PARTICULAR SOLUTION OF POLYHARMONIC SPLINE ASSOCIATED WITH REISSNER PLATE PROBLEMS

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ABSTRACT

In this paper, analytical particular solutions of the augmented polyharmonic spline (APS) associated with Reissner plate model are explicitly derived in order to apply the dual reciprocity method. In the derivations of the particular solutions, a coupled system of three second-ordered partial differential equations (PDEs), which governs problems of Reissner plates, is initially transformed into a single six-ordered PDE by the Hörmander operator decomposition technique. Then the particular solutions of the coupled system can be found by using the particular solution of the six-ordered PDE derived in the first author's previous study. These formulas are further implemented for solving problems of Reissner plates under arbitrary loadings. In the solution procedure, an arbitrary loading measured at some scattered points is first interpolated by the APS and a corresponding particular solution can then be approximated by using the prescribed formulas. After that the complementary homogeneous problem is formally solved by the method of fundamental solutions (MFS). Numerical experiments are carried out to validate these particular solutions.

Keywords: Augmented polyharmonic spline, Reissner plate, Hörmander operator decomposition technique, Method of fundamental solutions, Dual reciprocity method.

1. INTRODUCTION

The classical shear deformable thick plate theories of Reissner [1,2] and Mindlin [3] are widely used in solving plate problems by using many well developed numerical methods such as the finite element method (FEM), boundary element method (BEM) and method of fundamental solutions (MFS). In this paper, the particular solutions of thick plates under arbitrary loadings are studied in order to apply boundary-type numerical methods for solving plate problems.

In recent years, boundary-type numerical methods have become a popular research field because only discretizations on boundary are required. A comprehensive review of boundary-type numerical methods can be found in the article of Cheng and Cheng [4]. In these methods, the numerical discretization is performed on the boundary of the computational domain. These methods include the boundary element method (BEM) [4], method of fundamental solutions (MFS) [5,6], Trefftz methods (TM) [7] and others.

When the boundary-type numerical methods are applied to solve inhomogeneous problems, the dual reciprocity method (DRM) innovated by Nardini and Brebbia [8] in 1982 is required to convert the inhomogeneous problems into homogeneous ones. In the original DRM, an ad-hoc function $1 + r$ were exclusively used without an obvious reason. Golberg and Chen [9] improved the DRM by using the approximation theory in the field of mathematics. In the solution procedure, a numerical particular solution is approximated by a series of analytical solutions of the basis functions which are used to interpolate the right-hand-side function of the problem. Therefore, the application of the DRM depends on the availability of the particular solution associated with basis function as well as the partial differential operator of the problem.

In recent decades, significant achievements have been progressed in obtaining analytical particular solutions for various basis functions. Among these are the radial basis functions (RBF) [9-17], the trigonometric

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functions [18,19], the polynomials [20-27] and others. Although the trigonometric and polynomial functions are known to be very accurate, their applications are restricted to problems whose right-hand-side terms are given functions. For the cases when the right-hand-side terms are given on some scattered points, it is required to use the RBFs. This study focused on the analytical particular solution of the RBF of augmented polyharmonic spline (APS) [28] for problems of Reissner plates in bending.

Boundary-type numerical methods have also been applied for solving thick-plate problems. EL-Zafrany *et al*. [29] studied thick Reissner plates in bending by the BEM. On the other hand, Fadhil and EL-Zafrany [30] and AL-Hosani *et al*. [31] developed a BEM model for Reissner plates resting on foundations. Recently, Wen [32] and Wen *et al*. [33] applied the MFS for solving the problems of thick plates in the Laplace domain. However, their studies were limited to concentrated and uniform loadings. Therefore, Tsai [27] used the Chebyshev polynomials to obtain an approximated particular solution of an arbitrary loading which is an explicitly given function. For approximating an arbitrary loading on scattered points, Wen *et al*. [34] derived an analytical particular solution for the ad-hoc RBF $1 + r$.

In this paper, we derive the analytical particular solutions for thick-plate problems when the arbitrary loadings are represented by the APS. In the derivation, a coupled system of three second-ordered PDEs, which governs problems of Reissner plates, are initially transformed into a single six-ordered PDE by the Hörmander operator decomposition technique [35]. Then the particular solutions of the coupled PDE system can be found by using the particular of the six-ordered PDE derived in the first author's previous study [17]. After the particular solutions are found, the complementary homogeneous problems are formally solved by the MFS.

This paper is divided into 6 parts: The derivation of the Reissner plate model is summarized in section 2 and MFS-DRM formulation is introduced in section 3. Analytical particular solutions are derived in Section 4. Numerical experiments are carried out to validate the derived particular solutions in Section 5. Finally, the conclusions are drawn in Section 6.

2. REISSNER PLATE MODEL

In this article, the indices i, j, k and α, β, γ are presented in the range $\{1, 2, 3\}$ and $\{1, 2\}$, respectively.

As described in Fig. 1, we consider a plate of uniform thickness h with its middle plane being a domain Ω with boundary Γ in the $x_1 - x_2$ plane and thickness coordinate x_3. The plate is subjected to a normal loading with intensity $q(x_1, x_2)$. Then the equations of equilibrium are given by

$$\begin{cases} \dfrac{\partial \Upsilon_{\alpha\beta}}{\partial x_\beta} - Q_\alpha = 0 \\ \dfrac{\partial Q_\alpha}{\partial x_\alpha} + q = 0 \end{cases} \tag{1}$$

where Q_α is the transverse shear force and $\Upsilon_{\alpha\beta}$ is the bending or twisting moment.

In the Reissner thick plate theory [1,2], u_α represents an average slope angle and u_3 is the lateral deflection of the plate in the middle surface. According to the theory, the constitutive equations are given by:

$$\begin{cases} \Upsilon_{\alpha\beta} = \dfrac{D(1-\nu)}{2}\left(\dfrac{\partial u_\alpha}{\partial x_\beta} + \dfrac{\partial u_\beta}{\partial x_\alpha} + \dfrac{2\nu}{1-\nu}\dfrac{\partial u_\gamma}{\partial x_\gamma}\delta_{\alpha\beta}\right) + q\dfrac{\nu}{\lambda^2(1-\nu)}\delta_{\alpha\beta} \\ Q_\alpha = \dfrac{D(1-\nu)\lambda^2}{2}\left(u_\alpha + \dfrac{\partial u_3}{\partial x_\alpha}\right) \end{cases} \tag{2}$$

with

$$\lambda = \frac{\sqrt{10}}{h} \tag{3}$$

After combining Eqs. (1) and (2) and rearrangement, we can write the governing equations in the following form:

$$\tilde{L}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = -\tilde{E}q \tag{4}$$

with

$$\tilde{L} = \begin{pmatrix} \dfrac{D(1-\nu)}{2}(\nabla^2 - \lambda^2) + \dfrac{D(1+\nu)}{2}\dfrac{\partial^2}{\partial x_1^2} & \dfrac{D(1+\nu)}{2}\dfrac{\partial^2}{\partial x_1 \partial x_2} & -\dfrac{D\lambda^2(1-\nu)}{2}\dfrac{\partial}{\partial x_1} \\ \dfrac{D(1+\nu)}{2}\dfrac{\partial^2}{\partial x_1 \partial x_2} & \dfrac{D(1-\nu)}{2}(\nabla^2 - \lambda^2) + \dfrac{D(1+\nu)}{2}\dfrac{\partial^2}{\partial x_2^2} & -\dfrac{D\lambda^2(1-\nu)}{2}\dfrac{\partial}{\partial x_2} \\ \dfrac{D\lambda^2(1-\nu)}{2}\dfrac{\partial}{\partial x_1} & \dfrac{D\lambda^2(1-\nu)}{2}\dfrac{\partial}{\partial x_2} & \dfrac{D\lambda^2(1-\nu)}{2}\nabla^2 \end{pmatrix} \tag{5}$$

and

$$\tilde{E} = \begin{pmatrix} \dfrac{\nu}{(1-\nu)\lambda^2}\dfrac{\partial}{\partial x_1} \\ \dfrac{\nu}{(1-\nu)\lambda^2}\dfrac{\partial}{\partial x_2} \\ 1 \end{pmatrix} \tag{6}$$

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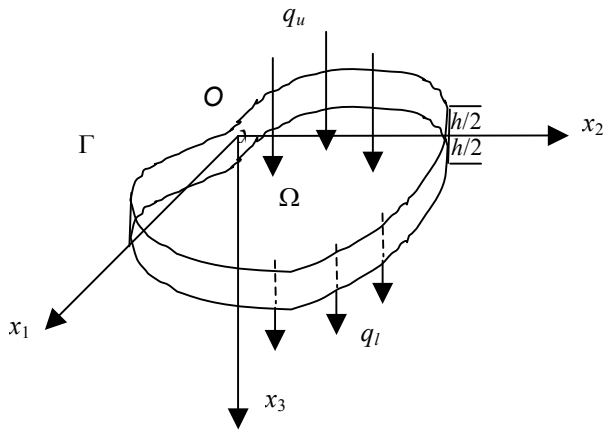


Fig. 1 Configuration of a thick plate

Equation (4) are a coupled system of three second-ordered PDEs for the three unknown function u_j, and therefore, three boundary conditions are required as follows:

$$\begin{cases} \mathbf{B}_1 \cdot \mathbf{u} = b_1(x_1, x_2) \\ \mathbf{B}_2 \cdot \mathbf{u} = b_2(x_1, x_2) \\ \mathbf{B}_3 \cdot \mathbf{u} = b_3(x_1, x_2) \end{cases} \tag{7}$$

where $b_i(x_1, x_2)$ are given functions. For example, the clamped boundary conditions are defined by

$$\begin{cases} \mathbf{B}_{\theta_n} \cdot \mathbf{u} = n_1 u_1 + n_2 u_2 = 0 \\ \mathbf{B}_{\theta_t} \cdot \mathbf{u} = -n_2 u_1 + n_1 u_2 = 0 \\ \mathbf{B}_w \cdot \mathbf{u} = u_3 = 0 \end{cases} \tag{8}$$

and the free boundary conditions are given as

$$\begin{cases} \mathbf{B}_{M_n} \cdot \mathbf{u} = n_1^2 \Upsilon_{11} + 2n_1 n_2 \Upsilon_{12} + n_2^2 \Upsilon_{22} = 0 \\ \mathbf{B}_{M_t} \cdot \mathbf{u} = -n_1 n_2 \Upsilon_{11} + (n_1^2 - n_2^2) \Upsilon_{12} + n_1 n_2 \Upsilon_{22} = 0 \\ \mathbf{B}_{Q_n} \cdot \mathbf{u} = n_1 Q_1 + n_2 Q_2 = 0 \end{cases} \tag{9}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{n} = (n_1, n_2)$ are the general displacement vector and the boundary normal vector, respectively.

3. MFS-DRM FORMULATION

In order to solve the well-posed problem of Eqs. (4) and (7), it is required to apply the principle of superposition as follows:

$$u_j = u_j^h + u_j^p \tag{10}$$

where the particular solution u_j^p satisfies

$$\tilde{L} \begin{pmatrix} u_1^p \\ u_2^p \\ u_3^p \end{pmatrix} = -\tilde{E}q \tag{11}$$

without specifying any boundary condition. In addition, the homogeneous solution u_j^h satisfies

$$\tilde{L} \begin{pmatrix} u_1^h \\ u_2^h \\ u_3^h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

with modified boundary conditions:

$$\begin{cases} \mathbf{B}_1 \cdot \mathbf{u}^h = b_1(x_1, x_2) - \mathbf{B}_1 \cdot \mathbf{u}^p \\ \mathbf{B}_2 \cdot \mathbf{u}^h = b_2(x_1, x_2) - \mathbf{B}_2 \cdot \mathbf{u}^p \\ \mathbf{B}_3 \cdot \mathbf{u}^h = b_3(x_1, x_2) - \mathbf{B}_3 \cdot \mathbf{u}^p \end{cases} \tag{13}$$

where $\mathbf{u}^h = (u_1^h, u_2^h, u_3^h)$ and $\mathbf{u}^p = (u_1^p, u_2^p, u_3^p)$.

In order to apply the DRM for solving the particular solution u_j^p, the loading intensity should be approximated by the APS [28] as follows:

$$\begin{aligned} & q(x_1, x_2) \cong \tilde{q}(x_1, x_2) \\ & = \sum_{n=1}^{N} a_n r_n^{2M} \ln r_n + \sum_{m_1+m_2 \le M}^{m_1+m_2 \le M} b_{(m_1, m_2)} x_1^{m_1} x_2^{m_2} \end{aligned} \tag{14}$$

where a_n and $b_{(m_1, m_2)}$ are unknown coefficients, and r_n is the distance between (x_1, x_2) and the n-th DRM field point, (ξ_{n1}, ξ_{n2}), as depicted in Fig. 2. In order to solve a_n and $b_{(m_1, m_2)}$, Eq. (14) should be collocated on the N DRM field points

$$q(\xi_{l1}, \xi_{l2}) = \sum_{n=1}^{N} a_n r_{ln}^{2M} \ln r_{ln} + \sum_{m_1+m_2 \le M}^{m_1+m_2 \le M} b_{(m_1, m_2)} \xi_{l1}^{m_1} \xi_{l2}^{m_2} \tag{15}$$

with the following constraint conditions

$$\sum_{n=1}^{N} a_n \xi_{n1}^{m_1} \xi_{n2}^{m_2} = 0 \quad \text{for all } m_1, m_2 \ge 0 \text{ and } m_1 + m_2 \le M \tag{16}$$

where r_{ln} is the distance between the l-th and n-th DRM field points. According to Duchon's theory of APS [28], the unknown coefficients, a_n and $b_{(m_1, m_2)}$, can be uniquely determined by Eqs. (15) and (16).

Then, the corresponding particular solution can be approximated by

$$u_j^p \cong \tilde{u}_j(x_1, x_2) = \sum_{n=1}^{N} a_n u_j^M(r_n) + \sum_{m_1+m_2 \le M}^{m_1+m_2 \le M} b_{(m_1, m_2)} u_j^{(m_1, m_2)}(x_1, x_2) \tag{17}$$

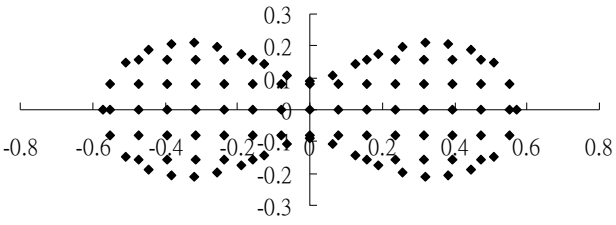


Fig. 2 Field points of the DRM

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where particular solutions $u_j^M(r)$ and $u_j^{(m_1,m_2)}(x_1,x_2)$ are defined, respectively, by

$$\tilde{L}\begin{pmatrix} u_1^M \\ u_2^M \\ u_3^M \end{pmatrix} = -\tilde{E}r^{2M}\ln r \tag{18}$$

and

$$\tilde{L}\begin{pmatrix} u_1^{(m_1,m_2)} \\ u_2^{(m_1,m_2)} \\ u_3^{(m_1,m_2)} \end{pmatrix} = -\tilde{E}x_1^{m_1}x_2^{m_2} \tag{19}$$

Also, the particular solutions of shear force and bending or twisting moment can be approximated, respectively, as

$$Q_\alpha^p \cong \tilde{Q}_\alpha(x_1,x_2) = \sum_{n=1}^{N} a_n Q_\alpha^M(r_n) + \sum_{m_1,m_2\geq 0}^{m_1+m_2\leq M} b_{(m_1,m_2)} Q_\alpha^{(m_1,m_2)}(x_1,x_2) \tag{20}$$

and

$$\Upsilon_{\alpha\beta}^p \cong \tilde{\Upsilon}_{\alpha\beta}(x_1,x_2) = \sum_{n=1}^{N}\alpha_n \Upsilon_{\alpha\beta}^M(r_n) + \sum_{m_1,m_2\geq 0}^{m_1+m_2\leq M} b_{(m_1,m_2)}\Upsilon_{\alpha\beta}^{(m_1,m_2)}(x_1,x_2) \tag{21}$$

where particular solutions, $Q_\alpha^M(r)$, $Q_\alpha^{(m_1,m_2)}(x_1,x_2)$, $\Upsilon_{\alpha\beta}^M(r)$, and $\Upsilon_{\alpha\beta}^{(m_1,m_2)}(x_1,x_2)$, are defined, respectively, by

$$Q_\alpha^M = \frac{D(1-\nu)\lambda^2}{2}\left(u_\alpha^M + \frac{\partial u_3^M}{\partial x_\alpha}\right) \tag{22}$$

$$Q_\alpha^{(m_1,m_2)} = \frac{D(1-\nu)\lambda^2}{2}\left(u_\alpha^{(m_1,m_2)} + \frac{\partial u_3^{(m_1,m_2)}}{\partial x_\alpha}\right) \tag{23}$$

$$\Upsilon_{\alpha\beta}^M(r) = \frac{D(1-\nu)}{2}\left(\frac{\partial u_\alpha^M}{\partial x_\beta} + \frac{\partial u_\beta^M}{\partial x_\alpha} + \frac{2\nu}{1-\nu}\frac{\partial u_\gamma^M}{\partial x_\gamma}\delta_{\alpha\beta}\right) + \frac{\delta_{\alpha\beta}\nu r^{2M}\ln r}{\lambda^2(1-\nu)} \tag{24}$$

$$\Upsilon_{\alpha\beta}^{(m_1,m_2)}(r) = \frac{D(1-\nu)}{2}\left(\frac{\partial u_\alpha^{(m_1,m_2)}}{\partial x_\beta} + \frac{\partial u_\beta^{(m_1,m_2)}}{\partial x_\alpha} + \frac{2\nu}{1-\nu}\frac{\partial u_\gamma^{(m_1,m_2)}}{\partial x_\gamma}\delta_{\alpha\beta}\right) + \frac{\delta_{\alpha\beta}\nu x_1^{m_1}x_2^{m_2}}{\lambda^2(1-\nu)} \tag{25}$$

These particular solutions are derived in the next section.

After the particular solutions are approximated by the DRM, the modified boundary conditions (13) become well-defined. Therefore, it is ready to use the MFS for solving the homogeneous solution. Formally, the homogeneous solution can be approximated by

$$u_j^h \cong \sum_{l=1}^{L}\sum_{k=1}^{3} c_l^k u_{jk}^*(x_1,x_2;s_{l1},s_{l2}) \tag{26}$$

where $u_{jk}^*(x_1,x_2;s_1,s_2)$ are the fundamental solutions defined by

$$\tilde{L}\begin{pmatrix} u_{11}^* & u_{12}^* & u_{13}^* \\ u_{21}^* & u_{22}^* & u_{23}^* \\ u_{31}^* & u_{31}^* & u_{33}^* \end{pmatrix} = -\begin{pmatrix} \delta(x_1-s_1,x_2-s_2) & 0 & 0 \\ 0 & \delta(x_1-s_1,x_2-s_2) & 0 \\ 0 & 0 & \delta(x_1-s_1,x_2-s_2) \end{pmatrix} \tag{27}$$

and (s_{l1}, s_{l2}) are L source points outside the plate domain as described in Fig. 3. In Eq. (26), the $3L$ unknowns c_l^k can be solved by collocated the modified boundary conditions (13) on L boundary points. More detailed of the MFS and explicit fundamental solutions and can be found in [27].

After both the particular solutions and the homogeneous solutions are solved, the principle of superposition (10) can be applied to obtain the desired solution.

4. ANALYTICAL PARTICULAR SOLUTION

Analytical explicit particular solutions, $u_j^M(r)$ and $u_j^{(m_1,m_2)}(x_1,x_2)$, are required to complete the whole procedure of the MFS-DRM. In this section, only the radial particular solutions, $u_j^M(r)$, $Q_\alpha^M(r)$, and $\Upsilon_{\alpha\beta}^M(r)$, will be derived since the polynomial particular solutions, $u_j^{(m_1,m_2)}(x_1,x_2)$, $Q_\alpha^{(m_1,m_2)}(x_1,x_2)$, and $\Upsilon_{\alpha\beta}^{(m_1,m_2)}(x_1,x_2)$ have been derived in [27].

The particular solutions $u_j^M(r)$, governed by Eqs. (18), can be obtained by using the Hörmander operator decomposition technology [35]. First of all, we assume

$$u_j^M(r) = \tilde{L}^{adj}\tilde{E}F^M(r) \tag{28}$$

where $F^M(r)$ is an unknown function to be determined and $\tilde{L}^{adj}$ is the adjoint operator defined by

$$\tilde{L}^{adj}\tilde{L} = \tilde{L}\tilde{L}^{adj} = \tilde{I}\det(\tilde{L}) = \begin{pmatrix} \det(\tilde{L}) & 0 & 0 \\ 0 & \det(\tilde{L}) & 0 \\ 0 & 0 & \det(\tilde{L}) \end{pmatrix} \tag{29}$$

where $\det(\tilde{L})$ is the determinant of $\tilde{L}$. From the above definition, it is clear that both $\det(\tilde{L})$ and $\tilde{L}^{adj}$ can be obtained by linear algebraic manipulation, respectively, as follows:

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$$\det(\tilde{L}) = \frac{D^3\lambda^2(1-\nu)^2}{4}(\nabla^2-\lambda^2)\nabla^2\nabla^2 \tag{30}$$

$$\tilde{L}^{adj} = \begin{pmatrix} 2\nabla^4 - \left[(1+\nu)\nabla^2+(1-\nu)\lambda^2\right]\frac{\partial^2}{\partial x_1^2} & -\left[(1+\nu)\nabla^2+(1-\nu)\lambda^2\right]\frac{\partial^2}{\partial x_1\partial x_2} & (1-\nu)\frac{\partial}{\partial x_1}(\nabla^2-\lambda^2) \\ -\left[(1+\nu)\nabla^2+(1-\nu)\lambda^2\right]\frac{\partial^2}{\partial x_1\partial x_2} & 2\nabla^4 - \left[(1+\nu)\nabla^2+(1-\nu)\lambda^2\right]\frac{\partial^2}{\partial x_2^2} & (1-\nu)\frac{\partial}{\partial x_2}(\nabla^2-\lambda^2) \\ -(1-\nu)\frac{\partial}{\partial x_1}(\nabla^2-\lambda^2) & -(1-\nu)\frac{\partial}{\partial x_2}(\nabla^2-\lambda^2) & \frac{(\nabla^2-\lambda^2)\left[2\nabla^2-(1-\nu)\lambda^2\right]}{\lambda^2} \end{pmatrix} \tag{31}$$

Substituting Eq. (28) into (18) and using Eqs. (30) & (31) result in

$$-\frac{D^3\lambda^2(1-\nu)^2}{4}(\nabla^2-\lambda^2)\nabla^2\nabla^2 F^M(r) = r^{2M}\ln r \tag{32}$$

The solution of Eq. (32) can be found in [17] as follows:

$$F^M(r) = \frac{4F_1^M(r)}{D^3\lambda^4(1-\nu)^2} + \frac{4F_2^M(r)}{D^3\lambda^6(1-\nu)^2} - \frac{4F_3^M(r)}{D^3\lambda^6(1-\nu)^2} \tag{33}$$

with

$$\begin{cases} F_1^M(r) = \dfrac{[M!]^2 r^{2M+4}}{16\left[(M+2)!\right]^2}\left(\ln r - \dfrac{1}{M+1} - \dfrac{1}{M+2}\right) \\ F_2^M(r) = \dfrac{[M!]^2 r^{2M+2}}{4\left[(M+1)!\right]^2}\left(\ln r - \dfrac{1}{M+1}\right) \\ F_3^M(r) = -4^M(M!)^2\dfrac{K_0(\lambda r)}{\lambda^{2M+2}} - (M!)^2\displaystyle\sum_{m=1}^{M+1}\frac{4^{M-m+1}}{\left[(m-1)!\right]^2}\frac{r^{2m-2}\ln r}{\lambda^{2M-2m+4}} - (M!)^2\sum_{m=1}^{M}\frac{r^{2m-2}4^{M-m+1}\displaystyle\sum_{n=m}^{M}n^{-1}}{\lambda^{2M-2m+4}\left[(m-1)!\right]^2} \end{cases} \tag{34}$$

which are the solutions for

$$\begin{cases} \nabla^2\nabla^2 F_1^M(r) = r^{2M}\ln r \\ \nabla^2 F_2^M(r) = r^{2M}\ln r \\ (\nabla^2-\lambda^2)F_3^M(r) = r^{2M}\ln r \end{cases} \tag{35}$$

Equations (28), (31), and (33) are sufficient for obtaining the required analytical particular solutions as:

$$u_\alpha^M = \frac{x_\alpha r^{2M}\left\{\begin{matrix} 4(M+1)(M+2)^2\nu + (3M+5)\lambda^2 r^2(1-\nu) \\ -2(M+1)(M+2)\left[4(M+1)(M+2)\nu + (1-\nu)\lambda^2 r^2\right]\ln r \end{matrix}\right\}}{16(M+1)^3(M+2)^2 D\lambda^2(1-\nu)} \tag{36}$$

$$u_3^M = \frac{r^{2M+2}\left\{\begin{matrix} 4(M+2)^3(2-\nu) - (2M+3)\lambda^2 r^2(1-\nu) \\ -(M+1)(M+2)\left[4(M+2)^2(2-\nu) - (1-\nu)\lambda^2 r^2\right]\ln r \end{matrix}\right\}}{16(M+1)^3(M+2)^3 D\lambda^2(1-\nu)} \tag{37}$$

Then, the corresponding shear force and bending or twisting moment can be obtained by using Eqs. (22) and (24), respectively, as follows:

$$Q_\alpha^M = \frac{x_\alpha r^{2M}\left\{1-2(M+1)\ln r\right\}}{4(M+1)^2} \tag{38}$$

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$$\Upsilon^{M}_{\alpha\beta}=\frac{\delta_{\alpha\beta}r^{2M}\left\{\begin{array}{l}(1-\nu)\left(4(M+1)(M+2)^2\nu+\left[3M+5+(4M^2+13M+11)\nu\right]\lambda^2r^2\right)\\-2(M+1)(M+2)\left[4(M+1)(M+2)\nu\left(1+(2M+1)\nu\right)+(1-\nu)\left(1+(2M+3)\nu\right)\lambda^2r^2\right]\ln r\end{array}\right\}}{16(M+1)^3(M+2)^2\lambda^2(1-\nu)}$$
$$+\frac{x_\alpha x_\beta r^{2M-2}\left\{-4(M+2)^2\nu+(2M+3)(1-\nu)\lambda^2r^2-2(M+1)(M+2)\left(4M(M+2)\nu+(1-\nu)\lambda^2r^2\right)\ln r\right\}}{8(M+1)^2(M+2)^2\lambda^2} \tag{39}$$

In practical implementation, M should be set larger than two since $\Upsilon^{1}_{\alpha\beta}$ is not defined as $r\to 0$. In addition, the following limit values can be used in the implementation.

$$u_i^M=Q_\alpha^M=\Upsilon^M_{\alpha\beta}=0 \quad \text{as} \quad r\to 0 \quad \text{when} \quad M\geq 2 \tag{40}$$

5. NUMERICAL RESULTS

In order to validate the proposed MFS-DRM formulation, three numerical cases are considered. Typically, D = λ = 2 and ν = 0.3 are selected in these numerical studies. Furthermore, in the MFS, sixty boundary points (L = 60) are uniformly located and source points are stipulated on a ten times larger artificial boundary as depicted in Fig. 3. This typical setting has been done without thorough explorations since we mainly concentrate on the validation of the derived particular solutions in this study. In order to analyze the accuracy of numerical results, the root mean square errors (RMSEs) are introduced as

$$u_{\mathrm{RMSE}}=\sqrt{\frac{\sum_{i=1}^{3}\sum_{l=1}^{K}(u^{NUM}{}_{i,l}-u^{ANA}{}_{i,l})^2}{3K}} \tag{41}$$

where $u^{NUM}{}_{i,l}$ is the numerical solution obtained by the MFS-DRM at the l-th point of the K considered positions and $u^{ANA}{}_{i,l}$ is the corresponding analytical solution of the problem. Here, K shall be chosen to be sufficient large to ensure the utilization of the RMSEs.

On the other hand, a usual way to validate a numerical method is to set up some analytical solution without the enforcement of the boundary conditions. In our numerical experiments, we solve the following Reissner plate problem:

$$\tilde{L}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}=-D\tilde{E}\left\{\sin x_1(\sinh x_2+x_2^2\sinh x_2)\right\} \tag{42}$$

The analytical solution of the problem is

$$u_1=\frac{\cos x_1\left\{-6\nu\left(-8x_2+(1+4x_2+2x_2^2)\cosh x_2-2(1+x_2)\sinh x_2\right)-\lambda^2(1-\nu)\left(48x_2-6(1+x_2)^2\cosh x_2+(9+9x_2+6x_2^2+2x_2^3)\sinh x_2\right)\right\}}{48\lambda^2(1-\nu)} \tag{43}$$

$$u_2=\frac{\sin x_1\left\{-6\nu\left(-8+2(1+x_2)\cosh x_2+(-1+4x_2+2x_2^2)\sinh x_2\right)-\lambda^2(1-\nu)\left(48+(-3-3x_2+6x_2^2+2x_2^3)\cosh x_2+3\sinh x_2\right)\right\}}{48\lambda^2(1-\nu)} \tag{44}$$

$$u_3=\frac{\sin x_1\left\{-6(2-\nu)\left(-8x_2+(1+4x_2+2x_2^2)\cosh x_2-2(1+x_2)\sinh x_2\right)-\lambda^2(1-\nu)\left(-48x_2+6(1+x_2)^2\cosh x_2-(9+9x_2+6x_2^2+2x_2^3)\sinh x_2\right)\right\}}{48\lambda^2(1-\nu)} \tag{45}$$

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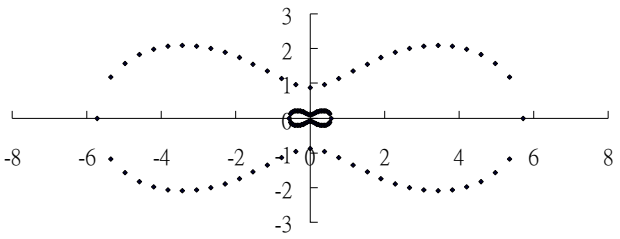


Fig. 3 Boundary and source points of the MFS

Example 1: We first solve a problem defined in $\left[-\frac{1}{2},\frac{1}{2}\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]$ subjected to clamped boundary conditions, which are set up according to the analytical solutions in Eqs. (43) ~ (45).

Table 1 gives the RMSEs for different M and N in which nice accuracy can be observed. In addition, it can be observed that the case of more DRM field points and higher APS order gives better accuracy. However, the coding for the case of $M = 4$ is much more difficult than that of $M = 2$.

This problem has been solved in the first author's previous study [35] using the Chebyshev polynomials. The accuracy of the Chebyshev polynomials is much higher than that of the results obtained by the present method. However, that method is only applicable to cases when the arbitrary loading is given by an explicit function. For approximating an arbitrary loading measured only on scattered points, the present method provides a very good solution.

Example 2: Then, the same problem is solved by changing the boundary conditions to free boundary conditions on one side.

Table 2 addresses the RMSEs of the numerical results in this example. In this example, only the solutions obtained by the APS with $M = 4$ are sufficient accurate. The solutions of $M = 2$ perform poorly. These results can be expected since free boundary conditions are imposed. Also, the coding for the case of $M = 4$ is more complicated than that for $M = 2$.

Example 3: Finally, we solve a clamped Reissner plate problem on a peanut defined by

$$r(\theta) = 0.4\sqrt{\cos 2\theta + \sqrt{1.1 - \sin^2 2\theta}} \qquad 0 \le \theta \le 2\pi \qquad (46)$$

where (r, θ) is the usual polar coordinate. Table 3 gives the RMSEs for this problem, which also performs nicely. This result demonstrates the applicability of the proposed method for irregular domains.

6. DISCUSSIONS AND CONCLUSIONS

In this study, the MFS-DRM formulation is developed for solving problems of Reissner plate in bending under arbitrary loading. In the solution procedure, the arbitrary loading term is approximated by augmented polyharmonic splines, which are constructed by polyharmonic splines and monomials. Then, the analytical particular solutions corresponding to the polyharmonic splines are derived. In the derivations, the three coupled second-ordered governing equations are transformed into a single sixth-order PDE by the Hörmander operator decomposition technology. Numerical experiments were carried out to validate the derived particular solutions with acceptable accuracy.

Table 1 The RMSEs for example 1

M \ N	$N = 5 \times 5$	$N = 10 \times 10$	$N = 15 \times 15$	$N = 20 \times 20$
$M = 2$	2.04E-05	9.16E-06	2.92E-06	1.34E-06
$M = 4$	6.47E-06	3.10E-07	3.98E-08	1.40E-08

Table 2 The RMSEs for example 2

M \ N	$N = 5 \times 5$	$N = 20 \times 20$	$N = 25 \times 25$	$N = 30 \times 30$
$M = 2$	1.07E-01	2.92E-02	2.52E-02	2.42E-02
$M = 4$	2.62E-03	1.57E-04	3.39E-05	2.01E-05

Table 3 The RMSEs for example 3

M \ N	$N = 25$	$N = 101$	$N = 225$	$N = 399$
$M = 2$	7.30E-05	5.21E-05	1.91E-05	1.01E-05
$M = 4$	1.09E-05	1.84E-07	1.46E-07	1.45E-07

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