MORE ON YET ANOTHER IDEAL VERSION OF THE BOUNDING NUMBER

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Abstract. This is a continuation of the paper [J. Symb. Log. 87 (2022), 1065–1092]. For an ideal $\mathcal{I}$ on ω we denote $\mathcal{D}_\mathcal{I} = \{f \in \omega^\omega : f^{-1}[\{n\}] \in \mathcal{I} \text{ for every } n \in \omega\}$ and write $f \leq_\mathcal{I} g$ if $\{n \in \omega : f(n) > g(n)\} \in \mathcal{I}$, where $f, g \in \omega^\omega$.

We study the cardinal numbers $\mathfrak{b}(\geq_\mathcal{I} \cap (\mathcal{D}_\mathcal{I} \times \mathcal{D}_\mathcal{I}))$ describing the smallest sizes of subsets of $\mathcal{D}_\mathcal{I}$ that are unbounded from below with respect to $\leq_\mathcal{I}$.

In particular, we examine the relationships of $\mathfrak{b}(\geq_\mathcal{I} \cap (\mathcal{D}_\mathcal{I} \times \mathcal{D}_\mathcal{I}))$ with the dominating number $\mathfrak{d}$. We show that, consistently, $\mathfrak{b}(\geq_\mathcal{I} \cap (\mathcal{D}_\mathcal{I} \times \mathcal{D}_\mathcal{I})) > \mathfrak{d}$ for some ideal $\mathcal{I}$, however $\mathfrak{b}(\geq_\mathcal{I} \cap (\mathcal{D}_\mathcal{I} \times \mathcal{D}_\mathcal{I})) \leq \mathfrak{d}$ for all analytic ideals $\mathcal{I}$. Moreover, we give example of a Borel ideal with $\mathfrak{b}(\geq_\mathcal{I} \cap (\mathcal{D}_\mathcal{I} \times \mathcal{D}_\mathcal{I})) = \mathrm{add}(\mathcal{M})$.

§1. Introduction. Let $\mathcal{I}$ be an ideal on ω. For $f, g \in \omega^\omega$ write $f \leq_\mathcal{I} g$ if $\{n \in \omega : f(n) > g(n)\} \in \mathcal{I}$. Moreover, denote $\mathcal{D}_\mathcal{I} = \{f \in \omega^\omega : f^{-1}[\{n\}] \in \mathcal{I} \text{ for every } n \in \omega\}$. Let $\mathfrak{b}(\geq_\mathcal{I} \cap (\mathcal{D}_\mathcal{I} \times \mathcal{D}_\mathcal{I}))$ (for simplicity denoted also by $\mathfrak{b}(\mathcal{I})$) be the smallest size of a set in $\mathcal{D}_\mathcal{I}$ not bounded from below with respect to the order $\leq_\mathcal{I}$ by any member of $\mathcal{D}_\mathcal{I}$:

$$\mathfrak{b}(\mathcal{I}) = \min\left\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{D}_\mathcal{I} \wedge \neg(\exists_{g \in \mathcal{D}_\mathcal{I}} \forall_{f \in \mathcal{F}}\, g \leq_\mathcal{I} f\right\}.$$

The cardinal $\mathfrak{b}(\mathcal{I})$ in the case of maximal ideals has been deeply studied by Canjar in 1980s in the context of smallest sizes of cofinal and coinitial subsets in ultrapowers $\omega^\omega/\mathcal{U}$ ordered by $\leq_\mathcal{I}$, where $\mathcal{U} = \mathcal{I}^\star$ (see [11–13]). In the case of Borel ideals, we have extensively examined $\mathfrak{b}(\mathcal{I})$ in [17].

This research is partially motivated by the study of ideal-QN-spaces. A topological space X is a QN-space if it does not distinguish pointwise and quasi-normal convergence of sequences of real-valued continuous functions defined on X. The research on QN-spaces was initiated by Bukovský, Recław and Repický in [9], who proved that the smallest size of non-QN-space is equal to the bounding number $\mathfrak{b}$. Studies of QN-spaces have been continued in papers [5, 6, 8–10, 28, 29, 33, 37]. Ideal variants of QN-spaces were introduced in [14] and studied in [7, 26, 30, 31, 34, 36]. For a given ideal $\mathcal{I}$, the cardinal number $\mathfrak{b}(\mathcal{I})$ characterizes the smallest size of a space which is not $\mathcal{I}$QN (see [30]).

The paper is organized as follows.

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In Section 2 we collect basics about ideals on ω and some known facts about the cardinal numbers $\mathfrak{b}(\mathcal{I})$. In particular, we recall a very useful combinatorial characterization of $\mathfrak{b}(\mathcal{I})$ from [17], which we use almost exclusively in the rest of the paper.

In the remaining part of the paper we answer two natural questions concerning $\mathfrak{b}(\mathcal{I})$.

The first question is about possible values of $\mathfrak{b}(\mathcal{I})$ for Borel ideals: in [17] it is shown that there are $\mathbf{\Sigma}^0_2$ ideals $\mathcal{I}$ with $\mathfrak{b}(\mathcal{I}) = \aleph_1$ provable in ZFC [17, Corollary 7.3 and Theorem 7.4] as well as $\mathbf{\Sigma}^0_2$ ideals $\mathcal{I}$ with $\mathfrak{b}(\mathcal{I}) = \mathfrak{b}$ provable in ZFC [17, Example 5.15]. However, it was unknown whether $\aleph_1$ and $\mathfrak{b}$ in the above can be replaced by other cardinals. In Section 3 we study $\mathfrak{b}(\mathcal{I})$ in the case of ideals with topological representation (such ideals were introduced by Sabok and Zapletal in [32]) obtaining a $\mathbf{\Pi}^0_3$ ideal with $\mathfrak{b}(\mathcal{I}) = \mathrm{add}(\mathcal{M})$ provable in ZFC.

The second question concerns comparison of $\mathfrak{b}(\mathcal{I})$ with the dominating number $\mathfrak{d}$. In general, $\aleph_1 \leq \mathfrak{b}(\mathcal{I}) \leq \mathfrak{c}$, for every ideal $\mathcal{I}$ [17, Theorem 4.2]. Moreover, the upper bound can be improved to $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{b}$ in the case of $\mathbf{\Pi}^0_4$ ideals $\mathcal{I}$ ([17, Corollary 6.9]; in fact, as shown in [17, Theorem 6.8], this holds even for a larger class of all so-called co-analytic weak P-ideals). On the other hand, Canjar proved in [12, 11–13] that under $\mathfrak{d} = \mathfrak{c}$ there is always a maximal ideal $\mathcal{I}$ with $\mathfrak{b}(\mathcal{I}) = \mathrm{cf}(\mathfrak{d})$ and that in the model obtained by adding λ Cohen reals to a model of GCH, $\mathfrak{b}(\mathcal{I})$ can be equal to any regular cardinal between $\aleph_1$ and $\lambda = \mathfrak{d}$, for some maximal ideal $\mathcal{I}$. However, it was unknown if it is consistent to have an ideal $\mathcal{I}$ with $\mathfrak{b}(\mathcal{I}) > \mathfrak{d}$. We show that the answer is affirmative (Section 4), however $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{d}$ for every analytic ideal $\mathcal{I}$ (Section 5). The latter result uses ideas developed by Debs and Saint Raymond in [15], providing a new method of proving statements about analytic ideals.

§2. Preliminaries. By ω we denote the set of all natural numbers. We identify a natural number n with the set $\{0, 1, \dots, n-1\}$. We write $A \subseteq^\star B$ if $A \setminus B$ is finite. For a set A and a finite or infinite cardinal number κ, we write $[A]^\kappa = \{B \subseteq A : |B| = \kappa\}$ and $[A]^{<\kappa} = \{B \subseteq A : |B| < \kappa\}$. Moreover, by $\mathrm{cf}(\kappa)$ we denote the cofinality of κ.

2.1. Ideals and σ-ideals. An *ideal on a set* X is a family $\mathcal{I} \subseteq \mathcal{P}(X)$ satisfying the following properties:

(i) if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$;
(ii) if $A \subseteq B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$;
(iii) $\mathcal{I}$ contains all finite subsets of X;
(iv) $X \notin \mathcal{I}$.

Note that, thanks to item (iii), $\bigcup \mathcal{I} = X$—we will use this observation several times in Section 5. A *σ-ideal on X* is an ideal satisfying one additional property:

(v) if $\langle A_n : n \in \omega \rangle \in \mathcal{I}^\omega$ then $\bigcup_{n\in\omega} A_n \in \mathcal{I}$.

Note that, unlike most authors, in our paper every σ-ideal on X already contains all countable subsets of X and has to be a proper subset of $\mathcal{P}(X)$ (i.e., cannot be equal to $\mathcal{P}(X)$). In this paper, by an *ideal* we mean ideal on some countable set, while all σ-ideals will be subsets of $\mathcal{P}(2^\omega)$.

The ideal of all finite subsets of ω is denoted by Fin. We say that an ideal $\mathcal{I}$ on X is *maximal* if $\mathcal{I} \subseteq \mathcal{J}$ implies $\mathcal{I} = \mathcal{J}$, for every ideal $\mathcal{J}$ on X.

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If $\mathcal{I}$ and $\mathcal{J}$ are ideals on X and Y, respectively, then we say that $\mathcal{I}$ and $\mathcal{J}$ are *isomorphic*, if there is a bijection $f : Y \to X$ such that

$$A \in \mathcal{I} \Leftrightarrow f^{-1}[A] \in \mathcal{J}$$

for every $A \subseteq X$. It is easy to see that many properties of ideals are preserved under isomorphisms of ideals.

For $\mathcal{A} \subseteq \mathcal{P}(X)$ we write

$$\mathcal{I}(\mathcal{A}) = \left\{ B \subseteq X : B \subseteq^{\star} \bigcup \mathcal{A}' \text{ for some } \mathcal{A}' \in [\mathcal{A}]^{<\omega} \right\}.$$

If $X \notin \mathcal{I}(\mathcal{A})$, then $\mathcal{I}(\mathcal{A})$ is an ideal, which we call *the ideal generated by* $\mathcal{A}$. Note that in this case $\mathcal{I}(\mathcal{A})$ is the smallest ideal containing $\mathcal{A}$.

By identifying subsets of a countable set X with their characteristic functions, we can equip $\mathcal{P}(X)$ with the topology of the Cantor space $\{0,1\}^X$ and therefore assign descriptive complexity to ideals on X. In particular, an ideal $\mathcal{I}$ is analytic if $\mathcal{I}$ is analytic as a subset of the space $\{0,1\}^X$.

By $\mathcal{M}$ ($\mathcal{N}$) we denote the σ-ideals of meager (null, respectively) subsets of 2^ω. If $\mathcal{A} \subseteq \mathcal{P}(2^\omega)$ and 2^ω cannot be covered by countably many members of $\mathcal{A}$, then by $\sigma\mathcal{A}$ we denote the *σ-ideal generated by* $\mathcal{A}$, i.e.,

$$\sigma\mathcal{A} = \left\{ B \subseteq 2^\omega : \exists_{\langle A_n \rangle \in \mathcal{A}^\omega} \, B \subseteq \bigcup_{n \in \omega} A_n \right\}.$$

For a σ-ideal I on 2^ω let $\overline{I}$ be the family of all compact subsets of 2^ω belonging to I. We will say that *I is generated by compact sets* if $I = \sigma\overline{I}$. In particular, $[2^\omega]^{<\omega}$, $\mathcal{M}$ and $\sigma\overline{\mathcal{N}}$ (i.e., the σ-ideal on 2^ω generated by compact null subsets of 2^ω) are generated by compact sets.

2.2. Some cardinal invariants. In our paper we will need the following cardinal invariants:

- the *pseudointersection number*:

$$\mathfrak{p} = \min \left\{ |\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \wedge \left(\forall_{\mathcal{A}_0 \in [\mathcal{A}]^{<\omega}} \bigcap \mathcal{A}_0 \neq \emptyset \right) \wedge \left(\forall_{S \in [\omega]^\omega} \exists_{A \in \mathcal{A}} \, |S \setminus A| = \omega \right) \right\};$$

- the *additivity of the σ-ideal of meager subsets of* 2^ω is given by

$$\mathrm{add}(\mathcal{M}) = \min \left\{ |\mathcal{A}| : \mathcal{A} \subseteq \mathcal{M} \wedge \bigcup \mathcal{A} \notin \mathcal{M} \right\};$$

- the *bounding number* $\mathfrak{b}$, which is the smallest size of an *$\leq_{\mathrm{Fin}}$-unbounded subset of* ω^ω, that is,

$$\mathfrak{b} = \min \left\{ |\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \neg(\exists_{g \in \omega^\omega} \forall_{f \in \mathcal{F}} \, f \leq_{\mathrm{Fin}} g \right\};$$

- the *dominating number* $\mathfrak{d}$, which is the smallest size of a *$\leq_{\mathrm{Fin}}$-dominating subset of* ω^ω, that is,

$$\mathfrak{d} = \min \left\{ |\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \forall_{g \in \omega^\omega} \exists_{f \in \mathcal{F}} \, g \leq_{\mathrm{Fin}} f \right\};$$

- the *ultrafilter number* $\mathfrak{u}$, which is the smallest size of a family generating a maximal ideal.

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It is known that

$$\aleph_1 \leq \mathfrak{p} \leq \mathrm{add}(\mathcal{M}) \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$$

(see [6, Section 9.2]). Moreover, $\mathfrak{b} \leq \mathfrak{u} \leq \mathfrak{c}$, however both $\mathfrak{d} < \mathfrak{u}$ and $\mathfrak{u} \leq \mathfrak{d}$ are consistent (see [4]).

The cardinal $\mathrm{add}(\mathcal{M})$ is a particular case of a more general invariant: if I is a σ-ideal on 2^ω, then

$$\mathrm{add}(I) = \min\left\{|\mathcal{A}| : \mathcal{A} \subseteq I \wedge \bigcup \mathcal{A} \notin I\right\}.$$

Note that $\mathrm{add}(I) \geq \aleph_1$. Moreover, if I is generated by compact sets, then

$$\mathrm{add}(I) = \min\left\{|\mathcal{A}| : \mathcal{A} \subseteq \overline{I} \wedge \bigcup \mathcal{A} \notin I\right\}.$$

2.3. A characterization of $\mathfrak{b}(\mathcal{I})$. Let $\mathcal{I}$ be an ideal on a set X. By $\widehat{\mathcal{P}}_\mathcal{I}$ we will denote the family of all sequences $\langle A_n : n \in \omega\rangle \in [\mathcal{P}(X)]^\omega$ such that $A_n \in \mathcal{I}$ for all $n \in \omega$ and $A_n \cap A_k = \emptyset$ whenever $n \neq k$. By $\mathcal{P}_\mathcal{I}$ we will denote the family of all sequences $\langle A_n : n \in \omega\rangle \in \widehat{\mathcal{P}}_\mathcal{I}$ such that $\bigcup\{A_n : n \in \omega\} = X$.

By [17, Theorem 3.10], the studied cardinal number $\mathfrak{b}(\mathcal{I})$ has a useful combinatorial characterization, which we will use almost exclusively in the rest of the paper without any reference:

$$\mathfrak{b}(\mathcal{I}) = \min\left\{|\mathcal{E}| : \mathcal{E} \subseteq \widehat{\mathcal{P}}_\mathcal{I} \wedge \forall_{\langle A_n\rangle \in \mathcal{P}_\mathcal{I}}\, \exists_{\langle E_n\rangle \in \mathcal{E}} \bigcup_{n\in\omega}\left(A_n \cap \bigcup_{i\leq n} E_i\right) \notin \mathcal{I}\right\}.$$

The following immediate observation will simplify some of our considerations.

Remark 2.1. If $\langle B_n : n \in \omega\rangle$ is a partition of a set X (that is $\bigcup_{n\in\omega} B_n = X$ and $B_n \cap B_m = \emptyset$ for all $n \neq m$) and $\langle A_i : i \in \omega\rangle$ is any sequence of subsets of X, then:

$$\bigcup_{i\in\omega}\left(A_i \setminus \bigcup_{n<i} B_n\right) = \bigcup_{n\in\omega}\left(B_n \cap \bigcup_{i\leq n} A_i\right).$$

§3. Ideals with topological representation. In this section we will deal with ideals having a topological representation in the sense of Sabok and Zapletal (see [32]).

Let X be a separable metrizable space with a countable dense set D and I be a σ-ideal on X (recall that in our paper every σ-ideal on X contains all singletons and is a proper subset of $\mathcal{P}(X)$). Following [32], we define an ideal on D by

$$\mathcal{J}_I = \{A \subseteq D : \mathrm{cl}_X(A) \in I\}$$

(here $\mathrm{cl}_X(A)$ denotes the closure of the set A in the space X). We say that an ideal *has a topological representation* if it is isomorphic to some $\mathcal{J}_I$ as above. In such case, we say that it is *represented on X by I*.

Note that $\mathcal{J}_I = \mathcal{J}_{\sigma\overline{I}}$, since $\mathcal{J}_I$ depends only on closed members of I. By [27, Proposition 2.1], two ideals represented by the same σ-ideal I, but defined on different countable dense subsets of X, are isomorphic. Moreover, each ideal with topological representation can be represented on the Cantor space 2^ω by a

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σ-ideal generated by some family of compact nowhere dense sets [27, Corollary 1.3]. Finally, every analytic ideal with topological representation is $\mathbf{\Pi^0_3}$, but not $\mathbf{\Sigma^0_2}$ [27, Theorem 1.4].

The basic examples of ideals having topological representation are NWD $= \mathcal{J}_{\mathcal{M}}$, NULL $= \mathcal{J}_{\sigma\overline{\mathcal{N}}}$ and CTBL $= \mathcal{J}_{[2^\omega]^{\leq\omega}}$. First two of them were introduces by Farah and Solecki in [16].

Before calculating the invariant $\mathfrak{b}(\mathcal{I})$ for ideals with topological representation, we need to introduce two notions.

DEFINITION 3.1. For a σ-ideal I on 2^ω we define

$$\mathrm{add}'(I) = \min\left\{|\mathcal{A}| :\ \mathcal{A} \subseteq \overline{I} \wedge \forall_{\langle B_n\rangle\in\overline{I}^\omega}\, \exists_{A\in\mathcal{A}}\, \forall_{n\in\omega}\, A \setminus B_n \neq \emptyset\right\}.$$

The coefficient $\mathrm{add}'(I)$ has been studied implicitly by Fremlin in [19, Definition 22A] and by Kankaanpää in [21]. Note that $\mathrm{add}'(I) = \mathrm{add}'(\sigma\overline{I})$, as $\mathrm{add}'(I)$ depends only on compact members of I.

PROPOSITION 3.2. *Let I be a σ-ideal on 2^ω generated by some family of compact sets. Then*

$$\aleph_1 \leq \mathrm{add}'(I) \leq \mathrm{add}(I).$$

PROOF. To show that $\aleph_1 \leq \mathrm{add}'(I)$, fix any countable $\mathcal{A} \subseteq \overline{I}$. Then $\mathcal{A} = \{B_n : n \in \omega\}$. Consider now the sequence $\langle B_n : n \in \omega\rangle \in \overline{I}^\omega$. Clearly, for each $A \in \mathcal{A}$ there is $n \in \omega$ with $A = B_n$, so $A \setminus B_n = \emptyset$. Thus, $\aleph_1 \leq \mathrm{add}'(I)$.

Now we show that $\mathrm{add}'(I) \leq \mathrm{add}(I)$. Since I is generated by some family of compact sets, we have

$$\begin{aligned}
\mathrm{add}(I) &= \min\left\{|\mathcal{A}| :\ \mathcal{A} \subseteq I \wedge \bigcup\mathcal{A} \notin I\right\}\\
&= \min\left\{|\mathcal{A}| :\ \mathcal{A} \subseteq \overline{I} \wedge \bigcup\mathcal{A} \notin I\right\}\\
&= \min\left\{|\mathcal{A}| :\ \mathcal{A} \subseteq \overline{I} \wedge \forall_{\langle B_n\rangle\in\overline{I}^\omega}\, \bigcup\mathcal{A} \setminus \bigcup_n B_n \neq \emptyset\right\}\\
&= \min\left\{|\mathcal{A}| :\ \mathcal{A} \subseteq \overline{I} \wedge \forall_{\langle B_n\rangle\in\overline{I}^\omega}\, \exists_{A\in\mathcal{A}}\, A \setminus \bigcup_n B_n \neq \emptyset\right\}.
\end{aligned}$$

Now it is easy to see that each family $\mathcal{A} \subseteq \overline{I}$ witnessing $\mathrm{add}(I)$ is also a witness for $\mathrm{add}'(I)$. Hence, $\mathrm{add}'(I) \leq \mathrm{add}(I)$. ⊣

PROPOSITION 3.3.

(a) $\mathrm{add}'(\mathcal{M}) = \mathrm{add}(\mathcal{M})$;
(b) $\mathfrak{p} \leq \mathrm{add}'(\mathcal{N}) = \mathrm{add}'(\sigma\overline{\mathcal{N}}) \leq \mathrm{add}(\mathcal{M})$;
(c) $\mathrm{add}'([2^\omega]^{\leq\omega}) = \aleph_1$.

PROOF. (a): The inequality $\mathrm{add}'(\mathcal{M}) \leq \mathrm{add}(\mathcal{M})$ follows from Proposition 3.2, while $\mathrm{add}'(\mathcal{M}) \geq \mathrm{add}(\mathcal{M})$ is [21, Lemma 3.5].

(b): Since $\mathrm{add}'(I)$ depends only on compact members of I, $\mathrm{add}'(\mathcal{N}) = \mathrm{add}'(\sigma\overline{\mathcal{N}})$. From Proposition 3.2 and [3, Theorem 3.1] we get $\mathrm{add}'(\sigma\overline{\mathcal{N}}) \leq \mathrm{add}(\sigma\overline{\mathcal{N}}) = \mathrm{add}(\mathcal{M})$, while $\mathfrak{p} \leq \mathrm{add}'(\mathcal{N})$ is shown in [19, Theorem 22G].

(c): It follows from Proposition 3.2 and the fact that $\mathrm{add}([2^\omega]^{\leq\omega}) = \omega_1$. ⊣

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DEFINITION 3.4. We say that a σ-ideal I on 2^ω is invariant over basic sets if given any $s \in 2^{<\omega}$ we have:

$$A \in I \Leftrightarrow f_s[A] \in I,$$

where $f_s : 2^\omega \to V_s$ is given by $f_s(x) = s^\frown x$ and $V_s = \{x \in 2^\omega : x | \operatorname{lh}(s) = s\}$ is the basic clopen set associated to s (here $\operatorname{lh}(s)$ is the length of s).

PROPOSITION 3.5. *The σ-ideals $\mathcal{M}, \mathcal{N}, \sigma\overline{\mathcal{N}}$ and $[2^\omega]^{<\omega}$ are invariant over basic sets.*

PROOF. In the case of $[2^\omega]^{<\omega}$, it suffices to observe that each f_s is a bijection (so $|A| = |f_s[A]|$). Since each f_s is also a homeomorphism, $\mathcal{M}$ is invariant over basic sets. Finally, since for each measurable $A \subseteq 2^\omega$ the measure of $f_s[A]$ is equal to the measure of A multiplied by $\frac{1}{2^{\operatorname{lh}(s)}}$, $\mathcal{N}$, and $\sigma\overline{\mathcal{N}}$ are invariant over basic sets. ⊣

Finally, we are ready to calculate the invariants $\mathfrak{b}(\mathcal{I})$ in the case of ideals with topological representation.

THEOREM 3.6. *Let I be an invariant over basic sets σ-ideal on 2^ω generated by some family of compact sets. Then*

$$\min\{\operatorname{add}'(I), \mathfrak{b}\} \leq \mathfrak{b}(\mathcal{J}_I) \leq \operatorname{add}'(I).$$

Moreover, if $\mathcal{J}_I$ is analytic, then

$$\mathfrak{b}(\mathcal{J}_I) = \min\{\operatorname{add}'(I), \mathfrak{b}\}.$$

PROOF. The case of analytic ideals will follow from the general case. Indeed, each analytic ideal with topological representation $\mathcal{J}_I$ is $\mathbf{\Pi^0_3}$ (by [27, Theorem 1.4]), so using [17, Corollary 6.9] we get $\mathfrak{b}(\mathcal{J}_I) \leq \mathfrak{b}$.

By [27, Proposition 2.1], without loss of generality we may assume that $\mathcal{J}_I$ is an ideal on $D = \{x \in 2^\omega : \exists_{k\in\omega}\, \forall_{n\geq k}\, x(n) = 0\}$. Note that $f_s[D] = D \cap V_s$, where V_s and f_s are as in Definition 3.4.

We start with $\mathfrak{b}(\mathcal{J}_I) \geq \min\{\operatorname{add}'(I), \mathfrak{b}\}$. Fix any $\kappa < \min\{\operatorname{add}'(I), \mathfrak{b}\}$. We need to show that $\kappa < \mathfrak{b}(\mathcal{J}_I)$. Let $\{\langle E^\alpha_n : n \in \omega\rangle : \alpha < \kappa\} \subseteq \widehat{\mathcal{P}}_{\mathcal{J}_I}$ be arbitrary. Since $\omega \cdot \kappa < \operatorname{add}'(I)$ (by Proposition 3.2), there is $\langle B_n : n \in \omega\rangle \in \overline{I}^\omega$ such that for each $n \in \omega$ and $\alpha < \kappa$ there exists $f_\alpha(n) \in \omega$ with $\operatorname{cl}_{2^\omega}(E^\alpha_n) \subseteq B_{f_\alpha(n)}$.

Since $\kappa < \mathfrak{b}$, there exists $g \in \omega^\omega$ such that $f_\alpha \leq^\star g$, for all $\alpha < \kappa$. Without loss of generality, we may assume that g is strictly increasing.

Let $\{q_n : n \in \omega\}$ be an enumeration of D and define $C_0 = \bigcup_{i\leq g(1)}(B_i \cap D) \cup \{q_0\}$ and $C_n = (\bigcup_{i\leq g(n+1)}(B_i \cap D) \cup \{q_n\}) \setminus \bigcup_{i<n} C_i$. Then $\langle C_n : n \in \omega\rangle \in \mathcal{P}_{\mathcal{J}_I}$, since

$$\operatorname{cl}_{2^\omega}(C_n) \subseteq \operatorname{cl}_{2^\omega}\left(\bigcup_{i\leq g(n+1)} B_i \cup \{q_n\}\right) = \bigcup_{i\leq g(n+1)} B_i \cup \{q_n\} \in I.$$

Fix now $\alpha < \kappa$. We need to show that $\bigcup_{n\in\omega}\left(C_n \cap \bigcup_{i\leq n} E^\alpha_i\right) \in \mathcal{J}_I$.

Let $k_\alpha \in \omega$ be such that $f_\alpha(m) \leq g(m)$, for every $m > k_\alpha$. Observe that for each $m > k_\alpha$ we get

$$E^\alpha_m \subseteq \operatorname{cl}_{2^\omega}(E^\alpha_m) \cap D \subseteq B_{f_\alpha(m)} \cap D \subseteq \bigcup_{n\leq g(m)} B_n \cap D \subseteq \bigcup_{n\leq m-1} C_n$$

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and consequently:

$$E^\alpha_m \cap \bigcup_{n\in\omega}\left(C_n \cap \bigcup_{i\le n} E^\alpha_i\right) \subseteq E^\alpha_m \cap \bigcup_{n\ge m} C_n = \emptyset,$$

since $\langle E^\alpha_n : n\in\omega\rangle \in \widehat{\mathcal{P}}_{\mathcal{J}_I}$. Hence,

$$\bigcup_{n\in\omega}\left(C_n \cap \bigcup_{i\le n} E^\alpha_i\right) \subseteq \left(\bigcup_{m\in\omega} E^\alpha_m\right) \cap \left(\bigcup_{n\in\omega}\left(C_n \cap \bigcup_{i\le n} E^\alpha_i\right)\right) \subseteq \bigcup_{m\le k_\alpha} E^\alpha_m \in \mathcal{J}_I.$$

Now we move to $\mathfrak{b}(\mathcal{J}_I) \le \mathrm{add}'(I)$. Let $\mathcal{A}\subseteq \overline{I}$ be the family of cardinality $\mathrm{add}'(I)$ such that for each $\langle B_n : n\in\omega\rangle \in \overline{I}^\omega$ one can find $A\in\mathcal{A}$ with $A\setminus B_n \neq \emptyset$, for all n.

Without loss of generality we may assume that $\mathrm{cl}_{2^\omega}(A\cap D) = A$ for each $A\in\mathcal{A}$. Indeed, given $A\in\mathcal{A}$, fix any countable $E\subseteq A$ such that $\mathrm{cl}_{2^\omega}(E) = A$ (this is possible as A is a closed subset of 2^ω) and enumerate $E = \{e_i : i\in\omega\}$. For each $i\in\omega$ find a sequence $\langle x_{i,j} : j\in\omega\rangle \in [D\cap B(e_i, \frac{1}{2^i})]^\omega$ converging to e_i (here $B(e_i,\frac{1}{2^i})$ denotes the open ball in 2^ω of radius $\frac{1}{2^i}$ centered at e_i). By defining $\hat{A} = A\cup\{x_{i,j} : i,j\in\omega\}$ we get a closed set such that $\hat{A} = \mathrm{cl}_{2^\omega}(\hat{A}\cap D)$. Moreover, as $A\subseteq\hat{A}$, the family $\{\hat{A} : A\in\mathcal{A}\}$ satisfies the same property as $\mathcal{A}$. Thus, we will assume from now on that $\mathrm{cl}_{2^\omega}(A\cap D) = A$, for every $A\in\mathcal{A}$.

Fix any bijection $h:\omega\to 2^{<\omega}$ and define $E^A_n = f_{h(n)}[A\cap D]$ for each $n\in\omega$ and $A\in\mathcal{A}$. Observe that

$$\mathrm{cl}_{2^\omega}(E^A_n) = \mathrm{cl}_{2^\omega}(f_{h(n)}[A\cap D]) \subseteq f_{h(n)}[\mathrm{cl}_{2^\omega}(A\cap D)] = f_{h(n)}[A] \in I,$$

since $f_{h(n)}$ is a homeomorphism and I is invariant over basic sets. Hence, each E^A_n belongs to $\mathcal{J}_I$. We claim that the family $\{\langle \hat{E}^A_n : n\in\omega\rangle : A\in\mathcal{A}\} \subseteq \widehat{\mathcal{P}}_{\mathcal{J}_I}$, where $\hat{E}^A_n = E^A_n \setminus \bigcup_{i<n} E^A_i$ for all $n\in\omega$ and $A\in\mathcal{A}$, witnesses $\mathfrak{b}(\mathcal{J}_I)\le \mathrm{add}'(I)$.

Fix any partition $\langle B_n : n\in\omega\rangle \in \mathcal{P}_{\mathcal{J}_I}$ of D.

Note that given any $n\in\omega$, we have $V_{h(n)}\cap\bigcup_{i<n}\mathrm{cl}_{2^\omega}(B_i) \subseteq \bigcup_{i<n}\mathrm{cl}_{2^\omega}(B_i)\in I$. Since I is invariant over basic sets and $f_{h(n)}$ is a bijection, the set $f^{-1}_{h(n)}[V_{h(n)}\cap \bigcup_{i<n}\mathrm{cl}_{2^\omega}(B_i)]$ belongs to $\overline{I}$. Hence, there is $A\in\mathcal{A}$ such that

$$A\setminus\left(f^{-1}_{h(n)}\left[V_{h(n)}\cap\bigcup_{i<n}\mathrm{cl}_{2^\omega}(B_i)\right]\right)\neq\emptyset$$

for all $n\in\omega$. To finish the proof, we will show that $B = \bigcup_{n\in\omega}\left(B_n\cap\bigcup_{i\le n}\hat{E}^A_n\right)$ is dense in 2^ω (hence, $\mathrm{cl}_{2^\omega}(B) = 2^\omega \notin I$ and $B\notin\mathcal{J}_I$).

Fix any $s\in 2^{<\omega}$. We need to find an element of B belonging to V_s. Since

$$\mathrm{cl}_{2^\omega}(A\cap D)\setminus f_s^{-1}\left[V_s\cap\bigcup_{i<h^{-1}(s)}\mathrm{cl}_{2^\omega}(B_i)\right] = A\setminus\left(f_s^{-1}\left[V_s\cap\bigcup_{i<h^{-1}(s)}\mathrm{cl}_{2^\omega}(B_i)\right]\right)\neq\emptyset,$$

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and $f_s^{-1}\left[V_s \cap \bigcup_{i<h^{-1}(s)} \mathrm{cl}_{2^\omega}(B_i)\right]$ is closed (as f_s is a homeomorphism), we can find $x_s \in (A \cap D) \setminus \left(f_s^{-1}[V_s \cap \bigcup_{i<h^{-1}(s)} B_i]\right)$. Then, using Remark 2.1, we get

$$f_s(x_s) \in V_s \cap \left(E^A_{h^{-1}(s)} \setminus \bigcup_{i<h^{-1}(s)} B_i\right) \subseteq V_s \cap \bigcup_{j\leq h^{-1}(s)} \left(\hat{E}^A_j \setminus \bigcup_{i<h^{-1}(s)} B_i\right)$$

$$\subseteq V_s \cap \bigcup_{j\leq h^{-1}(s)} \left(\hat{E}^A_j \setminus \bigcup_{i<j} B_i\right) \subseteq V_s \cap B.$$

This finishes the proof. ⊣

Next result gives some bounds for three well-known ideals having topological representation: $\mathrm{NWD} = \mathcal{J}_{\mathcal{M}}$, $\mathrm{NULL} = \mathcal{J}_{\sigma\overline{\mathcal{N}}}$, and $\mathrm{CTBL} = \mathcal{J}_{[2^\omega]^{\leq\omega}}$.

Corollary 3.7.

(a) $\mathfrak{b}(\mathrm{NWD}) = \mathrm{add}(\mathcal{M})$.
(b) $\mathfrak{p} \leq \mathfrak{b}(\mathrm{NULL}) \leq \mathrm{add}(\mathcal{M})$.
(c) $\mathfrak{b}(\mathrm{CTBL}) = \aleph_1$.

Proof. Follows from Theorem 3.6, Proposition 3.3 and the fact that $\aleph_1 \leq \mathrm{add}(\mathcal{M}) \leq \mathfrak{b}$ (by [2, Theorem 3.11]). ⊣

We end this section with calculation of $\mathfrak{b}(\mathcal{I})$ for one more well-known ideal defined on the rationals.

Recall that CONV is the ideal on $\mathbb{Q} \cap [0,1]$ generated by sequences in $\mathbb{Q} \cap [0,1]$ that are convergent in $[0,1]$. This is a $\mathbf{\Sigma^0_4}$ ideal contained in NWD, NULL, and CTBL (see [20, Section 3.4]).

Proposition 3.8. $\mathfrak{b}(\mathrm{CONV}) = \aleph_1$.

Proof. By [18, Proposition 4.1], $\mathfrak{b}(\mathcal{I}) \geq \aleph_1$ for every ideal $\mathcal{I}$, so we only need to show that $\mathfrak{b}(\mathcal{I}) \leq \aleph_1$.

Fix any family $\{A^\alpha_n : n \in \omega, \alpha < \aleph_1\} \subseteq \mathrm{CONV}$ such that:

- each A^α_n is a sequence in $\mathbb{Q} \cap [0,1]$ convergent in $[0,1]$;
- $\lim A^\alpha_n \neq \lim A^\beta_m$, for all $(\alpha, n), (\beta, m) \in \aleph_1 \times \omega$ such that $(\alpha, n) \neq (\beta, m)$;
- $A^\alpha_n \cap A^\alpha_m = \emptyset$ for all $\alpha < \aleph_1$ and $n, m \in \omega$ such that $n \neq m$.

We will show that this family witnesses $\mathfrak{b}(\mathrm{CONV}) \leq \aleph_1$.

Let $\langle B_n : n \in \omega\rangle \in \mathcal{P}_{\mathrm{CONV}}$. Since $\bigcup_{n\in\omega} \mathrm{cl}_{[0,1]}(B_n)$ is countable, there is $\alpha < \aleph_1$ such that $\lim A^\alpha_m \notin \bigcup_{n\in\omega} \mathrm{cl}_{[0,1]}(B_n)$, for all $m \in \omega$. In particular, $A^\alpha_m \cap B_n$ is finite for all $n, m \in \omega$.

Suppose to the contrary that $B = \bigcup_{n\in\omega} \left(B_n \cap \bigcup_{i\leq n} A^\alpha_i\right) \in \mathcal{I}$, i.e., there are sequences $D_0, \ldots, D_m \subseteq \mathbb{Q} \cap [0,1]$ convergent in $[0,1]$ and such that $B \subseteq^\star \bigcup_{i\leq m} D_i$. Find $j \in \omega$ such that $\lim A^\alpha_j \neq \lim D_i$ for all $i \leq m$. Observe that:

$$A^\alpha_j \setminus \bigcup_{n<j} B_n = A^\alpha_j \cap \bigcup_{n\geq j} B_n \subseteq \bigcup_{n\geq j} \left(B_n \cap \bigcup_{i\leq n} A^\alpha_i\right) \subseteq B.$$

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Now, since $A^\alpha_j \cap \bigcup_{n<j} B_n$ is finite, we get $A^\alpha_j \subseteq^\star B \subseteq^\star \bigcup_{i\le m} D_i$. Thus, there is $i \le m$ with $A^\alpha_j \cap D_i$ infinite. This contradicts $\lim A^\alpha_j \neq \lim D_i$ and finishes the proof. ⊣

§4. An ideal with $\mathfrak{b}(\mathcal{I}) > \mathfrak{d}$. In this section we will show that, consistently, there is an ideal $\mathcal{I}$ with $\mathfrak{b}(\mathcal{I}) > \mathfrak{d}$. It will follow from the following more general result.

LEMMA 4.1. *If $\kappa < \mathrm{cf}(\mathfrak{c}) \le \mathfrak{u} = \mathfrak{c} = \mathfrak{c}^\kappa$, then there exists an ideal $\mathcal{I}$ such that $\mathfrak{b}(\mathcal{I}) > \kappa$.*

PROOF. Without loss of generality we may assume that κ is infinite (as for finite κ this is trivial). Recall that for $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we write

$$\mathcal{I}(\mathcal{A}) = \left\{ B \subseteq \omega : B \subseteq^\star \bigcup \mathcal{A}' \text{ for some } \mathcal{A}' \in [\mathcal{A}]^{<\omega} \right\}.$$

In particular, $\mathcal{I}(\mathcal{A})$ is an ideal if and only if $\omega \notin \mathcal{I}(\mathcal{A})$.

Since $|\mathcal{P}(\omega)^{\kappa\times\omega}| = \mathfrak{c}^\kappa = \mathfrak{c}$, we can fix an enumeration $\{h_\alpha : \alpha < \mathfrak{c}\}$ of $\mathcal{P}(\omega)^{\kappa\times\omega}$ such that for each $h \in \mathcal{P}(\omega)^{\kappa\times\omega}$ and $\alpha < \mathfrak{c}$ there is $\alpha < \beta < \mathfrak{c}$ with $h_\beta = h$.

We will recursively define a sequence $\{\mathcal{I}_\alpha : \alpha < \mathfrak{c}\}$ of subsets of $\mathcal{P}(\omega)$ such that for each $\alpha < \mathfrak{c}$:

(i) $\mathcal{I}_\alpha$ is an ideal;
(ii) $\mathcal{I}_\alpha$ is generated by at most $|\alpha \cdot \kappa|$ sets;
(iii) if $\beta < \alpha$, then $\mathcal{I}_\beta \subseteq \mathcal{I}_\alpha$;
(iv) if $h_\alpha[\kappa \times \omega] = \{h_\alpha(\gamma, n) : \gamma < \kappa, n \in \omega\} \subseteq \bigcup_{\delta<\alpha} \mathcal{I}_\delta$, then there is $\langle B_n : n \in \omega\rangle \in \mathcal{P}_{\mathcal{I}_\alpha}$ such that $\bigcup_{n\in\omega} \left(B_n \cap \bigcup_{i\le n} h_\alpha(\gamma, i)\right) \in \mathcal{I}_\alpha$, for all $\gamma < \kappa$.

Assume that $\alpha < \mathfrak{c}$ and $\mathcal{I}_\beta$, for all $\beta < \alpha$, are already defined. Observe that $\hat{\mathcal{I}}_\alpha = \bigcup_{\delta<\alpha} \mathcal{I}_\delta$ is an ideal (as a union of an increasing sequence of ideals) generated by at most $|\alpha \cdot (\alpha \cdot \kappa)| = |\alpha \cdot \kappa|$ sets (since κ is infinite). Hence, if $h_\alpha(\gamma, n) \notin \hat{\mathcal{I}}_\alpha$ for some $(\gamma, n) \in \kappa \times \omega$, then just put $\mathcal{I}_\alpha = \hat{\mathcal{I}}_\alpha$ and observe that $\mathcal{I}_\alpha$ is as needed.

Assume now that $h_\alpha[\kappa \times \omega] \subseteq \hat{\mathcal{I}}_\alpha$. Recursively define a sequence $\langle C_n : n \in \omega\rangle$ of subsets of ω such that $C_n \notin \mathcal{I}(\hat{\mathcal{I}}_\alpha \cup \{C_j : j < n\})$ and $\mathcal{I}(\hat{\mathcal{I}}_\alpha \cup \{C_j : j \le n\})$ is an ideal. This is possible as $|\alpha \cdot (\alpha \cdot \kappa) + n| < \mathfrak{c} = \mathfrak{u}$ guarantees that $\mathcal{I}(\hat{\mathcal{I}}_\alpha \cup \{C_j : j < n\})$ is not maximal. Next, for each $n \in \omega$ put $B_n = (\{n\} \cup C_n) \setminus \bigcup_{j<n} B_j$ and note that $\langle B_n : n \in \omega\rangle$ is a partition of ω and $B_n \notin \mathcal{I}(\hat{\mathcal{I}}_\alpha \cup \{B_j : j < n\}) = \mathcal{I}(\hat{\mathcal{I}}_\alpha \cup \{C_j : j < n\})$, for each n. Moreover, $\mathcal{I}(\hat{\mathcal{I}}_\alpha \cup \{B_k : k \in \omega\})$ is an ideal, as otherwise we would have

$$\omega \in \mathcal{I}(\hat{\mathcal{I}}_\alpha \cup \{B_k : k \in \omega\}) = \bigcup_{n\in\omega} \mathcal{I}(\hat{\mathcal{I}}_\alpha \cup \{B_k : k < n\}) = \bigcup_{n\in\omega} \mathcal{I}(\hat{\mathcal{I}}_\alpha \cup \{C_k : k < n\}),$$

which contradicts the fact that $\omega \notin \mathcal{I}(\hat{\mathcal{I}}_\alpha \cup \{C_k : k \le n\})$ for each n.

Define

$$\mathcal{I}_\alpha = \mathcal{I}\left(\hat{\mathcal{I}}_\alpha \cup \{B_n : n \in \omega\} \cup \left\{ \bigcup_{n\in\omega} \left(B_n \cap \bigcup_{i\le n} h_\alpha(\gamma, i) \right) : \gamma < \kappa \right\}\right).$$

Obviously, items (iii) and (iv) are satisfied. What is more, item (ii) is also satisfied as $|\alpha \cdot \kappa + \omega + \kappa| = |\alpha \cdot \kappa|$. Thus, we only need to check that $\omega \notin \mathcal{I}_\alpha$.

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Assume to the contrary that $\omega \in \mathcal{I}_\alpha$. Then there are $k, m \in \omega$, $\gamma_0, \dots, \gamma_m < \kappa$ and $A \in \hat{\mathcal{I}}_\alpha$ such that

$$\omega = A \cup \bigcup_{j<k} B_j \cup \bigcup_{j\le m} \left(\bigcup_{n\in\omega} \left(B_n \cap \bigcup_{i\le n} h_\alpha(\gamma_j, i) \right) \right).$$

However, since $\langle B_n : n \in \omega \rangle$ is a partition of ω, we have

$$\begin{aligned} B_k = B_k \cap \omega &= B_k \cap \left(A \cup \bigcup_{j<k} B_j \cup \bigcup_{j\le m} \left(\bigcup_{n\in\omega} \left(B_n \cap \bigcup_{i\le n} h_\alpha(\gamma_j, i) \right) \right) \right) \\ &\subseteq A \cup \bigcup_{j\le m} \bigcup_{i\le k} h_\alpha(\gamma_j, i) \in \hat{\mathcal{I}}_\alpha, \end{aligned}$$

which contradicts $B_k \notin \mathcal{I}(\hat{\mathcal{I}}_\alpha \cup \{B_j : j < k\})$. Thus, $\omega \notin \mathcal{I}_\alpha$.

Once the recursive construction of $\{\mathcal{I}_\alpha : \alpha < \mathfrak{c}\}$ is completed, define $\mathcal{I} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{I}_\alpha$. Clearly, $\mathcal{I}$ is an ideal (by items (i) and (iii)). To finish the proof we need to show that $\mathfrak{b}(\mathcal{I}) > \kappa$. Fix any $\{\langle E_n^\gamma : n \in \omega \rangle : \gamma < \kappa\} \subseteq \widehat{\mathcal{P}}_\mathcal{I}$. Then for each $(\gamma, n) \in \kappa \times \omega$ there is $\beta(\gamma, n) < \mathfrak{c}$ such that $E_n^\gamma \in \mathcal{I}_{\beta(\gamma,n)}$. Denote $\beta = \sup\{\beta(\gamma, n) : (\gamma, n) \in \kappa \times \omega\}$. Since $|\kappa \times \omega| = \kappa < \mathrm{cf}(\mathfrak{c})$, we get that $\beta < \mathfrak{c}$. Thus, we can find $\beta < \alpha < \mathfrak{c}$ with $h_\alpha(\gamma, n) = E_n^\gamma$, for all $(\gamma, n) \in \kappa \times \omega$. By item (iii), $h_\alpha[\kappa \times \omega] \subseteq \bigcup_{\delta<\alpha} \mathcal{I}_\delta$. Using item (iv) for α we get $\langle B_n : n \in \omega \rangle \in \mathcal{P}_{\mathcal{I}_\alpha} \subseteq \mathcal{P}_\mathcal{I}$ such that $\bigcup_{n\in\omega} \left(B_n \cap \bigcup_{i\le n} E_i^\gamma \right) \in \mathcal{I}_\alpha \subseteq \mathcal{I}$, for all $\gamma < \kappa$. Therefore, $\mathfrak{b}(\mathcal{I}) > \kappa$ and the proof is completed. ⊣

THEOREM 4.2. *If* $\mathfrak{d} < \mathrm{cf}(\mathfrak{c}) \le \mathfrak{u} = \mathfrak{c} = \mathfrak{c}^\mathfrak{d}$, *then there exists an ideal* $\mathcal{I}$ *with* $\mathfrak{b}(\mathcal{I}) > \mathfrak{d}$. *In particular, it is consistent that there exists an ideal* $\mathcal{I}$ *such that* $\mathfrak{b}(\mathcal{I}) > \mathfrak{d}$.

PROOF. The first part follows directly from Lemma 4.1.

We will show that the assumptions $\mathfrak{d} < \mathrm{cf}(\mathfrak{c}) \le \mathfrak{u} = \mathfrak{c} = \mathfrak{c}^\mathfrak{d}$ are consistent (specifically, that they hold in the generic extension obtained by adding $\aleph_2$ random reals to the model of GCH).

Suppose that GCH is satisfied in the ground model V. Let

$$M(\aleph_2) = \{C \subseteq 2^{\aleph_2} : C \text{ is a closed set of positive measure}\}$$

(ordered by the inclusion modulo null sets) be a forcing which adds $\aleph_2$ random reals (here we consider the product measure on $2^{\aleph_2}$). Let G denote a $M(\aleph_2)$-generic filter.

It is known that $\mathfrak{d} = \aleph_1$ and $\mathfrak{u} = \mathfrak{c} = \aleph_2$ in $V[G]$ (see, e.g., [4, p. 474]), so $\mathfrak{d} < \mathrm{cf}(\mathfrak{c}) = \mathfrak{u} = \mathfrak{c}$. It remains to show that $\mathfrak{c} = \mathfrak{c}^\mathfrak{d}$ in $V[G]$.

Denote $\kappa = |M(\aleph_2)|$, $\lambda = \aleph_1$ and $\delta = \kappa^\lambda$. Since GCH holds in V, we have $\aleph_2 = 2^{\aleph_1}$, and consequently

$$\kappa = |M(\aleph_2)| = |[\aleph_2]^{<\aleph_0}|^{\aleph_0} = \aleph_2^{\aleph_0} = (2^{\aleph_1})^{\aleph_0} = 2^{\aleph_1} = \aleph_2$$

and

$$\delta = \aleph_2^{\aleph_1} = (2^{\aleph_1})^{\aleph_1} = \aleph_2.$$

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Hence, κ, λ, δ are infinite cardinals. As $M(\aleph_2)$ is ccc, using [25, Lemma IV.3.11 at p. 267] we obtain in $V[G]$ the inequality $2^\lambda \leq \delta$ which yields:

$$\mathfrak{c}^{\mathfrak{d}} = (2^{\aleph_0})^{\mathfrak{d}} = 2^{\mathfrak{d}} = 2^{\aleph_1} = 2^\lambda \leq \delta = \aleph_2 = \mathfrak{c}.$$

Since $\mathfrak{c} \leq \mathfrak{c}^{\mathfrak{d}}$, we obtain $\mathfrak{c} = \mathfrak{c}^{\mathfrak{d}}$, and the proof is finished. ⊣

§5. Dominating number and analytic ideals. In this section we show that $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{d}$ for every analytic ideal $\mathcal{I}$. We will apply ideas developed by Katětov in [22–24] and by Debs and Saint Raymond in [15]. We start with recalling several technical notions.

If $\mathcal{I}$ and $\mathcal{J}$ are ideals on X and Y, respectively, then we say that $\mathcal{I}$*is below* $\mathcal{J}$ *in the Katětov order* (in short: $\mathcal{I} \leq_K \mathcal{J}$), if there is a function $f : Y \to X$ such that $f^{-1}[A] \in \mathcal{J}$ for every $A \in \mathcal{I}$. If we can find a bijection with the above property, then we say that $\mathcal{J}$ *contains an isomorphic copy of* $\mathcal{I}$ and write $\mathcal{I} \sqsubseteq \mathcal{J}$.

If $\{X_t : t \in T\}$ is a family of sets, then $\sum_{t\in T} X_t = \{(t, x) : t \in T, x \in X_t\}$ is its disjoint sum. The vertical section of a set $A \subseteq \sum_{t\in T} X_t$ at a point $t \in T$ is defined by $A_{(t)} = \{x \in X_t : (t, x) \in A\}$.

Let $\mathcal{I}$ be an ideal on a countable set T and $\{\mathcal{I}_t : t \in T\}$ be a family of ideals. Following [22], we define the ideal

$$\mathcal{I} - \sum_{t\in T} \mathcal{I}_t = \{A \subseteq \sum_{t\in T} \left(\bigcup \mathcal{I}_t\right) : \{t \in T : A_{(t)} \notin \mathcal{I}_t\} \in \mathcal{I}\}.$$

In particular, if $\mathcal{J}$ is some ideal on S and $\mathcal{I}_t = \mathcal{J}$ for all $t \in T$, then we denote

$$\mathcal{I} \otimes \mathcal{J} = \mathcal{I} - \sum_{t\in T} \mathcal{I}_t = \{A \subseteq T \times S : \{t \in T : A_{(t)} \notin \mathcal{J}\} \in \mathcal{I}\}.$$

Following [23, 24], let $\{\mathrm{Fin}^\alpha : 1 \leq \alpha < \omega_1\}$ be the family of ideals given by

$$\mathrm{Fin}^\alpha = \begin{cases} \mathrm{Fin}, & \text{if } \alpha = 1, \\ \mathrm{Fin} \otimes \mathrm{Fin}^\beta, & \text{if } \alpha = \beta + 1, \\ \mathcal{I}_\alpha - \sum_{\beta<\alpha} \mathrm{Fin}^\beta, & \text{if } \alpha \text{ is a limit ordinal,} \end{cases}$$

where $\mathcal{I}_\alpha$ is the ideal on the ordinal α generated by the family $\{\beta : \beta < \alpha\}$ (in particular, $\mathcal{I}_\omega = \mathrm{Fin}$).

Finally, for ideals $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ we define

$$\mathfrak{b}(\mathcal{I}, \mathcal{J}, \mathcal{K}) = \min \left\{ |\mathcal{E}| : \mathcal{E} \subseteq \widehat{\mathcal{P}}_{\mathcal{K}} \wedge \forall_{\langle A_n\rangle\in\mathcal{P}_{\mathcal{J}}} \exists_{\langle E_n\rangle\in\mathcal{E}} \bigcup_{n\in\omega} \left(A_n \cap \bigcup_{i\leq n} E_i \right) \notin \mathcal{I} \right\}$$

(the above is an equivalent form of the coefficient defined in [35]—the equivalence is shown in [17, Proposition 3.9]). Note that $\mathfrak{b}(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \mathfrak{b}(\mathcal{I})$. Moreover, it is easy to see that if $\mathcal{K} \subseteq \mathcal{K}'$, then $\mathfrak{b}(\mathcal{I}, \mathcal{J}, \mathcal{K}) \geq \mathfrak{b}(\mathcal{I}, \mathcal{J}, \mathcal{K}')$ (see [17, Proposition 3.8]).

We are ready to prove some lemmas, which will imply that $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{d}$ for every analytic ideal $\mathcal{I}$.

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LEMMA 5.1. *If* $1 \leq \alpha < \omega_1$ *is an ordinal,* $\mathrm{Fin}^{\alpha+1} \not\sqsubseteq \mathcal{I}$, *but* $\mathrm{Fin}^{\alpha} \sqsubseteq \mathcal{I}$, *then* $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{d}$.

PROOF. If $\alpha = 1$ then this is true by [17, Theorems 4.2(4) and 4.2(9)].

Let now $1 < \alpha < \omega_1$ be any ordinal. We need to standardize the notation so that we will be able to proceed with the proof for both successor and limit ordinals α.

If $\alpha = \beta + 1$ is a successor ordinal, put $\gamma = \omega$ and write $\beta_\delta = \beta$ for all $\delta < \gamma$. Note that $\mathrm{Fin}^{\alpha} = \mathrm{Fin} \otimes \mathrm{Fin}^{\beta} = \mathcal{I}_\gamma - \sum_{\delta<\gamma} \mathrm{Fin}^{\beta_\delta}$.

On the other hand, if α is a limit ordinal, put $\gamma = \alpha$ and write $\beta_\delta = \delta$ for all $\delta < \gamma$. Note that $\mathrm{Fin}^{\alpha} = \mathcal{I}_\alpha - \sum_{\delta<\gamma} \mathrm{Fin}^{\delta} = \mathcal{I}_\gamma - \sum_{\delta<\gamma} \mathrm{Fin}^{\beta_\delta}$.

Now we can proceed with the proof not caring if α is a successor or a limit ordinal.

For each $\delta < \gamma$ let $\mathrm{Fin}^{\beta_\delta}(\omega)$ be any isomorphic copy of $\mathrm{Fin}^{\beta_\delta}$ on ω. Then Fin^{α} and $\mathcal{J} = \mathcal{I}_\gamma - \sum_{\delta<\gamma} \mathrm{Fin}^{\beta_\delta}(\omega)$ are isomorphic, so $\mathcal{J} \sqsubseteq \mathcal{I}$. Thus, without loss of generality we may assume that $\mathcal{I}$ is an ideal on $\gamma \times \omega$ such that $\mathcal{J} \subseteq \mathcal{I}$ (this can be done by considering the ideal $\{f[A] : A \in \mathcal{I}\} \supseteq \mathcal{J}$, instead of $\mathcal{I}$, where f is the bijection witnessing $\mathcal{J} \sqsubseteq \mathcal{I}$).

We will show that $\mathfrak{b}(\mathcal{I}, \mathcal{I}, \mathcal{J}) \leq \mathfrak{d}$. This will finish the proof as $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{b}(\mathcal{I}, \mathcal{I}, \mathcal{J})$ (by $\mathcal{J} \subseteq \mathcal{I}$).

Fix a strictly increasing sequence $\langle \gamma_i : i \in \omega \rangle \in \gamma^\omega$ converging to γ (in the order topology) such that $\gamma_0 = 0$. Let $\{f_\alpha : \alpha < \mathfrak{d}\} \subseteq \omega^\omega$ be a $\leq_{\mathrm{Fin}}$-dominating family. Without loss of generality we may assume that each f_α is strictly increasing and satisfies $f_\alpha(0) = 0$. Define $A^\alpha_n = (\gamma_{f_\alpha(n+1)} \setminus \gamma_{f_\alpha(n)}) \times \omega$ for all $\alpha < \mathfrak{d}$ and $n \in \omega$. Then $\{\langle A^\alpha_n : n \in \omega \rangle : \alpha < \mathfrak{d}\} \subseteq \widehat{\mathcal{P}}_{\mathcal{J}}$. We claim that this family witnesses $\mathfrak{b}(\mathcal{I}, \mathcal{I}, \mathcal{J}) \leq \mathfrak{d}$.

Fix any $\langle B_n : n \in \omega \rangle \in \mathcal{P}_{\mathcal{I}}$. There are two possibilities.

Assume first that there is $X \notin \mathcal{I}$ such that $(X \cap B_n)_{(\delta)} \in \mathrm{Fin}^{\beta_\delta}(\omega)$, for all $n \in \omega$ and $\delta < \gamma$. For $\delta < \gamma$ let $m_\delta \in \omega$ be such that $\gamma_{f_0(m_\delta)} \leq \delta < \gamma_{f_0(m_\delta+1)}$ (i.e., $\{\delta\} \times \omega \subseteq A^0_{m_\delta}$). Observe that

$$\begin{aligned}\left(\bigcup_{i\in\omega}\left(A^0_i \setminus \bigcup_{n<i} B_n\right)\right)_{(\delta)} &\supseteq \left(A^0_{m_\delta} \setminus \bigcup_{n<m_\delta} B_n\right)_{(\delta)}\\ &= \omega \setminus \bigcup_{n<m_\delta} (B_n)_{(\delta)}\\ &\supseteq X_{(\delta)} \setminus \bigcup_{n<m_\delta} (X \cap B_n)_{(\delta)}.\end{aligned}$$

Define $Y_\delta = \bigcup_{n<m_\delta} (X \cap B_n)_{(\delta)} \in \mathrm{Fin}^{\beta_\delta}(\omega)$ and $Y = \bigcup_{\delta<\gamma} \{\delta\} \times Y_\delta \in \mathcal{J} \subseteq \mathcal{I}$. Then we have

$$\mathcal{I} \not\ni X \setminus Y \subseteq \bigcup_{i\in\omega}\left(A^0_i \setminus \bigcup_{n<i} B_n\right) \subseteq \bigcup_{n\in\omega}\left(B_n \cap \bigcup_{i\leq n} A^0_i\right)$$

(the last inclusion follows from Remark 2.1).

Assume now that if $X \subseteq \gamma \times \omega$ satisfies $(X \cap B_n)_{(\delta)} \in \mathrm{Fin}^{\beta_\delta}(\omega)$ for all $n \in \omega$ and $\delta < \gamma$ then $X \in \mathcal{I}$. Recall that the ideal $\mathrm{Fin} \otimes \mathcal{J}$ (which is isomorphic to $\mathrm{Fin}^{\alpha+1}$) has three kinds of generators:

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- sets of the form $\{n\} \times \gamma \times \omega$, for $n \in \omega$ (*generators of the first type*);
- sets of the form $\bigcup_{n\in\omega}\{n\} \times \gamma_{g(n)} \times \omega$, for $g \in \omega^\omega$ (*generators of the second type*);
- sets $G \subseteq \omega \times \gamma \times \omega$ such that $G_{(n,\delta)} \in \mathrm{Fin}^{\beta_\delta}(\omega)$, for all $(n,\delta) \in \omega \times \gamma$ (*generators of the third type*).

Consider the function $h : \gamma \times \omega \to \omega \times \gamma \times \omega$ defined by the formula $h(\delta, j) = (n(\delta, j), \delta, j)$, where $n(\delta, j) \in \omega$ is given by $(\delta, j) \in B_{n(\delta,j)}$. Observe that:

- if $G = \{n\} \times \gamma \times \omega$ is a generator of he first type, then $h^{-1}[G] \subseteq B_n \in \mathcal{I}$, for every $n \in \omega$;
- if $G \subseteq \omega \times \gamma \times \omega$ is a generator of the third type, then $(h^{-1}[G] \cap B_n)_{(\delta)} \subseteq G_{(n,\delta)} \in \mathrm{Fin}^{\beta_\delta}(\omega)$, for all $(n,\delta) \in \omega \times \gamma$, so $h^{-1}[G] \in \mathcal{I}$ by our assumption.

On the other hand, $\mathrm{Fin}^{\alpha+1} \not\leq_K \mathcal{I}$ (by $\mathrm{Fin}^{\alpha+1} \not\sqsubseteq \mathcal{I}$ and [1, Example 4.1]), so there has to be a generator of the second type, $G = \bigcup_{n\in\omega}\{n\} \times \gamma_{g(n)} \times \omega$ for some $g \in \omega^\omega$, such that

$$\mathcal{I} \not\ni h^{-1}[G] \subseteq \bigcup_{n\in\omega} \left(B_n \cap (\gamma_{g(n)} \times \omega)\right).$$

Find $\alpha < \mathfrak{d}$ with $g \leq^\star f_\alpha$ and let $m \in \omega$ be such that $g(n) \leq f_\alpha(n)$ for all $n > m$. Note that

$$\bigcup_{n\in\omega}\left(B_n \cap \bigcup_{i\leq n} A^\alpha_i\right) \supseteq \bigcup_{n\in\omega}\left(B_n \cap (\gamma_{f_\alpha(n)} \times \omega)\right)$$
$$\supseteq \bigcup_{n>m}\left(B_n \cap (\gamma_{g(n)} \times \omega)\right) \supseteq h^{-1}[G] \setminus \bigcup_{n\leq m} B_n \notin \mathcal{I}$$

(as $\bigcup_{n\leq m} B_n \in \mathcal{I}$). This finishes the proof. ⊣

LEMMA 5.2. *For each $1 < \alpha < \omega_1$ there is a family $\{A_f : f \in \omega^\omega\} \subseteq \mathrm{Fin}^\alpha$ satisfying*:

- *for every $A \in \mathrm{Fin}^\alpha$ there is $f \in \omega^\omega$ with $A \subseteq A_f$;*
- *$A_f \subseteq A_g$ whenever $f \leq g$ (i.e., $f(n) \leq g(n)$ for all $n \in \omega$).*

PROOF. We will show it inductively.

This is clear for $\alpha = 2$ as witnessed by the family:

$$\left\{(f(0) \times \omega) \cup \bigcup_{i\in\omega}\{i\} \times f(i) : f \in \omega^\omega\right\} \subseteq \mathrm{Fin}^2.$$

Fix now any partition $\langle C_n : n \in \omega\rangle$ of ω into infinite sets and for each n let $h_n : \omega \to C_n$ be the increasing enumeration of C_n.

Assume that for some $1 < \alpha < \omega_1$ we have the needed family $\{A_f : f \in \omega^\omega\} \subseteq \mathrm{Fin}^\alpha$. We need to find the required family for $\alpha + 1$. Given $f \in \omega^\omega$, define

$$B_f = (f(0) \times \bigcup \mathrm{Fin}^\alpha) \cup \bigcup_{n\in\omega}\{n\} \times A_{f\circ h_n}.$$

Again it is clear that $\{B_f : f \in \omega^\omega\} \subseteq \mathrm{Fin}^{\alpha+1}$ is as needed.

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Finally, assume that $1 < \alpha < \omega_1$ is a limit ordinal and for each $\beta < \alpha$ we have the required family $\{A^\beta_f : f \in \omega^\omega\} \subseteq \mathrm{Fin}^\beta$. Find a bijection $g : \omega \to \alpha$ and a strictly increasing sequence $\langle \beta_i : i \in \omega\rangle$ converging to α (in the order topology). For each $f \in \omega^\omega$, define

$$B_f = \left(\sum_{\beta<\beta_{f(0)}} \bigcup \mathrm{Fin}^\beta\right) \cup \left(\bigcup_{n\in\omega} \{g(n)\} \times A^{g(n)}_{f\circ h_n}\right).$$

Once more it is clear that $\{B_f : f \in \omega^\omega\} \subseteq \mathrm{Fin}^\alpha$ is as needed. ⊣

LEMMA 5.3. *If $\alpha < \omega_1$ is a limit ordinal, $\mathrm{Fin}^\alpha \not\sqsubseteq \mathcal{I}$, but $\mathrm{Fin}^\beta \sqsubseteq \mathcal{I}$ for all $\beta < \alpha$, then $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{d}$.*

PROOF. Find an increasing sequence $\langle \beta_i : i \in \omega\rangle$ converging to α (in the order topology). Since $\mathrm{Fin}^\beta \sqsubseteq \mathcal{I}$ for all $\beta < \alpha$, we can assume that $\mathcal{I}$ is an ideal on ω and $\mathrm{Fin}^{\beta_i}(\omega) \subseteq \mathcal{I}$ for all i, where $\mathrm{Fin}^{\beta_i}(\omega)$ is some copy of Fin^{β_i} on ω. For each $i \in \omega$ let $\{B^i_f : f \in \omega^\omega\} \subseteq \mathrm{Fin}^{\beta_i}(\omega)$ be the family from Lemma 5.2.

Fix any partition $\langle C_n : n \in \omega\rangle$ of ω into infinite sets and for each n let $h_n : \omega \to C_n$ be the increasing enumeration of C_n. Let $\{f_\alpha : \alpha < \mathfrak{d}\} \subseteq \omega^\omega$ be a $\leq_{\mathrm{Fin}}$-dominating family of strictly increasing functions and recursively define $A^\alpha_n = B^n_{f_\alpha\circ h_n} \setminus \bigcup_{i<n} A^\alpha_i$, for all $\alpha < \mathfrak{d}$ and $n \in \omega$. Then $\{\langle A^\alpha_n : n \in \omega\rangle : \alpha < \mathfrak{d}\} \subseteq \widehat{\mathcal{P}}_\mathcal{I}$ as $\mathrm{Fin}^{\beta_i}(\omega) \subseteq \mathcal{I}$ for all i. We will show that $\{A^\alpha_n : n \in \omega, \alpha < \mathfrak{d}\}$ witnesses $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{d}$.

Fix any $\langle B_n : n \in \omega\rangle \in \mathcal{P}_\mathcal{I}$ and consider the function $h : \omega \to \alpha \times \omega$ defined by $h(i) = (\beta_n, i)$, where $n \in \omega$ is given by $i \in B_n$. Since $\mathcal{I}_\alpha - \sum_{\beta<\alpha} \mathrm{Fin}^\beta(\omega)$ is isomorphic to Fin^α and $\mathrm{Fin}^\alpha \not\leq_K \mathcal{I}$ (by $\mathrm{Fin}^\alpha \not\sqsubseteq \mathcal{I}$ and [1, Example 4.1]), there is $Y \in \mathcal{I}_\alpha - \sum_{\beta<\alpha} \mathrm{Fin}^\beta(\omega)$ such that $h^{-1}[Y] \notin \mathcal{I}$. Actually, for each $\beta < \alpha$ either $h^{-1}[\{\beta\} \times \omega] = \emptyset \in \mathcal{I}$ (if $\beta \neq \beta_i$ for all $i \in \omega$) or $h^{-1}[\{\beta\} \times \omega] \subseteq B_n \in \mathcal{I}$ for some $n \in \omega$ (if $\beta = \beta_n$). Hence, the set

$$X = h^{-1}\left[\bigcup\left\{\{\delta\} \times Y_{(\delta)} : Y_{(\delta)} \in \mathrm{Fin}^\delta(\omega)\right\}\right],$$

does not belong to $\mathcal{I}$, while $h[X] \subseteq h[h^{-1}[Y]] \subseteq Y \in \mathcal{I}_\alpha - \sum_{\beta<\alpha} \mathrm{Fin}^\beta(\omega)$. Observe that $X \cap B_n \subseteq (h[X])_{(\beta_n)} \in \mathrm{Fin}^{\beta_n}$ for all $n \in \omega$.

Then for each n there is $g_n \in \omega^\omega$ such that $X \cap B_n \subseteq B^n_{g_n}$. Define $g \in \omega^\omega$ by $g \restriction C_n = g_n \circ h_n^{-1}$. Find $\alpha < \mathfrak{d}$ with $g \leq^\star f_\alpha$ and let $m \in \omega$ be such that $g(i) \leq f_\alpha(i)$ whenever $i \geq m$. Let $k \in \omega$ be maximal such that $C_k \cap m \neq \emptyset$. Observe that for $n > k$ and any $j \in \omega$ we have

$$(f_\alpha \circ h_n)(j) = f_\alpha(h_n(j)) \geq g(h_n(j)) = g_n(j).$$

Hence, $X \cap B_n \subseteq B^n_{g_n} \subseteq B^n_{f_\alpha\circ h_n} \subseteq \bigcup_{i\leq n} A^\alpha_i$, whenever $n > k$. Thus,

$$\bigcup_{n\in\omega}\left(B_n \cap \bigcup_{i\leq n} A^\alpha_i\right) \supseteq \bigcup_{n>k}\left(B_n \cap \bigcup_{i\leq n} A^\alpha_i\right) \supseteq \bigcup_{n>k}(X \cap B_n) \notin \mathcal{I}$$

as $\mathcal{I} \not\ni X = \bigcup_{n\in\omega}(X \cap B_n)$ and $\bigcup_{n\leq k} B_n \in \mathcal{I}$. ⊣

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THEOREM 5.4. *If there is* $\alpha < \omega_1$ *such that* $\mathrm{Fin}^\alpha \not\sqsubseteq \mathcal{I}$, *then* $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{d}$.

PROOF. Let $\alpha < \omega_1$ be minimal such that $\mathrm{Fin}^\alpha \not\sqsubseteq \mathcal{I}$. If α is a successor ordinal, use Lemma 5.1. On the other hand, if α is a limit ordinal, we can apply Lemma 5.3. ⊣

Recall that for an ideal $\mathcal{I}$ on X we write $\mathcal{I}^\star = \{A \subseteq X : X \setminus A \in \mathcal{I}\}$ and call it the *dual filter of* $\mathcal{I}$.

COROLLARY 5.5. *If* $\mathcal{I}$ *is Borel-separated from* $\mathcal{I}^\star$ (*i.e., there is a Borel set* $S \subseteq 2^\omega$ *such that* $\mathcal{I} \subseteq S$ *and* $\mathcal{I}^\star \cap S = \emptyset$), *then* $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{d}$. *In particular,* $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{d}$ *for every analytic ideal* $\mathcal{I}$.

PROOF. We claim that there is $\alpha < \omega_1$ such that $\mathrm{Fin}^\alpha \not\sqsubseteq \mathcal{I}$. Indeed, by [15, Theorems 3.2 and 6.5 and Lemma 7.2], if $\mathrm{Fin}^\alpha \sqsubseteq \mathcal{I}$, then $\mathcal{I}$ is not $\mathbf{\Delta}^{\mathbf{0}}_{\mathbf{1}+\alpha}$-separated from $\mathcal{I}^\star$. Thus, the assumption that $\mathrm{Fin}^\alpha \sqsubseteq \mathcal{I}$, for all $\alpha < \omega_1$, would contradict the fact that $\mathcal{I}$ is Borel-separated from $\mathcal{I}^\star$. Now it suffices to apply Theorem 5.4.

For each analytic ideal, its dual filter is also analytic (by [15, Section 1.1]), so they can be Borel-separated (by the Lusin separation theorem). This proves the "In particular" part. ⊣

Actually, we do not know whether Corollary 5.5 can be strengthened by replacing $\mathfrak{d}$ with $\mathfrak{b}$. In particular, we do not know the answer to the following problem.

PROBLEM 1. *Does there consistently exist an analytic ideal* $\mathcal{I}$ *with* $\mathfrak{b}(\mathcal{I}) > \mathfrak{b}$?

REFERENCES

[1] P. BARBARSKI, R. FILIPÓW, N. MROŻEK, and P. SZUCA, *When does the Katětov order imply that one ideal extends the other?* ***Colloquium Mathematicum***, vol. 130 (2013), pp. 91–102.

[2] T. BARTOSZYŃSKI, *Invariants of measure and category*, ***Handbook of Set Theory, vols. 1–3*** (F. Matthew and K. Akihiro, editors), Springer, Dordrecht, 2010, pp. 491–555.

[3] T. BARTOSZYŃSKI and S. SHELAH, *Closed measure zero sets*. ***Annals of Pure and Applied Logic***, vol. 58 (1992), no. 2, pp. 93–110.

[4] A. BLASS, *Combinatorial cardinal characteristics of the continuum*, ***Handbook of Set Theory, vols. 1–3*** (F. Matthew and K. Akihiro, editors), Springer, Dordrecht, 2010, pp. 395–489.

[5] L. BUKOVSKÝ, *On* wQN_* *and* wQN^* *spaces*. ***Topology and Its Applications***, vol. 156 (2008), no. 1, pp. 24–27.

[6] ———, *The structure of the real line*, ***Instytut Matematyczny Polskiej Akademii Nauk***, Monografie Matematyczne (New Series) [Mathematics Institute of the Polish Academy of Sciences, Mathematical Monographs (New Series)], 71, Birkhäuser/Springer Basel AG, Basel, 2011.

[7] L. BUKOVSKÝ, P. DAS, and J. ŠUPINA, *Ideal quasi-normal convergence and related notions*. ***Colloquium Mathematicum***, vol. 146 (2017), no. 2, pp. 265–281.

[8] L. BUKOVSKÝ and J. HALEŠ, *QN-spaces, wQN-spaces and covering properties*. ***Topology and Its Applications***, vol. 154 (2007), no. 4, pp. 848–858.

[9] L. BUKOVSKÝ, I. RECŁAW, and M. REPICKÝ, *Spaces not distinguishing pointwise and quasinormal convergence of real functions*. ***Topology and Its Applications***, vol. 41 (1991), no. 1–2, pp. 25–40.

[10] ———, *Spaces not distinguishing convergences of real-valued functions*. ***Topology and Its Applications***, vol. 112 (2001), no. 1, pp. 13–40.

[11] M. CANJAR, *Countable ultraproducts without CH**. ***Annals of Pure and Applied Logic***, vol. 37 (1988), no. 1, pp. 1–79.

[12] R. M. CANJAR, *Model-theoretic properties of countable ultraproducts without the continuum hypothesis*, Ph.D. thesis, University of Michigan, ProQuest LLC, Ann Arbor, 1982.

[13] ———, *Cofinalities of countable ultraproducts: The existence theorem*. ***Notre Dame Journal of Formal Logic***, vol. 30 (1989), no. 4, pp. 539–542.

[14] P. DAS and D. CHANDRA, *Spaces not distinguishing pointwise and* $\mathcal{I}$*-quasinormal convergence*. ***Commentationes Mathematicae Universitatis Carolinae***, vol. 54 (2013), no. 1, pp. 83–96.

https://doi.org/10.1017/jsl.2024.70 Published online by Cambridge University Press

[15] G. Debs and J. S. Raymond, *Filter descriptive classes of Borel functions*. ***Fundamenta Mathematicae***, vol. 204 (2009), no. 3, pp. 189–213.

[16] I. Farah and S. Solecki, Two $F_{\sigma\delta}$ *ideals*. ***Proceedings of the American Mathematical Society***, vol. 131 (2003), no. 6, pp. 1971–1975.

[17] R. Filipów and A. Kwela, *Yet another ideal version of the bounding number*. ***The Journal of Symbolic Logic***, vol. 87 (2022), no. 3, pp. 1065–1092.

[18] R. Filipów and M. Staniszewski, *Pointwise versus equal (quasi-normal) convergence via ideals*. ***Journal of Mathematical***, vol. 422 (2015), no. 2, pp. 995–1006.

[19] D. H. Fremlin, ***Consequences of Martin's Axiom***, Cambridge Tracts in Mathematics, 84, Cambridge University Press, Cambridge, 1984.

[20] M. Hrušák, *Combinatorics of filters and ideals*, ***Set Theory and Its Applications***, Contemporary Mathematics, 533, American Mathematical Society, Providence, 2011, pp. 29–69.

[21] T. Kankaanpää, *Remarks on gaps in dense* $(\mathbb{Q})/\mathrm{nwd}$. ***Mathematical Logic Quarterly***, vol. 59 (2013), nos. 1–2, pp. 51–61.

[22] M. Katětov, *Products of filters*. ***Commentationes Mathematicae Universitatis Carolinae***, vol. 9 (1968), pp. 173–189.

[23] ———, ***On Descriptive Classes of Functions***, Theory of Sets and Topology (in honour of Felix Hausdorff, 1868–1942), VEB Deutsch. Verlag Wissensch., Berlin, 1972, pp. 265–278.

[24] ———, *On descriptive classification of functions*, ***General Topology and its Relations to Modern Analysis and Algebra, III (Proceedings of Third Prague Topological Symposium, 1971)***, (J. Novák, Z. Frolik and V. Pták, editors), Academia, Prague, 1972, pp. 235–242.

[25] K. Kunen, ***Set Theory***, Studies in Logic (London), 34, College Publications, London, 2011.

[26] A. Kwela, *Ideal weak QN-spaces*. ***Topology and Its Applications***, vol. 240 (2018), pp. 98–115.

[27] A. Kwela and M. Sabok, Topological representations. ***Journal of Mathematical Analysis and Applications***, vol. 422 (2015), no. 2, pp. 1434–1446.

[28] I. Recław, *Metric spaces not distinguishing pointwise and quasinormal convergence of real functions*. ***Bulletin of the Polish Academy of Sciences. Mathematics***, vol. 45 (1997), no. 3, pp. 287–289.

[29] M. Repický, *Spaces not distinguishing convergences*. ***Commentationes Mathematicae Universitatis Carolinae***, vol. 41 (2000), no. 4, pp. 829–842.

[30] ———, *Spaces not distinguishing ideal convergences of real-valued functions*. ***Real Analysis Exchange***, vol. 46 (2021), no. 2, pp. 367–394.

[31] ———, *Spaces not distinguishing ideal convergences of real-valued functions, II*. ***Real Analysis Exchange***, vol. 46 (2021), no. 2, pp. 395–421.

[32] M. Sabok and J. Zapletal, *Forcing properties of ideals of closed sets*. ***Journal of Symbolic Logic***, vol. 76 (2010), pp. 4011–4039.

[33] M. Sakai, *The sequence selection properties of* $C_p(X)$. ***Topology and Its Applications***, vol. 154 (2007), no. 3, pp. 552–560.

[34] V. Šottová and J. Šupina, *Principle* $\mathrm{S}_1(\mathcal{P},\mathcal{R})$: *ideals and functions*. ***Topology and Its Applications***, vol. 258 (2019), pp. 282–304.

[35] M. Staniszewski, *On ideal equal convergence II*. ***Journal of Mathematical Analysis and Applications***, vol. 451 (2017), no. 2, pp. 1179–1197.

[36] J. Šupina, *Ideal QN-spaces*. ***Journal of Mathematical Analysis and Applications***, vol. 435 (2016), no. 1, pp. 477–491.

[37] B. Tsaban and L. Zdomskyy, *Hereditarily Hurewicz spaces and Arhangel'skiĭ sheaf amalgamations*. ***Journal of the European Mathematical Society***, vol. 14 (2012), no. 2, pp. 353–372.

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