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Ends of graphs

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Abstract

It is shown how questions about ends of locally finite graphs can be reduced to questions about trees. Several applications are given; for example, locally finite connected graphs with infinitely many ends and automorphism groups that act transitively on the ends are classified.

Introduction

In their book *Groups Acting on Graphs*, Dicks and Dunwoody developed powerful techniques for forcing a group to act on a tree. One of their results is a method of building a tree from a graph Γ such that $\mathrm{Aut}(\Gamma)$ acts naturally on the tree. We call a tree constructed in this way from a graph Γ a *structure tree* of Γ. The aim of this paper is to show how this technique can be utilized to investigate connected graphs with infinitely many ends. We will give four applications where we utilize the fact that an end of a structure tree of Γ can be identified with an end of Γ. The first of these applications is to determine the kernel of the action of $\mathrm{Aut}(\Gamma)$ on the set of ends of Γ when Γ is locally finite and transitive. The second application is to what we call 'fixed-point theorems'. In [**15**] W. Woess asks for a classification of those connected locally finite graphs with infinitely many ends which have the property that the automorphism group acts transitively on the set of ends. The third application is to find such a classification. In the fourth one we investigate the structure of the set of ends.

In Section 1 we fix notation and state some general theorems needed in the sequel. The next section contains a brief account of the necessary results and terminology we need from [**2**]. In the third section we then state and prove the connections between the ends of Γ and the structure trees of Γ. The promised applications then follow in Sections 4–7.

1. *Notation and preliminaries*

Let Γ denote an undirected connected graph. We use $V\Gamma$ and $E\Gamma$ to denote the set of vertices and edges of Γ, respectively. We will use $d(-,-)$ to denote the distance between two vertices of a graph. The graph Γ is said to be *transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $V\Gamma$. Even if we are only working with undirected graphs it is useful to think of the edges as having directions. Thus an edge $e \in E\Gamma$ is represented by an ordered pair of vertices; that is, $e = (o(e), t(e))$. We call $o(e)$ the *origin* of e and $t(e)$ the

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terminus of e. Because our graphs are undirected, if $e = (o(e), t(e))$ is in $E\Gamma$ then the edge $e^* := (t(e), o(e))$ is also in $E\Gamma$. If $s \subseteq V\Gamma$ then define

$$\partial s := \{e \in E\Gamma \mid \text{exactly one of } o(e) \text{ and } t(e) \text{ is in } s\}.$$

The set ∂s is called the boundary of s.

Let $\mathscr{B}\Gamma$ denote the boolean ring generated by subsets s of $V\Gamma$ with $|\partial s| < \infty$. Indeed, because the intersection and symmetric difference of two such sets again have finite boundary and $\partial s = \partial s^*$ (where s^* denotes the complement of s) for any $s \subseteq V\Gamma$ we see that

$$\mathscr{B}\Gamma = \{s \subseteq V\Gamma \mid |\partial s| < \infty\}.$$

Define $\mathscr{B}_n\Gamma$ as the subring of $\mathscr{B}\Gamma$ generated by those $s \subseteq V\Gamma$ with $|\partial s| \leqslant n$. We say that s is *thin* if $s \notin \mathscr{B}_{n-1}\Gamma$, where $n = |\partial s|$. Clearly a thin set is connected.

A *line* in a graph is a sequence of distinct vertices $\{\alpha_i\}_{i\in\mathbb{Z}}$ such that α_i is adjacent to α_{i+1} for all i. A *half-line* is a sequence of distinct vertices $\{\alpha_i\}_{i\in\mathbb{N}}$ such that α_i is adjacent to α_{i+1} for all i. An *end* of a graph is defined as an equivalence class of half-lines: we say that two half-lines L_1 and L_2 are in the same end if there is a third half-line L_3 that contains infinitely many vertices of both L_1 and L_2. Equivalently, we could say that L_1 and L_2 are in different ends if and only if there is a finite subset $\Phi \subseteq V\Gamma$ such that there are distinct connected components C_1 and C_2 of $\Gamma\backslash\Phi$ such that C_1 contains infinitely many vertices of L_1 and C_2 contains infinitely many vertices of L_2. Yet another equivalent definition is to say that L_1 and L_2 are in the same end if and only if there are infinitely many disjoint paths connecting vertices on L_1 to vertices on L_2. Following [**5**] we define the 'thickness', denoted by $m_1(\epsilon)$, of an end ϵ as the maximum number of disjoint half-lines it contains. The following theorem summarizes various results from [**6**].

THEOREM 1. *Let Γ be a locally finite connected graph. Let $g \in \mathrm{Aut}(\Gamma)$. Then either there is a finite subgraph Φ invariant under g or we can find a line L and a natural number n, such that g^n induces a non-trivial translation on L. In the latter case the ends which are determined by L are invariant under g. Let L^+ be a half-line contained in L such that $L^+g^n \subset L^+$. Let $\mathscr{D}(g)$ denote the end of Γ that contains L^+, and call $\mathscr{D}(g)$ the 'direction' of g. Then the following hold*:

(i) *$\mathscr{D}(g) = \mathscr{D}(g^{-1})$ if and only if $m_1(\mathscr{D}(g)) = \infty$;*

(ii) *if $\mathscr{D}(g) \neq \mathscr{D}(g^{-1})$ then $m_1(\mathscr{D}(g)) < \infty$ and $m_1(\mathscr{D}(g)) = m_1(\mathscr{D}(g^{-1}))$.*

Definition 1. If $g \in \mathrm{Aut}(\Gamma)$ does not leave any non-empty finite subgraph invariant then we call g a *translation*. If, furthermore, g satisfies (ii) in the theorem above then we say that g is *hyperbolic*.

We will write $\mathscr{E}\Gamma$ for the set of ends of Γ. If $\epsilon \in \mathscr{E}\Gamma$ and there is $g \in \mathrm{Aut}(\Gamma)$ such that $\epsilon = \mathscr{D}(g)$ then we call ϵ a *direction* of Γ. The set of directions of Γ is denoted by $\mathscr{D}\Gamma$. Furthermore, let $\mathscr{F}\mathscr{E}\Gamma$ (respectively, $\mathscr{F}\mathscr{D}\Gamma$) be the set of those ends (directions) that have finite thickness.

If $s \in \mathscr{B}\Gamma$ then we say that an end ϵ of Γ belongs to s if s contains one half-line in ϵ. We see that if ϵ belongs to s then $L \cap s$ contains a half-line for every half-line L in ϵ. By slight abuse of notation we will write $\epsilon \in s$ if ϵ belongs to s.

We will also have occasion to refer to theorem 1 in [**7**]. Here is a version of that theorem which has been slightly rephrased to suit our needs better; a short proof can

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be found in [**11**]. Because we restrict ourselves to locally finite graphs we get a stronger conclusion than in the original theorem.

THEOREM 2. *Assume that Γ is a locally finite connected graph and s is an infinite connected element of $\mathscr{B}\Gamma$ such that s^* is also infinite. If a group G acts transitively on Γ then G contains a hyperbolic element g such that $\mathscr{D}(g) \in s$ and $\mathscr{D}(g^{-1}) \in s^*$.*

Let G be a group acting on a set Ω. We can make G into a topological group by taking as a basis of neighbourhoods around the identity the pointwise stabilizers of finite subsets of Ω. Another way to phrase this is to say that we endow Ω with the discrete topology and let G have the topology of pointwise convergence. This topology is called the permutation group topology on G. For the purpose of this paper we only need to know that if G is acting on a locally finite graph Γ then the setwise stabilizer in G of a finite subgraph in Γ is compact. For a proof of this and further information about this topology the reader is referred to W. Woess' survey paper [**15**].

2. *Structure trees*

In [**3**] and [**2**] a very powerful technique for building trees from graphs is developed. The aim of this section is to give a description of this machinery, and to establish the notation needed to make it work.

Let Ω be some non-empty set and let $\mathscr{P}(\Omega)$ denote the power set of Ω. We say that $E \subseteq \mathscr{P}(\Omega)$ is *nested* if for every $e, f \in E$ one of

$$e \cap f, \quad e \cap f^*, \quad e^* \cap f, \quad e^* \cap f^*$$

is empty (i.e. $e \subseteq f$, $e \subseteq f^*$, $e^* \subseteq f$ or $f \subseteq e$).

If E is nested and for all $\alpha, \beta \in \Omega$ the number of elements of E containing α but not β is finite then we say that E is a *tree set*. An element $\alpha \in \Omega$ can be treated as a function $E \to \mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the integers modulo 2. This function, denoted by $\alpha | E$, is defined by

$$\alpha | E(e) := \begin{cases} 1 & \text{if } \alpha \in e \\ 0 & \text{if } \alpha \notin e, \end{cases}$$

where $e \in E$. Then E is a tree set if and only if it is nested and for all $\alpha, \beta \in \Omega$ the functions $\alpha | E$ and $\beta | E$ agree almost everywhere. We say that a tree set E is *undirected* if $e^* \in E$ whenever $e \in E$. Clearly if E is a tree set then $E \cup \{e^* | e \in E\}$ is also a tree set, so that every tree set is a subset of an undirected tree set.

Let T be an undirected tree. If $e, f \in ET$ then define $e \leqslant f$ to hold if and only if there is an edge path

$$e = e_0, \ldots, e_n = f$$

such that $t(e_{i-1}) = o(e_i)$ and $e_{i-1} \neq e_i^*$ for all $i = 1, \ldots, n$. One shows easily that this relation is a partial order. It is also clear that * acts on ET as an order-reversing involution.

THEOREM 3 ([**2**], corollary II·1·10). *Suppose that E is a partially ordered set with an order-reversing involution *. Furthermore, suppose that for all $e, f \in E$ exactly one of*

$$e < f, \quad e = f, \quad e > f, \quad e < f^*, \quad e = f^*, \quad e > f^*$$

holds and that there are only finitely many $g \in E$ such that $e < g < f$. Then there is an undirected tree T: such that E is isomorphic (as a partially ordered set with an involution) to ET. Also, ET must satisfy the above conditions for any undirected tree T.

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If E satisfies the conditions in the theorem then we use $T(E)$ to denote the associated tree. It can be verified rather easily that an undirected tree set satisfies the conditions in the above theorem. Thus if E is an undirected tree set then there is an undirected tree $T := T(E)$ such that E can be identified with ET in a natural way. Then inclusion in E corresponds to the relation $\leqslant$ defined above on ET and taking complements in E corresponds to the operation *, as defined on ET. We also see that if some group G acts on Ω such that E is invariant under G then there is a natural action of G on T.

Definition 2. Let T be a tree. If $e \in ET$ then we say that e points towards $\alpha \in VT$ if $d(\alpha, o(e)) > d(\alpha, t(e))$. If $\alpha \in VT$ then let $\alpha | E$ denote the function from ET to $\mathbb{Z}_2$ that takes the value 1 on those edges that point towards α and the value 0 on those edges that do not point towards α.

Let E be an undirected tree set. Consider functions $\phi : E \to \mathbb{Z}_2$ that satisfy the following conditions:

(i) there is a point $\alpha \in \Omega$ such that ϕ and $\alpha | E$ agree almost everywhere;

(ii) if $e, f \in E$ such that $e \cap f = \varnothing$ then $\phi(e)\phi(f) = 0$.

Call the set of all such functions V.

Let T be an undirected tree. We construct a new directed tree T^d from T in the following way: the set of vertices of T^d is the union of VT and the set $\{\{e, e^*\} \mid e \in ET\}$, and the set of edges of T^d is the set $\{e^d \mid e \in ET\}$, where $e^d := (\{e, e^*\}, t(e))$. We can identify ET and ET^d in a natural way, in particular the vertices of T^d can be thought of as functions $E \to \mathbb{Z}_2$. If $\alpha \in VT^d$ and $\alpha = \{e, e^*\}$ then $\alpha | E$ agrees on $E \setminus \{e, e^*\}$ with the functions given by the endpoints of e, but takes the value zero on both e and e^*. Informally, T^d is a kind of barycentric division of T.

THEOREM 4 ([2], theorem II·1·5). *Let $E \subseteq \mathscr{P}(\Omega)$ be an undirected tree set and let $T := T(E)$. Suppose V is as above and let $\phi \in V$. Then there is a vertex $\alpha \in VT^d$ such that $\phi = \alpha | E$.*

Let us now again consider general graphs.

Definition 3. If $E \subseteq \mathscr{P}(V\Gamma)$ is an undirected tree set, invariant under $\mathrm{Aut}(\Gamma)$, then we call the tree $T(E)$ a *structure tree* of Γ.

THEOREM 5 ([2], theorem II·2·20). *Let Γ be a connected infinite graph and let G be a subgroup of* $\mathrm{Aut}(\Gamma)$. *There is a chain of G-invariant undirected tree sets $E_1 \subseteq E_2 \subseteq \ldots$ in $\mathscr{P}(V\Gamma)$ such that all elements in E_n are thin and E_n generates $\mathscr{B}_n\Gamma$ for all n.*

The condition that E_n generates $\mathscr{B}_n\Gamma$ implies that if there is an $s \in \mathscr{B}_n\Gamma$ separating two vertices (edges, ends) then there is an element $e \in E_n$ that separates them also. This will be very useful in what follows, as will the following lemma also.

LEMMA 1 ([2], lemma II·2·5). *If $\alpha, \beta \in V\Gamma$ then every descending chain in $\{e \in E_n \mid \alpha \in e\}$ is finite and every chain in $\{e \in E_n \mid \alpha \in e, \beta \notin e\}$ is finite.*

3. *Ends and structure trees*

In this section Γ will denote some connected graph.

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LEMMA 2. *Let $T := T(E)$ be some structure tree (undirected) of Γ and let ϵ be an end of T. Then there is an unique end in Γ that corresponds to ϵ in a natural way.*

Proof. Let $L = \{\alpha_0, \alpha_1, \ldots\}$ be some half-line in ϵ and $e_i \in E$ be the edge (α_i, α_{i+1}). Then $e_0 \supset e_1 \supset e_2 \supset \ldots$, so, by Lemma 1, $\bigcap_{i \geqslant 0} e_i = \varnothing$. Clearly we can find a half-line L that contains at least one vertex from infinitely many of the sets e_i. Assume that L' and L'' are two such half-lines. Find i_0 such that e_{i_0} contains vertices β_0' and β_0'' from L' and L'', respectively. Because e_{i_0} is connected we can find a path P_0 in e_{i_0} connecting β_0' to β_0''. Then we find i_1 such that $P_0 \cap e_{i_1} = \varnothing$. Take vertices β_1' and β_1'' from L' and L'', respectively, that are in e_{i_1}. We let P_1 denote a path in e_{i_1} connecting β_1' and β_1''. Then we go on to find a number i_2 such that $P_1 \cap e_{i_2} = \varnothing$ and so on. The paths P_0, $P_1, \ldots$ form an infinite family of disjoint paths connecting L' *to* L'', thus L' and L'' are in the same end. It follows that all half-lines L such that infinitely many of the sets e_i contain vertices from L are in the same end. This end is then the only end that is contained in e_i, for all i. ∎

A similar argument can be found in §4 in [**14**]. On the other hand it does not follow that every end in Γ corresponds in this way to an end in a structure tree T, but we have the following lemma.

LEMMA 3. *Let ϵ be an end of Γ and let $T = T(E)$ be a structure tree of Γ. Then there is either a vertex in T or an end in T that corresponds to ϵ.*

Proof. Assume that no end of T corresponds to ϵ, that is to say that there is no infinite strictly descending chain in E such that all the elements in the chain contain ϵ. Define $\epsilon | E : E \to \mathbb{Z}_2$ by

$$\epsilon | E(e) := \begin{cases} 1 & \text{if } \epsilon \in e, \\ 0 & \text{if } \epsilon \notin e. \end{cases}$$

First of all it is clear that if $e, f \in E$ and $e \cap f = \varnothing$ then $(\epsilon | E(e))(\epsilon | E(f)) = 0$. In order to prove our lemma we have to show that if $\alpha \in V\Gamma$ then $\alpha | E$ and $\epsilon | E$ are almost equal. Let

$$A := \{e \in E \mid \epsilon | E(e) = 1 \text{ and } \alpha | E(e) = 0\}.$$

Because E is nested, for all $e, f \in A$ one of

$$e \leqslant f, \quad f \leqslant e, \quad e^* \leqslant f, \quad f \leqslant e^*,$$

holds. But, because $\epsilon \in f$ and $\epsilon \notin e^*$, we can exclude the last possibility and the second to last one can also be excluded because $\alpha \in e^*$ but $\alpha \notin f$. Thus A is a chain. By assumption A cannot contain an infinite strictly descending chain and an infinite strictly ascending chain would contradict Lemma 1. Hence A must be finite. The elements of the set

$$\{e \in E \mid \epsilon | E(e) = 0 \text{ and } \alpha | E(e) = 1\}$$

are just the complements of the elements in A. Therefore $\alpha | E$ and $\epsilon | E$ agree almost everywhere. By Theorem 4 there is a vertex in T^d that corresponds to $\epsilon | E$. If that vertex was not in VT then there would be a set $e \in E$ such that

$$\epsilon | E(e) = \epsilon | E(e^*) = 0;$$

but this is clearly out of the question. ∎

Now we can ask whether one can recognize those vertices in a structure tree that correspond to ends in Γ. The next lemma contains a complete answer to this question.

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LEMMA 4. *Let Γ be a locally finite connected graph, and let $T := T(E)$ be a structure tree (undirected) of Γ. Define $\phi : V\Gamma \to VT$; $\alpha \mapsto \alpha|E$. If $\alpha \in VT$ has infinite valency or $\phi^{-1}(\alpha)$ is infinite then there is an end ϵ in Γ such that $\epsilon|E = \alpha$. Furthermore, if $\alpha \in VT$ corresponds to an end ϵ then $\phi^{-1}(\alpha)$ is infinite or α has infinite valency.*

Proof. Suppose that $\alpha \in VT$ has infinite valency. Let C be a component of $T\backslash\{\alpha\}$. If e is an edge in C pointing away from α then we know that e points towards the elements of $\phi(e)$. Thus every component of $T\backslash\{\alpha\}$ contains some elements of $\operatorname{Im}\phi$.

If $\alpha \in V\Gamma$ has infinite valency then we take an element β_C from $\phi^{-1}(C)$ for each component C of $T\backslash\{\alpha\}$. Let Λ be a minimal spanning tree in Γ for all the vertices β_C. By König's Lemma we can find a half-line L in Λ. Denote by ϵ the end of Γ that L belongs to. If e is a connected element of $\mathscr{B}\Gamma$ which contains all but finitely many of the vertices β_C then e must contain ϵ. This is because ∂e is finite and thus $e^* \cap \Lambda$ will also be finite. If $e \in E$ points away from α then e contains at most one of the vertices β_C and thus cannot contain ϵ. On the other hand, if e points towards α then e must contain ϵ. But now we have that α, as a function of E, is equal to $\epsilon|E$.

If $\phi^{-1}(\alpha)$ is infinite then we let Λ be a spanning tree of $\phi^{-1}(\alpha)$ and the result follows by the same argument.

Assume that $\alpha \in VT$ and $\alpha|E = \epsilon|E$ for some end ϵ of Γ. Suppose that α has finite valency and that $\phi^{-1}(\alpha)$ is finite. Take a half-line L that belongs to the end ϵ. The conditions imply that infinitely many of the vertices in L must be mapped by ϕ to some component C of $T\backslash\{\alpha\}$. If e is the edge in T that has α as its origin and points towards C then e must contain ϵ but e points away from α. Thus $\epsilon|E(e) = 1$ but $\alpha|E(e) = 0$, which contradicts the assumption that $\alpha|E = \epsilon|E$. ∎

In the following corollary we collect together the information we can now deduce about connections between actions of elements of $\operatorname{Aut}(\Gamma)$ on Γ and on structure trees of Γ.

COROLLARY 1. *Let Γ be a connected locally finite graph and $T := T(E)$ be some structure tree of Γ.*

(i) *If $g \in \operatorname{Aut}(\Gamma)$ acts like a translation on T then g acts like a translation on Γ and $m_1(\mathscr{D}(g)) < \infty$.*

(ii) *If $g \in \operatorname{Aut}(\Gamma)$ is a translation then either g acts as a translation on T or there is a unique vertex of T fixed by g and that vertex has infinite valency.*

(iii) *If $g \in \operatorname{Aut}(\Gamma)$ is a translation and $m_1(\mathscr{D}(g)) < \infty$ then there is an undirected tree set E_g such that g acts as a translation on $T(E_g)$.*

Proof. (i) Let L be the line in T that is invariant under g. Suppose that e is an edge in L. Then ∂e has an infinite orbit under g, and therefore g cannot fix any non-empty finite subgraph. There are two ends of T that are invariant under g, so there are also two ends of Γ invariant under g, which means that $m_1(\mathscr{D}(g)) < \infty$.

(ii) If $\mathscr{D}(g)$ does not appear as an end in T then there is a unique vertex α in T that corresponds to ϵ. Clearly α is fixed by g; and α must have infinite valency, because otherwise some power of g would fix an edge of T which is impossible.

(iii) Assume $\mathscr{D}(g) \neq \mathscr{D}(g^{-1})$. Then there is $s \in \mathscr{B}\Gamma$ such that $\mathscr{D}(g) \in s$ but $\mathscr{D}(g^{-1}) \notin s$. We can then find a tree set E_g (by Theorem 5) that contains an element e that separates $\mathscr{D}(g)$ and $\mathscr{D}(g^{-1})$. Clearly g acts on $T(E_g)$ as a translation. ∎

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4. *The action on the ends*

Let Γ be a connected locally finite graph. We say that $g \in \mathrm{Aut}(\Gamma)$ is *bounded* if there is a natural number k such that $d(\alpha, \alpha g) \leqslant k$ for all $\alpha \in \Gamma$. The bounded automorphisms of Γ form a subgroup $B(\Gamma)$ of $\mathrm{Aut}(\Gamma)$.

THEOREM 6. *Assume that Γ is connected, locally finite and transitive and, furthermore, that Γ has infinitely many ends. Then the kernel K of the action of $\mathrm{Aut}(\Gamma)$ on $\mathscr{E}\Gamma$ is the group $B(\Gamma)$.*

Proof. First we will show that $B(\Gamma) \leqslant K$. Take $g \in B(\Gamma)$. Assume that there is an end $\epsilon \in \mathscr{E}\Gamma$ such that $\epsilon \neq \epsilon g$. Then we can find a finite subgraph Φ *of* Γ such that ϵ and ϵg belong to different components of $\Gamma \backslash \Phi$. Let L_1 be a half-line in ϵ. Then $L_1 g$ is in ϵg. We can assume that both $L_1 \cap \Phi$ and $L_1 g \cap \Phi$ are empty. All paths connecting vertices in L_1 to vertices in $L_1 g$ must go through Φ. Because Γ is locally finite we have a contradiction.

Let $e_0 \in \mathscr{B}\Gamma$ be an infinite and co-infinite set such that the set

$$E := \{e_0 \,\mathrm{Aut}(\Gamma), e_0^* \,\mathrm{Aut}(\Gamma)\}$$

is a tree set. Put $T := T(E)$. Then $\mathrm{Aut}(\Gamma)$ is transitive on the edges of T, and (as pointed out in [3]) this fact together with the transitivity of Γ ensures that T has no leaves (i.e. there are no vertices of valency one). Take $h \in K$. Suppose h does not act trivially on T. We must then either have that h moves an end of T, which (by Lemma 2) would mean that h moves an end of Γ, or T has only two ends and h acts like a translation on T. But by Corollary 1 the latter would imply that h acts like a translation on Γ and a translation fixes at most two ends. Thus K acts trivially on T. For $e \in E$, set

$$\partial_v e := \{\alpha \in V\Gamma \mid \alpha \notin e \text{ but } \alpha \text{ is adjacent to a vertex in } e\}.$$

Because Γ is transitive we have that every vertex in Γ is contained in $\partial_v e$ for some $e \in E$. The sets $\partial_v e$ are all invariant under $\mathrm{Aut}(\Gamma)$ and all have the same diameter (the diameter of a set is the supremum of the distances between its elements). It follows that for all $h \in K$ and $\alpha \in V\Gamma$ we have

$$d(\alpha, \alpha h) \leqslant \mathrm{diam}(\partial_v e_0) < \infty.$$

Thus $K \leqslant B(\Gamma)$. ∎

In fact we have proved more: $B(\Gamma)$ is a subcartesian product of isomorphic finite groups (given by the action of $B(\Gamma)$ on its orbits). In particular this gives us:

COROLLARY 2 ([4], theorem 5). *The group $B(\Gamma)$ is locally finite.*

5. *Fixed-point theorems*

Throughout this section Γ is a locally finite, connected graph and H will be a subgroup of $\mathrm{Aut}(\Gamma)$. Consider the following properties:

(*a*) There is a non-empty finite subgraph of Γ that is invariant under H.

(*b*) There is an end of Γ fixed by H.

(*c*) There is a pair of ends of Γ that is invariant under H.

The first 'fixed-point theorem' was the following.

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PROPOSITION 1 ([12], proposition 3·4). *Let T be a tree and H a group acting on T. If H contains no translations then either (a) or (b) holds.*

This result has been generalized to locally finite graphs by W. Woess [13].

PROPOSITION 2 ([13], proposition 1). *Suppose that $H \leqslant \mathrm{Aut}(\Gamma)$ contains no hyperbolic elements. Then (a) or (b) holds and, furthermore, if (b) holds then there is a unique end fixed by H.*

We also have the following theorem (cf. [10]), which gives a kind of 'Tits alternative' for groups acting on trees.

THEOREM 7. *Let T be a tree and $H \leqslant \mathrm{Aut}(T)$. If H satisfies none of (a), (b), (c) then H contains a free group on two generators.*

Now we are able to use the results in Section 4 to prove the following generalization of this theorem.

THEOREM 8. *Let Γ be a connected, locally finite graph and $H \leqslant \mathrm{Aut}(\Gamma)$. If H satisfies none of (a), (b), (c) then H contains a free group on two generators.*

Proof. By Proposition 2 we can assume that H contains a hyperbolic element g. Let $T := T(E)$ be a structure tree of Γ such that g acts as a translation on T. Because we have this element g acting as a translation on T it is impossible that the action of H on T satisfies (a). If (b) or (c) were to hold for the action of H on T then they would also hold for the action of H on Γ, because the ends of T correspond to ends of Γ. Thus none of (a), (b), (c) holds for the action of H on T. Therefore, by Theorem 7, H contains a free group on two generators. ∎

H. A. Jung also has a proof of this theorem which works for a wider class of graphs. For example, his proof works for all countable graphs. By using arguments similar to the ones in the proof of the above theorem one is able to deduce the following theorem of W. Woess from the corresponding result for trees. W. Woess, in his paper [13], does indeed suggest that this should be possible.

THEOREM 9 ([13], theorem 1). *Let $H \leqslant \mathrm{Aut}(\Gamma)$. If H acts amenably on Γ then one of (a), (b), (c) holds.*

6. *End-transitive graphs*

We say that a graph Γ is *end-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on the ends of Γ. The aim of this section is to classify the connected locally finite end-transitive graphs that have infinitely many ends. This is an answer to problem 2 in [15].

Before we can state our theorem we need some additional notation. Let G be a group acting on some set Ω, and let $\alpha \in \Omega$. The orbits of G_α on Ω are called *suborbits*. The suborbit βG_α corresponds in a natural way to the orbit $(\alpha, \beta)G$ in $\Omega \times \Omega$. In the case of transitive groups this gives us a 1–1 correspondence between the G_α-orbits, for some fixed $\alpha \in \Omega$, and the G-orbits on $\Omega \times \Omega$. Orbits of G on $\Omega \times \Omega$ are called *orbitals*. We call a graph Δ a *poly-orbital* graph of Ω if $V\Delta$ is a union of G-orbits on Ω and $E\Delta$ is a union of orbitals (tacitly we exclude graphs where all the edges are of the form (α, α)).

Let T be a tree. There is a natural bipartition of T. Define $\mathrm{Aut}^+(T)$ as the subgroup of $\mathrm{Aut}(T)$ that stabilizes the blocks of this bipartition. We say that T is *semi-regular* if the vertices in each of the blocks of that bipartition all have the same valency.

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THEOREM 10. *Let Γ be a connected locally finite graph with infinitely many ends. Suppose that Γ is end-transitive. Then either there is an* $\mathrm{Aut}(\Gamma)$*-congruence π with finite classes such that Γ/π is isomorphic to a connected component of a poly-orbital graph arising from the action of* $\mathrm{Aut}^+(T)$ *on $V\!T$, where T is a semi-regular tree, or there is a non-empty finite subset of $V\Gamma$ invariant under* $\mathrm{Aut}(\Gamma)$*. In particular, if there is no non-empty* $\mathrm{Aut}(\Gamma)$*-invariant finite set of vertices then* $\mathrm{Aut}(\Gamma)$ *acts with finitely many orbits on $V\Gamma$.*

First we need to prove two lemmas. Let us fix some notation. Set $G := \mathrm{Aut}(\Gamma)$. If $\mathrm{Aut}(\Gamma)$ contains no hyperbolic elements then by Proposition 2 there is a non-empty finite $\mathrm{Aut}(\Gamma)$-invariant subset of $V\Gamma$. We may thus assume that $\mathrm{Aut}(\Gamma)$ contains a hyperbolic element. Let $e_0 \in \mathscr{B}\Gamma$ be a thin element, both infinite and co-infinite, such that $E := \{e_0 \mathrm{Aut}(\Gamma), e_0^* \mathrm{Aut}(\Gamma)\}$ is a tree set. Put $T := T(E)$. We can assume that some element of $\mathrm{Aut}(\Gamma)$ acts like a translation on T. We clearly have a 1–1 correspondence between the ends of T and the ends of Γ, and G acts transitively on $\mathscr{E}T$. We have a map

$$\phi : V\Gamma \to VT; \quad \alpha \mapsto \alpha \,|\, E.$$

The fibres of ϕ give us a G-congruence π on $V\Gamma$.

LEMMA 5. *The fibres of ϕ are finite and T is locally finite.*

Proof. This is an easy application of Lemma 4. Because no vertex in T corresponds to an end of Γ then for all vertices α in T we know that α has finite valency and $\phi^{-1}(\alpha)$ is finite. ∎

Consider the graph Γ/π. Define $P : \Gamma \to \Gamma/\pi$ to be the natural projection. Clearly there is a 1–1 correspondence between the ends of Γ and the ends of Γ/π. Because G has at most two orbits on the vertices of T we see that G has at most two orbits on the vertices of Γ/π. Because the π-classes are finite, this in turn implies that G has only finitely many orbits on $V\Gamma$. One can also note that the set $EP := \{eP \,|\, e \in E\}$ is a tree set and that $T(EP) = T(E)$. From now on we are only interested in the graph Γ/π and to simplify the exposition we will be replacing Γ with Γ/π and G will be any closed end-transitive subgroup of $\mathrm{Aut}(\Gamma)$. (We cannot be sure that EP is invariant under $\mathrm{Aut}(\Gamma/\pi)$ but the projection of $\mathrm{Aut}(\Gamma)$ on $\mathrm{Aut}(\Gamma/\pi)$ is end-transitive and closed.) The reader will easily convince himself that this makes no difference in what follows. It is thus possible to assume that ϕ is injective. We will be identifying vertices in Γ with their ϕ-images in T.

The next lemma is based on proposition 1 in [**9**]. For completeness a full proof of the lemma will be included.

LEMMA 6. *The stabilizer of a vertex $\alpha \in V\Gamma$ acts transitively on the ends of Γ.*

From now on most of the action will take place inside T. We will represent the set of ends as the set of half-lines that originate at α. We can define a metric t on $\mathscr{E}T$ by the formula

$$t(L_1, L_2) := 2^{-k},$$

where L_1 and L_2 are two half-lines starting at α and $k = |L_1 \cap L_2| - 1$. One easily sees that $\mathscr{E}T$ with this metric is a complete metric space.

Proof of Lemma 6. We first prove that the orbits of G_α on $\mathscr{E}T$ are all closed. Let Δ

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be an orbit of G_α on $\mathscr{E}T$. Assume that $\{\delta_i\}_{i\in\mathbb{N}}$ is a sequence in Δ that converges to an end δ. Now find a sequence $\{g_i\}_{i\in\mathbb{N}}$ such that $\delta_i = \delta_1 g_i$. Because Γ is a locally finite graph, G_α is compact in the permutation group topology induced on G_α by the action on Γ. Thus there is a convergent subsequence $\{g_{i_j}\}_{j\in\mathbb{N}}$, which converges to some element g in G_α. But this subsequence must also converge to g in the permutation group topology induced by the action on T. Thus we see that $\delta_1 g = \delta$. Therefore Δ is closed.

Above we showed that $\mathscr{E}T$ is a complete metric space, and so it is also a Baire space. Let $\{g_i\}_{i\in\mathbb{N}}$ be a complete set of coset representatives for G_α in G. Suppose that Δ is a G_α orbit on $\mathscr{E}T$. Then $\mathscr{E}T = \bigcup_{i\in\mathbb{N}} \Delta g_i$. By Baire's Theorem at least one of the sets Δg_i (therefore all) must have non-empty interior. Because Δ is a G_α orbit we can conclude that Δ is open. Thus all G_α orbits on $\mathscr{E}T$ are open.

Let $g\in G$ be a translation along some line L in T. Because G acts transitively on the undirected edges of T we can assume that $\alpha\in L$. Say $L = \{\alpha_i\}_{i\in\mathbb{Z}}$ with $\alpha_0 = \alpha$. Let ϵ_1 be the end represented by $\{\alpha_i\}_{i\geqslant 0}$ and ϵ_2 be the end represented by $\{\alpha_i\}_{i\leqslant 0}$. For $\beta\in V\Gamma$ define $E(\beta)$ as the set of ends that are represented by half-lines that go through β. These sets form a basis for the topology on $\mathscr{E}T$. Because $\epsilon_1 G_\alpha$ and $\epsilon_2 G_\alpha$ are open we can find a number j such that $E(\alpha_j)\subseteq \epsilon_1 G_\alpha$ and $E(\alpha_{-j})\subseteq \epsilon_2 G_\alpha$. Now we see by using the translation g that G_α must act transitively on the points that are adjacent to α in T and that G_α must act transitively on $E(\alpha_1)$. Hence G_α acts transitively on $\mathscr{E}T$. ∎

For simplicity let us assume that G acts transitively on Γ; the case where G has two orbits on Γ is similar. Let α be a vertex in Γ. Let $C_i(\alpha)$ denote the set of vertices in T that are at distance i (in T) from α. The real point of the last lemma is that it implies that G_α acts transitively on $C_i(\alpha)$ for all i. Let Φ be the set of vertices in Γ that are adjacent to α in Γ. Let F be the union of all the orbitals $(\alpha,\beta)\,\mathrm{Aut}^+(T)$ and $(\beta,\alpha)\,\mathrm{Aut}^+(T)$ where $\beta\in\Phi$. We see now that $E\Gamma = F$ and thus we have proved the theorem.

Remark 1. Conversely, one can note that the connected components of the poly-orbital graphs one gets from semi-regular trees are all end-transitive.

Remark 2. A graph Γ is said to be *distance-transitive* if for all vertices $\alpha,\beta,\gamma,\delta$ with $d(\alpha,\beta) = d(\gamma,\delta)$ we can find $g\in\mathrm{Aut}(\Gamma)$ such that $\alpha g = \gamma$ and $\beta g = \delta$. Clearly a locally finite regular tree is distance-transitive. We get another example of a distance-transitive graph by taking a locally finite semi-regular tree T, and looking at the orbital graph we get by saying that two vertices are adjacent if and only if their distance in T is 2; the connected components of this orbital graph are distance-transitive. Macpherson [**8**] has proved that every connected locally finite infinite distance-transitive graph is of one of the above types. A simplified version of Macpherson's argument can be found in chapter II, §3 in [**2**]. With very little addition the preceding argument will also give a proof of Macpherson's theorem.

7. *D-relations*

In the study of infinite permutation groups certain types of relations seem to occur naturally. These types of relations have been axiomatized and investigated by Adeleke and Neumann [**1**]. We are interested in what they call D-relations: these

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arise naturally on the set of ends of sufficiently homogeneous 'tree-like' structures. It would take us too far afield to describe precisely what these 'tree-like' structures are. The remarkable thing about these D-relations is that a set carrying a D-relation can be represented as the set of ends of a 'tree-like' structure. A quaternary relation D on a set Ω is a D-relation if it satisfies the following conditions:

D1. If $D(\alpha, \beta; \gamma, \delta)$ then $D(\beta, \alpha; \gamma, \delta)$ and $D(\gamma, \delta; \alpha, \beta)$.
D2. *If* $D(\alpha, \beta; \gamma, \delta)$ then $\neg D(\alpha, \gamma; \beta, \delta)$.
D3. *If* $D(\alpha, \beta; \gamma, \delta)$ then $D(\alpha, \beta; \gamma, \epsilon)$ or $D(\alpha, \epsilon; \gamma, \delta)$.
D4. *If* α, β, γ are all distinct then there is $\delta \in (\Omega \backslash \{\alpha, \beta, \delta\})$ such that $D(\alpha, \beta; \gamma, \beta)$.

A quaternary relation D defined on the ends of a tree T by $D(\alpha, \beta; \gamma, \delta)$ if and only if the lines having ends α, β and γ, δ respectively do not intersect will automatically satisfy D1, D2, D3. If, for instance, our tree T is transitive then this relation satisfies also D4. If we could define a D-relation on a subset of ends of a graph Γ then the results of Adeleke and Neumann would imply that this subset of ends of Γ could be represented as a set of ends of a tree. This would give us another way to construct a tree from the graph Γ.

Assume that $E \subseteq \mathscr{B}\Gamma$ is an undirected tree set. Define a quaternary relation D_E on $\mathscr{E}\Gamma$ as follows: $D_E(\alpha, \beta; \gamma, \delta)$ if and only if there is an element $e \in E$ with $\alpha, \beta \in e$ and $\gamma, \delta \in e^*$.

PROPOSITION 3. *The relation* D_E *defined above satisfies* D1, D2, D3.

Proof. D1. This follows straight from the definition.

D2. Assume that $D(\alpha, \beta; \gamma, \delta)$ holds and we have $e \in E$ such that $\alpha, \beta \in e$ and $\gamma, \delta \in e^*$. Suppose that $D(\alpha, \gamma; \beta, \delta)$ holds and is witnessed by some element $f \in E$. Then we have

$$\alpha \in e \cap f, \quad \beta \in e \cap f^*, \quad \gamma \in e^* \cap f \quad \text{and} \quad \delta \in e^* \cap f^*.$$

But this is impossible, because E is a tree set and therefore one of the intersections above must be empty. Hence $\neg D(\alpha, \gamma; \beta, \delta)$.

D3. Assume that $D(\alpha, \beta; \gamma, \delta)$ holds and is witnessed by some $e \in E$. Let ϵ be some end of Γ. Clearly, either $\epsilon \in e$ or $\epsilon \in e^*$. If $\epsilon \in e$ then $D(\alpha, \epsilon; \gamma, \delta)$ and if $\epsilon \in e^*$ then $D(\alpha, \beta; \gamma, \epsilon)$ holds. ∎

If we restrict this relation D_E to those ends of Γ that actually appear as ends of $T(E)$ then we see that D_E agrees with the relation D, as defined above on the ends of a tree.

Now let $\{E_n\}$ be a chain of undirected tree sets as described in Theorem 5. Define $E_\omega := \bigcup E_n$. We then get a tower of relations $D_{E_1} \subseteq D_{E_2} \subseteq \ldots$, with $D_{E_\omega} = \bigcup D_{E_n}$.

PROPOSITION 4. *Assume that* Γ *is locally finite and transitive. Then the restriction of* D_{E_ω} *to* $\mathscr{F}\mathscr{D}\Gamma$ *is a D-relation.*

Proof. The only thing we need to check is condition D4: the others follow straight from the previous proposition. Let $\alpha, \beta, \gamma \in \mathscr{F}\mathscr{D}\Gamma$ be distinct. We need to find $\delta \in \mathscr{F}\mathscr{D}\Gamma \backslash \{\alpha, \beta, \gamma\}$ such that there is a set $e_0 \in E_\omega$ containing γ and δ but neither α nor β. Because $\gamma \in \mathscr{F}\mathscr{D}\Gamma$, we have a translation g such that $\mathscr{D}(g) = \gamma$. Find $e \in E_\omega$ such that $\gamma \in e$ and $\mathscr{D}(g^{-1}) \notin e$. Then there is a natural number j such that $\alpha, \beta \notin eg^j$. Set $e_0 := eg^j$.

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It is clear that e_0 must contain some end δ_0 of Γ different from γ; whereupon we find $f \in \mathscr{B}e_0$ such that $\delta_0 \in f$ but $\gamma \notin f$. Now we apply Theorem 2 which tells us that there is a hyperbolic element h of $\mathrm{Aut}(\Gamma)$ such that $\mathscr{D}(g) \in f$. Put $\delta := \mathscr{D}(g)$. Because $\delta \in f$ and none of α, β, γ is in f we have that δ is not equal to any of α, β, γ and $D(\alpha, \beta; \gamma, \delta)$. ∎

COROLLARY 3. *Let Γ be locally finite and transitive. If* $\mathrm{Aut}(\Gamma)$ *is transitive on the ends of Γ then D_{E_ω} is a D-relation.*

Proof. Theorem 2 above gives us that $\mathscr{F}\mathscr{D}\Gamma$ is not empty. Therefore $\mathscr{E}\Gamma = \mathscr{F}\mathscr{D}\Gamma$. ∎

In the proof of the above proposition the only time we require the transitivity of Γ is when we apply Theorem 2. Woess points out in [13] that the conclusion of Jung's theorem holds if we only assume that for every vertex α and every infinite element e of $\mathscr{B}\Gamma$ we can find an element $g \in \mathrm{Aut}(\Gamma)$ such that $\alpha g \in e$. The above proposition and the corollary also hold if we replace transitivity by this weaker condition.

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Note added in proof. Since completing this paper the author has learnt that A. Nevo has also proved results similar to the ones in Section 6. His paper is due to appear in *Israel J. Math.*

REFERENCES

[1] S. A. ADELEKE and P. M. NEUMANN. Semilinearly ordered sets, betweenness relations and their automorphism groups. In preparation.
[2] W. DICKS and M. J. DUNWOODY. *Groups Acting on Graphs* (Cambridge University Press, 1989).
[3] M. J. DUNWOODY. Cutting up graphs. *Combinatorica* **2** (1982), 15–23.
[4] C. D. GODSIL, W. IMRICH, N. SEIFTER, M. E. WATKINS and W. WOESS. A note on bounded automorphisms of infinite graphs. *Graphs Combin.* **5** (1989), 333–338.
[5] R. HALIN. Über die Maximalzahl fremder unendlicher Wege in Graphen. *Math. Nachr.* **30** (1965), 119–127.
[6] R. HALIN. Automorphisms and endomorphisms of infinite locally finite graphs. *Abh. Math. Sem. Univ. Hamburg* **39** (1973), 251–283.
[7] H. A. JUNG. A note on fragments of infinite graphs. *Combinatorica* **1** (1981), 285–288.
[8] H. D. MACPHERSON. Infinite distance transitive graphs of finite valency. *Combinatorica* **2** (1982), 63–69.
[9] C. NEBBIA. Groups of isometries of a tree and the Kunze–Stein phenomenon. *Pacific J. Math.* **133** (1988), 141–149.
[10] C. NEBBIA. Amenability and Kunze–Stein property for groups acting on a tree. *Pacific J. Math.* **135** (1988), 371–380.
[11] P. M. SOARDI and W. WOESS. Amenability, unimodularity, and the spectral radius of random walks on infinite graphs. *Math. Z.* **205** (1990), 471–486.
[12] J. TITS. Sur le groupe des automorphismes d'un arbre. In *Essays on Topology and Related Topics. (Mémoires dédiés à G. de Rham)* (Springer-Verlag, 1970), pp. 188–211.
[13] W. WOESS. Amenable group actions on infinite graphs. *Math. Ann.* **284** (1989), 251–265.
[14] W. WOESS. Boundaries of random walks on graphs and groups with infinitely many ends. *Israel J. Math.* **68** (1989), 271–301.
[15] W. WOESS. Topological groups and infinite graphs. *Ann. Discrete Math.*, to appear.

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