underline{\mathrm{Hom}}(TY^1, Y^0) \\ (X,\rho) &\longmapsto ((X^0,\rho^0),\ (X^1,\rho^1),\ \text{extension class of } X). \end{aligned} \tag{4.18}$$

Here, we view $\underline{\mathrm{Hom}}(TY^1, Y^0)$ as a sheaf on $M_{\mathbb{X}^0} \times M_{\mathbb{X}^1}$ and (X^j, ρ^j) denotes the image point in $M_{\mathbb{X}^j}$. Proposition 4.3 proves the following proposition.

Proposition 4.4. *The map* (4.18) *is an isomorphism. It moreover has the following functoriality property. Given* $\zeta = (\zeta^0, \zeta^1) \in \mathrm{End}^0(\mathbb{X}^0) \times \mathrm{End}^0(\mathbb{X}^1)$, *the closed formal subscheme* $\mathcal{Z}(\zeta) \subseteq M_{\mathbb{X}}$ *where* $\rho^{-1}\zeta\rho$ *is a homomorphism agrees with the subfunctor of* ζ*-linear homomorphisms in*

$$(\mathcal{Z}(\zeta^0) \times \mathcal{Z}(\zeta^1)) \times_{(M_{\mathbb{X}^0} \times M_{\mathbb{X}^1})} \underline{\mathrm{Hom}}(TY^1, Y^0). \tag{4.19}$$

Here, an S*-valued point of* (4.19) *is called* ζ*-linear if*

$$(\rho^0)^{-1}\zeta^0\rho^0 \circ h = h \circ T((\rho^1)^{-1}\zeta^1\rho^1).$$

Consider now a regular semi-simple pair $\beta : (E_1, E_2) \to \mathrm{End}^0(\mathbb{X})$, write $\beta^j : (E_1, E_2) \to \mathrm{End}^0(\mathbb{X}^j)$, $j = 0, 1$ for its components. Then we have the three intersections $I(\beta, m)$, $I(\beta^0, m^0)$ and $I(\beta^1, m^1)$ that result from (3.8) applied to $\mathbb{X}$, $\mathbb{X}^0$ and $\mathbb{X}^1$, respectively. The above considerations provide a map

$$\begin{aligned} \Pi : I(\beta, m) &\longrightarrow \coprod_{m=m^0+m^1} I(\beta^0, m^0) \times I(\beta^1, m^1) \\ (X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_r} X_r, \rho) &\longmapsto ((X_0^0 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_r} X_r^0, \rho^0),\ (X_0^1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_r} X_r^1, \rho^1)). \end{aligned} \tag{4.20}$$

Proposition 4.5. *The map* (4.20) *is finite locally free. Its degree over* $I(\beta^0, m^0) \times I(\beta^1, m^1)$ *equals*

$$|\mathrm{Disc}_{E_1/F} \cdot \mathrm{Disc}_{E_2/F}|_F^{-\frac{n^0 \cdot n^1}{2}} |\operatorname{Res}(\mathrm{Inv}(\beta^0), \mathrm{Inv}(\beta^1))|_F^{-1} \cdot q^{n^1 \cdot |m^0| + n^0 \cdot |m^1|}. \tag{4.21}$$

The proof is given in the next section. Here we note the following description of $I(\beta, m)$, which is provided by Proposition 4.3 and Proposition 4.4. Namely

$$I(\beta, m) \subset \prod_{i=0}^{r} \underline{\mathrm{Hom}}_{O_F}(TY_i^1, Y_i^0)|_{\coprod_{m=m^0+m^1} I(\beta^0,m^0)\times I(\beta^1,m^1)}$$

is the subfunctor of those tuples $(\gamma_i : TY_i^1 \to Y_i^0)_{i=0}^r$ such that the diagram

$$\begin{array}{ccccccc} TY_0^1 & \xrightarrow{T\varphi_1^1} & TY_1^1 & \xrightarrow{T\varphi_2^1} & \cdots & \xrightarrow{T\varphi_r^1} & TY_r^1 \\ \downarrow{\scriptstyle \gamma_0} & & \downarrow{\scriptstyle \gamma_1} & & & & \downarrow{\scriptstyle \gamma_r} \\ Y_0^0 & \xrightarrow{\varphi_1^0} & Y_1^0 & \xrightarrow{\varphi_2^0} & \cdots & \xrightarrow{\varphi_r^0} & Y_r^0 \end{array} \tag{4.22}$$

commutes, such that γ_0 is O_{E_2}-linear, and such that γ_r is O_{E_1}-linear.

4.4 Degree computation

In this section we prove Propositions 4.1, 4.2 and 4.5 in parallel, the arguments being the same for all three cases. Our main interest lies in the RZ space intersection, so we use the terminology of π-divisible O_F-modules. Fix a pair $\beta : (E_1, E_2) \to \mathrm{End}^0(\mathbb{X})$, a tuple $m \in \mathbb{Z}_{\geq 0}^r$ and a decomposition $m = m^0 + m^1$. Let $Y_\bullet^1 \in I(\beta^1, m^1)(S)$ and $Y_\bullet^0 \in I(\beta^0, m^0)(S)$ be two fixed S-valued points where S is local artinian with residue field $\mathbb{F}$. Our aim is to compute the fiber

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

over $(Y_\bullet^0, Y_\bullet^1)$ of

$$I(\beta, m) \longrightarrow \coprod_{m=m^0+m^1} I(\beta^0, m^0) \times I(\beta^1, m^1). \tag{4.23}$$

The group $Y_\bullet^1$ only comes in via its Tate module, so we introduce the notation $T_\bullet := TY_\bullet^1$ and put $Y_\bullet = Y_\bullet^0$.

For the proofs in the analytic settings, one considers instead a pair $\alpha : (E_0, E_3) \to M_{2n}(F)$ (respectively, $\beta : (E_1, E_2) \to M_{2n}(F)$) that preserves a decomposition $V = V^0 \oplus V^1$. Fix lattice chains $\Lambda_\bullet \in \mathcal{L}(\alpha^0, m^0)$ and $T_\bullet \in \mathcal{L}(\alpha^1, m^1)$, (respectively, $\Lambda_\bullet \in \mathcal{L}(\beta^0, m^0)$ and $T_\bullet \in \mathcal{L}(\beta^1, m^1)$). Taking $S = \operatorname{Spec} \mathbb{F}$, the datum $\Lambda_\bullet$ is equivalent to that of a chain $Y_\bullet := \Lambda_\bullet \otimes_{O_F} (F/O_F)$ of étale π-divisible O_F-modules of type (α^0, m^0) (respectively, (β^0, m^0)). Similarly, $T_\bullet$ is the chain of Tate modules of $T_\bullet \otimes_{O_F} (F/O_F)$. In this way, all arguments about π-divisible O_F-modules in the following apply literally and provide the proofs of Propositions 4.1 and 4.2. We only formulate the case of (E_1, E_2), however, leaving it to the reader to substitute for (E_0, E_3).

Consider the following π-divisible O_F-modules over S,

$$\begin{aligned} H_i &= \operatorname{Hom}_{O_F}(T_i, Y_i), \quad i = 0, \dots, r, \\ C_i &= \operatorname{Hom}_{O_F}(T_{i-1}, Y_i), \quad i = 1, \dots, r, \end{aligned} \tag{4.24}$$

as well as

$$\begin{aligned} H_0^+ &= \operatorname{Hom}_{O_{E_2}}(T_0, Y_0), \quad H_0^- = \operatorname{Hom}_{O_{E_2}\text{-conj}}(T_0, Y_0), \\ H_r^+ &= \operatorname{Hom}_{O_{E_1}}(T_r, Y_r), \quad H_r^- = \operatorname{Hom}_{O_{E_1}\text{-conj}}(T_r, Y_r). \end{aligned} \tag{4.25}$$

Here, we used the notation

$$\operatorname{Hom}_{O_{E_2}\text{-conj}}(T_0, Y_0) = \{ f : T_0 \to Y_0 \mid f(at) = a^{\sigma_2} f(t) \ \forall \ a \in O_{E_2} \}$$

in the first line and an analogous notation in the second. We now define a homomorphism

$$\Phi = \begin{pmatrix} p_2^- & & & \\ -R_1 & L_1 & & \\ & \ddots & \ddots & \\ & & -R_r & L_r \\ & & & p_1^- \end{pmatrix} : \bigoplus_{i=0}^{r} H_i \longrightarrow H_0^- \oplus \bigoplus_{i=1}^{r} C_i \oplus H_r^- \tag{4.26}$$

such that $\ker(\Phi)$ precisely describes the tuples $(\gamma_0, \dots, \gamma_r)$ in (4.9), (4.12) and (4.22). The components R_i and L_i are the composition to the right and left in each square,

$$\begin{aligned} R_i : H_{i-1} &\longrightarrow C_i, \quad \gamma \longmapsto [Y_{i-1} \to Y_i] \circ \gamma, \quad i = 1, \dots, r \\ L_i : H_i &\longrightarrow C_i, \quad \gamma \longmapsto \gamma \circ [T_{i-1} \to T_i], \quad i = 1, \dots, r. \end{aligned} \tag{4.27}$$

The condition $L_i(\gamma_i) - R_i(\gamma_{i-1}) = 0$ precisely expresses commutativity of the following square.

$$\begin{array}{ccc} T_{i-1} & \longrightarrow & T_i \\ {\scriptstyle \gamma_{i-1}}\downarrow & & \downarrow{\scriptstyle \gamma_i} \\ Y_{i-1} & \longrightarrow & Y_i \end{array}$$

The kernels of $p_2^- : H_0 \to H_0^-$ and $p_1^- : H_r \to H_r^-$, on the other hand, should precisely be the O_{E_2}-linear (respectively, O_{E_1}-linear) elements. Pick elements ζ_i such that $O_{E_i} = O_F[\zeta_i]$. Define

$$\begin{aligned} p_2^- : H_0 &\longrightarrow H_0^-, \quad f \longmapsto [t \mapsto f(\zeta_2 t) - \zeta_2 f(t)], \\ p_1^- : H_r &\longrightarrow H_r^-, \quad f \longmapsto [t \mapsto f(\zeta_1 t) - \zeta_1 f(t)]. \end{aligned} \tag{4.28}$$

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

It is easily checked that p_2^- and p_1^- have image in H_0^- and H_r^-, respectively, as claimed. For example,

$$\begin{aligned} p_1^-(f)(\zeta_1 t) &= f(\zeta_1^2 t) - \zeta_1 f(\zeta_1 t) \\ &= (\zeta_1 + \zeta_1^{\sigma_1}) f(\zeta_1 t) - \zeta_1 \zeta_1^{\sigma_1} f(t) - \zeta_1 f(\zeta_1 t) \\ &= \zeta_1^{\sigma_1}[f(\zeta_1 t) - \zeta_1 f(t)] \\ &= \zeta_1^{\sigma_1} p_1^-(f)(t). \end{aligned}$$

This concludes the definition of (4.26).

Lemma 4.6. *Let E/F be an étale quadratic extension and T an O_E-lattice. Let Y be a π-divisible group with O_E-action, write $O_E = O_F[\zeta]$. Then $\mathrm{Hom}_{O_E}(T,Y)$ is again a π-divisible O_F-module and equals the kernel of*

$$p : \mathrm{Hom}_{O_F}(T,Y) \longrightarrow \mathrm{Hom}_{O_E\text{-}conj}(T,Y),\ \ f \longmapsto [t \mapsto f(\zeta t) - \zeta f(t)].$$

In particular, the following sequence is exact,

$$0 \longrightarrow \mathrm{Hom}_{O_E}(T,Y) \xrightarrow{\iota} \mathrm{Hom}_{O_F}(T,Y) \xrightarrow{p} \mathrm{Hom}_{O_E\text{-}conj}(T,Y) \longrightarrow 0. \tag{4.29}$$

Proof. Choosing a basis of T, it suffices to consider $T = O_E$. Then both $\mathrm{Hom}_{O_E}(O_E,Y)$ and $\mathrm{Hom}_{O_E\text{-conj}}(O_E,Y)$ are isomorphic to Y by the map $f \mapsto f(1)$. The choice of ζ also allows us to identify

$$\mathrm{Hom}_{O_F}(O_E,Y) \cong Y^2,\ \ f \longmapsto (f(1), f(\zeta)).$$

In these coordinates, p is given by $p(y_1,y_2) = y_2 - \zeta y_1$. Its kernel is isomorphic to Y via $(y_1, \zeta y_1) \mapsto y_1$ and, hence, a π-divisible O_F-module as claimed. Then the cokernel of ι is a π-divisible O_F-module and has the same height as $\mathrm{Hom}_{O_E\text{-conj}}(T,Y)$, so maps isomorphically onto it. □

In particular, $\ker(\Phi)$ precisely describes the fiber of (4.23). The maps R_i and L_i are isogenies, so we have quasi-isogenies $L_i^{-1}R_i : H_{i-1} \dashrightarrow H_i$ that allow to define a quasi-homomorphism

$$\widetilde{p_1^-} := p_1^- L_r^{-1} R_r L_{r-1}^{-1} R_{r-1} \cdots L_1^{-1} R_1 : H_0 \dashrightarrow H_r^-.$$

Lemma 4.7. *The homomorphism Φ is an isogeny if and only if the quasi-homomorphism*

$$\begin{pmatrix} p_2^- \\ \widetilde{p_1^-} \end{pmatrix} : H_0 \dashrightarrow H_0^- \oplus H_r^-$$

is a quasi-isogeny. If both are quasi-isogenies, then their degrees are related by

$$\deg(\Phi) = \deg \begin{pmatrix} p_2^- \\ \widetilde{p_1^-} \end{pmatrix} \cdot \prod_{i=1}^{r} \deg(L_i). \tag{4.30}$$

Proof. We may multiply Φ from the left by a quasi-isogeny of degree 1 in the following way:

$$\begin{pmatrix} I_{\frac{n}{2}} & & & & \\ & I_n & & & \\ & & \ddots & & \\ & & & -p_1^- L_r^{-1} & I_{\frac{n}{2}} \\ & & & I_n & \end{pmatrix} \begin{pmatrix} p_2^- & & & & \\ -R_1 & L_1 & & & \\ & \ddots & \ddots & & \\ & & & -R_r & L_r \\ & & & & p_1^- \end{pmatrix} = \begin{pmatrix} p_2^- & & & & \\ -R_1 & L_1 & & & \\ & \ddots & \ddots & & \\ & & R_{r-1} & L_{r-1} & \\ & & & p_1^- L_r^{-1} R_r & \\ & & & -R_r & L_r \end{pmatrix}.$$

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Iterating this operation leaves us with

$$\begin{pmatrix} p_1^- & & & \\ \widetilde{p_2^-} & & & \\ -R_1 & L_1 & & \\ & \ddots & \ddots & \\ & & -R_r & L_r \end{pmatrix}.$$

The degree of the diagonal here is the right-hand side of (4.30) and the claim follows. □

We now shift our attention to the framing object. The link with Lemma 4.7 will be given at the end of the section. Write

$$N = N(\mathbb{X}), \quad N^0 = N(\mathbb{X}^0), \quad N^1 = N(\mathbb{X}^1)$$

for the isocrystals of the respective O_F-modules. These are $\breve{F}$-vector spaces of dimensions $2n$, $2n^0$ and $2n^1$, respectively. In addition, put

$$P = N(\mathrm{Hom}_{O_F}(T\mathbb{X}^1, \mathbb{X}^0)) = \mathrm{Hom}(N^1, N^0)$$

which has $\breve{F}$-dimension $4n^0n^1$. Define the following subspaces of P, each of half that dimension,

$$\begin{aligned} P_1^+ &= \mathrm{Hom}_{E_1}(N^1, N^0), \quad P_1^- = \mathrm{Hom}_{E_1\text{-conj}}(N^1, N^0), \\ P_2^+ &= \mathrm{Hom}_{E_2}(N^1, N^0), \quad P_2^- = \mathrm{Hom}_{E_2\text{-conj}}(N^1, N^0). \end{aligned} \tag{4.31}$$

We also consider the projection maps to $P_i^\pm$, for $i = 1, 2$,

$$\begin{aligned} q_i^+ &: P \longrightarrow P_i^+, \quad f \longmapsto [t \mapsto f(\zeta_i t) - \zeta_i^{\sigma_i} f(t)], \\ q_i^- &: P \longrightarrow P_i^-, \quad f \longmapsto [t \mapsto f(\zeta_i t) - \zeta_i^{\sigma_i} f(t)]. \end{aligned} \tag{4.32}$$

The given points $(Y_\bullet^j, \rho_\bullet^j, \varphi_\bullet^j) \in I(\beta^j, m^j)(S)$ define lattices in the above $\breve{F}$-vector spaces. Let $M(-)$ denote the integral Dieudonné module functor over the special point of S. Set

$$\begin{aligned} M_1^0 &= N(\rho_r^0)(M(Y_r^0)), \quad M_2^0 = N(\rho_0^0)(M(Y_0^0)), \\ M_1^1 &= N(\rho_r^1)(M(Y_r^1)), \quad M_2^1 = N(\rho_0^1)(M(Y_0^1)). \end{aligned} \tag{4.33}$$

Then $M_1^0 \subset N^0$ and $M_1^1 \subset N^1$ are O_{E_1}-stable, whereas M_2^0 and M_2^1 are O_{E_2}-stable. We note for later use that there is an inclusion $M_2^0 \subseteq M_1^0$ of index $|m^0|$ because of the chain of isogenies

$$Y_0^0 \xrightarrow{\varphi_1^0} Y_1^0 \xrightarrow{\varphi_2^0} \cdots \xrightarrow{\varphi_r^0} Y_r^0$$

that satisfies $\rho_0^0 = \rho_r^0 \circ \varphi_r^0 \circ \cdots \circ \varphi_1^0$. In the same way, $M_2^1 \subseteq M_1^1$ with index $|m^1|$.

Passing to Hom-spaces, these lattices provide Dieudonné lattices in P and the $P_i^\pm$,

$$\begin{aligned} \Lambda_i &= \mathrm{Hom}_{O_F}(M_i^1, M_i^0), \\ \Lambda_i^+ &= \mathrm{Hom}_{O_{E_i}}(M_i^1, M_i^0), \\ \Lambda_i^- &= \mathrm{Hom}_{O_{E_i}\text{-conj}}(M_i^1, M_i^0). \end{aligned} \tag{4.34}$$

Just like in (4.29), the projections from (4.32) restrict to surjections $q_i^\pm : \Lambda_i \twoheadrightarrow \Lambda_i^\pm$. Let us adopt the following notion. Given two $O_{\breve{F}}$-lattices or O_F-lattices Λ and Λ', we write $f : \Lambda \dashrightarrow \Lambda'$ for homomorphisms $f : \Lambda[\pi^{-1}] \to \Lambda'[\pi^{-1}]$ together with the datum of the two lattices. We call such morphisms quasi-homomorphisms. We call a quasi-homomorphism f a quasi-isogeny if the map $f : \Lambda[\pi^{-1}] \to \Lambda'[\pi^{-1}]$ is bijective. In this case, we define the degree of f by $\deg(f) = q^{[\Lambda' : f(\Lambda)]}$. For example, Λ_1 and Λ_2 are both lattices in the $\breve{F}$-vector space P. The inclusion maps define a

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

quasi-isogeny $\Lambda_2 \dashrightarrow \Lambda_1$ with

$$\begin{aligned}\deg(\Lambda_2 \dashrightarrow \Lambda_1) &= q^{[\Lambda_1:\Lambda_2]} \\ &= q^{[M_1^0:M_2^0]\dim_{\breve{F}}(N^1)-[M_1^1:M_2^1]\dim_{\breve{F}}(N^0)} \\ &= q^{2n^1|m^0|-2n^0|m^1|}.\end{aligned} \tag{4.35}$$

Proposition 4.8. *Assume that β is regular semi-simple. Then the quasi-homomorphism*

$$(q_1^-, q_2^-) : \Lambda_2 \dashrightarrow \Lambda_1^- \times \Lambda_2^-$$

is a quasi-isogeny of degree

$$|\mathrm{Disc}_{E_1/F} \cdot \mathrm{Disc}_{E_2/F}|_F^{-\frac{n^0\cdot n^1}{2}} \,|\operatorname{Res}(\mathrm{Inv}(\beta^0), \mathrm{Inv}(\beta^1))|_F^{-1} \cdot q^{n^1\cdot|m^0|-n^0\cdot|m^1|}. \tag{4.36}$$

Here, we have used the notation $|x|_F := |\mathrm{Nm}_{E_3/F}(x)|_F^{1/2}$ for elements $x \in E_3$.

We first prove two lemmas.

Lemma 4.9. *If one of the six maps appearing below is a quasi-isogeny, then all of them are. In this case, the degrees of*

$$\begin{gathered}(q_1^-, q_2^-) : \Lambda_1 \dashrightarrow \Lambda_1^- \times \Lambda_2^-, \\ q_2^-|_{\Lambda_1^+} : \Lambda_1^+ \dashrightarrow \Lambda_2^- \quad \text{and} \quad q_2^+|_{\Lambda_1^-} : \Lambda_1^- \dashrightarrow \Lambda_2^+\end{gathered} \tag{4.37}$$

all agree. Similarly, the degrees of

$$\begin{gathered}(q_1^-, q_2^-) : \Lambda_2 \dashrightarrow \Lambda_1^- \times \Lambda_2^-, \\ q_1^-|_{\Lambda_2^+} : \Lambda_2^+ \dashrightarrow \Lambda_1^- \quad \text{and} \quad q_1^+|_{\Lambda_2^-} : \Lambda_2^- \dashrightarrow \Lambda_1^+\end{gathered} \tag{4.38}$$

all agree.

Proof. We prove the claim for the first triple of maps. Consider the exact sequence from (4.29) to obtain the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda_1^+ & \longrightarrow & \Lambda_1 & \xrightarrow{q_1^-} & \Lambda_1^- & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q_2^-} & & \downarrow{\scriptstyle \binom{q_1^-}{q_2^-}} & & \| & & \\ 0 & \longrightarrow & \Lambda_2^- & \longrightarrow & \Lambda_1^- \times \Lambda_2^- & \xrightarrow{\mathrm{pr}} & \Lambda_1^- & \longrightarrow & 0 \end{array}$$

(Note that $q_1^- : \Lambda_1 \to \Lambda_1^-$ is surjective even if E_1/F is ramified; the argument is as for the map p in (4.29).) The claim follows from this diagram for the two maps (q_1^-, q_2^-) and $q_2^-|_{\Lambda_1^+}$ in (4.37). Consider now any quasi-isogeny $z \in \mathrm{End}_F^0(\mathbb{X}^0)$ that Galois commutes with both E_1 and E_2 in the sense that $z\zeta_i = \zeta_i^{\sigma_i} z$ for both $i = 1, 2$. (Such elements always exist; a specific choice is given by the element $\mathbf{z}$ from [HL20, (2.4.2)]. It Galois commutes with both E_i by [HL20, Proposition 2.4.2] and is invertible because β^0 is regular semi-simple. The latter implies $\mathbf{t}$ in [HL20, Proposition 2.4.2] to be invertible by definition, then [HL20, Proposition 2.4.1] implies $\mathbf{z}$ invertible.) The element z intertwines $q_2^-|_{\Lambda_1^+}$ and $q_2^+|_{\Lambda_1^-}$ in the sense that the following diagram commutes.

$$\begin{array}{ccc} \Lambda_1^+ & \xrightarrow{f \longmapsto z\circ f} & \Lambda_1^- \\ \downarrow{\scriptstyle q_2^-} & & \downarrow{\scriptstyle q_2^+} \\ \Lambda_2^- & \xrightarrow{f \longmapsto z\circ f} & \Lambda_2^+ \end{array}$$

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Since z is a quasi-isogeny and since the horizontal maps have the same degree, this proves the claimed shared properties of the first triple. The case of the second triple is the same by symmetry. Moreover, Λ_1 and Λ_2 are lattices in the same isocrystal, so the first three maps are quasi-isogenies if and only if the second triple is. □

Lemma 4.10. *The endomorphism $q_1^+ q_2^-|_{\Lambda_1^+} : \Lambda_1^+ \to \Lambda_1^+$ is a quasi-isogeny of degree*

$$|\mathrm{Disc}_{E_1/F} \cdot \mathrm{Disc}_{E_2/F}|_F^{-n^0 \cdot n^1} \,|\, \mathrm{Res}\,(\mathrm{Inv}(\beta^0), \mathrm{Inv}(\beta^1))|_F^{-2}.$$

Proof. For any $f \in \Lambda_1^+$, we have $\beta_1^0(x) \circ f = f \circ \beta_1^1(x)$ for every $x \in E_1$. From the definition in (4.32) we obtain that

$$\begin{aligned} q_1^+(q_2^-(f)) &= \underbrace{(f \circ \beta_2^1(\zeta_2) - \beta_2^0(\zeta_2) \circ f)}_{\text{definition of } q_2^-(f) \text{ in (4.32)}} \circ \beta_1^1(\zeta_1) \\ &\quad - \beta_1^0(\zeta_1^{\sigma_1}) \circ \underbrace{(\beta_2^0(\zeta_2^{\sigma_2}) \circ f - f \circ \beta_2^1(\zeta_2^{\sigma_2}))}_{\text{equals } q_2^-(f), \text{ see (4.41)}} \\ &= f \circ w(\beta^1) - w(\beta^0) \circ f \end{aligned} \tag{4.39}$$

with the two elements

$$\begin{aligned} w(\beta^0) &= (\beta_2^0(\zeta_2) \circ \beta_1^0(\zeta_1) + \beta_1^0(\zeta_1^{\sigma_1}) \circ \beta_2^0(\zeta_2^{\sigma_2})), \\ w(\beta^1) &= (\beta_2^1(\zeta_2) \circ \beta_1^1(\zeta_1) + \beta_1^1(\zeta_1^{\sigma_1}) \circ \beta_2^1(\zeta_2^{\sigma_2})). \end{aligned} \tag{4.40}$$

We have also used the simple observation

$$f \circ \zeta_i - \zeta_i \circ f = \zeta_i^{\sigma_i} \circ f - f \circ \zeta_i^{\sigma_i} \tag{4.41}$$

for all maps f, which results from $f \circ \mathrm{tr}(\zeta_i) = \mathrm{tr}(\zeta_i) \circ f$. The elements $w(\beta^0)$ and $w(\beta^1)$ are precisely the elements $\mathbf{w}$ for the two pairs β^0 and β^1 from [HL20, (2.4.1)]. They have the property that each of them is both O_{E_1}- and O_{E_2}-linear, i.e. $w(\beta^i)$ centralizes β^i, see [HL20, Proposition 2.4.2]. Note that $\mathrm{End}^0_{E_1}(\mathbb{X}^1) \cong M_{n^1}(E_1)$, also write $D = \mathrm{End}^0_{E_1}(\mathbb{X}^0)$. Thus, we may view

$$w(\beta^0) \in D, \quad w(\beta^1) \in M_{n^1}(E_1)$$

and $q_1^+ q_2^-|_{\Lambda_1^+} = w(\beta^1) \otimes 1 - 1 \otimes w(\beta^0) \in M_{n^1}(E_1) \otimes_{E_1} D$. We recall a basic fact about the resultant. Assume we are given $A \in M_n(k)$ and $B \in M_m(k)$ where k is any field. Then

$$\det\,[A \otimes 1 - 1 \otimes B : M_{n\times m}(k) \longrightarrow M_{n\times m}(k), \quad f \longmapsto Af - fB] = \mathrm{Res}(P_A, P_B),$$

where P_A and P_B denote the characteristic polynomials of A and B. (This identity is clear if A and B are diagonalizable. The general case follows from a Zariski density argument and by passing to the algebraic closure of k.) Put differently, the determinant of $A \otimes 1 - 1 \otimes B \in M_n(k) \otimes_k M_m(k)$, viewed as $(nm \times nm)$-matrix is $\mathrm{Res}(P_A, P_B)$. Since determinants may be computed after scalar extension, these considerations also apply to central simple algebras and we obtain

$$\begin{aligned} \deg(q_1^+ q_2^-|_{\Lambda_1^+}) &= |\mathrm{Nrd}_{M_{n^1}(D)/E_1}(w(\beta^1) \otimes 1 - 1 \otimes w(\beta^0))|_{E_1}^{-1} \\ &= |\mathrm{Res}(\mathrm{charred}(w(\beta^0)), \mathrm{charred}(w(\beta^1)))|_{E_1}^{-1}, \\ &= |\mathrm{Res}(\mathrm{charred}(w(\beta^0)), \mathrm{charred}(w(\beta^1)))^2|_F^{-1}. \end{aligned} \tag{4.42}$$

It is shown in [HL20, Proposition 2.4.1] that the two characteristic polynomials here are related to the invariants of β^0 and β^1 by a linear transformation. More precisely, one has $w(\beta^i) = d\mathbf{s}_{\beta^i} + c$ in $E_3 \otimes_F M_{2n^1}(F)$ and in $E_3 \otimes_F D_{1/2n^0}$, respectively, where c, $d \in E_3$, are the following two

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

elements,

$$c = -(\zeta_1 \otimes \zeta_2^{\sigma_2} + \zeta_1^{\sigma_1} \otimes \zeta_2), \quad d = (\zeta_1 - \zeta_1^{\sigma_1}) \otimes (\zeta_2 - \zeta_2^{\sigma_2}).$$

(These a priori lie in $E_1 \otimes_F E_2$ but are clearly $\sigma_1 \otimes \sigma_2$-invariant.) It follows that if $\mu_1, \dots, \mu_{n^i} \in \bar{F}$ are the eigenvalues of $\mathrm{Inv}(\beta^i)$ with respect to a homomorphism $\rho : E_3 \to \bar{F}$, then $d\mu_1 + c, \dots, d\mu_{n^i} + c$ are the eigenvalues of $\mathrm{charred}(w(\beta^i))$ with respect to ρ. The resultant term in (4.42) is thus equal to

$$|((\zeta_1 - \zeta_1^{\sigma_1})(\zeta_2 - \zeta_2^{\sigma_2}))^{2n^0 \cdot n^1} \mathrm{Res}(\mathrm{Inv}(\beta^0), \mathrm{Inv}(\beta^1))^2|_F^{-1}. \tag{4.43}$$

Since $|\zeta_i - \zeta_i^{\sigma_i}|_F = |\mathrm{Disc}_{E_i/F}|_F^{1/2}$, the lemma is proved. □

Proof of Proposition 4.8. Lemmas 4.9 and 4.10 already show that $(q_1^-, q_2^-) : \Lambda_2 \dashrightarrow \Lambda_1^- \times \Lambda_2^-$ is a quasi-isogeny. From Lemma 4.9, we obtain that

$$\begin{aligned}
&\deg\left((q_1^-, q_2^-) : \Lambda_2 \dashrightarrow \Lambda_1^- \times \Lambda_2^-\right)^2 \\
&\quad = \deg(\Lambda_2 \dashrightarrow \Lambda_1) \deg\left((q_1^-, q_2^-) : \Lambda_1 \dashrightarrow \Lambda_1^- \times \Lambda_2^-\right) \deg\left((q_1^-, q_2^-) : \Lambda_2 \dashrightarrow \Lambda_1^- \times \Lambda_2^-\right) \\
&\quad = \deg(\Lambda_2 \dashrightarrow \Lambda_1) \deg(q_2^- : \Lambda_1^+ \dashrightarrow \Lambda_2^-) \deg(q_1^+ : \Lambda_2^- \dashrightarrow \Lambda_1^+) \\
&\quad = \deg(\Lambda_2 \dashrightarrow \Lambda_1) \deg(q_1^+ q_2^- : \Lambda_1^+ \dashrightarrow \Lambda_1^+).
\end{aligned} \tag{4.44}$$

Substituting (4.35) and Lemma 4.10, this means that $\deg((q_1^-, q_2^-) : \Lambda_2 \dashrightarrow \Lambda_1^- \times \Lambda_2^-)^2$ equals

$$|\mathrm{Disc}_{E_1/F} \cdot \mathrm{Disc}_{E_2/F}|_F^{-n^0 \cdot n^1} |\operatorname{Res}(\mathrm{Inv}(\beta^0), \mathrm{Inv}(\beta^1))|_F^{-2} \cdot q^{2n^1 \cdot |m^0| - 2n^0 \cdot |m^1|}. \tag{4.45}$$

Taking the square root proves Proposition 4.8. □

Proof of Propositions 4.1, 4.2 and 4.5 and Theorems 2.12 and 3.12. We take up the setting of Lemma 4.7. Recall that this lemma concerned the quasi-homomorphism

$$\begin{pmatrix} p_2^- \\ \widetilde{p_1^-} \end{pmatrix} : \mathrm{Hom}(T_0, Y_0) \dashrightarrow \mathrm{Hom}_{O_{E_2}-\mathrm{conj}}(T_0, Y_0) \times \mathrm{Hom}_{O_{E_1}-\mathrm{conj}}(T_r, Y_r). \tag{4.46}$$

Composing with the quasi-isogenies ρ_0^0, ρ_0^1, ρ_r^0 and ρ_r^1, and applying the Dieudonné module functor $M(-)$ over the special fiber, (4.46) is sent to $(q_1^-, q_2^-) : \Lambda_2 \dashrightarrow \Lambda_1^- \times \Lambda_2^-$. Thus, Proposition 4.8 shows that (4.46) is a quasi-isogeny and that its degree is given by (4.36). Then Lemma 4.7 states that Φ is an isogeny. Clearly,

$$\prod_{i=1}^{r} \deg(L_i) = q^{2n^0 |m^1|}.$$

Combining (4.30) with (4.36), we obtain

$$\deg(\Phi) = |\mathrm{Disc}_{E_1/F} \cdot \mathrm{Disc}_{E_2/F}|_F^{-\frac{n^0 \cdot n^1}{2}} |\operatorname{Res}(\mathrm{Inv}(\beta^0), \mathrm{Inv}(\beta^1))|_F^{-1} \cdot q^{n^1 \cdot |m^0| + n^0 \cdot |m^1|}. \tag{4.47}$$

This finishes the proof of all our results. □

Appendix A. Local intersection theory

This appendix provides some background on intersection theory on regular local formal schemes. Its first aim is to give a precise definition of cycles, correspondences and the action of the latter on the former. Its second aim is to isolate some cases in which intersection numbers may be computed as lengths. These allow us to work with our non-standard definition of Hecke correspondences in §§ 3 and 4 of the paper.

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

The most important input in the following is Serre's vanishing conjecture, Theorem A.4, which is due (independently) to Roberts [Rob85] and Gillet and Soulé [GS85]. In fact, we present the theory of cycles (in our context) as a special case of [GS85], which thus becomes our main reference. None of the statements that follow is new.

A.1 Cycle intersection

Let $M = \operatorname{Spf} A$ be the formal spectrum of a regular complete local ring of dimension n. We write $M^{(c)}$ for the set of integral closed formal subschemes $Z \subseteq M$ of codimension c. These are the same as integral closed subschemes of $\operatorname{Spec} A$ and we write $\eta_Z \in \operatorname{Spec} A$ for the generic point of Z. Denote by

$$Z^c(M) := \bigoplus_{Z \in M^{(c)}} \mathbb{Z} \cdot [Z] \tag{A.1}$$

the group of cycles of codimension c. The next definition is the standard way of defining cycles from coherent sheaves: take the maximal dimensional irreducible components of its support with appropriate multiplicities.

Definition A.1.

(1) For a finite-type A-module E with $\operatorname{codim} \operatorname{Supp}(E) \geq c$, we define

$$[E]^c := \sum_{Z \in M^{(c)}} \ell_{A_{\eta_Z}}(E_{\eta_Z}) \cdot [Z] \in Z^c(M). \tag{A.2}$$

Here, ℓ is our notation for the length of an artinian module, whereas A_{η_Z} and E_{η_Z} denote the localizations at η_Z.

(2) Given a bounded complex of finite-type A-modules $E^\bullet$, we write $\operatorname{Supp}(E^\bullet) := \bigcup_{i \in \mathbb{Z}} \operatorname{Supp}(H^i(E^\bullet))$. If $\operatorname{codim} \operatorname{Supp}(E^\bullet) \geq c$, we define

$$[E^\bullet]^c := \sum_{i \in \mathbb{Z}} (-1)^i [H^i(E^\bullet)]^c \in Z^c(M). \tag{A.3}$$

In fact, every such complex $E^\bullet$ is quasi-isomorphic to a perfect complex because A is regular. (A complex is perfect if it is bounded and if all its terms are finite free.) For $Y \subseteq \operatorname{Spec} A$ closed, let $K_0^Y(M)$ denote the K-group of perfect complexes acyclic outside Y, cf. [GS85, § 1]. Let $K_0'(Y)$ denote the K-group of coherent sheaves with support on Y. We write $(E^\bullet)$ (respectively, (E)) for the class of a perfect complex (respectively, coherent sheaf) in these K-groups. Then, by [GS85, Lemma 1.9],

$$\begin{aligned} K_0^Y(M) &\xrightarrow{\cong} K_0'(Y), \\ (E^\bullet) &\longmapsto \sum_{i \in \mathbb{Z}} (-1)^i (H^i(E^\bullet)). \end{aligned} \tag{A.4}$$

It is clear that (A.3) descends to a map $K_0'(Y) \to Z^c(M)$ whenever $\operatorname{codim} Y \geq c$ and that (A.2) then comes from composition with (A.4). There is an associative, commutative cup product

$$\begin{aligned} \cup : K_0^Y(M) \times K_0^Z(M) &\longrightarrow K_0^{Y \cap Z}(M), \\ (E_1^\bullet), (E_2^\bullet) &\longmapsto \left(E_1^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_M} E_2^\bullet \right). \end{aligned} \tag{A.5}$$

Every cycle $z \in Z^c(M)$ may be viewed as an element of $K_0'(\operatorname{Supp} z)$ by linear extension from the map $M^{(c)} \ni Z \mapsto \mathcal{O}_Z$. Applying the cup product and composing with the cycle map of Definition A.1 provides an associative, commutative intersection product of cycles whenever intersections are non-degenerate.

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Definition A.2. Given $Z_1 \in M^{(c_1)}$ and $Z_2 \in M^{(c_2)}$ such that $\operatorname{codim}(Z_1 \cap Z_2) \geq c_1 + c_2$, put

$$[Z_1] \cdot [Z_2] := \left[\mathcal{O}_{Z_1} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_M} \mathcal{O}_{Z_2}\right]^{c_1+c_2} \in Z^{c_1+c_2}(M). \tag{A.6}$$

This is feasible because, as mentioned before, the $\mathcal{O}_{Z_i}$ are quasi-isomorphic to perfect complexes. Write $(Z^{c_1}(M) \times Z^{c_2}(M))_{\text{proper}}$ for the set of those pairs (z_1, z_2) with $\operatorname{codim}(\operatorname{Supp}(z_1) \cap \operatorname{Supp}(z_2)) \geq c_1 + c_2$. Then (A.6) extends linearly to a map

$$(Z^{c_1}(M) \times Z^{c_2}(M))_{\text{proper}} \longrightarrow Z^{c_1+c_2}(M).$$

Remark A.3. Formula (A.6) may be written more concretely as

$$[Z_1] \cdot [Z_2] := \sum_{Z \in M^{(c_1+c_2)}} \sum_{i \geq 0} (-1)^i \ell_{A_{\eta_Z}}(\operatorname{Tor}_i^{A_{\eta_Z}}(\mathcal{O}_{Z_1,\eta_Z}, \mathcal{O}_{Z_2,\eta_Z})) \cdot [Z] \in Z^{c_1+c_2}(M). \tag{A.7}$$

Theorem A.4 (Serre's vanishing conjecture [GS85, Theorem 5.4]). *Let $E_1^\bullet$ and $E_2^\bullet$ be two perfect complexes of A-modules with* $\operatorname{codim} \operatorname{Supp}(E_1^\bullet) + \operatorname{codim} \operatorname{Supp}(E_2^\bullet) > n = \dim A$. *Then*

$$\left[E_1^\bullet \overset{\mathbb{L}}{\otimes} E_2^\bullet\right]^n = 0.$$

Corollary A.5. *Let $E_1^\bullet$ and $E_2^\bullet$ be two perfect complexes of A-modules with* $\operatorname{codim} \operatorname{Supp}(E_i^\bullet) \geq c_i$. *Further, assume* $\operatorname{codim}(\operatorname{Supp}(E_1^\bullet) \cap \operatorname{Supp}(E_2^\bullet)) \geq c_1 + c_2$. *Then*

$$[E_1^\bullet]^{c_1} \cdot [E_2^\bullet]^{c_2} = \left[E_1^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_M} E_2^\bullet\right]^{c_1+c_2}. \tag{A.8}$$

Proof. The left-hand side of (A.8) is defined by viewing $[E_1^\bullet]^{c_1}$ and $[E_2^\bullet]^{c_2}$ as elements of $K_0'(\operatorname{Supp}(E_i^\bullet))$ by linear extension of the map $M^{(c)} \ni Z \mapsto \mathcal{O}_Z$. In $K_0'(\operatorname{Supp}(E_i^\bullet))$ we may write

$$[E_i^\bullet]^{c_i} = (E_i^\bullet) + \varepsilon_i$$

for some coherent sheaf ε_i with $\operatorname{codim} \operatorname{Supp} \varepsilon_i > c_i$. Applying Theorem A.4 to the localizations A_{η_Z} for $Z \in M^{(c_1+c_2)}$ we obtain

$$\left[\varepsilon_1 \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_M} E_2^\bullet\right]^{c_1+c_2} = \left[E_1^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_M} \varepsilon_2\right]^{c_1+c_2} = \left[\varepsilon_1 \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_M} \varepsilon_2\right]^{c_1+c_2} = 0. \qquad \square$$

A.2 Correspondences

Let W denote a complete discrete valuation ring. Assume from now on that M is local, formally of finite type, and formally smooth of relative dimension $n-1$ over $\operatorname{Spf} W$. We define an intersection number as follows.

Definition A.6. The degree $\deg : Z^n(M) \xrightarrow{\sim} \mathbb{Z}$ is defined as $\deg(\mathcal{O}_Z) = \ell_W(\mathcal{O}_Z)$. Composing with the intersection product (A.6) defines the intersection number of properly intersecting cycles,

$$(\ ,\ ) : (Z^c(M) \times Z^{n-c}(M))_{\text{proper}} \longrightarrow \mathbb{Z}, \quad (z_1, z_2) = \deg(z_1 \cdot z_2). \tag{A.9}$$

The formal smoothness provides that $M \times_{\operatorname{Spf} W} M$ is again regular and of dimension $2n-1$. Consider the following diagram.

$$\begin{array}{ccc} M \times_{\operatorname{Spf} W} M & \xrightarrow{p_2} & M \\ \downarrow{\scriptstyle p_1} & & \\ M & & \end{array}$$

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

There is a pullback of cycles

$$p_2^* : Z^c(M) \longrightarrow Z^c(M \times_{\mathrm{Spf}\, W} M), \quad [Z] \longmapsto [p_2^{-1}(Z)].$$

There is also a pushforward of cycle whose support is finite (via p_1) over M,

$$p_{1,*} : Z^c(M \times_{\mathrm{Spf}\, W} M)_{p_1\text{-finite}} \longrightarrow Z^{c-n+1}(M), \quad [Z] \longmapsto \deg(Z \to p_1(Z))[p_1(Z)].$$

These considerations similarly apply to the various projection maps

$$p_{ij} : M \times_{\mathrm{Spf}\, W} M \times_{\mathrm{Spf}\, W} M \longrightarrow M \times_{\mathrm{Spf}\, W} M.$$

Definition A.7.

(1) Denote by $\mathrm{Corr}(M) \subseteq Z^{n-1}(M \times_{\mathrm{Spf}\, W} M)$ all those elements whose support is finite over M via both p_1 and p_2. Define the bilinear map

$$\begin{aligned} * : \mathrm{Corr}(M) \times \mathrm{Corr}(M) &\longrightarrow \mathrm{Corr}(M), \\ (C_1, C_2) &\longmapsto C_1 * C_2 := p_{13,*}(p_{12}^*(C_1) \cdot p_{23}^*(C_2)). \end{aligned} \tag{A.10}$$

Note that the intersection $\mathrm{Supp}\, p_{12}^*(C_1) \cap \mathrm{Supp}\, p_{23}^*(C_2)$ is finite over M (via both p_1 and p_3) and, hence, has the expected codimension.

(2) Define the bilinear map

$$\begin{aligned} \mathrm{Corr}(M) \times Z^c(M) &\longrightarrow Z^c(M), \\ (C, z) &\longmapsto C * z := p_{1,*}(C \cdot p_2^*(z)). \end{aligned} \tag{A.11}$$

Again, the occurring intersection is of the expected codimension and finite over M (via both p_1 and p_2.)

Let R be a formal scheme, let $p_1, p_2 : R \to M$ be two finite maps and assume that R is pure of dimension n. Then also $(p_1, p_2) : R \to M \times_{\mathrm{Spf}\, W} M$ is finite and R defines the correspondence $[R] := [\mathcal{O}_R]^{n-1}$.

Proposition A.8.

(1) *Definitions* (A.10) *and* (A.11) *make* $\mathrm{Corr}(M)$ *into a ring and* $Z^c(M)$ *into a left* $\mathrm{Corr}(M)$*-module.*

(2) *Moreover, if* $C_1 = [R_1]$ *and* $C_2 = [R_2]$ *for finite maps* $R_i \to M \times_{\mathrm{Spf}\, W} M$ *from purely* n*-dimensional formal schemes* R_i*, then simply*

$$C_1 * C_2 = [R_1 \times_{p_2, M, p_1} R_2]. \tag{A.12}$$

(3) *Similarly, if* $C \in \mathrm{Corr}(M)$ *has the form* $C = [R]$ *and* $z = [Z] \in Z^c(M)$*, then*

$$C * z = p_{1,*}[R \times_{p_2, M} Z]. \tag{A.13}$$

Proof. Statement (1) claims associativity of (A.10) and (A.11). Using the projection formula [GS85, (1.7.2)], this reduces to associativity of the cycle intersection for

$$p_{12}^* C_1 \cdot p_{23}^* C_2 \cdot p_{34}^* C_3 \quad \text{and} \quad p_{12}^* C_1 \cdot p_{23}^* C_2 \cdot p_3^* z$$

on M^4 and M^3, respectively, where $C_i \in \mathrm{Corr}(M)$ and $z \in Z^c(M)$. For example, the first expression may be rewritten as

$$\begin{aligned} p_{14,*}(p_{12}^* C_1 \cdot p_{23}^* C_2 \cdot p_{34}^* C_3) &= p_{13,*} p_{124,*}(p_{124}^*(p_{12}^* C_1) \cdot p_{23}^* C_2 \cdot p_{34}^* C_3) \\ &= p_{13,*}(p_{12}^* C_1 \cdot p_{124,*}(p_{23}^* C_2 \cdot p_{34}^* C_3)) \\ &= p_{13,*}(p_{12}^* C_1 \cdot p_{23}^*(C_2 * C_3)) \\ &= C_1 * (C_2 * C_3). \end{aligned}$$

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

We omit further details.

Now let $R_1, R_2 \to M \times_{\operatorname{Spf} W} M$ be as in statement (2). By definition, of the two sides in (A.12), we need to see the equality

$$p_{13,*}\left[\mathcal{O}_{p_{12}^{-1}(R_1)} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{M\times M\times M}} \mathcal{O}_{p_{23}^{-1}(R_2)}\right]^{2n-2} = p_{13,*}[\mathcal{O}_{R_1} \otimes_{p_2^*,\, \mathcal{O}_M,\, p_1^*} \mathcal{O}_{R_2}]^{2n-2}. \tag{A.14}$$

In fact, we claim this before applying $p_{13,*}$ and we show the slightly stronger statement that all higher Tor-terms on the left-hand side of (A.14) have support in codimension $\geq 2n-1$. To see this, replace both $\mathcal{O}_{R_i}$ by perfect complexes $E_i^\bullet$ on $M \times_{\operatorname{Spf} W} M$. Then there is an isomorphism of the following kind, which may be checked term by term,

$$(E_1^\bullet \otimes_W A) \otimes_{A\otimes_W A\otimes_W A} (A \otimes_W E_2^\bullet) \cong (E_1^\bullet \otimes_{p_2^*,A,p_1^*} E_2^\bullet).$$

The maps $p_1^*, p_2^* : A \to A \otimes_W A$ are flat, so the $E_i^\bullet$ provide flat resolutions of $\mathcal{O}_{R_i}$ as A-modules. Thus, we find that

$$\mathcal{O}_{p_{12}^{-1}R_1} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{M\times M\times M}} \mathcal{O}_{p_{23}^{-1}R_2} \cong \mathcal{O}_{R_1} \overset{\mathbb{L}}{\otimes}_{p_2^*,\mathcal{O}_M,p_1^*} \mathcal{O}_{R_2}. \tag{A.15}$$

However, A is an integral domain, so the various projection maps $R_1, R_2 \to M$ are generically flat. Thus, the higher Tor-terms on the right-hand side of (A.15) have support in dimension $< \dim M$ as claimed, proving part (2). The proof of part (3) is identical and omitted. □

We finally recall that intersection numbers may often be computed as lengths. This relies on the following fact about Cohen–Macaulay rings.

Lemma A.9 ([Sta18, Tag 02JN]). *Let R be a Cohen–Macaulay noetherian local ring. A sequence of elements $x_1, \dots, x_r \in R$ is regular if and only if* $\dim(R/(x_1, \dots, x_r)) = \dim(R) - r$.

Corollary A.10.

(1) *Let $Z_1, Z_2 \subseteq M$ be closed formal subschemes. Assume that Z_1 is Cohen–Macaulay and that Z_2 is defined by a regular sequence. Assume further that* $\operatorname{codim} Z_1 + \operatorname{codim} Z_2 = n$ *and that $Z_1 \cap Z_2$ is artinian. Then*

$$([Z_1], [Z_2]) = \ell_W(\mathcal{O}_{Z_1\cap Z_2}).$$

(2) *Assume that R is Cohen–Macaulay of pure dimension n and that $p_1, p_2 : R \to M$ are two finite maps. Let $Z_1, Z_2 \subseteq M$ be closed formal subschemes that are defined by regular sequences and such that* $\operatorname{codim} Z_1 + \operatorname{codim} Z_2 = n$. *Assume further that $R \times_{M\times_{\operatorname{Spf} W} M} (Z_1 \times_{\operatorname{Spf} W} Z_2)$ is artinian. Then*

$$([Z_1], [R] * [Z_2]) = \ell_W(R \times_{M\times_{\operatorname{Spf} W} M} (Z_1 \times_{\operatorname{Spf} W} Z_2)).$$

Proof.

(1) This is standard. Assume $Z_2 = V(x_1, \dots, x_c)$ for a regular sequence $x_1, \dots, x_c$. By Lemma A.10 and our dimension assumptions, $x_1|_{Z_1}, \dots, x_c|_{Z_1}$ define a regular sequence on Z_1. Resolving $\mathcal{O}_{Z_2}$ by the Koszul complex of $(x_1, \dots, x_1)$ shows that $\operatorname{Tor}_i^{\mathcal{O}_M}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}) = 0$ for $i > 0$. Thus, $\mathcal{O}_{Z_1\cap Z_2}$ represents $\mathcal{O}_{Z_1} \otimes^{\mathbb{L}}_{\mathcal{O}_M} \mathcal{O}_{Z_2}$. By Corollary A.5, $\ell_W(\mathcal{O}_{Z_1\cap Z_2})$ then agrees with $([Z_1], [Z_2])$.

(2) By the projection formula [GS85, (1.7.2)] or by Proposition A.8,

$$([Z_1], [R] * [Z_2]) = \deg([R] \cdot p_1^*[Z_1] \cdot p_2^*[Z_2]).$$

Corollary A.5 provides the first equality in

$$p_1^*[Z_1] \cdot p_2^*[Z_2] = \left[\mathcal{O}_{p_1^{-1}(Z_1)} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{M\times M}} \mathcal{O}_{p_2^{-1}(Z_2)}\right]^n = [Z_1 \times_{\operatorname{Spf} W} Z_2]^n.$$

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

The second equality follows because $p_1^{-1}(Z_1)$ and $p_2^{-1}(Z_2)$ are Tor-independent. Now we may apply statement (1) for the formal scheme $M \times_{\mathrm{Spf}\, W} M$, finishing the proof. □

Appendix B. Partial Satake transformation for GL_n

B.1 Partial Satake transformation

Let $G = GL_n$ and let $P = LU \subset G$ be a parabolic with Levi factor L and unipotent radical U. Let δ_L be the modular character of P, meaning it is the character such that $d(lul^{-1}) = \delta_L(l)\, du$ for $u \in U$, $l \in L$ and every left Haar measure du on U. Up to conjugacy, the situation is that of a standard parabolic: let $n_\bullet = (n_1, \dots, n_k)$ be a partition of n and let $P_{n_\bullet}$ be the block upper triangular matrices with block size given by $n_\bullet$. Let $L_{n_\bullet}$ be the Levi of block-diagonal matrices and let $U_{n_\bullet}$ be the unipotent radical. For $l = (l_1, l_2, \dots, l_k) \in L_{n_\bullet}$ with $l_i \in \mathrm{GL}_{n_i}(F)$, the modular character is given by

$$\delta_{L_{n_\bullet}}(l) = \prod_{i=1}^{k} |\det l_i|_F^{-n_1-\cdots-n_{i-1}+n_{i+1}+\cdots+n_k}. \tag{B.1}$$

The $\mathbb{C}$-coefficient (partial) Satake transformation is the following map of spherical Hecke algebras

$$\begin{aligned} &\mathcal{S} : \mathbb{C}[G(O_F)\backslash G(F)/G(O_F)] \longrightarrow \mathbb{C}[L(O_F)\backslash L(F)/L(O_F)], \\ &f \longmapsto \mathcal{S}f; \quad \mathcal{S}f(l) := \delta_L(l)^{\frac{1}{2}} \int_{U(F)} f(lu)\, du, \end{aligned} \tag{B.2}$$

where du is the Haar measure normalized by $U(O_F)$.

Lemma B.1. *The partial Satake transformation* (B.2) *is a map of* $\mathbb{C}$*-algebras.*

Proof. The integral

$$\mathcal{S}[f * g](l) = \delta_L(l)^{\frac{1}{2}} \cdot \int_{U(F)} \int_{G(F)} f(y^{-1}lu) g(y)\, dy\, du$$

is invariant under $y \mapsto yk$ for $k \in G(O_F)$. The Iwasawa decomposition provides a P-equivariant isomorphism

$$P(F)/P(O_F) \cong G(F)/G(O_F).$$

The measure on the two spaces here is given by the left invariant Haar measure $dl\, du$ on $P = LU$. Therefore,

$$\mathcal{S}[f * g](l) = \delta_L(l_2^{-1}l)^{\frac{1}{2}} \delta_L(l_2)^{\frac{1}{2}} \int_{U(F)} \int_{U(F)} \int_{L(F)} f(u_2^{-1} l_2^{-1} lu) g(l_2 u_2)\, du\, du_2\, dl_2.$$

Applying the change of variables $u \longmapsto l^{-1} l_2 u_2 l_2^{-1} lu$, we conclude that

$$\mathcal{S}[f * g](l) = \int_{L(F)} \Bigg(\underbrace{\delta_L(l_2^{-1}l)^{\frac{1}{2}} \int_U f(l_2^{-1}lu)\, du}_{\mathcal{S}[f](l_2^{-1}l)} \cdot \underbrace{\delta_L(l_2)^{\frac{1}{2}} \int_U g(l_2 u_2)\, du_2}_{\mathcal{S}[g](l_2)} \Bigg) dl_2 = (\mathcal{S}[f] * \mathcal{S}[g])(l)$$

as desired. □

We next specialize to the usual Satake transformation, which is the case of the partition of n as $1 + 1 + \cdots + 1$. We write T for the diagonal torus and U for the unipotent radical, now consisting of upper triangular unipotent matrices. Since T is abelian, we have $T(O_F)\backslash T(F)/T(O_F) = T(F)/T(O_F)$.

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Denote by $x_i : T \longrightarrow \{0,1\}$ the characteristic function of the subset

$$O_F^\times \times \cdots O_F^\times \times \underbrace{\pi O_F^\times}_{i\text{th component}} \times O_F^\times \cdots \times O_F^\times \subset T.$$

Then each x_i is invertible in $\mathbb{C}[T(F)/T(O_F)]$ and there is the explicit description

$$\mathbb{C}[T(F)/T(O_F)] = \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]. \tag{B.3}$$

The monomial $x_1^{m_1} \cdots x_n^{m_n}$ on the right-hand side is simply the characteristic function of the subset $\pi^{m_1} O_F^\times \times \pi^{m_2} O_F^\times \times \cdots \times \pi^{m_n} O_F^\times$.

Put $\mathrm{Mat}_{n\times n}^{\det \neq 0}(O_F) := \mathrm{Mat}_{n\times n}(O_F) \cap \mathrm{GL}_n(F)$. Let $W = N(T)/T$ be the Weyl group. For any two O_F-modules Λ_1 and Λ_2, we write $\Lambda_1 \underset{k}{\subset} \Lambda_2$ to denote a (not necessarily strict) inclusion $\Lambda_1 \subseteq \Lambda_2$ with $|\Lambda_2/\Lambda_1| = q^k$, where $q = |O_F/\pi O_F|$.

Definition B.2. For any $0 \le k \le n$, respectively, for $0 \le k$, define the following functions on $G(F)$:

$$\mathbf{S}_k(g) = \begin{cases} 1 & \text{if } \pi O_F^n \underset{n-k}{\subset} gO_F^n \underset{k}{\subset} O_F^n; \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbf{T}_k(g) = \begin{cases} 1 & \text{if } gO_F^n \underset{k}{\subset} O_F^n; \\ 0 & \text{otherwise.} \end{cases} \tag{B.4}$$

For any $-n \le k < 0$ and for $k < 0$, put $\mathbf{S}_k(g) := \mathbf{S}_{-k}(g^{-1})$ and $\mathbf{T}_k(g) := \mathbf{T}_{-k}(g^{-1})$, respectively. In particular, $\mathbf{S}_n$, $\mathbf{S}_0$ and $\mathbf{S}_{-n}$ are the characteristic functions of $\pi G(O_F)$, $G(O_F)$ and $\pi^{-1}G(O_F)$, respectively. For example, $(\mathbf{S}_{-n} * f)(g) = f(\pi g)$ for every Hecke function f.

Our first result in this appendix is the following theorem.

Theorem B.3. *Let $G^{(k)}$ be the preimage of $\pi^k O_F^\times$ under the map $\det : \mathrm{Mat}_{n\times n}(O_F) \longrightarrow O_F$ and let*

$$\mathcal{H}^{(k)} := \mathbb{C}[G(O_F)\backslash G^{(k)}/G(O_F)], \quad \mathcal{H} := \bigoplus_{k=0}^{\infty} \mathcal{H}^{(k)}. \tag{B.5}$$

The following statements hold.

(1) *The Satake transformation defines an isomorphism from $\mathcal{H}^{(k)}$ to the vector space of degree k homogeneous symmetric polynomials, and consequently from $\mathcal{H}$ to the algebra of symmetric polynomials*

$$\mathcal{S} : \mathcal{H}^{(k)} \xrightarrow{\cong} \mathbb{C}[x_1, \ldots, x_n]^{W=\mathrm{id},\, \deg=k}; \quad \mathcal{S} : \mathcal{H} \xrightarrow{\cong} \mathbb{C}[x_1, \ldots, x_n]^{W=\mathrm{id}}. \tag{B.6}$$

(2) *Either of the sets $\{\mathbf{S}_i\}_{0\le i\le n}$ or $\{\mathbf{T}_i\}_{0\le i\le n}$ generates $\mathcal{H}$ as a $\mathbb{C}$-algebra.*

(3) *Either of the sets $\{\mathbf{S}_{-n}\} \cup \{\mathbf{S}_i\}_{0\le i\le n}$ or $\{\mathbf{S}_{-n}\} \cup \{\mathbf{T}_i\}_{0\le i\le n}$ generates $\mathbb{C}[G(O_F)\backslash G(F)/G(O_F)]$ as a $\mathbb{C}$-algebra.*

Note that part (3) follows immediately from part (2). The proof of parts (1) and (2) is given in the next few subappendices and will be complete after Appendix B.5. Part (1) is actually well-known and may be found in more general form in [Gro98], but we include a proof for convenience. We begin here with the observation that it is sufficient to prove surjectivity in (1).

Lemma B.4. *Assume $\mathcal{S}$ maps $\mathcal{H}$ surjectively onto $\mathbb{C}[x_1, \ldots, x_n]^{W=\mathrm{id}}$. Then $\mathcal{S} : \mathcal{H} \cong \mathbb{C}[x_1, \ldots, x_n]^{W=\mathrm{id}}$ is bijective.*

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Proof. Since $\det(g\cdot U)=\det(g)$, we have $\mathcal{S}(\mathcal{H}^{(k)})\subset\mathbb{C}[x_1,\dots,x_n]^{\deg=k}$ for any integer k. By counting double cosets via the Cartan decomposition,

$$\begin{aligned}\#(G(O_F)\backslash G^{(k)}/G(O_F)) &= \#\{k_1\le k_2\le\cdots\le k_n : k_1+k_2+\cdots+k_n=k\}\\ &= \dim\mathbb{C}[x_1,\dots,x_n]^{W=\mathrm{id},\deg=k},\end{aligned}$$

we conclude that the $\mathbb{C}$-vector spaces $\mathcal{H}^{(k)}$ and $\mathbb{C}[x_1,\dots,x_n]^{W=\mathrm{id},\deg=k}$ are of the same dimension and the lemma follows. □

B.2 Satake transformation in terms of orbital integrals

In this subappendix we prove the inclusion

$$\mathcal{S}(\mathcal{H})\subset\mathbb{C}[x_1,\dots,x_n]^{W=\mathrm{id}}. \tag{B.7}$$

Let $U_{n-1}:=U$ and inductively define $U_{i-1}:=[U_i,U_i]$ as the derived subgroup, meaning the minimal normal subgroup such that U_i/U_{i-1} is abelian. Explicitly,

$$U_i=\{(a_{ij})_{n\times n} : a_{k,k}=1 \text{ and } a_{k,k+j}=0 \text{ for } j<0 \text{ and for } 0<j<n-i\}.$$

Then

$$U_i/U_{i-1}\cong\{(a_{1,1+n-i},a_{2,2+n-i},\dots,a_{i,n})\}\cong F^{\oplus i}$$

and for any $t=\mathrm{diag}(t_1,\dots,t_n)\in T$, the map

$$U\longrightarrow U;\quad u\longmapsto t^{-1}utu^{-1}$$

induces linear transformations on U_i/U_{i-1} with eigenvalues $\{t_k^{-1}t_{k+n-i}-1\}_{1\le k\le i}$. Assume that the t_i are pairwise different. Then, denoting by du the Haar measure on U, this implies that

$$d(t^{-1}utu^{-1})=\prod_{i=1}^{n-1}\prod_{k=1}^{i}|t_k^{-1}t_{k+n-i}-1|_F\cdot du=|\det t|_F^{-\frac{n-1}{2}}\cdot\delta_T(t)^{-\frac12}\prod_{1\le i<j\le n}|t_i-t_j|_F\,du.$$

Note that the product of $|t_i-t_j|_F$ for $1\le i<j\le n$ is equal to $|\mathrm{Disc}(\mathrm{char}_t)|_F^{\frac12}$ where char_t is the characteristic polynomial $\det(\lambda\cdot\mathrm{id}-t)$ of t, and where Disc denotes the discriminant. We have

$$\begin{aligned}\mathcal{S}[f](t)&=\delta_T(t)^{\frac12}\int_{U(F)}f(t(t^{-1}utu^{-1}))d(t^{-1}utu^{-1})\\ &=|\det t|_F^{-\frac{n-1}{2}}|\mathrm{Disc}(\mathrm{char}_t)|_F^{\frac12}\int_{U(F)}f(utu^{-1})\,du.\end{aligned}$$

Since $\mathrm{GL}_n(F)=\mathrm{GL}_n(O_F)U(F)T(F)$ by the Iwasawa decomposition and since f is $\mathrm{GL}_n(O_F)$-invariant, the occurring orbital integral is

$$\int_{U(F)}f(utu^{-1})\,du=\int_{\mathrm{GL}_n(F)/T(F)}f(hth^{-1})\,dh.$$

This expression only depends on t up to conjugation. Therefore, the Satake transformation is Weyl-invariant as claimed in (B.7).

B.3 Satake transformation of minuscule Hecke functions

We next determine the images $\mathcal{S}[\mathbf{S}_k]$. Write $V_n=F^n$ and let $V_n\supset V_{n-1}\supset\cdots\supset V_0=\{0\}$ be the unique U-invariant filtration such that $V_i/V_{i-1}\cong F$. For any $t=\mathrm{diag}(t_1,t_2,\dots,t_n)$ and any $u\in U$, the linear transformation $tu:V_n\longrightarrow V_n$ induces scaling by t_i on V_i/V_{i-1}. Therefore, $O_F^n\supseteq$

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

$tuO_F^n \supseteq \pi \cdot O_F^n$ implies that $t_i \in O_F^\times \cup \pi \cdot O_F^\times$. This shows that

$$\mathcal{S}[\mathbf{S}_k] = \sum_{\substack{I \subset \{1,\dots,n\} \\ \#I = k}} C_I \cdot \prod_{i \in I} x_i$$

for some constants C_I. The polynomial on the right-hand side is W-invariant by (B.7), so the constants C_I are actually independent of I.

Now we determine $C_{\{n-k+1,\dots,n\}}$. Consider the element $t = \mathrm{diag}(1, 1, \dots, 1, \pi, \dots, \pi)$ and note that $\mathbf{S}_k(tu) = 1$ if and only if $u \in U(O_F)$. Therefore, $\mathcal{S}[\mathbf{S}_k](t) = \delta_T(t)^{\frac{1}{2}} = q^{\frac{kn-k^2}{2}}$ which implies that

$$\mathcal{S}[\mathbf{S}_k] = q^{\frac{kn-k^2}{2}} \sum_{\substack{I \subset \{1,\dots,n\} \\ \#I = k}} \prod_{i \in I} x_i$$

is (a non-zero scalar multiple of) an elementary symmetric polynomial. It is well-known that these generate $\mathbb{C}[x_1, \dots, x_n]^{W=\mathrm{id}}$, so

$$\mathcal{S}(\mathcal{H}) = \mathbb{C}[x_1, \dots, x_n]^{W=\mathrm{id}}.$$

At this point, we have shown part (2) of Theorem B.3 for the set $\{\mathbf{S}_k\}_{0 \le k \le n}$.

B.4 Generating symmetric polynomials

The next result provides an alternative generating set for the algebra of symmetric polynomials. Define

$$s_k(x_1, \dots, x_n) := \sum_{\substack{I \subset \{1,2,\dots,n\} \\ \#I = k}} \prod_{i \in I} x_i \quad \text{and} \quad b_m(x_1, \dots, x_n) := \sum_{\substack{m = m_1 + \cdots + m_n \\ m_1, \dots, m_n \ge 0}} \prod_{i=1}^{n} x_i^{m_i}. \tag{B.8}$$

It was already stated and used that $\{s_k\}_{0 \le k \le n}$ is a set of generators of $\mathbb{C}[x_1, \dots, x_n]^{W=\mathrm{id}}$.

Proposition B.5. *The set $\{b_k(x_1, \dots, x_n)\}_{0 \le k \le n}$ generates $\mathbb{C}[x_1, \dots, x_n]^{W=\mathrm{id}}$ as a $\mathbb{C}$-algebra.*

Proof. Note that $b_0 = s_0 = 1$. We claim that, for all $k \ge 1$,

$$\sum_{i=0}^{k} (-1)^i b_i s_{k-i} = 0. \tag{B.9}$$

To prove this, consider the polynomial expansion

$$\prod_{i=1}^{n} (X - x_i) = \sum_{i=0}^{n} (-1)^i s_i X^{n-i}$$

and that of its reciprocal

$$\prod_{i=1}^{n} (X - x_i)^{-1} = \sum_{i=0}^{\infty} b_i X^{-i-n}.$$

The claim follows by multiplying these two equations together. The proposition now follows by induction from the identities $s_k = -\sum_{i=1}^{k} (-1)^i b_i s_{k-i}$, $k \ge 1$. □

B.5 Satake transformation of $\mathbf{T}_m$

Proposition B.6. *We have*

$$\mathcal{S}[\mathbf{T}_m] = q^{\frac{m(n-1)}{2}} b_m(x_1, \dots, x_n).$$

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Proof. Consider a tuple $m_\bullet = (m_1, \dots, m_n)$ of integers $m_i \geq 0$ and put $|m_\bullet| = m_1 + \cdots + m_n$. Let

$$t = \mathrm{diag}(\pi^{m_1}, \pi^{m_2}, \dots, \pi^{m_n}) \in T, \quad u = (a_{ij})_{1 \leq i,j \leq n} \in U.$$

We have

$$\det(tu) = \det(t) = \pi^{|m_\bullet|}.$$

Note that $u \in U$ implies that $a_{ij} = 0$ if $j < i$ and $a_{ii} = 1$ for $1 \leq i \leq n$. We have

$$\begin{aligned}\int_{U(F)} \mathbf{T}_m(tu)\, du &= \mathbf{1}_{\pi^m O_F^\times}(\pi^{|m_\bullet|}) \cdot \prod_{i=1}^n \prod_{j=i+1}^n \int_F \mathbf{1}_{O_F}(\pi^{m_i} a_{ij})\, da_{ij} \\ &= \mathbf{1}_{\pi^m O_F^\times}(\pi^{|m_\bullet|}) \cdot \prod_{i=1}^n |\pi^{m_i}|^{-(n-i)}.\end{aligned}$$

Therefore, if $|m_\bullet| = m$, then

$$\mathcal{S}[\mathbf{T}_m](t) = \underbrace{\prod_{i=1}^n |\pi^{m_i}|^{\frac{n+1}{2}-i}}_{\delta_T(t)} \cdot \prod_{i=1}^n |\pi^{m_i}|^{-(n-i)} = \prod_{i=1}^n |\pi^{m_i}|^{\frac{1-n}{2}} = |\pi^m|^{\frac{1-n}{2}},$$

which implies

$$\mathcal{S}[\mathbf{T}_m] = |\pi^m|^{\frac{1-n}{2}} \sum_{m = m_1 + \cdots + m_n} \prod_{i=1}^n x_i^{m_i} = q^{\frac{m(n-1)}{2}} b_m(x_1, \dots, x_n),$$

where b_m is the polynomial from (B.8). □

Combining Proposition B.6 with Proposition B.5 and the bijectivity of the Satake transformation proves that $\{\mathbf{T}_m\}_{0 \leq m \leq n}$ is a set of generators for $\mathcal{H}$. The proof of Theorem B.3 is now complete.

B.6 Partial Satake transformation of $\mathbf{T}_m$

We finally come to the second result of this appendix, namely an explicit formula for the partial Satake transformation of $\mathbf{T}_m$. Suppose the partition in question is $n = n_1 + n_2$ and put $g = (g_1, g_2) \in L = \mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F)$. To avoid confusion, we write $\mathbf{T}_{\mathrm{GL}_n, m}$ for the function $\mathbf{T}_m$ on $GL_n(F)$.

Proposition B.7. *For any integer $m \geq 0$, we have*

$$\mathcal{S}[\mathbf{T}_{\mathrm{GL}_n, m}] = \sum_{m_1 + m_2 = m} q^{\frac{m_1 n_2 + m_2 n_1}{2}} \cdot \mathbf{T}_{\mathrm{GL}_{n_1}, m_1} \boxtimes \mathbf{T}_{\mathrm{GL}_{n_2}, m_2}.$$

Proof. For any

$$\mathbf{u} := \begin{pmatrix} I_{n_1} & u \\ & I_{n_2} \end{pmatrix} \in U_{(n_1, n_2)}(F),$$

we have

$$\mathbf{T}_{\mathrm{GL}_n, m}(g \cdot \mathbf{u}) = 1 \iff \begin{cases} (g_1, g_2) \in \mathrm{Mat}_{n_1 \times n_1}(O_F) \times \mathrm{Mat}_{n_2 \times n_2}(O_F), \\ g_1 u \in \mathrm{Mat}_{n_1 \times n_2}(O_F), \text{ and} \\ \det g_1 \cdot \det g_2 \in \pi^m O_F^\times. \end{cases}$$

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Therefore, for any $g=(g_1,g_2)\in \mathrm{Mat}_{n_1\times n_1}(O_F)\times \mathrm{Mat}_{n_2\times n_2}(O_F)$ with $\det g_1\cdot \det g_2\in \pi^m O_F^\times$, we find

$$\mathcal{S}(\mathbf{T}_m)(g)=\frac{|\det(g_1)|_F^{\frac{n_2}{2}}}{|\det(g_2)|_F^{\frac{n_1}{2}}}\cdot\int_{\mathrm{Mat}_{n_1\times n_2}(F)}\mathbf{1}_{\mathrm{Mat}_{n_1\times n_2}(O_F)}(g_1\cdot u)\,du=|\det(g_1)|_F^{-\frac{n_2}{2}}\cdot|\det(g_2)|_F^{-\frac{n_1}{2}}$$

as desired. □

Acknowledgements

We thank M. Rapoport for his continued interest in our work and several helpful discussions. We are grateful to U. Hartl for correspondence on π-divisible groups. We furthermore thank N. Hultberg, W. Zhang and Z. Zhang for comments on an earlier version of this article. We also thank the referee for their detailed suggestions.

Conflicts of interest

None.

Journal Information

Compositio Mathematica is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.

References

Clo90 L. Clozel, *The fundamental lemma for stable base change*, Duke Math. J. **61** (1990), 255–302.

Dra83 P. K. Draxl, *Skew fields*, London Mathematical Society Lecture Note Series, vol. 81 (Cambridge University Press, Cambridge, 1983).

GS85 H. Gillet and C. Soulé, *K-théorie et nullité des multiplicités d'intersection*, C. R. Math. Acad. Sci. Paris Sér. I **300** (1985), 71–74.

Gro98 B. H. Gross, *On the Satake isomorphism*, in *Galois representations in arithmetic algebraic geometry*, LMS Lecture Notes, vol. 254 (Cambridge University Press, Cambridge 1998), 223–238.

Guo96 J. Guo, *On a generalization of a result of Waldspurger*, Canad. J. Math. **48** (1996), 105–142.

HS19 U. Hartl and R. K. Singh, *Local shtukas and divisible local Anderson modules*, Canad. J. Math. **71** (2019), 1163–1207.

HL20 B. Howard and Q. Li, *Intersections in Lubin–Tate space and biquadratic fundamental lemmas*, Preprint (2020), arXiv:2010.07365.

HM23 N. Hultberg and A. Mihatsch, *A linear AFL for quaternion algebras*, Canad. J. Math. (First View), 1–23.

Lab90 J.-P. Labesse, *Fonctions élémentaires et lemme fondamental pour le changement de base stable*, Duke Math. J. **61** (1990), 519–530.

Li22a Q. Li, *A proof of the linear arithmetic fundamental lemma for GL_4*, Canad. J. Math. **74** (2022), 381–427.

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press

Li22b Q. Li, *An intersection formula for CM cycles in Lubin–Tate spaces*, Duke Math. J. **171** (2022), 1923–2011.

Li25 Q. Li, *A proof of the biquadratic linear AFL for GL_4*, Preprint (2025), arXiv:2505.22625.

LM23 Q. Li and A. Mihatsch, *Arithmetic transfer for inner forms of GL_{2n}*, Preprint (2023), arXiv:2307.11716.

Liu21 Y. Liu, *Fourier–Jacobi cycles and arithmetic relative trace formula (with an appendix by Chao Li and Yihang Zhu)*, Camb. J. Math. **9** (2021), 1–147.

Mes72 W. Messing, *The crystals associated to Barsotti-Tate groups: with applications to Abelian schemes*, Lecture Notes in Mathematics, vol. 264 (Springer, Berlin–New York, 1972).

Mih22 A. Mihatsch, *Local constancy of intersection numbers*, Algebra Number Theory **16** (2022), 505–519.

MZ24 A. Mihatsch and W. Zhang, *On the arithmetic fundamental lemma conjecture over a general p-adic field*, J. Eur. Math. Soc. **26** (2024), 4831–4901.

Pog17 T. Poguntke, *Group schemes with $\mathbb{F}_q$-action.*, Bull. Soc. Math. France **145** (2017), 345–380.

RSZ17 M. Rapoport, B. Smithling and W. Zhang, *On the arithmetic transfer conjecture for exotic smooth formal moduli spaces*, Duke Math. J. **166** (2017), 2183–2336.

RSZ18 M. Rapoport, B. Smithling and W. Zhang, *Regular formal moduli spaces and arithmetic transfer conjectures*, Math. Ann. **370** (2018), 1079–1175.

RSZ20 M. Rapoport, B. Smithling and W. Zhang, *Arithmetic diagonal cycles on unitary Shimura varieties*, Compos. Math. **156** (2020), 1745–1824.

RSZ21 M. Rapoport, B. Smithling and W. Zhang, *On Shimura varieties for unitary groups*, Pure Appl. Math. Q. **17** (2021), 773–837.

Rob85 P. Roberts, *The vanishing of intersection multiplicities of perfect complexes*, Bull. Amer. Math. Soc. **13** (1985), 127–130.

Sta18 The Stacks Project Authors, *Stacks Project* (2018), https://stacks.math.columbia.edu.

Zha12a W. Zhang, *Gross–Zagier formula and arithmetic fundamental lemma*, in *Fifth international congress of Chinese mathematicians*, vol. 1 (American Mathematical Society, Providence, RI, 2012), 447–459.

Zha12b W. Zhang, *On arithmetic fundamental lemmas*, Invent. Math. **188** (2012), 197–252.

Zha18 W. Zhang, *Periods, cycles, and L-functions: a relative trace formula approach*, in *Proc. int. congress of mathematicians*, Rio de Janeiro 2018, Invited Lectures, vol. II (World Scientific Publishing, Hackensack, NJ, 2018), 487–521.

Zha21a Z. Zhang, *Maximal parahoric arithmetic transfers, resolutions and modularity*, Duke Math. J. **174** (2025), 1–129.

Zha21b W. Zhang, *Weil representation and arithmetic fundamental lemma*, Ann. of Math. (2) **193** (2021), 863–978.

Zha22 W. Zhang, *More arithmetic fundamental lemma conjectures: the case of Bessel subgroups*, Pure Appl. Math. Q. **18** (2022), 2279–2336.

Qirui Li qiruili@postech.ac.kr
Mathematical Science Building, 77 Cheongam-Ro, Nam-Gu Pohang, Gyeongbuk 37673, Korea

Andreas Mihatsch mihatsch@zju.edu.cn
School of Mathematical Sciences, Zhejiang University, 866 Yuhangtang Rd, Hangzhou, 310058, P. R. China

https://doi.org/10.1112/S0010437X24007577 Published online by Cambridge University Press