Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **9**, No. 3, pp. 680-697

DOI: 10.4208/aamm.2014.m783
June 2017

Solution of Boundary Value Problems Using Dual Reciprocity Boundary Element Method

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Received 6 October 2014; Accepted (in revised version) 21 April 2016

Abstract. In this work we utilize the boundary integral equation and the Dual Reciprocity Boundary Element Method (DRBEM) for the solution of the steady state convection-diffusion-reaction equations with variable convective coefficients in two-dimension. The DRBEM is a numerical method to transform the domain integrals into the boundary only integrals by using the fundamental solution of Helmholtz equation. Some examples are calculated to confirm the accuracy of the approach. The results obtained by the analytic solutions are in good agreement with ones provided by the DRBEM technique.

AMS subject classifications: 65M10, 78A48

Key words: Convection-diffusion-reaction equation, dual reciprocity method, radial basis functions, fundamental solution.

1 Introduction

Combining of two dimensional (2D) steady state convection equation and diffusion equation with a linear reaction term leads to the 2D steady state convection-diffusion-reaction (SSCDR) equation. Many branches of science and engineering, especially in transport and fluid dynamics problems, governed by these type of equations and their combinations [1–5]. Modeling chemical, biochemical, physical, environmental and biological phenomena mathematically are samples of these transport problems.

Some researchers attempted to solve destabilizing effects caused by the presence of the first order derivatives in the CDR equations. Authors used different methods for the problem such as finite difference, finite volume and especially various types of finite element methods as well as combinations of them [5–16]. Today, we are still required better numerical technique for more accurate approximating partial differential equations

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(PDEs) that include first order spatial derivatives (convective term) and hence it is an attractive and challenging issue in fluid dynamics problems [17–19].

The aforementioned numerical methods have been employed as computational tools for discretization in computational fluid dynamics. The boundary element method (BEM) is another technique [20–24] for solving efficiently different types of PDEs such as linear and non-linear, homogeneous and non-homogeneous, time-dependent and steady state equations in different engineering problems. Bozkaya and Tezer-Sezgin [25] developed a BEM formulation for the solution of magneto-hydrodynamic flow in a semi-infinite duct. Hosseinzadeh et al. [26] recently utilized the constant and continuous linear BEMs to obtain the numerical solution of the coupled equations in velocity and induced magnetic field for the steady magneto-hydrodynamic (MHD) flow through a pipe of rectangular and circular sections having arbitrary conducting walls. Damanpack et al. [27] applied BEM to the bending analysis of thin functionally graded plates. Pisa and Aliabadi analysed bond-line cracks in thin walled aircraft structures by using the BEM [28].

The main advantage of this approach is to transform the domain integrals arising from PDEs into the boundary integrals by using the Green's second identity and a fundamental solution, since it reduces the dimension of the problem by one unit and then saves computational time and memory. However the main disadvantages of the BEM application to non-homogeneous and more general PDEs are due to the facts that the explicit fundamental solutions are not available and thus additional internal domain discretization is needed [23, 24, 29].

The DRBEM which was originally introduced by Nardini and Brebbia [30] is thus the most successful method for keeping above advantages. The DRBEM rearranges the PDE terms into an equation to find the fundamental solutions and then approximates the nonhomogeneous term b by interpolating functions $f(r)$, called radial basis functions (RBFs) in terms of its values at the collocation points, where r is the distance between a source point and the field point. Dehghan and Mirzaei [31, 32] have applied the boundary integral equation and DRBEM for solving 1D Cahn-Hilliard equation and 2D sine-Gordon equation respectively. Yun and Ang [33] treated the DRBEM for axisymmetric thermo-elastostatic analysis of non-homogeneous materials. In [34] a numerical method based on the boundary integral equation and dual reciprocity method for solving the 1D advection-diffusion equations is presented. Dehghan and Ghesmati [35] applied the dual reciprocity boundary integral equation technique to solve the nonlinear Klein-Gordon equation. A DRBEM formulation utilized for the solution of incompressible magneto-hydrodynamic (MHD) flow equations by Bozkaya and Tezer-Sezgin [36]. Purbolaksono and Aliabadi employed the DRBEM to evaluate large deformations of shear deformable plates [37]. Authors in [38] employed the DRBEM to evaluate the particular solution of Poisson equation with multi-quadratic functions. Tanaka and Chen [39] presented combined application of DRBEM and differential quadrature method to time-dependent diffusion problems.

Although many works have been done regarding to the DRBEM solving the different equations but efficient numerical schemes for solving general equations are scarce. This

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paper is presented to solve those equations with mixed different boundary conditions and variable convective coefficients as well as dependency of the internal and boundary node numbers. The layout of this paper is as follows: In Section 2, the governing equation and boundary condition of problem are presented. The discretization of the steady state convection-diffusion-reaction equation using dual reciprocity boundary element method is described in Section 3. The numerical and analytical of three examples are compared in Section 4. And, last section summarizes the conclusion of numerical results.

2 Problem statement

Let us consider a temperature field $u(x,y)$ in a homogeneous isotropic domain $\Omega \subset \Re^2$ bounded by a piece-wise smooth boundary $\Gamma = \partial\Omega$ such that $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \cap \Gamma_N = \emptyset$ with Dirichlet and Neumann boundary conditions respectively (see Fig. 1).

The mathematical equation governing non-homogeneous steady state convection-diffusion-reaction type problems in 2D is:

$$D\nabla^2 u - v\cdot\nabla u - ku - g = 0 \quad \text{in } \Omega. \tag{2.1}$$

With respect to a particular problem, u can demonstrate a temperature or the concentration of a chemical species. $D>0$ is the diffusion coefficient of the temperature or the species, $v = v(x,y)$ is the velocity field of fluid flow, $k>0$ is a constant reaction coefficient and $g = g(x,y)$ is a function depending only on the x and y coordinates. Boundary conditions of the problem are given as:

$$u(x,y) = \bar{u}(x,y) \qquad \text{on } \Gamma_D, \tag{2.2a}$$
$$q(x,y) = \partial u(x,y)/\partial n(x,y) = \bar{q} \qquad \text{on } \Gamma_N, \tag{2.2b}$$

where n is the outward normal vector to the boundary Γ, $\bar{u}$ and $\bar{q}$ are known boundary functions.

3 DRBEM formulation

There are several analytical and numerical methods for solving the above steady state boundary value problem. We employ the boundary integral equation and DRBEM for Eq. (2.1) subject to boundary conditions Eq. (2.2) to find the BEM solution. Because the velocity field may be variable and the fundamental solution of Helmholtz operator is available, the convective and non-homogeneous terms are shifted to the right hand side (RHS) and then the Eq. (2.1) can be rewritten as:

$$D\nabla^2 u - ku = v_x\frac{\partial u}{\partial x} + v_y\frac{\partial u}{\partial y} + g(x,y) = b(x,y). \tag{3.1}$$

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Now to expand the new non-homogeneity of the above equation (i.e., function $b(x,y)$) in terms of its values at each nodes based on the idea of the DRBEM, we utilize a set of radial basis functions (RBFs) f_k such that

$$b(x,y)=\sum_{k=1}^{N+L} f_k(r_k(x,y))\alpha_k, \tag{3.2}$$

where α_k is a set of undetermined interpolation coefficients, N is the number of collocation nodes on the boundary, L is the number of collocation points in the domain and r_k is defined as the distance between the node under consideration and the node k.

$$r_k(x,y)=\sqrt{(x-x_k)^2+(y-y_k)^2}. \tag{3.3}$$

The vector α_k is obtained from Eq. (3.2) as

$$\alpha_k=\sum_{k=1}^{N+L} F_{km}^{-1}b_m, \tag{3.4}$$

where F^{-1} is the inverse of matrix f. The radial basis function used in the dual reciprocity technique verifies the equation

$$\nabla^2\hat{u}_k-k\hat{u}_k=f_k. \tag{3.5}$$

There are different types of RBFs, e.g., polynomials $(c_0+c_1r+c_2r^2+\cdots)$, thin plate Splines $(r^2\ln r)$, multi-quadrics $(\sqrt{r^2+c^2})$, etc. Although the simple RBF, means $f_k=1+r_k$, is effective and has good results with most of problems [40–42], but it may not be used in this work. For more information about the RBFs the reader may refers to [43–45]. The authors tested several polynomial functions and chose the following particular solution that is the better ones based our experiments

$$\hat{u}_k=1+r_k^3, \tag{3.6}$$

and then the corresponding RBF can be obtained by substitution of $\hat{u}_k$ into Eq. (3.5)

$$f_k(x,y)=9r_k-k(1+r_k^3). \tag{3.7}$$

For simplicity we consider $D=1$. Substituting Eq. (3.5) into Eq. (3.2), we have

$$b(x,y)=\sum_{k=1}^{N+L}(\nabla^2\hat{u}_k-k\hat{u}_k)\alpha_k. \tag{3.8}$$

Let us consider W as the weighting function and the fundamental solution of Helmholtz equation

$$\nabla^2W-kW=-\delta(P(x,y),Q(\xi,\eta)),$$

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δ is the Dirac delta function and P, Q are a field point and a source point respectively. The fundamental solution and its normal derivative are given as

$$W=\frac{1}{2\pi}K_0(\sqrt{k}r_k), \tag{3.9a}$$

$$\frac{\partial W}{\partial n}=W'=-\frac{\sqrt{k}}{2\pi}K_1(\sqrt{k}r_k)\frac{\partial r}{\partial n}, \tag{3.9b}$$

where K_0 and K_1 are the modified Bessel's function of the second kind of the order zero and one, respectively. Applying the weighted residual technique to Eq. (3.1) and using Eq. (3.8), gives

$$\int_\Omega(\nabla^2u-ku)d\Omega=\int_\Omega W\Big[\sum_{k=1}^{N+L}(\nabla^2\hat{u}_k-k\hat{u}_k)\alpha_k\Big], \tag{3.10}$$

using the Green's second identity on both sides of the above equation, the resulting boundary integral equation is obtained as Eq. (3.10) which is a boundary only integral equation

$$\Big[\int_\Gamma W\frac{\partial u}{\partial n}-\frac{\partial W}{\partial n}u\Big]d\Gamma-\epsilon_iu_i=\sum_{k=1}^{N+L}\Big[\int_\Gamma\Big[W\frac{\partial\hat{u}_k}{\partial n}-\frac{\partial W}{\partial n}\hat{u}_k\Big]d\Gamma-\epsilon_i\hat{u}_{ik}\Big]\alpha_k, \tag{3.11}$$

where ϵ_i in the equation depends upon the location of the source point, $\epsilon_i(P)=1$ for $P\in\Omega$ and $\epsilon_i(P)=0.5$ for $P\in\Gamma$ and $\partial\hat{u}/\partial n$ is the normal derivative of the function $\hat{u}$ as

$$\hat{u}'=\frac{\partial\hat{u}}{\partial n}=3r_k[(x-x_k)n_x+(y-y_k)n_y], \tag{3.12}$$

using the constant element approximation, the discretized form of the Eq. (3.11) is given by [43]

$$D\Big[\sum_{j=1}^{N}W_{ij}u'_j-\sum_{j=1}^{N}W'_{ij}u_j-\epsilon_iu_i\Big]=\sum_{k=1}^{N+L}\alpha_k\Big[\sum_{j=1}^{N}W_{ij}\hat{u}'_jk-\sum_{j=1}^{N}W'_{ij}\hat{u}_{jk}-\epsilon_i\hat{u}_{ik}\Big]. \tag{3.13}$$

If we define a matrix H_{ij} as

$$H_{ik}=\sum_{j=1}^{N}W_{ij}\hat{u}'_{jk}-\sum_{j=1}^{N}W'_{ij}\hat{u}_{jk}-\epsilon_i\hat{u}_{ik}, \tag{3.14}$$

then Eq. (3.13) can be expressed as

$$\Big[\sum_{j=1}^{N}W_{ij}u'_j-\sum_{j=1}^{N}W'_{ij}u_j-\epsilon_iu_i\Big]=\sum_{k=1}^{N+L}H_{ik}\alpha_k. \tag{3.15}$$

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Employing Eq. (3.4), the RHS of Eq. (3.15)

$$\text{RHS} = \sum_{m=1}^{N+L} G_{im} b_m, \tag{3.16}$$

in which

$$G_{im} = \sum_{k=1}^{N+L} H_{ik} F_{km}^{-1}. \tag{3.17}$$

Substituting Eq. (3.16) into Eq. (3.15), giving rise to

$$\Big[\sum_{j=1}^{N} W_{ij} u_j' - \sum_{j=1}^{N} W_{ij}' u_j - \epsilon_i u_i\Big] = \sum_{m=1}^{N+L} G_{im} b_m, \tag{3.18}$$

as defined previous in Eq. (3.1) $b(x,y)$ is a function of the first order derivative of $u(x,y)$. We can utilize a global interpolation function for $u(x,y)$ to approximate the values of its derivatives at all the collocation points in similar way to that of for $b(x,y)$ [46]. We choose

$$u(x,y) = \sum_{k=1}^{N+L} q_k(r_k(x,y))\mu_k, \tag{3.19}$$

where q_k is the RBF and we choose it the same as the RBF f_k to increase computer efficiency [47]. To approximate the convective term, differentiating Eq. (3.19) and then substituting Eq. (3.4) into the resulted equation, leads to

$$\frac{\partial \vec{u}}{\partial x} = \frac{\partial \vec{F}}{\partial x}\vec{\mu} = \frac{\partial \vec{F}}{\partial x}\vec{F}^{-1}\vec{u}, \tag{3.20a}$$

$$\frac{\partial \vec{u}}{\partial y} = \frac{\partial \vec{F}}{\partial y}\vec{\mu} = \frac{\partial \vec{F}}{\partial y}\vec{F}^{-1}\vec{u}. \tag{3.20b}$$

The nodal values of $b(x,y)$ at the collocation points can be expressed as

$$b_m = (v_x)_m \sum_{n=1}^{N+L} R_{mn} u_n + (v_y)_m \sum_{n=1}^{N+L} S_{mn} u_n + M_m, \tag{3.21}$$

where

$$\vec{R} = \frac{\partial \vec{F}}{\partial x}\vec{F}^{-1}, \tag{3.22a}$$

$$\vec{S} = \frac{\partial \vec{F}}{\partial y}\vec{F}^{-1}, \tag{3.22b}$$

$$M_m = g(x_m, y_m). \tag{3.22c}$$

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Consequently, the final discretized equation of the problem by combining Eqs. (3.18) and (3.21) are expressed in the following equation

$$D\Big[\sum_{j=1}^{N} W_{ij}u'_j - \sum_{j=1}^{N} W'_{ij}u_j - \epsilon_i u_i\Big] = \sum_{n=1}^{N+L} Y_{in}u_n + \sum_{m=1}^{N+L} G_{im}M_m. \tag{3.23}$$

The new matrix Y_{in} in the above equation is defined as

$$Y_{in} = \sum_{m=1}^{N+L} G_{im}(v_x)_m R_{mn} + \sum_{m=1}^{N+L} G_{im}(v_y)_m S_{mn}. \tag{3.24}$$

Introducing the boundary conditions Eq. (2.2) into Eq. (3.23) and rearranging by taking known variables to the right hand side of the equation, and unknown variables to the left hand side, we can get a linear algebra system of equations that may be solved by employing the usual Gauss elimination method or the efficient LU decomposition technique.

4 Numerical results

In this section, the numerical results of several transport problems are presented. Comparisons between numerical results and corresponding analytical solutions can be given for three examples to evaluate accuracy and efficiency of the method described in this paper.

1. A two-dimensional SSCDR equation without convective term and with the constant reaction coefficient and non-homogeneous term,

2. A one-dimensional SSCDR equation in which the convective coefficient is a linear function of the x coordinate,

3. A two-dimensional SSCDR equation in which the convective coefficient is a function of the second order in y-direction coordinate, and it is symmetric or asymmetric about y-axis. In all examples, the numbers of boundary nodes are selected $NB = 50,100,200$ and the internal nodes are selected $NI = 254,513$ and 1521. The program code is written in PGI (Portland Group Inc) Visual Fortran.

4.1 Example 1

In the first example, we consider an non-homogeneous steady state convection-reaction transport problem in two-dimension and on square domain, $\Omega = (0,1)\times(0,1)$ (see Fig. 1) with the constant coefficients, $D = 1$, $k = 0.5,2$

$$\nabla^2 u - ku = g, \tag{4.1}$$

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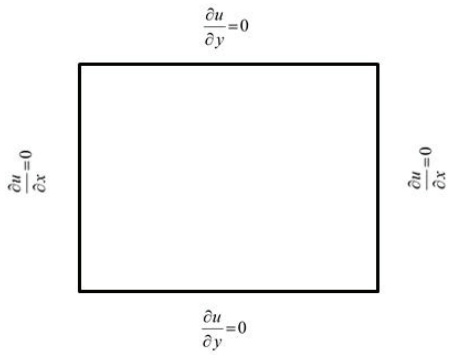


Figure 1: Domain and boundary conditions for Example 1.

where

$$g(x,y)=2(x-y)+k\left[\frac{x^2-y^2}{2}-\frac{x^3-y^3}{3}\right]. \tag{4.2}$$

The boundary conditions are as following:

$$\frac{\partial u}{\partial n}=0 \qquad \text{on } \Gamma_N. \tag{4.3}$$

The analytical solution is

$$u=\frac{y^2-x^2}{2}-\frac{y^3-x^3}{3}. \tag{4.4}$$

Figs. 2 and 3 show the relative errors

$$RE=\left|\frac{\text{Numerical}-\text{Analytical}}{\text{Analytical}}\right|$$

of the scalar function $u(x)$ on a bound of the domain for different reactive coefficients $(k=0.5,2.0)$. Because the domain and boundary conditions are symmetric, the result is presented just for one bound. Figs. 4 and 5 present the relative errors of the function $u(x)$ along a horizontal face, $y=0.025$, with $k=0.5$ and $k=2.0$ when the DRBEM is used. Different boundary and internal nodes were tested. As can be seen the error is decreased as the number of boundary nodes is increased and increasing the internal nodes does not change the result. The first and second numbers of all figures' legends are the number of boundary and internal nodes respectively. For example, in the legend of 2, $50-254$ means that $NB=50$ and $NI=254$. Also, it can be observed that increasing the reaction coefficient do not change the result significantly. In this example, when the boundary nodes are 50 or 100 or 200, and the internal nodes are increased (from 254 to 1521), there are no any changes in the values of boundaries solution for two values k. but the relative errors of the function for $k=0.5$ are higher than those of the function for $k=2.0$ in the domain. The relative errors on boundaries and in the domain are below 10 and 3.2 percent respectively. It can be concluded that we can obtain a solution with minimum errors just by choosing minimum and maximum number of internal and boundary nodes respectively.

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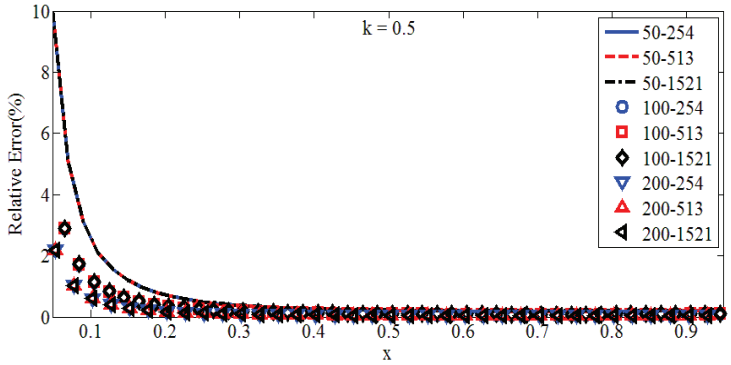


Figure 2: Relative errors between numerical and analytical solutions for the function $u(x)$ along one of the boundaries for reactive coefficients $k=0.5$.

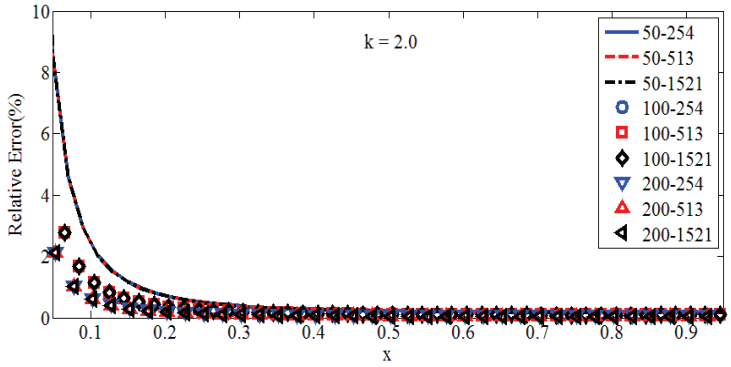


Figure 3: Relative errors between numerical and analytical solutions for the function $u(x)$ along one of the boundaries for reactive coefficients $k=2.0$.

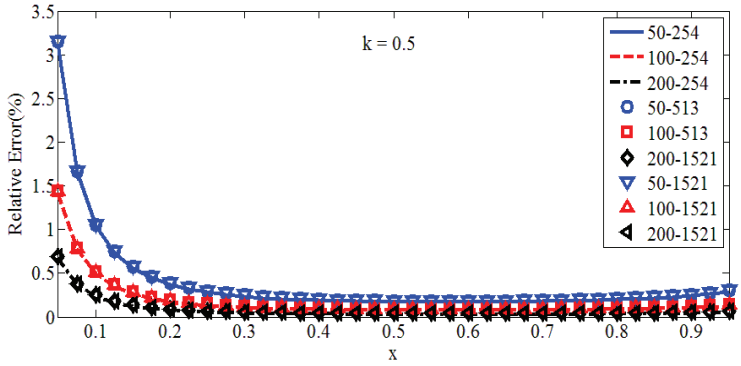


Figure 4: Relative errors of function $u(x)$ for a horizontal face $y=0.025$ with $k=0.5$.

4.2 Example 2

To check the efficiency of the DRBEM method for solving SSCDR problems, in this section we presents a $1D$ SSCDR equation in a square domain, $\Omega=(0,1)\times(0,1)$ as shown in

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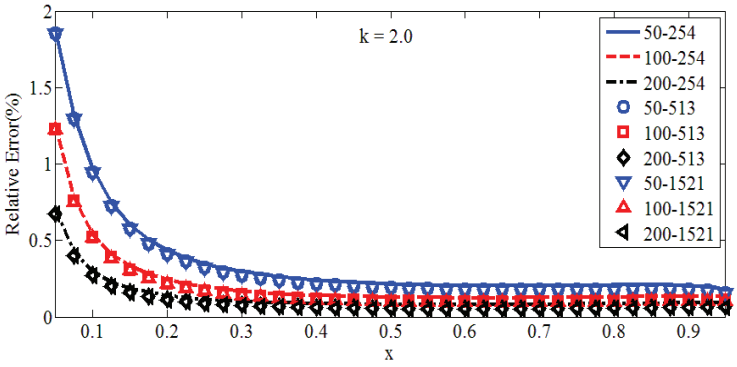


Figure 5: Relative errors of function $u(x)$ for a horizontal face $y=0.025$ with $k=2.0$.

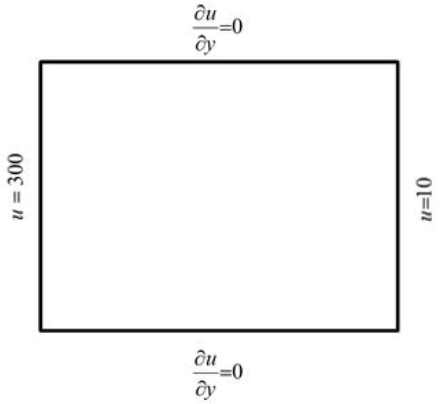


Figure 6: Domain and boundary conditions for Example 2.

Fig. 6, in which the convective coefficient is a linear function of the x coordinate and other coefficients are constant, $D=1$, $k=1.5$. Then the governing equation is given by

$$\nabla^2 u-(B_1+kx)\frac{\partial u}{\partial x}-ku=0, \tag{4.5}$$

where $B_1=\ln(u_b/u_a)-k/2$.

The problem boundary conditions are

$$u=u_a=300, \quad x=0; \quad 0\le y\le 1, \tag{4.6a}$$
$$u=u_b=10, \quad x=1; \quad 0\le y\le 1, \tag{4.6b}$$
$$\frac{\partial u}{\partial y}=0, \quad y=0; \quad 0\le x\le 1, \tag{4.6c}$$
$$\frac{\partial u}{\partial y}=0, \quad y=1; \quad 0\le x\le 1. \tag{4.6d}$$

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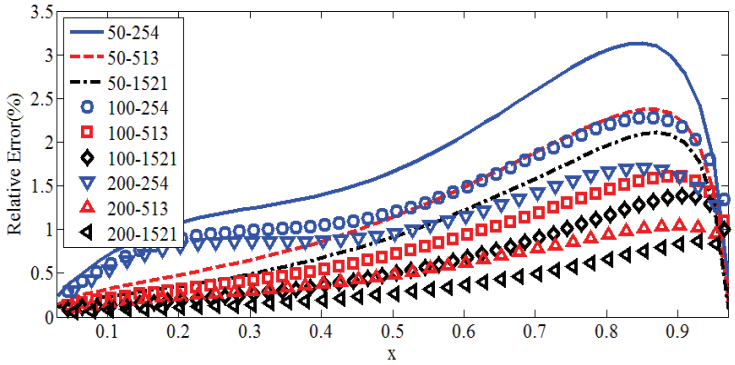


Figure 7: Relative errors between numerical and analytical solutions for the function $u(x)$ along one of the boundaries for reactive coefficient $k=1.5$.

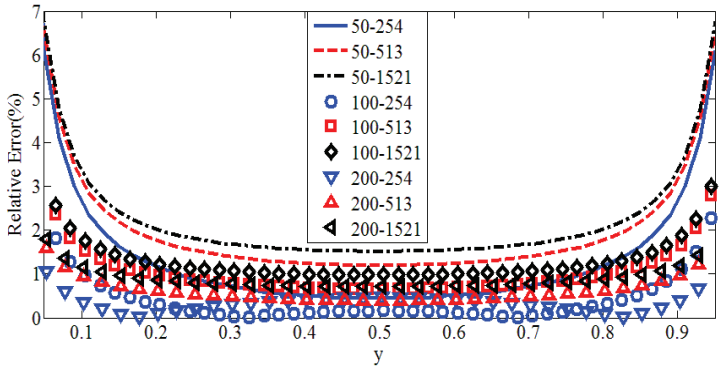


Figure 8: Relative errors between numerical and analytical solutions for the normal derivative at the vertical boundary $x=1.0$ for reactive coefficient $k=1.5$.

A particular solution of the above equation can be obtained

$$u=u_a\left[\exp\left(\frac{k}{2}x^2\right)+\left(\ln\frac{u_b}{u_a}-\frac{k}{2}\right)\right]. \tag{4.7}$$

Fig. 7 depicts the relative errors of resulted horizontal distribution of scalar function $u(x)$ along the face $y=0$ of the domain for the reactive coefficient $k=1.5$. It appears that the discrepancy between the DRBEM results and exact solutions are acceptable. The relative errors of the normal derivative associated to the method were analyzed and quantified with respect to the analytical solution for vertical side $x=1$ as demonstrated in Fig. 8. The relative errors of function $u(x)$ along a horizontal line $y=0.025$ are shown in Fig. 9.

It can be seen from the Figs. 7, 8 and 9 that the relative errors are generally low (almost less than 3 percent) and decreased with increasing boundary and internal nodes. In the Fig. 8, the trend of the normal derivative of the function is some different, so that the accuracy is lowered with enhancing internal nodes and decreasing boundary nodes. The relative error at the two ends $(y=0$ and $y=1)$ is higher than that of the middle. It's

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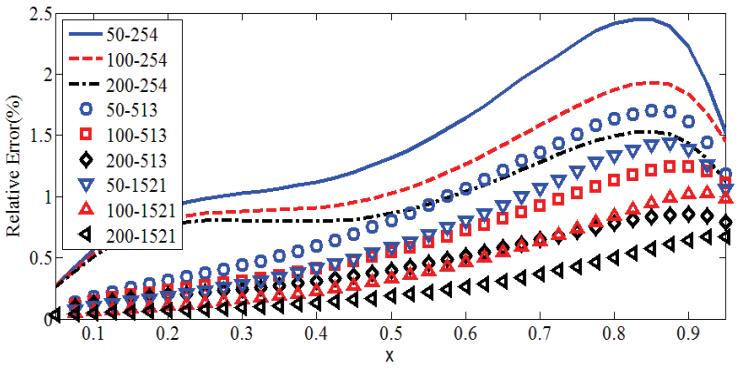


Figure 9: Relative errors of function $u(x)$ for a horizontal line $y=0.025$ with $k=1.5$.

worth noting that the effect of boundary nodes is more than that of internal nodes. The maximum and minimum relative errors are corresponding to minimum and maximum boundary nodes respectively and errors decreased on the boundaries from maximum around 7 to the 0.05 percent and in the domain from 2.5 to 0.1 percent.

4.3 Example 3

This example is a 2D SSCDR problem on the square domain $\Omega=(0,1)\times(0,1)$ as shown in Fig. 10. The convective coefficient is a second order function of y-coordinate. The other coefficients are $D=1$, $k=9.5$. The equation is given by

$$\nabla^2 u-\frac{\mu^2}{B_2}(y-A)^2\frac{\partial u}{\partial x}-ku=0, \tag{4.8}$$

where $\mu=k-B_2^2$ and $B_2=\ln(10/300)$. Here, the value of parameter A is assumed by two values as $A=0.5$ and $A=0.25$ to make respectively the profile of the convective coefficient

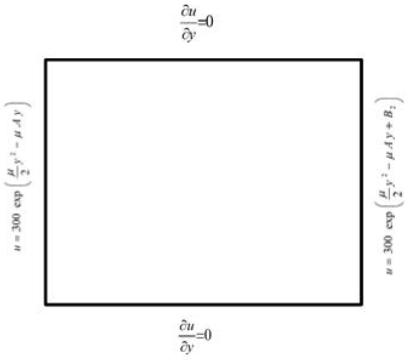


Figure 10: Domain and boundary conditions for Example 3.

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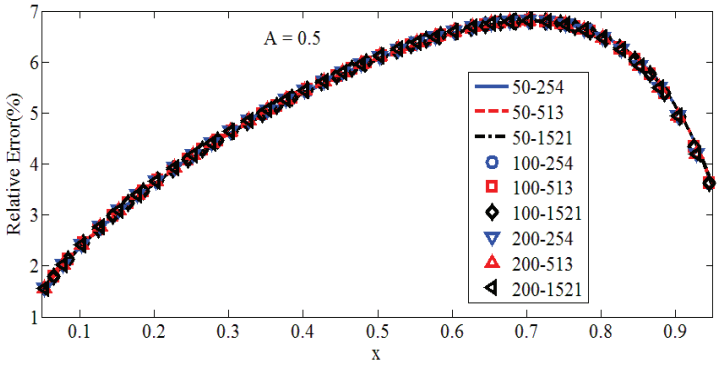


Figure 11: Relative errors between numerical and analytical solutions for the function $u(x)$ at the horizontal boundary $y=0$ for $A=0.5$.

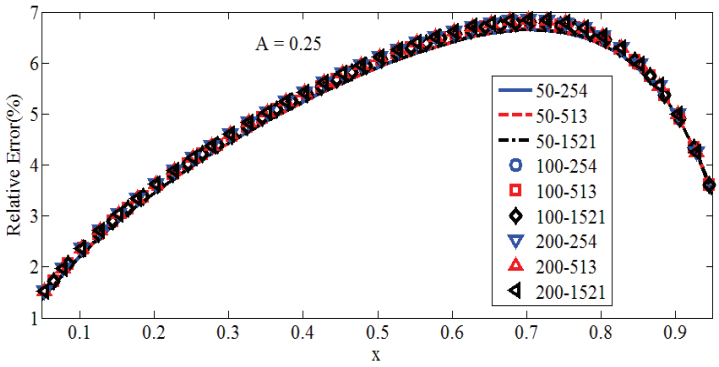


Figure 12: Relative errors between numerical and analytical solutions for the function $u(x)$ at the horizontal boundary $y=0$ for $A=0.25$.

and the function $u(x,y)$ symmetric and non-symmetric with respect to the y-coordinate.

The boundary conditions corresponding to the problems are defined as

$$u=u_a=300\exp\left(\frac{\mu}{2}y^2-\mu Ay\right), \qquad x=0; \qquad 0\le y\le 1, \tag{4.9a}$$

$$u=u_b=300\exp\left[\frac{\mu}{2}y^2-\mu Ay+B_2\right], \qquad x=1; \qquad 0\le y\le 1, \tag{4.9b}$$

$$\partial u/\partial y=300\mu A\exp(B_2x), \qquad y=0; \qquad 0\le x\le 1, \tag{4.9c}$$

$$\partial u/\partial y=300\mu(1-A)\exp\left[B_2x+\mu\left(\frac{1}{2}-A\right)\right], \qquad y=1; \qquad 0\le x\le 1. \tag{4.9d}$$

A particular solution of the above equation can be obtained

$$u=300\exp\left[\frac{\mu}{2}y^2-\mu Ay+B_2x\right]. \tag{4.10}$$

Figs. 11 and 12 present the relative errors between the obtained horizontal $u(x,y)$ profile and the corresponding analytical solution along the face $y=0$ for $A=0.5$ and $A=0.25$

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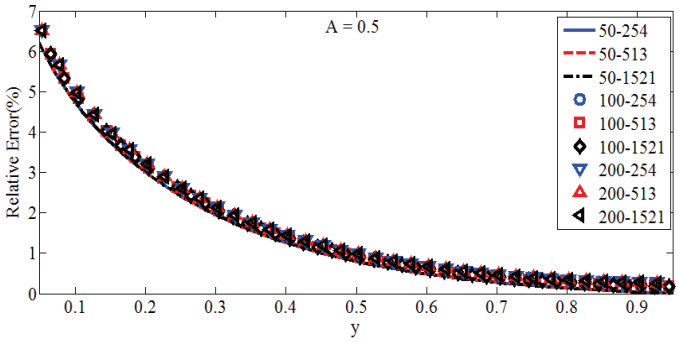


Figure 13: Relative errors between Numerical and analytical solutions for the normal derivative along the vertical boundary $x=1$ for $A=0.5$.

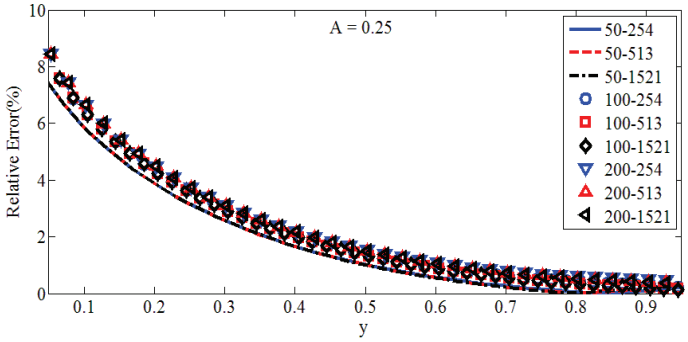


Figure 14: Relative errors between numerical and analytical solutions for the normal derivative along the vertical boundary $x=1$ for $A=0.25$.

respectively. With respect to the accuracy achieved, it can be seen from Figs. 13 and 14 for $A=0.5$ and $A=0.25$ respectively, good numerical results for the normal derivative of the function $u(x)$ have been obtained. The differences between the analytical solution and the numerical solution for a horizontal line $y=0.025$ of the domain can be shown from Figs. 15 and 16 for $A=0.5$ and $A=0.25$ respectively. All results of the Figs. 11 to 16 show that the accuracy of the solutions remains approximately constant with changing the number of boundary and internal nodes, but for different values of parameter A the accuracy of the function u in the domain and also its normal derivatives on the boundaries isn't the same. Finally, we can say that this example is almost independent by mesh numbers and it is concluded that the minimum number of nodes i.e., $NB=50$ and $NI=254$ is sufficient for obtaining the nearly accurate solutions by the DRBEM.

5 Conclusions

In this study, the formulation of the boundary integral equation and the dual reciprocity boundary element method were used for solving the steady state convection-diffusion-

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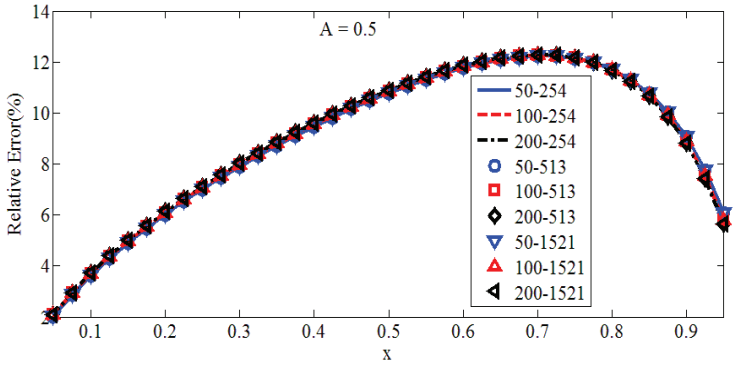


Figure 15: Relative errors of function $u(x)$ for a horizontal line $y=0.025$ for $A=0.5$.

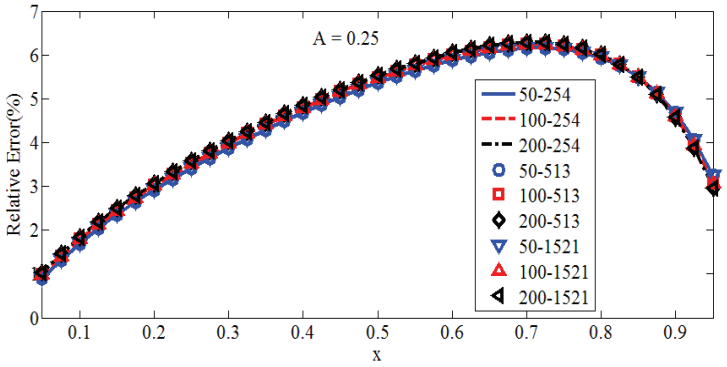


Figure 16: Relative errors of function $u(x)$ for a horizontal line $y=0.025$ for $A=0.25$.

reaction equation on two-dimensional domain. Discretized form of the equation was obtained by using the proposed scheme and its procedure was described in detail. Moreover, the significant advantage and robustness of the DRBEM permitted us to implement it for SSCDR transport problems with variable convective coefficient. The fundamental solution of Laplace operator and the linear radial basis function were utilized to approximate the nonhomogeneous term of the governing equation. The accuracy and efficiency of this technique were evaluated by applying it to several examples with different coefficients and good agreement between numerical results and the corresponding analytical solutions was achieved. The number of nodes on boundary and in domain was changed and it is shown that the effect of boundary nodes on the results is more than that of the internal nodes. It can be resulted that the errors near or at the boundaries are greater than the other location of the domain. It can be concluded that the number of nodes on the boundary and in the internal domain depends on the physics of the problem and boundary conditions. In this paper, we checked three examples with the same nodes number, but the most effectiveness is in the Example 2, no error changes in the Example 3, and no effect of the internal nodes in the Example 1.

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Acknowledgments

The authors would gratefully like to thank the reviewers for their comments that helped us to improve the manuscript.

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