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On uniform interpolation sets

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1. *Introduction.* A well-known result in the interpolation theory of integral functions (see Whittaker (**16, 17**), Pólya (**14**), Iyer (**2**), Pfluger (**10**)) states that an integral function of at most type $\kappa < \frac{1}{2}\pi$ of order 2 bounded at the lattice points $m+in$ $(m, n = 0, \pm 1, \pm 2, \ldots)$ is necessarily constant. That the value $\frac{1}{2}\pi$ cannot be increased is shown by the Weierstrass σ-function. The result has, however, been generalized in several ways; the lattice points being replaced by more general sets and the boundedness condition by one of restricted rate of growth.

Pfluger (**11, 12**) has shown that a set $\{z_n\}$ satisfying:

(i) for any θ_1, θ_2 with $0 < \theta_2 - \theta_1 \leqslant 2\pi$

$$\frac{n(r, \theta_1, \theta_2)}{r^\rho} \to \frac{\rho\tau(\theta_2 - \theta_1)}{2\pi}$$

as $r \to \infty$, where $n(r, \theta_1, \theta_2)$ denotes the number of z_n in the region $0 \leqslant |z| \leqslant r$, $\theta < \arg z \leqslant \theta_2$;

(ii) when ρ is an integer $\sum_{|z_n| \leqslant r} 1/z_n^\rho$ is convergent as $r \to \infty$;

(iii) for some fixed positive δ, if $m \neq n$

$$|z_m - z_n| \geqslant \delta \inf\{|z_m|^{1-\frac{1}{2}\rho}, |z_n|^{1-\frac{1}{2}\rho}\};$$

is such that an integral function at most of type $\kappa < \tau$ of order ρ bounded on the set is necessarily constant. We shall describe a set satisfying conditions (i), (ii), (iii) as belonging to $P(\rho, \tau)$.

Iyer (**3, 6**) described a set $\{z_n\}$ as being 'effective for ρ, τ' if for every integral function $f(z)$ of type $\kappa < \tau$ of order ρ the equation

$$\limsup \frac{\log |f(z_n)|}{|z_n|^\rho} = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^\rho} \tag{1·1}$$

is satisfied. He determined a number of conditions which must be satisfied by such a set, but the only sufficient set of conditions that he gave depends on the canonical product of the z_n, rather than on their geometrical distribution directly. Finally, he showed that a function at most of type $\kappa < \tau$ of order ρ bounded at the points of such a set is necessarily constant.

Several results for non-measurable sets $\{z_n\}$ have been obtained by Noble (**8**). One of these (Theorem 2, Corollary 3) states that if an integral function $f(z)$ satisfies both

$$\liminf_{r \to \infty} \frac{\log M(r, f)}{r^\rho} < \tau \tag{1·2}$$

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and
$$\limsup \frac{\log |f(z_n)|}{|z_n|^\rho} < \tau \tag{1·3}$$
for a set $\{z_n\}$ of $P(\rho, \tau)$, then
$$\limsup \frac{\log |f(z_n)|}{|z_n|^\rho} = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^\rho}. \tag{1·4}$$

Hence sets belonging to $P(\rho, \tau)$ are effective in the sense of Iyer.

The methods used by these writers have been applied to the related problem of inferring properties of a function regular in a sector, given its properties on certain sets of points in the sector. Many of the results obtained require very heavy conditions on the function on the boundary of the sector which, in some circumstances, make knowledge of the behaviour of the function on the set of points superfluous. An interesting result which does not have this drawback is the following theorem due, in a slightly weaker form, to Miss Cartwright.

THEOREM A (Cartwright (1), Iyer (5)). *Suppose that $f(z)$ is regular in the region $S(\alpha, l)$, defined by $|\arg z| \leqslant \alpha \leqslant \pi$, $|z| \geqslant l$, and satisfies*

(i) $\liminf_{r\to\infty} \frac{\log M(r,\alpha,f)}{r^2} < \frac{1}{2}\pi, \quad M(r,\alpha,f) = \sup_{|\theta|\leqslant\alpha} |f(r e^{i\theta})|;$

(ii) $\limsup_{r\to\infty} \frac{\log |f(r e^{\pm i\alpha})|}{r^2} \leqslant \gamma < \frac{1}{2}\pi$

and if $\alpha < \frac{1}{2}\pi$, suppose, in addition, that $\gamma < \frac{1}{2}\pi \sin^2\alpha$;

(iii) *$|f(m+in)| \leqslant A$ for all lattice points $m+in$ in $S(\alpha, l)$; then*
$$|f(r e^{i\theta})| = O(1) \quad as \quad r\to\infty$$
uniformly for $|\theta| \leqslant \beta < \alpha$, where $\sin^2(\alpha-\beta) > 2\gamma/\pi$.

The methods of Iyer and Noble give the following theorem (cf. Iyer (5)).

THEOREM B (Noble (9), Theorem A with $d = 0$, $\theta = 1$). *Suppose that $f(z)$ is regular in a sector $S(\alpha, 0)$ and satisfies*

(i) $\liminf_{r\to\infty} \frac{\log M(r,\alpha,f)}{r^\rho} < \tau;$

(ii) $\limsup_{r\to\infty} \frac{\log |f(r e^{\pm i\alpha})|}{r^\rho} \leqslant \gamma < \tau;$

(iii) $\limsup_{z_n \in S(\alpha,0)} \frac{\log |f(z_n)|}{|z_n|^\rho} \leqslant \kappa < \tau;$

where $\{z_n\}$ is a set of $P(\rho, \tau)$, then
$$\limsup_{r\to\infty} \frac{\log M(r,\alpha)}{r^\rho} \leqslant \sup\{\gamma, \kappa\}.$$

2. *Statement of results.* In this paper we consider interpolation sets which are similar to those of $P(\rho, \tau)$, but which are defined by means of a covering property instead of a density condition. We say that a set $\{z_n\}$ belongs to $B[\rho, \Delta]$ if the union of the discs
$$|z - z_n| \leqslant \Delta |z_n|^{1-\frac{1}{2}\rho} \tag{2·1}$$

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covers the complete complex plane. We shall show, in Theorem 3, that such a set has a subset belonging to $P(\rho,\tau)$ for every $\tau < \pi/\Delta^2 3\sqrt{3}$, and therefore a set of $B[\rho,\Delta]$ will be effective for $\rho, \pi/\Delta^2 3\sqrt{3}$. We can, however, say more than this; our principle interpolation result, which is intermediate to, and in certain cases an improvement on, Theorems A and B, is

THEOREM 1. *Suppose that $f(z)$ is regular in $S(\alpha,l)$ and satisfies*

$$\text{(i)} \quad \liminf_{r\to\infty} \frac{\log M(r,\alpha,f)}{r^\rho} < \tau, \tag{2·2}$$

$$\text{(ii)} \quad \limsup_{r\to\infty} \frac{\log|f(re^{\pm i\alpha})|}{r^\rho} \leqslant \gamma < \tau, \tag{2·3}$$

$$\text{(iii)} \quad \limsup_{z_n\in S(\alpha,l)} \frac{\log|f(z_n)|}{H(|z_n|)} \leqslant \kappa < \infty \tag{2·4}$$

for a set $\{z_n\}$ belonging to $B[\rho,\Delta]$, where $\tau = \pi/\Delta^2 3\sqrt{3}$ and $H(r)$ is a positive non-decreasing function of r with $r^{-\rho}H(r)$ non-increasing and having limit c, possibly zero, with $c\kappa < \tau$. If $\alpha < \pi/\rho$ suppose, in addition, that

$$\sin^2[\tfrac{1}{2}\rho\alpha] > \frac{\gamma}{\tau}\left\{1-\frac{c\kappa(\tau-\gamma)}{\gamma(\tau-c\kappa)}\right\}. \tag{2·5}$$

Then

$$\limsup_{r\to\infty} \frac{\log M(r,\beta,f)}{H(r)} \leqslant \kappa\left\{1+\frac{c\kappa}{\tau-c\kappa}\right\} + K\lim_{r\to\infty}\frac{1}{H(r)} \tag{2·6}$$

for all β in the range $0 < \beta < \alpha$ which satisfy

$$\sin^2[\rho(\alpha-\beta)/2] > \frac{\gamma}{\tau}\left\{1-\frac{c\kappa(\tau-\gamma)}{\gamma(\tau-c\kappa)}\right\}. \tag{2·7}$$

Furthermore, the conclusion remains unchanged if in (iii) *the set $\{z_n\}$ is replaced by a set $\{z_n'\}$ where for some finite positive numbers d, δ*

$$|z_n'-z_n| \leqslant d|z_n|^{1-\frac{1}{2}\rho} \tag{2·8}$$

and for $m \neq n$

$$|z_m'-z_n'| \geqslant \delta\inf\{|z_m'|^{1-\frac{1}{2}\rho}, |z_n'|^{1-\frac{1}{2}\rho}\}. \tag{2·9}$$

If $c\kappa = 0$ and $H(r)$ unbounded, the theorem gives

$$\limsup_{r\to\infty} \frac{\log M(r,\beta,f)}{H(r)} \leqslant \kappa, \tag{2·10}$$

which is the best one could expect, and if $H(r)$ is bounded, i.e. $|f(z_n)| \leqslant A$ for $z_n \in S(\alpha,l)$, then $f(z)$ is bounded in $S(\beta,l)$. The angles α, β in these cases are such that

$$\sin^2[\rho\alpha/2], \quad \sin^2[\rho(\alpha-\beta)/2] > \gamma/\tau.$$

If $c\kappa > 0$, we may assume that $H(r) \sim r^\rho$ and we then obtain

$$\limsup_{r\to\infty} \frac{\log M(r,\beta,f)}{r^\rho} \leqslant \kappa\left\{\frac{\tau}{\tau-\kappa}\right\}, \tag{2·11}$$

where α, β satisfy

$$\sin^2[\rho\alpha/2], \quad \sin^2[\tfrac{1}{2}\rho(\alpha-\beta)] > \frac{\gamma}{\tau}\left\{1-\frac{\kappa(\tau-\gamma)}{\gamma(\tau-\kappa)}\right\}. \tag{2·12}$$

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This result, although not necessarily best possible, gives a smaller estimate for the type of $f(z)$ in $S(\beta, l)$ than that of Theorem B, provided

$$\kappa < \gamma\tau/(\tau+\gamma).$$

Otherwise the result is inferior to that of Theorem B.

For integral functions we have

THEOREM 2. *Suppose that $f(z)$ is an integral function which satisfies both*

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{r^\rho} < \tau \tag{2·13}$$

and

$$\limsup \frac{\log |f(z_n)|}{H(|z_n|)} \leqslant \kappa < \infty \tag{2·14}$$

for $\{z_n\}$ belonging to $B[\rho, \Delta]$, where $\tau = \pi/\Delta^3 3\sqrt{3}$ and $H(r)\uparrow$, $r^{-\rho}H(r)\downarrow \to c$ as $r\to\infty$, with $c\kappa < \tau$. Then

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{H(r)} \leqslant \kappa. \tag{2·15}$$

Theorem 2 only provides additional information in the cases where $H(r)\to\infty$, $r^{-\rho}H(r)\to 0$. There are a number of cases of special interest. For example, if

$$H(r) = (\log r)^2 \quad \text{then} \quad \log M(r) = O\{(\log r)^2\},$$

and if we have, instead of (2·14),

$$|f(z_n)| \leqslant A\,|z_n|^\kappa \tag{2·16}$$

the conclusion is that $f(z)$ is a polynomial of degree at most κ. Theorem 2 is an almost immediate corollary to Theorem 1, for it follows from the result of Noble, mentioned in the introduction, that the type of $f(z)$ does not exceed $c\kappa$; also the case for $H(r)$ bounded is contained in the result of Pfluger.

In section 3 we consider further classes of interpolation sets and relationships between them, which we use in establishing our general results. In section 4 we exhibit the properties of certain generalised canonical products. These have independent interest but are primarily intended for use in section 5, which contains the proofs of Theorems 1 and 3. Finally, in section 6, we apply some of our results to the problem of integral functions with gap power series.

3. *Sets of points.* We consider a further two classes of interpolation sets. The first is a generalization of the lattice points: we say that a set $\{z_n\}$ belongs to $U^*(2, \frac{1}{2}\pi/k^2)$ if it has a representation $\{z_{m,n}\}$ $(m, n = 0, \pm 1, \pm 2, \ldots)$ which satisfies

$$|z_{m,n} - k(m+in)| \leqslant \Delta < \infty, \tag{3·1}$$

and for all $(p,q) \neq (m,n)$

$$|z_{p,q} - z_{m,n}| \geqslant \delta > 0. \tag{3·2}$$

The second is a subclass, containing the more uniform members, of $P(\rho, \tau)$, the points of a set not varying too much from some, conveniently chosen, symmetric set: to be precise, we shall say that a set $\{z_n\}$ belongs to $U(\rho, \tau)$ if it has a representation $\{z_{p,q}\}$ $(p = 1, 2, \ldots, q = 0, 1, \ldots, p-1)$ which satisfies, for some positive numbers, δ, Δ,

$$\left| z_{p,q} - \left(\frac{p^2}{2\rho\tau}\right)^{1/\rho} \exp\left\{\frac{2\pi i q}{p}\right\} \right| \leqslant \Delta \left(\frac{p^2}{2\rho\tau}\right)^{(1-\rho/2)/\rho} \tag{3·3}$$

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and for $(p,q) \neq (r,s)$

$$|z_{p,q}-z_{r,s}| \geqslant \delta \inf\{|z_{p,q}|^{1-\rho/2}, |z_{r,s}|^{1-\rho/2}\}. \tag{3·4}$$

Obviously, a set belonging to $U(\rho,\tau)$ also belongs to $B[\rho,d]$ for some finite d, but we can make more precise statements, as in

THEOREM 3. *A set $\{z_n\}$ belonging to $B[\rho,\Delta]$ has a subset belonging to $U(\rho,\tau)$ for every $\tau < \pi/\Delta^2 3\sqrt{3}$.*

The same will obviously still be true if we replace the set $\{z_n\}$ by a set $\{z_n'\}$ satisfying (2·8) and (2·9). Conversely we have

THEOREM 4. *A set of $U(\rho,\tau)$ has a subset $\{z_n'\}$ satisfying* (2·8) *and* (2·9), *where $\{z_n\}$ is a set of $B[\rho,\Delta]$, for every $\Delta > (\pi/\tau 3\sqrt{3})^{\frac{1}{2}}$.*

It can be seen, from Theorem 3, that to establish Theorem 1 it is sufficient to prove it for the case where $\{z_n\}$ is a set of $U(\rho,\tau)$, and, from Theorem 4, that the statement of Theorem 1 includes this result.

4. *Sequences of canonical products.* We consider, in this section, any set $\{z_{m,n}\}$ $(m,n = 0, \pm 1, \pm 2, \ldots,)$ which satisfies equations (3·1) and (3·2) with $k = 1$. For each pair (M,N) we form the integral function $\sigma_{M,N}(z)$ defined by

$$\sigma_{M,N}(z) = (z-z_{M,N}) \exp\{-\tfrac{1}{2}\alpha_{M,N}(z-z_{M,N})^2\} \prod_{(m,n)\neq(M,N)} E\left(\frac{z-z_{M,N}}{z_{m,n}-z_{M,N}}, 2\right), \tag{4·1}$$

where

$$\alpha_{M,N} = \lim_{r\to\infty} \sum_{0<|z_{m,n}-z_{M,N}|\leqslant r} \frac{1}{(z_{m,n}-z_{M,N})^2} \tag{4·2}$$

(we shall show later that this always exists), and $E(u,2) = (1-u)\exp\{u+\frac{1}{2}u^2\}$ is the Weierstrass primary factor of genus 2.

In the special case $z_{m,n} = m+in$, $\sigma_{M,N}(z)$ becomes $\sigma(z-M-iN)$, where $\sigma(z)$ is the Weierstrass σ-function, with zeros at the lattice points $m+in$,

$$\begin{aligned}\sigma(z) &= z \prod_{m+in\neq 0} E\left(\frac{z}{m+in}, 2\right) \\ &= z \prod_{m,n=1}^{\infty} \left(1-\frac{z^4}{(m+in)^4}\right).\end{aligned} \tag{4·3}$$

It might, therefore, be expected that, for large $|z-z_{M,N}|$, $\sigma_{M,N}(z)$ and $\sigma(z-M-iN)$ would be comparable in modulus, outside some region containing the zeros of either function. In fact, if we denote by (A_η) the set of discs $|z-z_{m,n}| \leqslant \eta < \frac{1}{4}\delta$, the main properties of the set of functions $\{\sigma_{M,N}(z)\}$ are contained in

THEOREM 5. *For the $\sigma_{M,N}(z)$ described above, we have*

(i) *$\sigma_{M,N}(z_{m,n}) = 0$ for all (m,n), (M,N) and $\sigma_{M,N}(z)$ has no zeros outside the set $\{z_{m,n}\}$. Also, for every $\epsilon > 0$, we can determine $r_0 = r_0(\epsilon,\eta,\delta,\Delta)$, independent of (M,N), such that*

(ii) $$\log|\sigma_{M,N}(z)| \leqslant (\tfrac{1}{2}\pi+\epsilon)|z-z_{M,N}|^2 \tag{4·4}$$

for $|z-z_{M,N}| \geqslant r_0$,

(iii) $$\log|\sigma_{M,N}(z)| \geqslant (\tfrac{1}{2}\pi-\epsilon)|z-z_{M,N}|^2 \tag{4·5}$$

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for $|z-z_{M,N}| \geqslant r_0$ *and* z *outside* (A_η), *and*

(iv) $$\log|\sigma'_{M,N}(z_{m,n})| \geqslant (\tfrac{1}{2}\pi-\epsilon)\,|z_{m,n}-z_{M,N}|^2 \tag{4·6}$$

for $|z_{m,n}-z_{M,N}| \geqslant r_0$. *Furthermore*

(v) $$|\sigma_{M,N}(z)| \geqslant \omega > 0 \tag{4·7}$$

for z *outside* (A_η), *and*

(vi) $$|\sigma'_{M,N}(z_{m,n})| \geqslant \omega' > 0 \tag{4·8}$$

for all (m,n), *where* ω, ω' *are both independent of* (M,N).

Proof of Theorem 5. Part (i) of the theorem follows immediately from the definition of $\sigma_{M,N}(z)$.

The proof of parts (ii), (iii), (iv) is based on a comparison of $\sigma_{M\,N}(z)$ with $\sigma(z-z_{M\,N})$. We denote by (B_η) (different for each pair (M,N)) the set of discs

$$|z-(m+in+z_{M,N})| \leqslant \eta \quad (m,n = 0, \pm 1, \pm 2, \ldots,).$$

We then have

LEMMA 1. *If* z *is outside* (A_η) *and* (B_η), *and* $\epsilon > 0$, *then we can determine* $r_1 = r_1(\epsilon,\eta,\Delta)$ *independent of* (M,N) *such that*

$$\left|\log\left|\prod_{\mathscr{E}'}\left\{\frac{1-\dfrac{z-z_{M,N}}{z_{m+M,n+N}-z_{M,N}}}{1-\dfrac{z-z_{M,N}}{m+in}}\right\}\right|\right| \leqslant \epsilon\,|z-z_{M,N}|^2, \tag{4·9}$$

where $\mathscr{E}'$ *is the set of* (m,n) *satisfying* $0 < |m+in| \leqslant 2|z-z_{M,N}|$.

Proof. Write $|z-z_{M,N}| = r$, and let $\mathscr{E}_1$ be the set of (m,n) satisfying either

$$0 < |m+in| \leqslant r^{\frac{1}{2}} \quad \text{or} \quad |m+M+i(n+N)-z| \leqslant r^{\frac{1}{2}}.$$

Now $$\frac{1-\dfrac{z-z_{M,N}}{m+in}}{1-\dfrac{z-z_{M,N}}{z_{m+M,\,n+N}-z_{M,N}}} = 1+\frac{(z-z_{M,N})\,(m+in+z_{M,N}-z_{m+M,n+N})}{(m+in)\,(z_{m+M,n+N}-z)},$$

and if (m,n) belongs to $\mathscr{E}'-\mathscr{E}_1$ we have, writing $\delta_{p,q} = z_{p,q}-(p+iq)$,

$$\begin{aligned}|(m+in)\,(z_{m+M,\,n+N}-z)| &= |(m+in)\{m+in-(z-M-iN)+\delta_{m+M,\,n+N}\}| \\ &\geqslant \inf_{\zeta\notin\mathscr{E}^*}|\zeta\{\zeta-(z-M-iN)+\delta_{m+M,\,n+N}\}|,\end{aligned}$$

where $\mathscr{E}^*$ is the set of ζ satisfying either $|\zeta| \leqslant r^{\frac{1}{2}}$, or $|\zeta-(z-M-iN)| \leqslant r^{\frac{1}{2}}$. Now $\zeta\{\zeta-(z-M-iN)+\delta_{m+M,n+N}\}$ is a quadratic in ζ with no zeros outside $\mathscr{E}^*$ and hence its minimum modulus outside $\mathscr{E}^*$ occurs on the boundary of $\mathscr{E}^*$. This gives

$$|(m+in)\,(z_{m+M,\,n+N}-z)| \geqslant (r^{\frac{1}{2}}-\Delta)\,(r-r^{\frac{1}{2}}-2\Delta) \tag{4·10}$$

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and therefore

$$\left|\frac{(z-z_{M,N})(m+in+z_{M,N}-z_{m+M,n+N})}{(m+in)(z_{m+M,n+N}-z)}\right| \leqslant \frac{r2\Delta}{(r^{\frac{1}{2}}-\Delta)(r-r^{\frac{1}{2}}-2\Delta)} \leqslant \frac{4\Delta}{r^{\frac{1}{2}}} \tag{4·11}$$

for $r \geqslant r_2(\Delta)$, and if $r \geqslant r_3(\epsilon, \Delta)$ we have

$$\left|\log\left|1+\frac{(z-z_{M,N})(m+in+z_{M,N}-z_{m+M,n+N})}{(m+in)(z_{m+M,n+N}-z)}\right|\right| \leqslant \frac{\epsilon}{32}.$$

There are at most $\pi(2r+1)^2$ members of $\mathscr{E}'$ and hence for sufficiently large r

$$\left|\log\left|\prod_{\mathscr{E}'-\mathscr{E}_1}\left\{\frac{1-\dfrac{z-z_{M,N}}{z_{m+M,n+N}-z_{M,N}}}{1-\dfrac{z-z_{M,N}}{m+in}}\right\}\right|\right| \leqslant \tfrac{1}{2}\epsilon r^2. \tag{4·12}$$

If $(m,n) \in \mathscr{E}_1$ and z is outside (A_η) and (B_η) we have the inequalities

$$1 \leqslant |m+in| \leqslant 2r;$$

$$\delta \leqslant |z_{m+M,n+N}-z_{M,N}| \leqslant |m+in|+2\Delta \leqslant 2r+2\Delta;$$

$$\eta \leqslant |z_{m+M,n+N}-z| \leqslant |z-z_{M,N}|+|m+in|+2\Delta \leqslant 3r+2\Delta;$$

$$\eta \leqslant |m+in+z_{M,N}| \leqslant 3r,$$

and therefore

$$\frac{\eta\delta}{2r(3r+2\Delta)} \leqslant \left|\frac{(m+in+z_{M,N}-z)(z_{m+M,n+N}-z_{M,N})}{(m+in)(z_{m+M,n+N}-z)}\right| \leqslant \frac{3r(2r+2\Delta)}{\eta}. \tag{4·13}$$

This gives for $r \geqslant r_4(\Delta)$

$$\left|\log\left|\frac{(m+in+z_{M,N}-z)(z_{m+M,n+N}-z_{M,N})}{(m+in)(z_{m+M,n+N}-z)}\right|\right| \leqslant \log\frac{9r^2}{\eta\delta} = 2\log\frac{3r}{\eta\delta}. \tag{4·14}$$

There are at most $2\pi(r^{\frac{1}{2}}+1)^2$ members of $\mathscr{E}_1$ and therefore

$$\left|\log\left|\prod_{\mathscr{E}_1}\left\{\frac{1-\dfrac{z-z_{M,N}}{z_{m+M,n+N}-z_{M,N}}}{1-\dfrac{z-z_{M,N}}{m+in}}\right\}\right|\right| \leqslant 4\pi(r^{\frac{1}{2}}+1)^2\log\frac{3r}{\eta\delta} \leqslant \tfrac{1}{2}\epsilon r^2 \tag{4·15}$$

for $r \geqslant r_5(\epsilon, \eta, \delta, \Delta)$, which together with (4·12) establishes the lemma.

LEMMA 2. *If* $|z-z_{M,N}| \geqslant 4(\Delta+1)$, *there exists a constant* $a_1 = a_1(\Delta)$ *such that*

$$\left|\log\left|\prod_{\mathscr{G}}\left\{\frac{\left(1-\dfrac{z-z_{M,N}}{z_{m+M,n+N}-z_{M,N}}\right)\exp\left\{\dfrac{z-z_{M,N}}{z_{m+M,n+N}-z_{M,N}}\right\}}{\left(1-\dfrac{z-z_{M,N}}{m+in}\right)\exp\left\{\dfrac{z-z_{M,N}}{m+in}\right\}}\right\}\right|\right| \leqslant a_1|z-z_{M,N}|,$$

where $\mathscr{G}$ *is the set of* (m,n) *satisfying* $|m+in| > 2|z-z_{M,N}|$.

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Proof. We have, under the given conditions, both

$$\left|\frac{z-z_{M,N}}{z_{m+M,n+N}-z_{M,N}}\right| \leqslant \frac{r}{2r-2\Delta} \leqslant \frac{2}{3}; \quad \left|\frac{z-z_{M,N}}{m+in}\right| \leqslant \tfrac{1}{2},$$

and therefore

$$\left|\log\left\{\frac{\left(1-\dfrac{z-z_{M,N}}{z_{m+M,n+N}-z_{M,N}}\right)\exp\left\{\dfrac{z-z_{M,N}}{z_{m+M,n+N}-z_{M,N}}\right\}}{\left(1-\dfrac{z-z_{M,N}}{m+in}\right)\exp\left\{\dfrac{z-z_{M,N}}{m+in}\right\}}\right\}\right|$$

$$= \left|\sum_{p=2}^{\infty}\frac{(z-z_{M,N})^p}{p}\left\{\frac{1}{(z_{m+M,n+N}-z_{M,N})^p}-\frac{1}{(m+in)^p}\right\}\right| \tag{4·16}$$

$$\leqslant \sum_{p=2}^{\infty}\frac{r^p 2\Delta(|m+in|+2\Delta)^{p-1}}{(|m+in|-2\Delta)^p\,|m+in|^p}$$

$$= \frac{2\Delta}{|m+in|+2\Delta}\sum_{p=2}^{\infty}\left[\frac{r(|m+in|+2\Delta)}{(|m+in|-2\Delta)\,|m+in|}\right]^p. \tag{4·17}$$

Now $$\frac{r(|m+in|+2\Delta)}{(|m+in|-2\Delta)\,|m+in|} \leqslant \frac{r\frac{5}{4}|m+in|}{\frac{3}{4}|m+in|^2} = \frac{5r}{3|m+in|} \leqslant \tfrac{5}{6},$$

and therefore

$$\sum_{p=2}^{\infty}\frac{r^p 2\Delta(|m+in|+2\Delta)^{p-1}}{(|m+in|-2\Delta)^p\,|m+in|^p} \leqslant \frac{2\Delta r^2(|m+in|+2\Delta)}{(|m+in|-2\Delta)^2\,|m+in|^2}\cdot\frac{1}{1-\frac{5}{6}}$$

$$\leqslant \frac{12\Delta r^2\frac{5}{4}|m+in|}{\frac{9}{16}|m+in|^4}$$

$$= \frac{80\Delta r^2}{3|m+in|^3}. \tag{4·18}$$

Lemma 2 follows from equations (4·17) and (4·18), since

$$\sum_{|m+in|\geqslant r}\frac{1}{|m+in|^3} = O\left(\frac{1}{r}\right).$$

LEMMA 3.

$$\alpha_{M,N} = \lim_{r\to\infty}\sum_{0<|z_{m,n}-z_{M,N}|\leqslant r}\frac{1}{(z_{m,n}-z_{M,N})^2}$$

exists and is bounded.

Proof. Since $\sum_{r_1<|m+in|\leqslant r_2}\frac{1}{(m+in)^2} = 0$, we have

$$\left|\left\{\sum_{0<|m+in|\leqslant r_2}-\sum_{0<|m+in|\leqslant r_1}\right\}\frac{1}{(z_{m+M,n+N}-z_{M,N})^2}\right|$$

$$= \left|\sum_{r_1<|m+in|\leqslant r_2}\left\{\frac{1}{(z_{m+M,n+N}-z_{M,N})^2}-\frac{1}{(m+in)^2}\right\}\right|$$

$$\leqslant \sum_{r_1<|m+in|\leqslant r_2}\frac{2\Delta\,.\,2(|m+in|+\Delta)}{(|m+in|-2\Delta)^2\,|m+in|^2}$$

$$\leqslant \sum_{|m+in|>r_1}\frac{8\Delta}{|m+in|^3}$$

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for $r_1 \geqslant r(\Delta)$. Consequently

$$\left| \sum_{r_1 < |m+in| \leqslant r_2} \frac{1}{(z_{m+M,\,n+N} - z_{M,\,N})^2} \right| \leqslant \frac{8\Delta K}{r_1} \to 0 \tag{4·19}$$

as $r_1 \to \infty$. This shows that

$$\beta_{M,\,N} = \lim_{r \to \infty} \sum_{0 < |m+in| \leqslant r} \frac{1}{(z_{m+M,\,n+N} - z_{M,\,N})^2}$$

exists. Since the number of (m, n) satisfying $|m+in| \leqslant r_1$ is at most $\pi(r_1+1)^2$, we have

$$\left| \sum_{0 < |m+in| \leqslant r_1} \frac{1}{(z_{m+M,\,n+N} - z_{M,\,N})^2} \right| \leqslant \frac{\pi(r_1+1)^2}{\delta^2}. \tag{4·20}$$

This together with (4·19) shows that $\beta_{M,\,N}$ is bounded. Now consider

$$\sum_{0 < |z_{p,q} - z_{M,N}| \leqslant r} \frac{1}{(z_{p,\,q} - z_{M,\,N})^2} - \sum_{0 < |m+in| \leqslant r+2\Delta} \frac{1}{(z_{m+M,\,n+N} - z_{M,\,N})^2}.$$

Since every element of the first sum is a member of the second, and that the number of elements contained in the second sum but not the first is at most

$$[\pi(r+2\Delta+1)^2 - \pi(r-1)^2],$$

it follows that

$$\left| \sum_{0 < |z_{p,q} - z_{M,N}| \leqslant r} \frac{1}{(z_{p,\,q} - z_{M,\,N})^2} - \sum_{0 < |m+in| \leqslant r+2\Delta} \frac{1}{(z_{m+M,\,n+N} - z_{M,\,N})^2} \right| \leqslant \frac{2\pi(\Delta+1)(r+1)}{r^2} \to 0$$

as $r \to \infty$. Hence since $\beta_{M,\,N}$ exists so does $\alpha_{M,\,N}$ and $\alpha_{M,\,N} = \beta_{M,\,N}$. This completes the proof of the lemma.

Now

$$\left| (z - z_{M,\,N}) \left\{ \frac{1}{z_{m+M,\,n+N} - z_{M,\,N}} - \frac{1}{m+in} \right\} \right| \leqslant \inf \left\{ \frac{2\Delta r}{(|m+in| - 2\Delta)\,|m+in|}, \frac{2r}{\delta} \right\}$$

and, therefore, writing

$$\Sigma_{\mathscr{E}'} = \sum_{0 < |m+in| \leqslant 4\Delta} + \sum_{4\Delta < |m+in| \leqslant 2r}$$

and using the estimate $2r/\delta$ in the first sum and $2\Delta r/(|m+in| - 2\Delta)\,|m+in|$ in the second, we have

$$\begin{aligned} &\left| \Sigma_{\mathscr{E}'} (z - z_{M,\,N}) \left\{ \frac{1}{z_{m+M,\,n+N} - z_{M,\,N}} - \frac{1}{m+in} \right\} \right| \\ &\quad \leqslant \frac{2r\pi(4\Delta+1)^2}{\delta} + 4\Delta r \, \Sigma_{\mathscr{E}'} \frac{1}{|m+in|^2} \\ &\quad \leqslant K_1(\delta, \Delta) r + K_2(\Delta) r \log r, \end{aligned}$$

since

$$\sum_{0 < |m+in| \leqslant r} \frac{1}{|m+in|^2} = O(\log r) \quad \text{as} \quad r \to \infty.$$

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Hence we can find an a_2 such that, for $r \geqslant r(\delta, \Delta)$,

$$\left|\log\left|\prod_{\mathscr{E}'} \frac{\exp\left\{\dfrac{z-z_{M,N}}{z_{m+M,n+N}-z_{M,N}}\right\}}{\exp\left\{\dfrac{z-z_{M,N}}{m+in}\right\}}\right|\right| \leqslant a_2 r \log r. \tag{4·21}$$

Also

$$\prod_{m+in\neq 0}\left\{\frac{\exp\left\{\frac{1}{2}\left(\dfrac{z-z_{M,N}}{z_{m+M,n+N}-z_{M,N}}\right)^2\right\}}{\exp\left\{\frac{1}{2}\left(\dfrac{z-z_{M,N}}{m+in}\right)^2\right\}}\right\} = \exp\{\tfrac{1}{2}\alpha_{M,N}(z-z_{M,N})^2\}. \tag{4·22}$$

Combining the results of Lemmas 1 and 2 with equations (4·21) and (4·22) gives, for any z outside (A_η) and (B_η),

$$\begin{aligned}\left|\log\left|\frac{\sigma_{M,N}(z)}{\sigma(z-z_{M,N})}\right|\right| &\leqslant \epsilon\,|z-z_{M,N}|^2 + (a_1+a_2)\,|z-z_{M\ N}|\log|z-z_{M,N}| \\ &\leqslant 2\epsilon\,|z-z_{M,N}|^2 \end{aligned} \tag{4·23}$$

for $|z-z_{M,N}| \geqslant r_1(\epsilon, \eta, \delta, \Delta)$. Hence, on rechoosing ϵ, we have, using the properties of the σ-function, for any z outside (A_η) and (B_η),

$$\exp\{(\tfrac{1}{2}\pi - \tfrac{1}{2}\epsilon)\,|z-z_{M,N}|^2\} < |\sigma_{M,N}(z)| < \exp\{(\tfrac{1}{2}\pi + \tfrac{1}{2}\epsilon)\,|z-z_{M,N}|^2\} \tag{4·24}$$

for $|z-z_{M,N}| \geqslant r_2(\epsilon, \eta, \delta, \Delta)$.

Equation (4·24) holds for any z outside $(A_{\frac{1}{4}\eta})$ and $(B_{\frac{1}{4}\eta})$ for $|z-z_{M,N}| \geqslant r_2(\epsilon, \frac{1}{4}\eta, \delta, \Delta)$. Now if two discs of $(A_{\frac{1}{4}\eta})$ and $(B_{\frac{1}{4}\eta})$ intersect, then the corresponding disc of (A_η) contains the disc of $(B_{\frac{1}{4}\eta})$ in its interior. Hence we can apply the maximum and minimum modulus principles to those discs of $(B_{\frac{1}{4}\eta})$ which do not intersect with $(A_{\frac{1}{4}\eta})$ to yield

$$\exp\{(\tfrac{1}{2}\pi - \tfrac{1}{2}\epsilon)\,(|z-z_{M,N}| - \tfrac{1}{2}\eta)^2\} < |\sigma_{M,N}(z)| < \exp\{(\tfrac{1}{2}\pi + \tfrac{1}{2}\epsilon)\,(|z-z_{M,N}| + \tfrac{1}{2}\eta)^2\} \tag{4·25}$$

for all z outside (A_η) with $|z-z_{M,N}| \geqslant r_2(\epsilon, \frac{1}{4}\eta, \delta, \Delta)$, that is

$$\exp\{(\tfrac{1}{2}\pi - \epsilon)\,|z-z_{M,N}|^2\} < |\sigma_{M,N}(z)| < \exp\{(\tfrac{1}{2}\pi + \epsilon)\,|z-z_{M,N}|^2\} \tag{4·26}$$

for $|z-z_{M,N}| \geqslant r_3(\epsilon, \eta, \delta, \Delta)$. The left-hand inequality of (4·26) is conclusion (iii) of Theorem 5, while the right-hand inequality together with the maximum modulus principle applied to the discs of (A_η) gives conclusion (ii).

To obtain Theorem 5 (iv), we set, temporarily,

$$\zeta(z) = \sigma_{M,N}(z)/(z-z_{m,n}) \quad (z \neq z_{m,n})$$

and

$$\zeta(z_{m,n}) = \sigma'_{M,N}(z_{m,n});$$

then $\zeta(z)$ is an integral function and

$$\left|\frac{\log|\zeta(z)|}{|z-z_{M,N}|^2} - \tfrac{1}{2}\pi\right| \leqslant \left|\frac{\log|\sigma_{M,N}(z)|}{|z-z_{M,N}|^2} - \tfrac{1}{2}\pi\right| + \left|\frac{\log|z-z_{m,n}|}{|z-z_{M,N}|^2}\right|. \tag{4·27}$$

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Now if z is any point on $|z-z_{m,n}| = \eta$, and if $|z_{m,n}-z_{M,N}| \geqslant r_3(\frac{1}{2}\epsilon, \eta, \delta, \Delta)$, it follows that

$$\left|\frac{\log|\zeta(z)|}{|z-z_{M,N}|^2} - \tfrac{1}{2}\pi\right| < \tfrac{1}{2}\epsilon + \frac{|\log\eta|}{(|z_{m,n}-z_{M,N}|-\eta)^2}. \tag{4·28}$$

Since $\zeta(z)$ is regular and has no zeros in $|z-z_{m,n}| \leqslant \eta$, we have

$$\left|\frac{\log|\zeta(z_{m,n})|}{|z_{m,n}-z_{M,N}|^2} - \tfrac{1}{2}\pi\right| < \epsilon \tag{4·29}$$

if $|z_{m,n}-z_{M,N}| \geqslant r_4(\epsilon, \eta, \rho, \Delta)$, as required.

For $|z-z_{M,N}|$, $|z_{m,n}-z_{M,N}| \geqslant r_0(\frac{1}{4}\pi, \eta, \delta, \Delta) = r_0$ (say) results (v) and (vi) of the theorem follow from parts (iii) and (iv). For $|z-z_{M,N}| \leqslant r_0$ with z outside (A_η) and $0 < |m+in| \leqslant 2r_0$,

$$\left|\left\{\frac{z_{m+M,\,n+N}-z}{z_{m+M,\,n+N}-z_{M,N}}\right\}\exp\left\{\frac{z-z_{M,N}}{z_{m+M,\,n+N}-z_{M,N}}+\frac{1}{2}\left(\frac{z-z_{M,N}}{z_{m+M,\,n+N}-z_{M,N}}\right)^2\right\}\right|$$

$$\geqslant \frac{\eta}{2(r_0+\Delta)}\exp\left\{-\frac{r_0}{\delta}-\frac{1}{2}\frac{r_0^2}{\delta^2}\right\}. \tag{4·30}$$

If $|m+in| > 2r_0 \geqslant 8\Delta$,

$$\left|\log\left[\left(1-\frac{z-z_{M,N}}{z_{m+M,\,n+N}-z_{M,N}}\right)\exp\left\{\frac{z-z_{M,N}}{z_{m+M,\,n+N}-z_{M,N}}+\frac{1}{2}\left(\frac{z-z_{M,N}}{z_{m+M,\,n+N}-z_{M,N}}\right)^2\right\}\right]\right|$$

$$= \left|\sum_{p=3}^{\infty}\frac{1}{p}\left(\frac{z-z_{M,N}}{z_{m+M,\,n+N}-z_{M,N}}\right)^p\right|$$

$$\leqslant \sum_{p=3}^{\infty}\frac{r_0^p}{p(|m+in|-2\Delta)^p}$$

$$\leqslant \frac{K_3 r_0^3}{|m+in|^3}$$

and since $\sum_{m+in\neq 0} 1/|m+in|^3$ is convergent, we have

$$\left|\prod_{|m+in|<2r_0} E\left(\frac{z-z_{M,N}}{z_{m+M,\,n+N}-z_{M,N}}, 2\right)\right| \geqslant \exp\{-K_4 r_0^3\}. \tag{4·31}$$

Combining equations (4·30) and (4·31) gives, since $\alpha_{M,N}$ is bounded ($\leqslant K_5$, say), for z outside (A_η)

$$\log|\sigma_{M,N}(z)| \geqslant -\pi(2r_0+1)^2\left\{\frac{r_0}{\delta}+\frac{1}{2}\frac{r_0^2}{\delta^2}+\log\frac{2(r_0+\Delta)}{\eta}\right\}-K_4 r_0^3-K_5 r_0^2, \tag{4·32}$$

which establishes part (v) of our theorem.

Now $\zeta(z) = \sigma_{M,N}(z)/(z-z_{m,n})$ is such that $\zeta(z_{m,n}) = \sigma'_{M,N}(z_{m,n})$, and $|\zeta(z)| \geqslant \omega/\eta$ for any z on the boundary of a disc of (A_η) and is non-zero in its interior. An application of the minimum modulus principle gives result (vi).

5. *Proof of Theorem* 1. We first establish the result for the case where $\rho = 2, \tau = \frac{1}{2}\pi$ and, in (2·4), $\{z_n\}$ is a set $\{z_{m,n}\}$ of $U^*(2, \frac{1}{2}\pi)$. The starting-point of our proof for this

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case is a generalization of that used by Pfluger (10) and Cartwright (1) for lattice point interpolation. In what follows, we suppose that $0 < \alpha - \beta \leqslant \frac{1}{2}\pi$ and

$$\sin^2(\alpha-\beta) > \frac{2\gamma}{\pi}\left\{1-\frac{c\kappa(\frac{1}{2}\pi-\gamma)}{\gamma(\frac{1}{2}\pi-c\kappa)}\right\}. \tag{5·1}$$

We can assume, without loss of generality, that no $z_{m,n}$ lies within a distance $\eta < \frac{1}{4}\delta$ of the boundary of $S(\alpha, l)$, for we can construct a sequence belonging to $U^*(2, \frac{1}{2}\pi)$ which agrees with the given one in $S(\alpha, l)$, except within a distance η of the boundary which has this property.

With $\sigma_{M,N}(z)$ defined as in section 4, let

$$\phi_{M,N}(z) = \sum_{z_{m,n} \in S(\alpha, l)} \frac{f(z_{m,n})}{(z-z_{m,n})\,\sigma'_{M,N}(z_{m,n})}, \tag{5·2}$$

where $z_{M,N}$ is any member of the sequence $\{z_{m,n}\}$ lying in $S(\beta, R_0)$, for some R_0 to be determined. Now $\phi_{M,N}(z_{m,n}) = f(z_{m,n})$ and so

$$\Phi_{M,N}(z) = \frac{f(z)-\phi_{M,N}(z)}{\sigma_{M,N}(z)} \tag{5·3}$$

is regular in $S(\alpha, l)$. Let $\{r_p\}$, $r_p \uparrow \to \infty$, be a sequence satisfying

$$\limsup_{p\to\infty} \frac{\log M(r_p, \alpha, f)}{r_p^2} = h < \tfrac{1}{2}\pi \tag{5·4}$$

and let $C(r_p)$ be a contour lying in the region $r_p - 1 \leqslant r \leqslant r_p$, joining $\arg z = \pm\alpha$, such that the shortest distance from $C(r_p)$ to any point $z_{m,n}$ is at least η. If z is any point on $C(r_p)$ we have, from (4·5) and (4·7),

$$\left|\frac{1}{\sigma_{M,N}(z)}\right| \leqslant K_1(\eta, \epsilon)\exp\{-(\tfrac{1}{2}\pi-\epsilon)\,|z-z_{M,N}|^2\}, \tag{5·5}$$

and therefore

$$\left|\frac{f(z)}{\sigma_{M,N}(z)}\right| \leqslant K_2(\eta, \epsilon)\exp\{-(\tfrac{1}{2}\pi-\epsilon)\,(r_p-|z_{M,N}|-1)^2+(h+\epsilon)r_p^2\}, \tag{5·6}$$

and, with $\epsilon < \frac{1}{4}(\frac{1}{2}\pi - h)$,

$$\left|\frac{f(z)}{\sigma_{M,N}(z)}\right| \leqslant K_3(M, N, \eta)\exp\{-\tfrac{1}{2}(\tfrac{1}{2}\pi-h)r_p^2\}. \tag{5·7}$$

For z on the boundary of $S(\alpha, l)$

$$\left|\frac{f(z)}{\sigma_{M,N}(z)}\right| \leqslant K_4(\epsilon, \eta)\exp\{(\gamma+\epsilon)\,|z|^2-(\tfrac{1}{2}\pi-\epsilon)\,|z-z_{M,N}|^2\} \tag{5·8}$$

and since, with $0 \leqslant A < B$,

$$\begin{aligned} A\,|r\,e^{i\theta}|^2 - B|\,r\,e^{i\theta} - R\,e^{i\phi}|^2 &= (A-B)r^2 + 2BR\cos(\theta-\phi) - BR^2 \\ &= -(B-A)\left[r - \frac{BR\cos(\theta-\phi)}{B-A}\right]^2 + \frac{B^2R^2\cos^2(\theta-\phi)}{B-A} - BR^2 \\ &\leqslant \frac{BR^2}{B-A}[A - B\sin^2(\theta-\phi)], \end{aligned}$$

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it follows that

$$\left|\frac{f(z)}{\sigma_{M,N}(z)}\right| \leqslant K_4(\epsilon,\eta)\exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)\,|z_{M,N}|^2}{\frac{1}{2}\pi-\gamma-2\epsilon}[\gamma+\epsilon-(\tfrac{1}{2}\pi-\epsilon)\sin^2(\alpha-\beta)]\right\}. \tag{5·9}$$

Also, in view of equation (5·1), we can choose $\epsilon=\epsilon_0$ so small that

$$\frac{\frac{1}{2}\pi-\epsilon_0}{\frac{1}{2}\pi-\gamma-2\epsilon_0}[\gamma+\epsilon_0-(\tfrac{1}{2}\pi-\epsilon_0)\sin^2(\alpha-\beta)] < \frac{\frac{1}{2}\pi c\kappa}{\frac{1}{2}\pi-c\kappa},$$

which gives

$$\left|\frac{f(z)}{\sigma_{M,N}(z)}\right| \leqslant K_4(\epsilon_0,\eta)\exp\left\{\frac{\frac{1}{2}\pi c\kappa}{\frac{1}{2}\pi-c\kappa}\,|z_{M,N}|^2\right\}. \tag{5·10}$$

For z outside (A_η)

$$\left|\frac{\phi_{M,N}(z)}{\sigma_{M,N}(z)}\right| \leqslant \frac{1}{\eta}\sum_{S(\alpha,l)}\left|\frac{f(z_{m,n})}{\sigma'_{M,N}(z_{m,n})}\right|. \tag{5·11}$$

From (4·6) and (4·8) we deduce that for any ϵ satisfying $0<\epsilon<\frac{1}{4}(\frac{1}{2}\pi-c\kappa)$,

$$|\sigma'_{M,N}(z_{m,n})| \geqslant K_5(\epsilon)\exp\{(\tfrac{1}{2}\pi-\epsilon)\,|z_{m,n}-z_{M,N}|^2\} \tag{5·12}$$

and, since for $z_{m,n}$ in $S(\alpha,l)$

$$|f(z_{m,n})| \leqslant K_6(\epsilon)\exp\{(\kappa+\epsilon)\,H(|z_{m,n}|)\}$$

we have

$$\left|\frac{f(z_{m,n})}{\sigma'_{M,N}(z_{m,n})}\right| \leqslant K_7(\epsilon)\exp\{(\kappa+\epsilon)\,H(|z_{m,n}|)-(\tfrac{1}{2}\pi-\epsilon)\,|z_{m,n}-z_{M,N}|^2\}. \tag{5·13}$$

If we now write $|z_{M,N}|=R(M,N)=R$, $|z_{m,n}-z_{M,N}|=r(m,n,M,N)=r$ and $H(r)/r^2=\mu(r)\downarrow\to c$, we have

$$\begin{aligned}
&(\kappa+\epsilon)\,H(|z_{m,n}|)-(\tfrac{1}{2}\pi-\epsilon)\,|z_{m,n}-z_{M,N}|^2\\
&\quad\leqslant (\kappa+\epsilon)\,H(R+r)-(\tfrac{1}{2}\pi-\epsilon)r^2\\
&\quad= (\kappa+\epsilon)\,(R+r)^2\mu(R+r)-(\tfrac{1}{2}\pi-\epsilon)r^2\\
&\quad= \frac{(\frac{1}{2}\pi-\epsilon)\,(\kappa+\epsilon)\mu(R+r)R^2}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R+r)}-\{\tfrac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R+r)\}\left[r-\frac{(\kappa+\epsilon)\mu(R+r)R}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R+r)}\right]^2\\
&\quad\leqslant \frac{(\frac{1}{2}\pi-\epsilon)\,(\kappa+\epsilon)\mu(R)R^2}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R)}\\
&\quad= \frac{(\frac{1}{2}\pi-\epsilon)\,(\kappa+\epsilon)H(R)}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R)}.
\end{aligned} \tag{5·14}$$

Also

$$\begin{aligned}
&(\kappa+\epsilon)\,H(R+r)-(\tfrac{1}{2}\pi-\epsilon)r^2+\tfrac{1}{2}(\tfrac{1}{2}\pi-c\kappa)r^2\\
&\quad= (\kappa+\epsilon)\mu(R+r)\,(R+r)^2-\tfrac{1}{2}(\tfrac{1}{2}\pi-2\epsilon+c\kappa)r^2\\
&\quad= \frac{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)\,(\kappa+\epsilon)\mu(R+r)R^2}{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)-(\kappa+\epsilon)\mu(R+r)}\\
&\qquad-\{\tfrac{1}{2}(\tfrac{1}{2}\pi-2\epsilon+c\kappa)-(\kappa+\epsilon)\mu(R+r)\}\left[r-\frac{(\kappa+\epsilon)\mu(R+r)R}{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)-(\kappa+\epsilon)\mu(R+r)}\right]^2\\
&\quad\leqslant 0
\end{aligned}$$

for all r satisfying

$$r \geqslant \frac{R}{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)-(\kappa+\epsilon)\mu(R+r)}\left[(\kappa+\epsilon)\mu(R+r)+\left\{\frac{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)\,(\kappa+\epsilon)\mu(R+r)}{\frac{1}{2}(\frac{1}{2}\pi-2\epsilon+c\kappa)-(\kappa+\epsilon)\mu(R+r)}\right\}^{\frac{1}{2}}\right],$$

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or, equivalently,

$$\frac{r}{[H(r)]^{\frac{1}{2}}} \geqslant \frac{1}{\frac{1}{2}(\frac{1}{2}\pi - 2\epsilon + c\kappa) - (\kappa+\epsilon)\mu(R+r)} \times \left[(\kappa+\epsilon)\frac{\mu(R+r)}{[\mu(R)]^{\frac{1}{2}}} + \left\{\frac{\frac{1}{2}(\frac{1}{2}\pi - 2\epsilon + c\kappa)(\kappa+\epsilon)}{\frac{1}{2}(\frac{1}{2}\pi - 2\epsilon + c\kappa) - (\kappa+\epsilon)\mu(R+r)} \frac{\mu(R+r)}{\mu(R)}\right\}^{\frac{1}{2}}\right]. \quad (5{\cdot}15)$$

Now the right-hand side of (5·15) is bounded (suppose by K_8) for all sufficiently large R and all $r \geqslant 0$, and therefore

$$(\kappa+\epsilon)H(R+r) - (\tfrac{1}{2}\pi - \epsilon)r^2 + \tfrac{1}{2}(\tfrac{1}{2}\pi - c\kappa)r^2 \leqslant 0 \quad (5{\cdot}16)$$

provided $r \geqslant K_8[H(R)]^{\frac{1}{2}}$, for all sufficiently large R.

$$\sum_{S(\alpha,l)} \left|\frac{f(z_{m,n})}{\sigma'_{M,N}(z_{m,n})}\right| = \{\Sigma_1 + \Sigma_2\}\left|\frac{f(z_{m,n})}{\sigma'_{M,N}(z_{m,n})}\right|,$$

where Σ_1 is the sum over the terms $z_{m,n}$ lying within the disc $|z - z_{M,N}| \leqslant K_8[H(R)]^{\frac{1}{2}}$ (at most $K_9 H(R)$ in number) and Σ_2 is the sum over the remaining terms. We have, from (5·14) and the above, that

$$\Sigma_1\left|\frac{f(z_{m,n})}{\sigma'_{M,N}(z_{m,n})}\right| \leqslant K_9 H(R)\exp\left\{\frac{(\frac{1}{2}\pi - \epsilon)(\kappa+\epsilon)H(R)}{\frac{1}{2}\pi - \epsilon - (\kappa+\epsilon)\mu(R)}\right\}. \quad (5{\cdot}17)$$

Since for the relevant terms

$$(\kappa+\epsilon)H(R+r) - (\tfrac{1}{2}\pi - \epsilon)r^2 \leqslant -\tfrac{1}{2}(\tfrac{1}{2}\pi - c\kappa)r^2,$$

$$\Sigma_2\left|\frac{f(z_{m,n})}{\sigma'_{M,N}(z_{m,n})}\right| \leqslant \sum_{(m,n)\neq(M,N)} \exp\{-\tfrac{1}{2}(\tfrac{1}{2}\pi - c\kappa)\,|z_{m,n} - z_{M,N}|^2\}. \quad (5{\cdot}18)$$

Now $\exp\{-\frac{1}{2}(\frac{1}{2}\pi - c\kappa)\,|z_{m,n} - z_{M,N}|^2\}$

$$\leqslant \frac{1}{\pi(\frac{1}{2}\delta)^2}\iint_{|re^{i\theta} - (z_{m,n} - z_{M,N})| \leqslant \frac{1}{2}\delta} \exp\{-\tfrac{1}{2}(\tfrac{1}{2}\pi - c\kappa)(r - \tfrac{1}{2}\delta)^2\} r\,dr\,d\theta,$$

and, since the regions of integration do not overlap, it follows that the right-hand side of (5·18) does not exceed

$$\frac{1}{\pi(\frac{1}{2}\delta^2)}\int_{\frac{1}{2}\delta}^{\infty}\int_0^{2\pi} \exp\{-\tfrac{1}{2}(\tfrac{1}{2}\pi - c\kappa)(r - \tfrac{1}{2}\delta)^2\} r\,d\theta\,dr \leqslant \frac{8}{(\frac{1}{2}\pi - \kappa)\delta^2}, \quad (5{\cdot}19)$$

and therefore

$$\left|\frac{\phi_{M,N}(z)}{\sigma_{M,N}(z)}\right| \leqslant K_{10}(\eta)H(R)\exp\left\{\frac{(\frac{1}{2}\pi - \epsilon)(\kappa+\epsilon)H(R)}{\frac{1}{2}\pi - \epsilon - (\kappa+\epsilon)\mu(R)}\right\}. \quad (5{\cdot}20)$$

Hence for z on $C(r_p)$ we have

$$|\Phi_{M,N}(z)| \leqslant K_{10}H(R)\exp\left\{\frac{(\frac{1}{2}\pi - \epsilon)(\kappa+\epsilon)H(R)}{\frac{1}{2}\pi - \epsilon - (\kappa+\epsilon)\mu(R)}\right\} + K_3(M,N,\eta)\exp\{-\tfrac{1}{2}(\tfrac{1}{2}\pi - h)r_p^2\}, \quad (5{\cdot}21)$$

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and on the boundary of $S(\alpha, l)$

$$|\Phi_{M,N}(z)| \leqslant K_{10}H(R)\exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)(\kappa+\epsilon)H(R)}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R)}\right\}+K_4(\epsilon_0,\eta)\exp\left\{\frac{\frac{1}{2}\pi c\kappa R^2}{\frac{1}{2}\pi-c\kappa}\right\}. \quad (5{\cdot}22)$$

The right-hand side of (5·21) tends to

$$K_{10}H(R)\exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)(\kappa+\epsilon)H(R)}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R)}\right\}$$

as $p\to\infty$, for every fixed (M,N) such that $z_{M,N}$ is in $S(\beta,R_0)$, and since $\Phi_{M,N}(z)$ is regular in the region enclosed by these curves it follows for all z in $S(\alpha,l)$, and in particular in the disc $|z-z_{M,N}|\leqslant 1+\delta+\Delta$ provided $R_0\geqslant 1+\delta+\Delta+l$, that

$$|\Phi_{M,N}(z)| \leqslant K_{10}H(R)\exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)(\kappa+\epsilon)H(R)}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R)}\right\}+K_4\exp\left\{\frac{\frac{1}{2}\pi c\kappa}{\frac{1}{2}\pi-c\kappa}R^2\right\}. \quad (5{\cdot}23)$$

From Theorem 5 (ii) we see that

$$|\sigma_{M,N}(z)| \leqslant K_{11}(\delta,\Delta) \quad (5{\cdot}24)$$

in the disc $|z-z_{M,N}|\leqslant 1+\Delta+\delta$. Combining this with (5·20) gives

$$|\Phi_{M,N}(z)| \leqslant K_{12}H(R)\exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)(\kappa+\epsilon)H(R)}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R)}\right\} \quad (5{\cdot}25)$$

on a contour, lying inside $|z-z_{M,N}|\leqslant 1+\Delta=\delta$ and such that the shortest distance from it to any point $z_{m,n}$ of the set is at least η, containing the disc $|z-z_{M,N}|\leqslant 1+\Delta$ in its interior. This together with (5·23) and (5·24) gives

$$|f(z)| \leqslant K_{13}H(R)\exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)(\kappa+\epsilon)H(R)}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(R)}\right\}+K_4\exp\left\{\frac{\frac{1}{2}\pi c\kappa}{\frac{1}{2}\pi-c\kappa}R^2\right\} \quad (5{\cdot}26)$$

on the contour. Since $f(z)$ is regular in the domain bounded by this contour the same inequality holds there, and in particular in the disc $|z-z_{M,N}|\leqslant 1+\Delta\cdot S(\beta,R_0)$ is covered by the union of these discs, and hence it follows that, for all z in $S(\beta,R_0)$,

$$|f(z)| \leqslant K_{13}H(|z|+1+\Delta)\exp\left\{\frac{(\frac{1}{2}\pi-\epsilon)(\kappa+\epsilon)H(|z|+1+\Delta)}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(|z|-1-\Delta)}\right\}$$
$$+K_4\exp\left\{\frac{\frac{1}{2}\pi c\kappa}{\frac{1}{2}\pi-c\kappa}(|z|+1+\Delta)^2\right\}. \quad (5{\cdot}27)$$

If $c\kappa>0$ then $H(r)\sim cr^2$ and since $(\frac{1}{2}\pi-\epsilon)/[\frac{1}{2}\pi-\epsilon-c(\kappa+\epsilon)]>\frac{1}{2}\pi/(\frac{1}{2}\pi-c\kappa)$ we have from (5·27)

$$\frac{\log M(r,\beta,f)}{H(r)} \leqslant \frac{\log[2(K_{13}+K_4)\{1+H(r+1+\Delta)\}]}{H(r)}$$
$$+\frac{(\frac{1}{2}\pi-\epsilon)(\kappa+\epsilon)H(r+1+\Delta)}{[\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)\mu(r-1-\Delta)]H(r)}. \quad (5{\cdot}28)$$

Now
$$1 \leqslant \frac{H(r+1+\Delta)}{H(r)} = \frac{\mu(r+1+\Delta)}{\mu(r)}\frac{(r+1+\Delta)^2}{r^2}$$

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and since $\mu(r+\Delta+1)/\mu(r) \leqslant 1$ it follows that

$$\lim_{r\to\infty} \frac{H(r+1+\Delta)}{H(r)} = 1.$$

We therefore have, if $H(r)$ unbounded,

$$\limsup_{r\to\infty} \frac{\log M(r,\beta,f)}{H(r)} \leqslant \frac{(\frac{1}{2}\pi-\epsilon)(\kappa+\epsilon)}{\frac{1}{2}\pi-\epsilon-(\kappa+\epsilon)c}, \tag{5·29}$$

and this is true for every sufficiently small $\epsilon > 0$. Therefore

$$\limsup_{r\to\infty} \frac{\log M(r,\beta,f)}{H(r)} \leqslant \frac{\frac{1}{2}\pi\kappa}{\frac{1}{2}\pi-c\kappa} = \kappa\left[1+\frac{c\kappa}{\frac{1}{2}\pi-c\kappa}\right], \tag{5·30}$$

and if $H(r)$ bounded (5·27) shows that for z in $S(\beta, R_0)$

$$|f(z)| \leqslant K. \tag{5·31}$$

These last two inequalities establish the case of Theorem 1 under consideration. The simple linear transformation $\zeta = (2\tau/\pi)^{\frac{1}{2}}z$ now establishes the result in the case where $\{z_n\}$ is a set of $U^*(2,\tau)$, for general τ with $\rho = 2$.

We now complete the proof of Theorem 1, by establishing the result for the case where $\{z_n\}$ is a set of $U(\rho,\tau)$ and then proving Theorem 3. Consider the effect of the transformation

$$\zeta = (2\tau/\pi)^{\frac{1}{2}}z^{\frac{1}{2}\rho}$$

on the members of the set of points $\{z_{p,q}\}$ of equations (3·3) and (3·4) which lie in some fixed sector of aperture less than $4\pi/\rho$. The points $z_{p,q}$ are transformed into points $\zeta_{p,q}$ satisfying

$$\left|\zeta_{p,q} - \frac{p}{(\rho\pi)^{\frac{1}{2}}}\exp\left\{\frac{\rho\pi iq}{p}\right\}\right| \leqslant \Delta' < \infty \tag{5·32}$$

and, provided both $z_{p,q}$ and $z_{r,s}$ lie in the sector,

$$|\zeta_{p,q} - \zeta_{r,s}| \geqslant \delta' > 0. \tag{5·33}$$

Also these transformed points lie in a fixed sector of aperture less than 2π. Next consider the set of points

$$\xi_{p,q} = \frac{p}{(\rho\pi)^{\frac{1}{2}}}\exp\left\{\frac{\rho\pi iq}{p}\right\}$$

in some sector which are transforms of the members of the set of points

$$\left(\frac{p^2}{2\rho\tau}\right)^{1/\rho}\exp\left\{\frac{2\pi iq}{p}\right\}$$

which lie in a sector of aperture less than $4\pi/\rho$. With each point $\xi_{p,q}$ we can associate a 'quadrilateral' $A_{p,q}$

$$\frac{p-\frac{1}{2}}{(\rho\pi)^{\frac{1}{2}}} < |\xi| \leqslant \frac{p+\frac{1}{2}}{(\rho\pi)^{\frac{1}{2}}}, \quad \frac{\rho\pi(q-\frac{1}{2})}{p} < \arg\xi \leqslant \frac{\rho\pi(q+\frac{1}{2})}{p}$$

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of area 1, with $\xi_{p,q}$ at its centre. It would appear, from a consideration of the covering properties of these quadrilaterals and those of the set of squares

$$m-\tfrac{1}{2} < \Re z \leqslant m+\tfrac{1}{2}, \quad n-\tfrac{1}{2} < \Im z \leqslant n+\tfrac{1}{2},$$

that the set $\{\xi_{p,q}\}$ agrees in the sector with the points of a sequence of $U^*(2, \frac{1}{2}\pi)$. It is, however, difficult to set up a rule for pairing the members of the sets such that (3·1), with $k = 1$, would be satisfied, and we prove instead that $\{\xi_{p,q}\}$ contains a subset agreeing with the members of a set of $U^*(2, \frac{1}{2}\bar{\pi}/k^2)$ in the sector, for every $k > 1$. This together with (5·32) and (5·33) will show that the same is true of $\{\zeta_{p,q}\}$.

To do this we suppose the sector to be covered by the union of squares of the type

$$rkB < \Re\xi \leqslant (r+1)kB, \quad skB < \Im\xi \leqslant (s+1)kB$$

for the appropriate range of integers r, s where $B = B(\rho, k)$ is an integer, each containing B^2 points of the form $k[m+\frac{1}{2}+i(n+\frac{1}{2})]$. It can easily be shown that if

$$l = \min\left\{\frac{1}{(\rho\pi)^{\frac{1}{2}}}, \inf_{p>\rho}\left[\frac{2p-1}{(\rho\pi)^{\frac{1}{2}}}\sin\left(\frac{\rho\pi}{2p}\right)\right]\right\}, \tag{5·34}$$

a continuous curve of length less than l can intersect at most six of the quadrilaterals $A_{p,q}$. Now

$$\left(\frac{\rho}{\pi}\right)^{\frac{1}{2}} \leqslant \frac{2p-1}{(\rho\pi)^{\frac{1}{2}}}\sin\left(\frac{\pi\rho}{2p}\right)$$

for $p \geqslant \rho$, and therefore

$$l \geqslant \min\{(\rho\pi)^{-\frac{1}{2}}, (\rho/\pi)^{\frac{1}{2}}\},$$

which gives

$$1/l < (\rho\pi)^{\frac{1}{2}} + (\pi/\rho)^{\frac{1}{2}}.$$

Hence the perimeter of a square of side kB can intersect at most $24kB[(\rho\pi)^{\frac{1}{2}} + (\pi/\rho)^{\frac{1}{2}}]$ of the quadrilaterals. Therefore the number of points $\xi_{p,q}$ in each square is at least

$$k^2B^2 - 24kB[(\rho\pi)^{\frac{1}{2}} + (\pi/\rho)^{\frac{1}{2}}]$$

and this is at least B^2 if

$$B \geqslant \frac{24k[(\rho\pi)^{\frac{1}{2}} + (\pi/\rho)^{\frac{1}{2}}]}{k^2-1}.$$

Consequently we can allot to each point $k(m+\frac{1}{2}+i(n+\frac{1}{2}))$ a point $\xi_{p,q}$ such that

$$|\xi_{p,q} - k[m+\tfrac{1}{2}+i(n+\tfrac{1}{2})]| \leqslant \frac{24\sqrt{2}k[(\rho\pi)^{\frac{1}{2}} + (\pi/\rho)^{\frac{1}{2}}]}{k^2-1}, \tag{5·35}$$

which is equivalent to (3·1), for all $k > 1$.

Suppose, initially, that $\frac{1}{2}\rho\alpha < \pi$. The transformation $\zeta = (2\tau/\pi)^{\frac{1}{2}} z^{\frac{1}{2}\rho}$ takes a function $f(z)$ satisfying the conditions of Theorem 1, with $\{z_n\}$ a set of $U(\rho, \tau)$, into a function $F(\zeta)$ satisfying

(i) $$\liminf_{R\to\infty} \frac{\log M(R, \frac{1}{2}\rho\alpha, F)}{R^2} < \tfrac{1}{2}\pi;$$

(ii) $$\limsup_{R\to\infty} \frac{\log|F(Re^{\pm\frac{1}{2}i\rho\alpha})|}{R^2} \leqslant \frac{\pi\gamma}{2\tau} < \tfrac{1}{2}\pi;$$

(iii) $$\limsup_{\zeta_n \in S(\rho\alpha, l')} \frac{\log|F(\zeta_n)|}{H(|\zeta_n|)} \leqslant \kappa < \infty$$

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for a set $\{\zeta_n\}$ belonging to $U^*(2, \frac{1}{2}\pi/k^2)$, for every $k > 1$, where $H(R)$ is a positive non-decreasing function of R with $R^{-2}H(R)$ non-increasing and having limit $\pi c/2\tau$, with $c\kappa\pi/2\tau < \frac{1}{2}\pi$. We now choose $k\,(>1)$ sufficiently near to 1 to ensure that the inequalities

$$\sin^2[\tfrac{1}{2}\rho\alpha], \quad \sin^2[\tfrac{1}{2}\rho(\alpha-\beta)] > k^2\frac{\gamma}{\tau}\left\{1-\frac{c\kappa(\tau-\kappa^2\gamma)}{\gamma(\tau-k^2c\kappa)}\right\},$$

$$\liminf_{R\to\infty}\frac{\log M(R, \frac{1}{2}\rho\alpha, F)}{R^2}, \quad \frac{c\kappa\pi}{2\tau}, \quad \frac{\pi\gamma}{2\tau} < \tfrac{1}{2}\pi/k^2$$

all hold. It then follows, from the cases of the theorem already established, that

$$\limsup_{R\to\infty}\frac{\log M(R, \frac{1}{2}\rho\beta, F)}{H(R)} \leqslant \kappa\left(1+\frac{k^2c\kappa}{\tau-k^2c\kappa}\right)+K\lim_{R\to\infty}\frac{1}{H(R)},$$

and since this is so for every $k > 1$ it follows that

$$\limsup_{R\to\infty}\frac{\log M(R, \frac{1}{2}\rho\beta, F)}{H(R)} \leqslant \kappa\left(\frac{1+c\kappa}{\tau-c\kappa}\right)+K\lim_{R\to\infty}\frac{1}{H(R)}.$$

This, on transforming back, gives the result for $\frac{1}{2}\rho\alpha < \pi$.

For the case $\frac{1}{2}\rho\alpha > \pi$ we first note that the conditions of the theorem, with $\{z_n\}$ a set belonging to $U(\rho, \tau)$, imply those of Theorem B, and hence

$$\limsup_{r\to\infty}\frac{\log M(r, \alpha, f)}{r^\rho} \leqslant \sup\{c\kappa, \gamma\}.$$

If $0 < \gamma < c\kappa$ we have nothing further to prove, and therefore we may assume

$$\limsup_{r\to\infty}\frac{\log M(r, \alpha, f)}{r^\rho} \leqslant \gamma.$$

We may also, without loss of generality, assume that $\frac{1}{2}\rho(\alpha-\beta) < \frac{1}{2}\pi$. Now choose a θ such that

$$4\pi/\rho > \theta > 2(\alpha-\beta),$$

and consider any sector of aperture θ contained in $S(\alpha, l)$. The conditions (suitably modified to allow for the loss of symmetry) of the special case of Theorem 1 proved above are satisfied by this sector and repeated application of this result completes the proof of the theorem for $\{z_n\}$ belonging to $U(\rho, \tau)$.

Proof of Theorem 3. Consider the effect of the transformation $\zeta = z^{\frac{1}{2}\rho}$ on the set of discs $U_n \equiv |z-z_n| \leqslant \Delta\,|z_n|^{\frac{1}{2}(1-\rho)}$. Write $z_n^{\frac{1}{2}\rho} = \zeta_n$; we then have

$$\zeta(U_n) \equiv |\zeta-\zeta_n+o(1)| \leqslant \tfrac{1}{2}\rho\Delta \tag{5·36}$$

as $|z_n| \to \infty$. We now restrict ourselves to considering the transforms of those discs which intersect some fixed sector of aperture $\theta < 4\pi/\rho$. The transforms of the discs cover a sector of aperture $\frac{1}{2}\rho\theta$, and if $\Delta_1 > \Delta$ the part of the sector outside $|\zeta| \leqslant R_0$, $R_0 = R_0(\rho, \Delta, \Delta_1)$ is covered by

$$\bigcup_n\{|\zeta-\zeta_n| \leqslant \tfrac{1}{2}\rho\Delta_1\}.$$

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If $\Delta_2 > \Delta_1$ we can choose a subset $\{\zeta_{n'}\}$ of the ζ_n such that this region is covered by

$$\bigcup_{n'}\{|\zeta-\zeta_{n'}| \leqslant \tfrac{1}{2}\rho\Delta_2\},$$

and for $m' \neq n'$ $$|\zeta_{n'}-\zeta_{m'}| \geqslant \tfrac{1}{2}\rho(\Delta_2-\Delta_1) > 0.$$

As in the proof of Theorem 1 for $\{z_n\}$ belonging to $U(\rho,\tau)$, we suppose the sector to be covered by a lattice of squares of side

$$\frac{k\rho\Delta_2(3\sqrt{3})^{\frac{1}{2}}B}{2\sqrt{2}},$$

where $k > 1$ and $B = B(k)$ is an integer. It follows from a result of Kershner ((7), Lemma 5) that the minimum number of circles of radius d necessary to cover a square of side A exceeds $2(A^2-2\pi d^2)/d^2 3\sqrt{3}$, and hence the number of $\zeta_{n'}$ in each of the squares of side $k\rho\Delta_2(3\sqrt{3})^{\frac{1}{2}}B/2\sqrt{2}$, lying wholly within $|\zeta| \geqslant R_0$, must exceed

$$\frac{8}{\rho^2\Delta_2^2 3\sqrt{3}}\left\{\left[\frac{k\rho\Delta_2(3\sqrt{3})^{\frac{1}{2}}B}{2\sqrt{2}}-\frac{\rho\Delta_2}{2}\right]^2-\frac{\pi\rho^2\Delta_2^2}{2}\right\}$$

and this is at least B^2 if

$$B \geqslant \frac{k\sqrt{2}+\sqrt{[2(2\pi-1)(\kappa^2-1)+2]}}{(k^2-1)(3\sqrt{3})^{\frac{1}{2}}}.$$

It follows that $\{\zeta_n\}$ has a subset agreeing with a set of $U^*(2, 4\pi/k^2\rho^2\Delta_2^2 3\sqrt{3})$ in the sector, for every $k > 1$ and $\Delta_2 > \Delta_1 > \Delta$. It follows from this, by a method similar to that used in the proof of Theorem 1, that $\{z_n\}$ has a subset belonging to $U(\rho,\tau)$ for every τ satisfying

$$\tau < \frac{\rho^2}{4}\frac{4\pi}{k^2\rho^2\Delta_2^2 3\sqrt{3}} = \frac{\pi}{k^2\Delta_2^2 3\sqrt{3}},$$

and therefore for every τ satisfying

$$\tau < \frac{\pi}{\Delta^2 3\sqrt{3}}.$$

This completes the proof of the theorem.

It is easy to see that the constant $\pi/\Delta^2 3\sqrt{3}$ of Theorem 3 cannot be increased by considering the set of points at the vertices of a lattice of equilateral triangles which cover the plane, which belongs to $U^*(2, \pi/d^2\sqrt{3})$, where d is the length of the sides of the triangles, and also to $B[2, d/\sqrt{3}]$.

The proof of Theorem 4 follows from similar considerations.

6. *Gap power series.* In this section we consider integral functions having a power series representation,

$$f(z) = \sum_{k=1}^{\infty} a_{n_k} z^{n_k}, \tag{6·1}$$

which satisfies the Fabry gap condition,

$$n_k/k \to \infty \quad \text{as} \quad k \to \infty. \tag{6·2}$$

Several authors (e.g. Pólya (13), Turan (15)) have obtained results which show that under this condition the behaviour of $f(z)$ is angularly uniform.

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Turan, using a very general inequality on series, obtained

$$[M(r,f)]^{1+\epsilon} \leqslant \frac{16\rho\pi}{\beta-\alpha}[M(2r)]^{2\epsilon} M(r,\alpha,\beta,f) \tag{6·3}$$

for $r \geqslant r_0(\epsilon)$, where $M(r,\alpha,\beta,f) = \sup_{\alpha<\theta<\beta} |f(r e^{i\theta})|$. If $f(z)$ is of finite order ρ this gives, for any sequence $\{r_n\}$, $r_n\uparrow \to \infty$,

$$\lim \sup_{\inf} \frac{\log M(r_n,f)}{r_n^\rho} = \lim \sup_{\inf} \frac{\log M(r_n,\alpha,\beta,f)}{r_n^\rho} \tag{6·4}$$

and if, also, $f(z)$ is bounded in any sector then it is constant.

The results immediately above do not depend on the aperture $\beta-\alpha$ of the sector. It might be asked how the behaviour of $f(z)$ on a set of points in some sector, of arbitrarily small aperture, determines the behaviour of the function in the whole plane. For example, it might be conjectured, for the type of function under discussion, that boundedness at the members of a set of $P(\rho,\tau)$ lying within some sector, together with the assumption that the function was at most of type $\kappa < \tau$ of order ρ would imply that the function was constant. A result of this kind is contained in

THEOREM 6. *Suppose that $f(z)$ is an integral function at most of minimum type of order ρ satisfying* (6·1) *and* (6·2). *Suppose, also, that*

$$\limsup_{z_n \in S(\alpha,\beta)} \frac{\log|f(z_n)|}{\log|z_n|} \leqslant \kappa < \infty,$$

where $\{z_n\}$ is a set of $B[\rho,\Delta]$, for some $\Delta < \infty$. Then $f(z)$ is a polynomial of degree at most κ.

Proof of Theorem 6. It follows from (6·3) that

$$\frac{\log M(r,\alpha,f)}{\log M(r,f)} \geqslant 1+\epsilon-\frac{K}{\log M(r,f)} - 2\epsilon\frac{\log M(2r,f)}{\log M(r,f)}. \tag{6·5}$$

Now, since $f(z)$ is of finite order, we must have

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{\log M(2r,f)} > 0, \tag{6·6}$$

which, together with (6·5), gives (assuming $M(r,f)$ to be unbounded)

$$\limsup_{r\to\infty} \frac{\log M(r,\alpha,f)}{\log M(r,f)} = 1. \tag{6·7}$$

Consequently, there exists a sequence $\{r_n\}$ with $r_n\uparrow \to \infty$ such that

$$\log M(r_n,\alpha,f) \geqslant (1-\epsilon)\log M(r_n,f). \tag{6·8}$$

Now it follows from Theorem 1 that, for any $\beta < \alpha$ with $r \geqslant r_0(\epsilon,\beta)$,

$$\log M(r,\beta,f) \leqslant (\kappa+\epsilon)\log r, \tag{6·9}$$

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which gives
$$\log M(r_n,f) \leqslant \frac{\kappa+\epsilon}{1-\epsilon}\log r_n. \tag{6·10}$$

Now $\log M(r,f)$ is a convex function of $\log r$, and hence for $r_{n-1} \leqslant r \leqslant r_n$ we have

$$\log M(r,f) \leqslant \frac{\log r-\log r_{n-1}}{\log r_n-\log r_{n-1}}\log M(r_n,f)+\frac{\log r_n-\log r}{\log r_n-\log r_{n-1}}\log M(r_{n-1},f) \tag{6·11}$$

$$\leqslant \frac{\kappa+\epsilon}{1-\epsilon}\log r. \tag{6·12}$$

This implies
$$\limsup_{r\to\infty}\frac{\log M(r,f)}{\log r} \leqslant \kappa, \tag{6·13}$$

which is equivalent to the result of the theorem.

Theorem 6 might be combined with

THEOREM 7. *Suppose that $f(z)$ is an integral function at most of type $\gamma < \tau$ of order ρ satisfying* (6·1) *and* (6·2). *Suppose, also, that*

$$\limsup_{z_n\in S(\alpha,l)}\frac{\log|f(z_n)|}{|z_n|^\rho} = 0, \tag{6·14}$$

where $\{z_n\}$ is a set of $B[\rho,\Delta]$, $\tau = \pi/\Delta^2 3\sqrt{3}$ and $\sin[\frac{1}{2}\rho\alpha] > \gamma/\tau$. Then $f(z)$ is at most of minimum type of order ρ.

It should be noted that this result is stronger than that obtained by a direct application of Theorem 1, as this would require $\sin^2[\frac{1}{2}\rho\alpha] > \gamma/\tau$, instead of the weaker $\sin[\frac{1}{2}\rho\alpha] > \gamma/\tau$.

Proof of Theorem 7. The proof is based on the following

LEMMA. *Suppose that $f(z)$ satisfies conditions* (i), (ii), (iii) *of Theorem* 1, *with $c = 0$. If $\alpha < \pi/\rho$ suppose, in addition, that $\sin[\frac{1}{2}\rho\alpha] > \gamma/\tau$. Then if β $(0 < \beta < \alpha)$ satisfies $\sin[\frac{1}{2}\rho(\alpha-\beta)] > \gamma/\tau$,*

$$\limsup_{r\to\infty}\frac{\log M(r,\beta,f)}{r^\rho} \leqslant \sup\left\{0, \gamma\left(1-\frac{2\eta\tau}{\tau-\gamma}\right)\right\}, \tag{6·15}$$

where $\eta = \sin[\frac{1}{2}\rho(\alpha-\beta)]-\gamma/\tau$.

The proof follows closely that of Theorem 1, the difference occurring in the estimate for $|f(z)/\sigma_{M,N}(z)|$ on the boundary of $S(\alpha,l)$ in (5·7)–(5·9).

It follows from the lemma that the type of $f(z)$ in $S(\beta,l)$ is at most $\theta\gamma$, where $\theta < 1$, and combining this with (6·4) shows that $f(z)$ is at most of type $\theta\gamma$ of order ρ. Repeated application of this argument, with $\theta = 1-2\eta$ = constant, shows that $f(z)$ is at most of minimum type of order ρ.

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REFERENCES

(1) CARTWRIGHT, M. L. On functions bounded at the lattice points in an angle. *Proc. London Math. Soc.* (2), **43** (1937), 26–32.
(2) IYER, V. G. A note on integral functions of order 2 bounded at the lattice points. *J. London Math. Soc.* **11** (1936), 247–9.
(3) IYER, V. G. On effective sets of points in relation to integral functions. *Trans. American Math. Soc.* **42** (1937), 358–65.
(4) IYER, V. G. (Correction to (3).) *Trans. American Math. Soc.* **43** (1938), 494.
(5) IYER, V. G. Some theorems on functions regular in an angle. *Quart. J. Math.* (Oxford) **9** (1938), 206–15.
(6) IYER, V. G. Determinative sets of points for classes of integral functions. *Math. Z.* **44** (1939), 195–200.
(7) KERSHNER, R. The number of circles covering a set. *American J. Math.* **61** (1939), 665–71.
(8) NOBLE, M. E. Non-measurable interpolation sets. I. *Proc. Cambridge Philos. Soc.* **47** (1951), 713–32.
(9) NOBLE, M. E. Non-measurable interpolation sets. II. *Proc. Cambridge Philos. Soc.* **47** (1951), 733–40.
(10) PFLUGER, A. On analytic functions bounded at the lattice points. *Proc. London Mach. Soc.* (2), **42** (1937), 305–15.
(11) PFLUGER, A. Über Interpolation ganzer Funktionen. *Comment. Math. Hel.* **14** (1941–42), 314–49.
(12) PFLUGER, A. Über ganze Funktionen ganzer Ordnung. *Comment. Math. Hel.* **18** (1945–46), 177–203.
(13) PÓLYA, G. Untersuchungen über Lücken und Singularitäten von Potenzreihen. *Math. Z.* **29** (1929), 549–640.
(14) PÓLYA, G. Bemerkung zu der Lösung der Aufgabe 105. *Jber. Deutsch. Math. Verein* **43** (1933), 67–69.
(15) TURAN, P. On a property of lacunary power series. *Acta Sci. Math. Szeged* **14** (1952), 209–18.
(16) WHITTAKER, J. M. On the flat regions of integral functions of finite order. *Proc. Edinburgh Math. Soc.* **2** (1930), 111–28.
(17) WHITTAKER, J. M. On the fluctuation of integral and meromorphic functions. *Proc. London Math. Soc.* (2), **37** (1934), 383–401.

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