# Convex polygons as carriers

G. C. SHEPHARD

## 1. *Introduction*

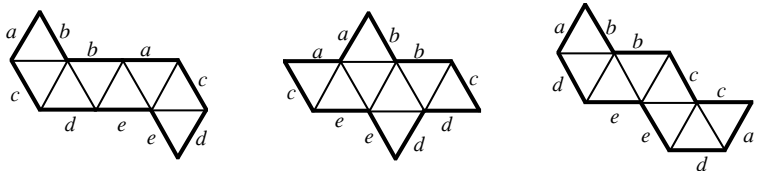
We shall use the word 'polyhedron' to mean a connected, simply-connected 3-polytope of positive (non-zero) volume. The idea of the *net* of a polyhedron $P$ is well known. For example, the regular octahedron has eleven distinct nets, three of which are shown in Figure 1. A net consists of three parts:

(a) A plane connected and simply-connected polygon $Q$ (denoted by heavy lines in the diagrams), known as the *carrier* of the net;

(b) A set of lines known as *fold-lines* in the interior of $Q$;

(c) A labelling of the edges of $Q$.

If one cuts $Q$ out of paper or similar material, folds it along the fold-lines, and then pastes together edges with matching labels, one obtains a model of the polyhedron $P$. We say that a net is *convex* if, and only if, its carrier $Q$ is convex.



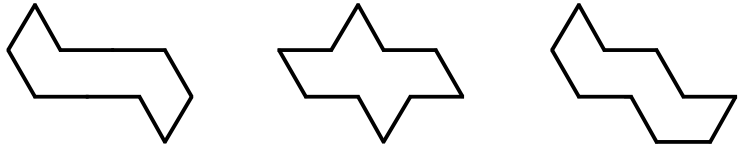

(a) (b) (c)

FIGURE 1

Examples of nets can be found in many books on polyhedra, and hundreds of examples can be found on the Internet. In particular, [1] contains many illustrations of nets along with historical notes. Sometimes nets of distinct polyhedra have the same carriers but differ in their fold-lines, or in their labelling. Examples of the former appear below, and an example of the latter is shown in [2, Figure 2].

The carriers corresponding to the nets of Figure 1 are shown in Figure 2. Notice that each carrier is a *simple* polygon, that is, it is a simple closed loop with no crossing points. Vertices are the points where two (non-collinear) edges meet. Without the latter condition, and if we admit polygons in which vertices can occur as points on the edges (so the internal angle there is $\pi$) then the problem changes in character; see the remarks below about the results for the square and other polygons in [3]. The basic problem is this: *Given a polyhedron $P$, it is simple to construct a net, and therefore, if it is not self-overlapping, a carrier $Q$ for $P$, but, given a polygon $Q$, is it possible to insert fold-lines and labels so as to obtain a net for some polyhedron $P$?*

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(a) (b) (c)

FIGURE 2

In 2007 O'Rourke posed the same problem. Using a different terminology he asks 'Which polygons can fold to a polyhedron?' He remarks that rarely can an open problem on the frontier of mathematical research be stated so succinctly. This problem has been partially answered, see below.

However the problem can be made more tractable by placing restrictions on the types of net to be considered. We introduce the term edge-to-edge in analogy with the theory of tilings. A net is *edge-to-edge* if, in constructing the polyhedron, each edge of $Q$ is pasted to another edge of $Q$ of equal length. Alternatively we may say that a net is edge-to-edge if no fold-line meets an edge of the carrier at a point other than a vertex of the carrier. Notice that we do not exclude the possibility that the sum of several angles of the carrier is $2\pi$ and that these are brought together at a point (in the interior of a face) when the polyhedron is constructed. Hence:

(E) *Edge Condition*: The carrier $Q$ of an edge-to-edge net of a polyhedron must be a $2n$-gon ($n \geqslant 2$) and its edges must form $n$ pairs of equal lengths.

The carriers in Figures 2(b) and (c) correspond to edge-to-edge nets; the fold-lines and labels in Figure 1 show how the short edges are to be pasted together. But in Figure 2(a) the two long edges (labelled *ba* and *ed* in Figure 1) are not joined to each other, and it is easy to see that it is not possible to choose fold-lines and labels so that they are joined. So in this case the carrier does not correspond to an edge-to-edge net.

We now ask: 'Is every polygon which satisfies Condition (E) the carrier of an edge-to-edge net of a polyhedron?' The answer is 'No', for consider a rectangle size $a \times b$ where $a \neq b$. Then joining the $a$ edges together and also the $b$ edges produces a toroidal surface which is not simply-connected, so it cannot be the surface of a polyhedron as defined above.

In Section 3 we shall introduce conditions which ensure that many polygons satisfying (E) are the (edge-to-edge) carriers of polyhedra, but before doing this we shall consider a special case, namely when $Q$ is a regular polygon with an even number of edges. Of course, every regular polygon with an even number of edges satisfies Condition (E); the edges can be paired off in many different ways. This makes the problem easier to deal with. Also it suggests an approach to the more general case in which $Q$ is not regular. We shall do this in Section 3.

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## 2. *Convex Polygons*

A casual examination of many nets shows that the great majority of them (nets of both convex and non-convex polyhedra) have non-convex carriers. The only regular polyhedron with convex carriers is the tetrahedron: here there exist two nets and the corresponding carriers are a triangle and a parallelogram. Both nets have vertices on the edges of the carrier, and so are not edge-to-edge. However, there exist many other convex polygons which are carriers of edge-to-edge nets of polyhedra. Some are shown in [2], where slight modifications of a tetrahedron lead to nets of polyhedra which are combinatorially equivalent to $n$-pyramids, cyclic polyhedra, $n$-bipyramids, $n$-wedges, prisms and antiprisms. All of these satisfy condition (E).

Are there any edge-to-edge nets with regular polygons as carriers that satisfy condition (E)? If the word 'edge-to-edge' is omitted, then the answer is 'yes'. In [3, Chapter 25] it is shown how a square (or any convex polygon) is the (non-edge-to-edge) carrier of an infinite number of nets of convex polyhedra. More generally, two Russian mathematicians [4] have shown that every polygon is the carrier of some polyhedron though such a polyhedron could have a huge (and unknown) number of faces. An obvious way to find edge-to-edge nets whose carriers are regular polygons is to examine the polyhedra with convex nets in [2], and see if they can be modified in such a way as to transform the carriers into regular polygons. However it appears that this cannot be done. Consequently the author of this article conjectured many years ago that no regular polygon is the edge-to-edge carrier of a polyhedron. It turns out that this is incorrect. However we suspect that the conjecture would be correct if we only consider *convex* polyhedra since although there exist regular polygons as carriers, all the known corresponding edge-to-edge nets give rise to non-convex polyhedra.

Some simple examples of edge-to-edge nets with regular polygons as carriers are shown in Figure 3. These are a square, a regular hexagon and a regular 10-gon; it should be clear how an analogous construction can be applied to any regular $2n$-gon with $n \geqslant 2$. As before, the thinner lines inside the polygons are the fold-lines; the thicker line segment in the centre of the figure (which we shall call the *root*), is also a fold line. This segment is a perpendicular bisector of the diameter shown as a dashed line. It is important that its length is less that half the circumradius of the polygon; in the figures it is about one-third of the circumradius. It is denoted differently from the other fold-lines to indicate that in the polyhedra that arise from these nets, this segment corresponds to a *concave* edge – all other folds correspond to convex edges. These polyhedra may be said to be *almost convex*.

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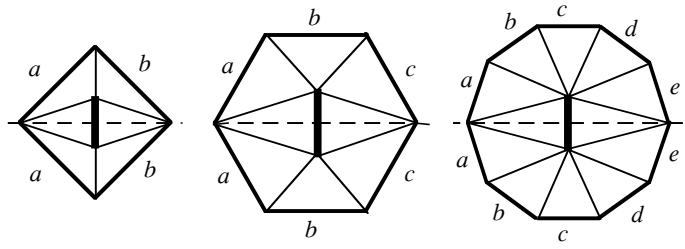



FIGURE 3

Photographs of polyhedra corresponding to the nets in Figure 3 are shown in Figure 4. The polyhedra which arise from the nets like those shown in Figure 3 are examples of *stack polyhedra*. These are defined as clusters of tetrahedra, built up one at a time, each adjoined tetrahedron having exactly one face in common with a face of the rest of the cluster. Here the stack polyhedra are very special; in every case they consist of a sequence of tetrahedra all of which have one edge (the root) in common. We shall refer to such a stack polyhedron as a *fan polyhedron*.

Clearly, such polyhedra, consisting of the union of $n$ tetrahedra, exist for all $2n$ where $n \geqslant 2$.

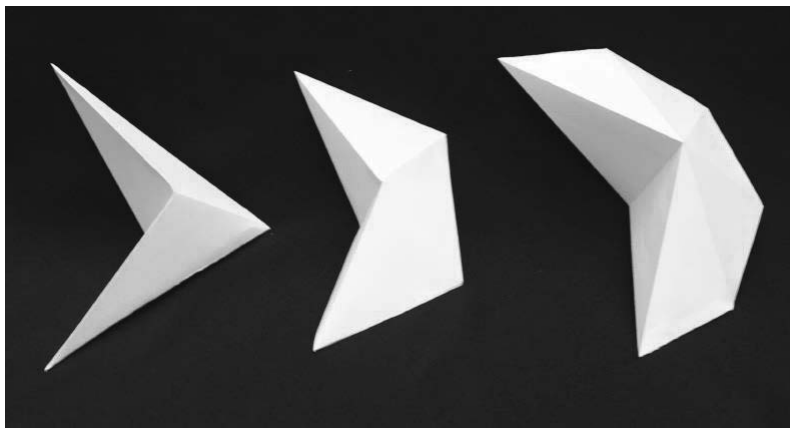


FIGURE 4

The polyhedra with nets like those shown in Figure 3 are not the only ones with regular polygons as carriers of edge-to-edge nets.

*Theorem*: Every regular $2n$-gon is the carrier of at least $1 + p^{(>2)}(n)$ combinatorially distinct polyhedra where $p^{(>2)}(n)$ denotes the number of partitions of $n$ into more than two parts.

For example, if $n = 5$ there are $p^{(>2)}(5) = 4$ partitions of this kind, namely [1, 1, 1, 1, 1], [1, 1, 1, 2], [1, 1, 3] and [1, 2, 2]. Each of these corresponds to a partition of $2n = 10$ into even parts, namely

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[2, 2, 2, 2, 2], [2, 2, 2, 4], [2, 2, 6], [2, 4, 4]

and we shall show that each of these leads to an edge-to-edge net of a polyhedron whose carrier is a 10-gon. These nets are shown in Figure 5. As before, the heavy lines in the interior of the polygons denote fold-lines which are concave in the polyhedra.

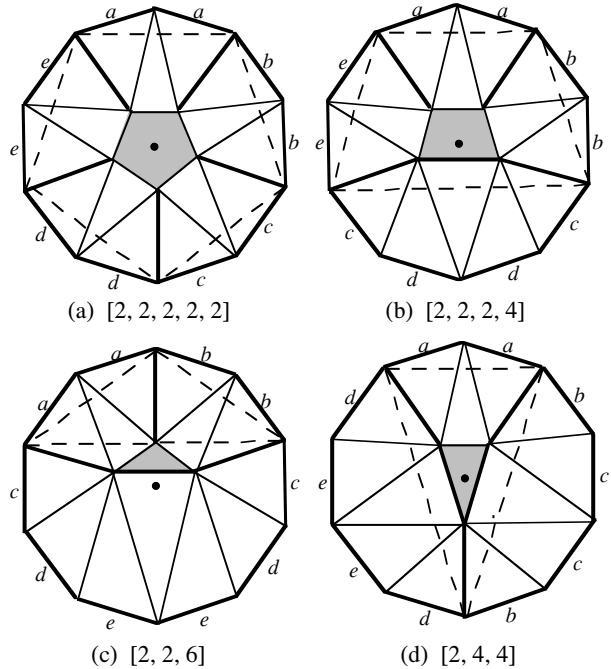



FIGURE 5

In general, if the partition is $[r_1, r_2, \ldots, r_t]$ with $\sum_{i=1}^{t} r_i = 2n$ we select $t$ vertices $V_1', V_2', \ldots, V_t'$ of $Q$ so that $V_1'$ and $V_2'$ are separated by $r_1$ edges of $Q$, $V_2'$ and $V_3'$ are separated by $r_2$ edges of $Q$, and so on. Finally $V_t'$ and $V_1'$ are separated by $r_t$ edges of $Q$. The convex hull of $V_1', V_2', \ldots, V_t'$ is a $t$-gon which we denote by $R'$. (In the nets of Figure 5 the polygons $R'$ are indicated by dashed lines). Now choose a constant $\lambda < \frac{1}{2}$ and construct another $t$-gon $R$ with vertices $V_1, V_2, \ldots, V_t$ where $V_i$ lies on $OV_i'$ (here $O$ is the centre of $Q$) and $|OV_i| = \lambda |OV_i'|$ for each $i = 1, \ldots, t$. The convex hull of $V_1, V_2, \ldots, V_t$ is a $t$-gon similar and similarly situated to $R'$. We shall denote this polygon by $R$ and call it the *root polygon* for the given $Q$ and partition. (In Figure 5, the root polygons are shaded.) The root polygon is a $t$-gon where $t$ is the number of parts in the chosen partition. In the case of a partition into 2 parts the root polygon is a 2-gon, that is a line segment counted twice.

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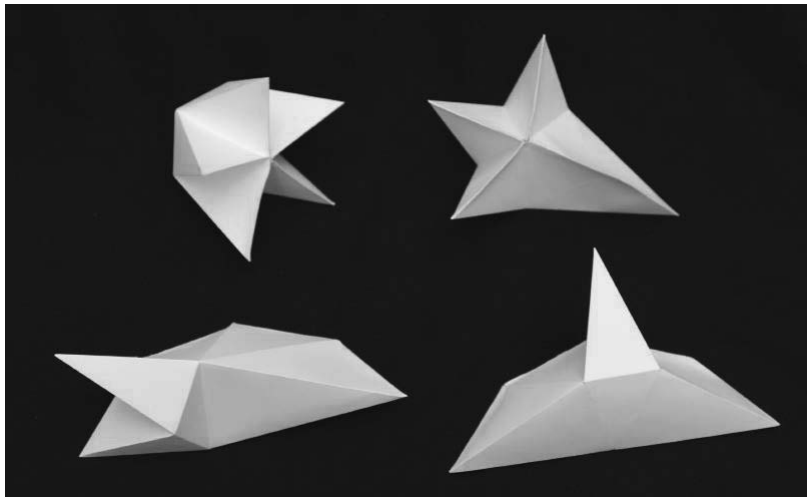


FIGURE 6

The edges of the root polygon are fold-lines, and it should be clear from the examples for the 10-gon in Figure 5 how $n + t$ additional fold-lines are drawn from the vertices of the root polygon to the vertices of $Q$. We claim that the result of this construction is, for each partition, the net of a polyhedron. The polyhedra obtained from these nets may be described thus: construct a pyramid on the root polygon as base, and to its sloping faces attach fan polyhedra of lengths $\left[\frac{1}{2}r_1, \frac{1}{2}r_2, \ldots, \frac{1}{2}r_t\right]$. The sum of the face angles of the tetrahedra in a fan at the root is less than 180°, as can be seen, for example, in Figure 5 (but note this condition does not hold in Figure 3). This implies that each adjoined fan lies on one side of a face of the pyramid on the core and hence there are no self-intersections. Photographs of the four polyhedra whose nets appear in Figure 5 are shown in Figure 6.

The above construction yields $p^{(>2)}(n)$ combinatorially distinct polyhedra. One more arises from the partition of $2n$ into either one or two parts. Both yield a net with a core line similar to that in Figure 3. Hence we have one more net, and in total we have $1 + p^{(>2)}(n)$ nets. The theorem is therefore proved.

Of course there may exist more edge-to-edge polyhedra with regular polygons as carriers but we know of none. The number $1 + p^{(>2)}(n)$ is only a lower bound.

## 3. *Carriers which are not regular polygons*

We now consider the general question of whether there exist edge-to-edge convex nets whose carriers are not regular polygons. We have already mentioned that Edge Condition (E) is not a sufficient condition for a polygon to be the carrier of the edge-to-edge net of a polyhedron, but clearly it is necessary. So we shall, from now on, assume (E) is satisfied and the $2n$

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edges of $Q$ are paired off in some way. Allocate the same labels to both edges in a pair, so that, in all, $n$ labels will be needed. The fundamental question is the following: with labels assigned to the edges of $Q$ in this way, what are the necessary and sufficient conditions for it to be possible to insert fold-lines so that the labelled polygon becomes an edge-to-edge net of a polyhedron? We cannot answer this question completely, but we can state a necessary condition.

Read off the labels in order as we go clockwise round $Q$. We shall obtain what we shall call an edge-word for $Q$. For example an *edge-word* of the polygon $Q$ in Figure 5(d) is *abccbdeeda*. The edge-word is only defined within a cyclic permutation since we can start reading it off from any edge of $Q$.

(I) *Identity Condition*: The edge-word of a carrier satisfies the Identity Condition if its value can be reduced to the identity $i$ using the relations

$$a^2 = b^2 = c^2 = \ldots = i.$$

Notice that this condition does not depend on which cyclic permutation of the edge-word is chosen; an edge-word satisfies (I) if and only if every cyclic permutation of it also satisfies (I). We make the following conjecture:

*Conjecture*: A polygon $Q$ which satisfies the Edge Condition (E) and whose edge-word satisfies (I), is the carrier of an edge-to-edge net of a (non-self-intersecting) polyhedron.

We know of no counter-example to this conjecture, but a general proof is missing. However we can prove it in two special cases:

(i) Firstly suppose that the edge-word consists of a sequence of letters, followed by the same sequence in the reverse order, that is, it is *palindromic*. Then Condition (I) will automatically be satisfied. (This holds for all the carriers in Figure 3 and also for edge-word *abcdeedcba* of the carrier in Figure 7(a)). In assembling a polyhedron from such a net the edge $AB$ is pasted to the edge $A'B'$ and together they will form one edge of a tetrahedron whose opposite edge is the core line $XY$. For this tetrahedron to exist, the lengths of fold-lines that join $X$ and $Y$ to $A$ and $B$ as well as the lengths $|AB|$ and $|XY|$ must obey certain inequalities (the triangle inequalities and others). These conditions are not easy to state, but they will automatically be satisfied if the following condition holds:

(S) The labelled carrier is symmetrical about a diameter, that is, a line joining two of its vertices.

In other words, not only does reflection in the diameter map every edge onto another edge, but both these edges carry the same label. Notice that (S) implies (E) because (S) implies the edge-word is of the form considered in (i). The polygon in Figure 7(a) satisfies Condition (S), the diameter about which the polygon is symmetrical is indicated by a dashed line. We deduce that any polygon satisfying (E) and (S) corresponds to a polyhedron built up from tetrahedra and so is the carrier of an edge-to-edge net of a stack

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polyhedron. As with the polyhedra arising from the nets in Figure 5, this will be a fan polyhedron.

A photograph of the polyhedron to which the carrier in Figure 7(a) corresponds appears on the left of Figure 8.

(ii) Suppose that condition (i) on the edge-word does not hold. Then it will be possible to divide the edge-word into maximal *subwords* each equal to the identity. For example, in Figure 7(b) the edge-word is *aabccbdeffedgg* and it is divided thus:

$$aa \mid bccb \mid deffed \mid gg.$$

(Here 'maximal' means that the subwords cannot be extended by a letter at the beginning and end, and still remain equal to the identity). The lengths of the subwords form a partition of $2n$ (in this case [2, 4, 6, 2]) which can then be used as in the construction of the carriers of Figure 5. (Notice that an edge-word satisfying (I) cannot be divided into two maximal subwords, because the condition for maximality fails to hold for either of the parts.)

This partition [2, 4, 6, 2] is illustrated in the polygon of Figure 7(b). As in the discussion of case (i) the edges $AB = A'B'$ and $BC = B'C'$ belong to tetrahedra each of which has the core $XY$ as an opposite edge. As in the previous case, the required inequalities are satisfied if $A$ and B are the images of $A'$ and $B'$ under reflection in the line joining $C$ to the midpoint of the chord $AA'$. This is the perpendicular bisector of $AA'$, and also the perpendicular bisector of $XY$. These conditions will hold for all the subwords of the edge-word if (C) is true:

(C) $Q$ is concyclic, that is, all its vertices lie on a circle.

The net shown in Figure 7(b) has a carrier that satisfies (C).

Note that if (C) holds then the root polygon is also concyclic, and all the perpendicular bisectors of its edges are concurrent at the centre of the circle. Then, as in case (i), symmetry about these perpendiculars is sufficient to ensure that we obtain nets of tetrahedra such as $ABXY$ and $BCXY$ and so the whole is the net of a polyhedron.

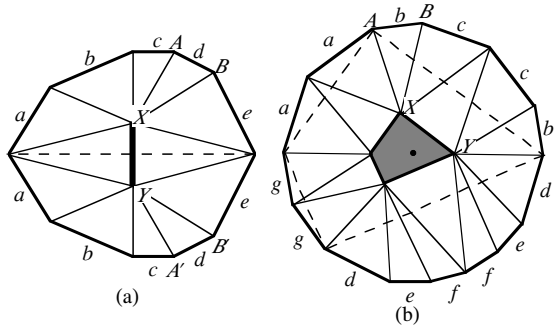



FIGURE 7

The polyhedron whose net is Figure 7(b) is shown on the right of

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Figure 8. Its close resemblance to the carriers shown in Figure 5 will be apparent.

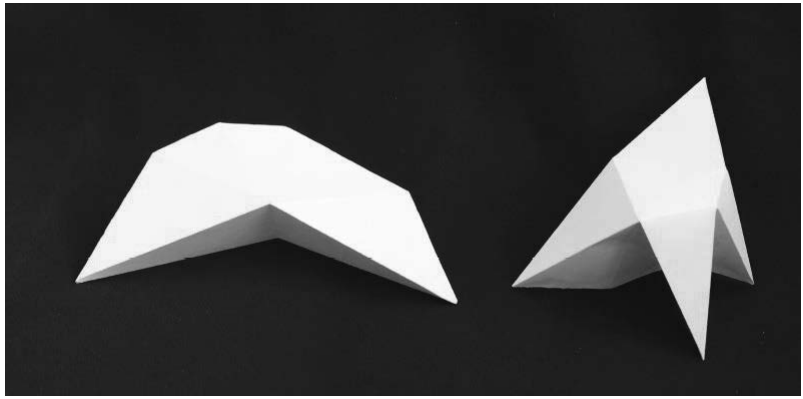


FIGURE 8

We deduce:

*Theorem*: A convex polygon $Q$ which satisfies the Edge Condition (E) whose edge-word satisfies (I) and which also satisfies either (S) or (C), is the carrier of an edge-to-edge net of a polyhedron.

As in the case of regular polygons (compare Figures 5 and 7), the fact that the sum of the face angles of the tetrahedra in a fan at the root is less than $\pi$ ensures that the resulting polyhedron is not self-intersecting.

Finally, we remark that in recent years there have been major advances in solving the problems of which polygons are carriers of polyhedra, but, as we see from the conjectures above there are still open problems, especially concerning edge-to-edge nets. For example, does the above theorem remain true without the (somewhat artificial) conditions (S) or (C)? For non-edge-to-edge nets the results of [4] do not end the field for investigation for they make no assertions as to the number of faces of the resulting polyhedra.

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