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Veech's dichotomy and the lattice property

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Abstract. Veech showed that if a translation surface has a stabilizer which is a lattice in $\mathrm{SL}(2, \mathbb{R})$, then any direction for the corresponding constant slope flow is either completely periodic or uniquely ergodic. We show that the converse does not hold: there are translation surfaces that satisfy Veech's dichotomy but for which the corresponding stabilizer subgroup is not a lattice. The construction relies on work of Hubert and Schmidt.

1. *Introduction*

In a celebrated paper [**Ve1**], Veech gave precise information about the billiard flow in regular polygons by establishing a link with the properties of the affine automorphism group of the related translation surfaces. Fix a regular polygon P in the plane. A starting point in P and an angle θ determine a billiard trajectory. Veech showed that for any angle θ, one of two possibilities occurs:

(1) all trajectories in direction θ either are periodic or, if they hit a vertex in forward time, must also hit a vertex in backward time;
(2) no trajectory in direction θ hits a vertex in forward and backward time, and all infinite trajectories are uniformly distributed.

A regular polygon has the property that its angles are rational multiples of π. We call a polygon with this property a rational polygon. Associated to each rational polygon there is a translation surface, and the billiard trajectories correspond to geodesics on this translation surface [**FoKi**]; general references on translation surfaces are [**Vo, MaTa**]. A direction in which all infinite trajectories are dense is said to be *minimal*. A direction in which all infinite trajectories are uniformly distributed is said to be *uniquely ergodic*. A direction in which all orbits are periodic or saddle connections is said to be *completely periodic*. A translation surface satisfies the *Veech dichotomy* if each direction is either completely periodic or uniquely ergodic.

Veech found a sufficient condition for the Veech dichotomy in terms of the affine automorphism group of the translation surface. An affine automorphism of a translation

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surface is a selfmap with constant derivative. Mapping an automorphism to its derivative gives a homomorphism from the group of affine automorphisms to $G = \mathrm{SL}(2, \mathbb{R})$. The image of the affine automorphism group under this map is called the Veech group. Veech showed that the Veech dichotomy holds when the Veech group is a lattice in G. We call such translation surfaces *lattice surfaces*, although they are also referred to as *Veech surfaces* in the literature. Examples of lattice polygons known prior to Veech include the square (see [**HW**]) and the arithmetic examples in [**Gu**]. At the current time there are many known examples of translation surfaces which have the lattice property and also many known examples of translation surfaces which do not have the lattice property. This raises the question of whether there might be other ways to establish the Veech dichotomy, or if the lattice property is in fact equivalent to the Veech dichotomy. The purpose of this note is to show the following.

THEOREM 1. *There is a translation surface which satisfies the Veech dichotomy but is not a lattice surface.*

The proof of Theorem 1 relies on the previous work of several authors. In the remainder of this introduction we will review these ideas and reduce the proof of the theorem to some observations about loci of branched covers.

A useful technique for constructing new translation surfaces from old is the branched cover construction. If $\pi : \widetilde{S} \to S$ is a branched cover and S has a translation structure, then $\widetilde{S}$ has a natural translation structure where the charts are obtained by pulling back the charts for S. The inverse images of cone points of S are cone points of $\widetilde{S}$ and, in addition, the ramification points of the map π become cone points for $\widetilde{S}$. The relation between the Veech group of S and that of $\widetilde{S}$ was studied by Hubert and Schmidt [**HuSc**], building on previous work of several authors (see [**Ve2, GuJu, Vo, GuHuSc**]). Hubert and Schmidt found examples where S is a lattice surface but the Veech group of $\widetilde{S}$ is not finitely generated. For their construction, they defined the notion of a *non-periodic connection point* on a translation surface; this is a point p which has an infinite orbit under the Veech group, and which has the property that if one continues any segment from a cone point to p, then one must reach a singularity. Hubert and Schmidt showed that there are surfaces in genus two and three containing non-periodic connection points, and that if $\pi : \widetilde{S} \to S$ is branched over non-periodic connection points in S, then the Veech group of $\widetilde{S}$ is infinitely generated. They also observed that these examples satisfy the following weak form of the Veech dichotomy: every saddle connection direction is completely periodic. According to [**FoKi**], in the remaining directions the flow is minimal. These examples constitute the starting point for our analysis. To summarize, we have the following.

PROPOSITION 2. (Hubert–Schmidt) *There are lattice surfaces of genus two and three that contain infinitely many non-periodic connection points. If X is such a translation surface and $\widetilde{X}$ is a translation surface obtained by forming a cover of X branched only at non-periodic connection points, then $\widetilde{X}$ is not a lattice surface, yet has the property that any direction is either completely periodic or minimal.*

We will show that, provided the branching takes place *over a single point*, the flow in the minimal directions is in fact uniquely ergodic and thus these surfaces satisfy the full

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Veech dichotomy even though their Veech groups are not lattices. The next proposition shows that such a surface exists.

PROPOSITION 3. *For a surface S of genus $g \geq 1$ and $d \geq 3$, there is a surface $\widetilde{S}$ and a branched covering map $\pi : \widetilde{S} \to S$ of degree d where the branch locus is a single point.*

This proposition is not new (see e.g. [**FaKr**, §IV.9]), but we include a direct proof for the reader's convenience. Taking $g = 2$ and $d = 3$, the construction gives a surface of genus 5. Fixing the branch point at an aperiodic connection point as in Proposition 2, we obtain the surface of smallest genus for which our arguments show that it satisfies the Veech dichotomy without the lattice property. It would be interesting to know the minimal genus in which this phenomenon can occur.

In order to prove that minimal directions on such a branched cover are uniquely ergodic, we will use a fundamental result of Masur [**Ma**]. Masur's theorem involves a moduli space for translation structures which is most easily described in the language of quadratic differentials. A translation structure on a surface S gives rise to a conformal structure on S. If we are given a Riemann surface, then the information needed to reconstruct the translation structure is contained in a holomorphic quadratic differential with finite area. (The particular quadratic differentials that come from translation structures correspond to squares of Abelian differentials.) A useful way to construct a moduli space for quadratic differentials on a surface of a fixed genus g is to start with the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. For each surface S in $\mathcal{M}_g$, the collection of holomorphic quadratic differentials on S with finite area is a vector space. The unit-area quadratic differentials correspond to a sphere in this vector space. We define $\mathcal{Q}_g$ to be the total space of this sphere bundle over $\mathcal{M}_g$. Also of importance for us will be $\mathcal{M}_{g,b}$ and $\mathcal{Q}_{g,b}$, which are, respectively, the moduli space of and the quadratic differentials over Riemann surfaces of genus g with b marked points.

There is a natural action of G on $\mathcal{Q}_g$. We set

$$a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \quad \text{and} \quad r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

A trajectory $\{a_t x : t \geq 0\}$ is called *divergent* if the map $t \mapsto a_t x$ is proper; in other words, a divergent trajectory eventually leaves any compact subset of X. Masur established a relation between the unique ergodicity of the translation flow in direction θ and the dynamics of the geodesic trajectory $\{a_t r_\theta(q) : t \geq 0\}$, which is stated as follows.

PROPOSITION 4. (Masur) *If $q \in \mathcal{Q}$ and the translation flow in the direction θ is not uniquely ergodic, then $\{a_t r_\theta(q) : t \geq 0\}$ is divergent.*

Veech used Masur's theorem in his proof that the lattice property implies the Veech dichotomy. When q is a lattice surface with Veech group Γ, we can identify the G-orbit of q with the coset space G/Γ, and we observe that the inclusion from the coset space to $\mathcal{Q}_g$ is a proper map. This coset space can be viewed as the unit tangent bundle of the hyperbolic orbifold $M = (SO(2, \mathbb{R}))\backslash(G/\Gamma)$, and we can identify the flow a_t with the geodesic flow on this tangent bundle. A hyperbolic orbifold of finite volume can be expressed as a union of a compact set and a finite number of cusps. Veech showed that

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the saddle connection directions correspond to geodesics which diverge to cusps and that all other geodesics return infinitely often to the compact piece. In particular, for lattice surfaces Masur's condition is both necessary and sufficient for unique ergodicity. We summarize this discussion in the following proposition.

PROPOSITION 5. (Veech) *For a lattice surface q, there is a compact subset $C \subset G/\Gamma$ such that for every minimal direction θ, $a_t r_\theta(q)$ returns to C infinitely often.*

Let $\pi : \widetilde{S} \to S$ be a branched cover and let $\Sigma \subset S$ be the branch points. Below we will introduce the 'locus of branched covers' corresponding to π. This is a moduli space parametrizing the Riemann surfaces $\widetilde{X}$ with quadratic differentials, for which there is a Riemann surface X and a holomorphic map $p : \widetilde{X} \to X$ pulling back the quadratic differential on X to that on $\widetilde{X}$. Furthermore, the cover p is *topologically equivalent* to the cover π in that there are diffeomorphisms $h : S \to X$ and $\widetilde{h} : \widetilde{S} \to \widetilde{X}$ so that $h\pi = p\widetilde{h}$. We will need the following three propositions.

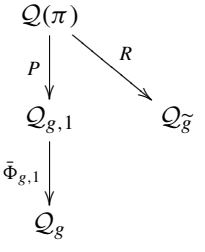
PROPOSITION 6. *For a branched cover $\pi : \widetilde{S} \to S$ with b branch points, the collection of holomorphic branched covers that are topologically equivalent to π can be parametrized by a topological space $\mathcal{Q}(\pi)$ with a G-action. We have continuous maps $P : \mathcal{Q}(\pi) \to \mathcal{Q}_{g,b}$ and $R : \mathcal{Q}(\pi) \to \mathcal{Q}_{\tilde{g}}$ which are G-equivariant.*

PROPOSITION 7. *The map P is proper.*

There is a surjective continuous 'forgetful' map $\bar{\Phi}_{g,n} : \mathcal{Q}_{g,n} \to \mathcal{Q}_g$ which maps a surface together with a collection of marked points to the surface alone.

PROPOSITION 8. *The forgetful map $\bar{\Phi}_{g,1} : \mathcal{Q}_{g,1} \to \mathcal{Q}_g$ is proper for $g \geq 2$.*

Proof of Theorem 1 assuming Propositions 3, 6, 7 and 8. Using Propositions 2 and 3, let $\pi : \widetilde{X} \to X$ be a cover branched at a single non-periodic connection point x. Let θ be a direction for which the flow on $\widetilde{X}$ is minimal. We need to show that the flow in direction θ is in fact uniquely ergodic. By Masur's theorem we need to show that $\{a_t r_\theta(\widetilde{X}) : t \geq 0\}$ is not divergent when viewed as a path in $\mathcal{Q}_{\tilde{g}}$. The pair $(\widetilde{X}, X)$ corresponds to a point in $\mathcal{Q}(\pi)$ which maps by P to the surface X in $\mathcal{Q}_{g,1}$ with the marked point x.



This pair maps via $\bar{\Phi}_{g,1}$ to the surface X in $\mathcal{Q}_g$. The path $\{a_t r_\theta(X) : t \geq 0\}$ returns infinitely often to a compact set $C \subset \mathcal{Q}_g$ by Proposition 5. The map P is proper by Proposition 7, and $\bar{\Phi}_{g,1}$ is proper by Proposition 8. Therefore $C' = (\bar{\Phi}_{g,1} \circ P)^{-1}(C)$ is compact and the path $a_t r_\theta((\widetilde{X}, X))$ returns infinitely often to C'. Applying R, we see that $a_t r_\theta(\widetilde{X}) \subset \mathcal{Q}_{\tilde{g}}$ returns infinitely often to the compact set $R(C')$, so it follows from Masur's theorem that the flow in direction θ on $\widetilde{X}$ is uniquely ergodic. □

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To explain intuitively the maps in Proposition 6 and why they should be proper, note that we might naively expect to be able to define a map $\mathcal{Q}_{g,b} \to \mathcal{Q}_{\tilde{g}}$ by 'placing the branch points of π at the marked points and pulling back q via π'. This definition may not make sense, however, because of the different equivalence relations imposed in the definitions of $\mathcal{Q}_{g,b}$ and $\mathcal{Q}_{\tilde{g}}$. To make this idea meaningful, we resolve the ambiguity by passing to finite covers. An intuitive explanation of why the map in Proposition 8 is proper is that the forgetful map 'forgets the location of the marked point', and thus the fiber is the compact underlying surface; when $b \geq 2$, the fiber of $\bar{\Phi}_{g,b}$ is the b-fold power of the surface with the diagonals removed, and is not compact. This explains why it is important to branch over a single point. By recent work of Cheung *et al* [**CHM**], surfaces obtained by degree-two covers of genus-two surfaces, which necessarily have more than one branch point, admit minimal non-uniquely-ergodic directions.

In the remainder of the paper, we make this intuitive picture rigorous by constructing moduli spaces of branched coverings of quadratic differentials by quadratic differentials and establishing the properties we need. The paradigm that we follow for the construction of moduli spaces is quite standard. Moduli spaces of branched covers of Riemann surfaces were constructed by [**Nat**] using a similar paradigm. A novel feature of our construction is that we use the language of orbifold fibrations which is very well suited to our purposes. Along the way we will prove Propositions 3, 6, 7 and 8.

2. *Orbifolds and moduli spaces*

An issue that arises in the construction of moduli spaces is the fact that some of the spaces which arise are quotients of non-free group actions. This problem is often resolved by considering Teichmüller spaces, which play the role of universal covering spaces of moduli spaces; for more information on the general construction of Teichmüller spaces, see [**E**]. However, such an approach will not work for us because we are interested in showing that certain maps are proper, and this property is not preserved after passing to the universal cover. Instead, we shall make use of the notions of *orbifold* and *orbifold fibration*, as introduced in [**Th**, §13.2].

First, we recall these notions. An orbifold is a topological space X with a system of 'orbifold charts' $\phi_j : U_j \to X$ where, for each index j, there is an open set U_j in Euclidean space and a finite group Γ_j acting smoothly on U_j such that ϕ_j induces an injective map $U_j/\Gamma_j \to X$ (the complete definition prescribes how these charts fit together; see [**Th**] or the definition of orbifold in Wikipedia). Orbifolds are produced by group actions on manifolds.

PROPOSITION 9. *If Γ acts properly discontinuously on a manifold M, then the quotient has an orbifold structure.*

See [**Th**, Proposition 13.2.1]. A key point in the proof is that for any point $x \in M$, we can find a neighborhood U_x so that if $\gamma \in \Gamma$ satisfies $\gamma(U_x) \cap U_x \neq \emptyset$, then $\gamma(x) = x$ and $\gamma(U_x) = U_x$. The orbifold charts arise by composing the inclusion $\iota : U_x \to M$ with the projection $M \to M/\Gamma$. If I_x denotes the isotropy group of x, then I_x is finite and the chart induces a homeomorphism from U_x/I_x to its image in M/Γ.

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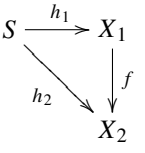
A map $p : X \to Y$ between orbifolds is an *orbifold fibration with generic fiber* F if each point $y \in Y$ has a neighborhood $U = \widetilde{U}/\Lambda$ so that $p^{-1}(U) = (\widetilde{U} \times F)/\Lambda$, where the action of Λ on $\widetilde{U} \times F$ preserves the factors and the action of Λ on the first factor agrees with the previous action of Λ on $\widetilde{U}$; see [**Th**, Definition 13.4.1]. An orbifold covering space is an orbifold fibration for which the fiber is discrete.

PROPOSITION 10. *Suppose* $p : E \to B$ *is an orbifold fibration with generic fiber* F, *and that* Γ *acts on* E *and* B *so that* p *is equivariant. Assume that* Γ *acts properly discontinuously on* E. *Let* Γ_0 *be the subgroup of* Γ *acting trivially on* B, *and assume that* Γ/Γ_0 *acts properly discontinuously on* B *and that the induced map* $q : E/\Gamma_0 \to B$ *is an orbifold fibration. Then the induced map* $\bar{p} : E/\Gamma \to B/\Gamma$ *is an orbifold fibration with generic fiber* F/Γ_0.

Proof. For each $x \in B$ we have a neighborhood U_x as discussed above. By assumption, the projection $q : E/\Gamma_0 \to B$ is a fibration. Let $V_x \subset E/\Gamma_0$ denote the set $q^{-1}(U_x)$. By replacing U_x with a smaller set if necessary, we can assume that V_x is diffeomorphic to $U_x \times (F/\Gamma_0)$. Let I_x denote the isotropy group of x in Γ/Γ_0. The group I_x is finite and acts on $U_x \times (F/\Gamma_0)$, preserving the projection onto the first factor. Since the action of Γ/Γ_0 on B is faithful, a generic point has a trivial stabilizer and so the generic fiber is F/Γ_0. □

We will be dealing with closed surfaces with marked points. In order to apply a result of [**EE**] in Proposition 11, we assume that when we are dealing with the torus, at least one point is marked, and when we are dealing with the sphere, at least three points are marked.

We begin by sketching the construction of the standard moduli space, following [**E**]. Let S be a fixed smooth surface of genus g. Consider the collection of pairs (X, h) where X is a Riemann surface of genus g and h is a diffeomorphism from S to X. Two pairs (X_1, h_1) and (X_2, h_2) are equivalent if there is a holomorphic map $f : X_1 \to X_2$ such that $fh_1 = h_2$.



Let $\mathrm{Conf}(S)$ denote the set of equivalence classes of pairs. We can identify this set with the set of conformal structures on S. Let $\mathrm{Diffeo}(S)$ denote the group of orientation-preserving diffeomorphisms of S. There is a natural action of $\mathrm{Diffeo}(S)$ on the set $\mathrm{Conf}(S)$, where the diffeomorphism $\gamma \in \mathrm{Diffeo}(S)$ sends (X, h) to $(X, h\gamma)$. Set-theoretically, the moduli space $\mathcal{M}_g = \mathcal{M}(S)$ is the quotient $\mathrm{Conf}(S)/\mathrm{Diffeo}(S)$.

In fact, the space $\mathcal{M}(S)$ has an orbifold structure, but this structure is not apparent from the quotient construction above. In order to establish the existence of the orbifold structure, it is useful to form the quotient in two stages. Let $\mathrm{Diffeo}_0(S) \subset \mathrm{Diffeo}(S)$ denote the normal subgroup of diffeomorphisms isotopic to the identity. The quotient $\mathrm{Conf}(S)/\mathrm{Diffeo}_0(S)$ is Teichmüller space, and will be denoted by $\mathcal{T}_g$ or $\mathcal{T}(S)$. Teichmüller space has a natural smooth manifold structure and, in fact, is

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diffeomorphic to $\mathbb{R}^n$ (see [**E**, §11]). We define the mapping class group to be $\mathrm{Mod}(S) = \mathrm{Diffeo}(S)/\mathrm{Diffeo}_0(S)$. The group $\mathrm{Mod}(S)$ acts properly discontinuously on $\mathcal{T}(S)$. We can recover the moduli space as $\mathcal{M}(S) = \mathcal{T}(S)/\mathrm{Mod}(S)$. Since it is the quotient of a smooth manifold by a properly discontinuous smooth action, $\mathcal{M}(S)$ has the structure of an orbifold (see Proposition 9).

A modification of this construction produces moduli spaces of Riemann surfaces with finite sets of marked points [**E**, §5]. In this case, we start with a surface S and a finite set $\Sigma \subset S$ with cardinality b. Let $\mathrm{Diffeo}(S, \Sigma) \subset \mathrm{Diffeo}(S)$ be the subgroup of diffeomorphisms that fix Σ. Let $\mathrm{Diffeo}_0(S, \Sigma)$ be the normal subgroup consisting of diffeomorphisms which are isotopic to the identity relative to Σ. We define $\mathcal{T}(S, \Sigma)$ to be $\mathrm{Conf}(S)/\mathrm{Diffeo}_0(S, \Sigma)$. Once again, $\mathcal{T}(S, \Sigma)$ has a natural smooth manifold structure and is diffeomorphic to $\mathbb{R}^m$. Define $\mathrm{Mod}(S, \Sigma) = \mathrm{Diffeo}(S, \Sigma)/\mathrm{Diffeo}_0(S, \Sigma)$. $\mathrm{Mod}(S, \Sigma)$ acts properly discontinuously on $\mathcal{T}(S, \Sigma)$ and we let

$$\mathcal{M}(S, \Sigma) = \mathcal{T}(S, \Sigma)/\mathrm{Mod}(S, \Sigma).$$

This space has a natural orbifold structure. The topology of the spaces $\mathcal{T}(S, \Sigma)$ and $\mathcal{M}(S, \Sigma)$ and the isomorphism type of the groups $\mathrm{Mod}(S, \Sigma)$ all depend only on g and b, hence are usually written as $\mathcal{T}_{g,b}$, $\mathcal{M}_{g,b}$ and $\mathrm{Mod}_{g,b}$.

The inclusion $\mathrm{Diffeo}(S, \Sigma) \subset \mathrm{Diffeo}(S)$ defines a surjective map

$$\Phi_{g,b} : \mathcal{M}_{g,b} = \mathrm{Conf}(S)/\mathrm{Diffeo}(S, \Sigma) \to \mathcal{M}_g = \mathrm{Conf}(S)/\mathrm{Diffeo}(S).$$

We call $\Phi_{g,b}$ the *forgetful map*. This map can be shown to be an orbifold fibration in general, but we only need the special case described in the following result.

PROPOSITION 11. *For $g \geq 2$, the map $\Phi_{g,1} : \mathcal{M}_{g,1} \to \mathcal{M}_g$ is an orbifold fibration and the generic fiber is the surface with genus g.*

Proof. Let S be a surface of genus g, and let Σ be a subset with a single element z. Let $\Phi = \Phi_{g,1}$ and let $\widetilde{\Phi}$ be the natural map from $\mathcal{T}_{g,1}$ to $\mathcal{T}_g$. The map $\widetilde{\Phi}$ was shown in [**B**] to be a fibration with fiber $\widetilde{S}$.

Since the inclusion of $\mathrm{Diffeo}(S, \Sigma)$ into $\mathrm{Diffeo}(S)$ maps $\mathrm{Diffeo}_0(S, \Sigma)$ into $\mathrm{Diffeo}_0(S)$, it defines a homomorphism from $\mathrm{Mod}_{g,b}$ to Mod_g. This homomorphism is surjective, since any diffeomorphism is isotopic to one which fixes Σ. Write $\Gamma = \mathrm{Mod}_{g,1}$ and let Γ_0 denote the kernel of the homomorphism from Γ to Mod_g. An element $\gamma_0 \in \Gamma_0$ is represented by a diffeomorphism which fixes z and is isotopic to the identity. The track of the point z under such an isotopy is a loop based at z, so each isotopy from γ_0 to the identity gives rise to an element of $\pi_1(S)$. The group $\mathrm{Diffeo}_0(S)$ is contractible by [**EE**], so any two isotopies from γ_0 to the identity are connected by a one-parameter family of isotopies from γ_0 to the identity. It follows that the corresponding element of $\pi_1(S)$ is well-defined, and we obtain a homomorphism from Γ_0 to $\pi_1(S)$. Using the isotopy extension theorem, we can show that this map is an isomorphism. If we act by Γ_0, then the induced projection on the quotient space $\mathcal{T}_{g,1}/\Gamma_0 \to \mathcal{T}_g$ is a fibration with fiber S (see [**EF**]), known as the universal curve. According to Proposition 10, the fibration $\Phi : \mathcal{T}_{g,1}/\Gamma \to \mathcal{T}_g/\Gamma$ is an orbifold fibration with generic fiber S. □

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We can modify the previous construction to produce the moduli space of unit-area quadratic differentials, $\mathcal{Q}(S) = \mathcal{Q}_g$. Consider pairs (X, h) where X is a Riemann surface with a holomorphic quadratic differential of area one and $h : S \to X$ is a diffeomorphism. Two pairs (X_1, h_1) and (X_2, h_2) are equivalent if there is a holomorphic map $f : X_1 \to X_2$ such that $fh_1 = h_2$ and *f preserves the quadratic differential*. We denote by $\mathrm{Quad}(S)$ the set of equivalence classes. There is a natural action of $\mathrm{Diffeo}(S)$ on $\mathrm{Quad}(S)$, and the quotient (as a set) is $\mathcal{Q}_g$. As before, it is useful to deal with an intermediate quotient. We define $\widetilde{\mathcal{Q}}_g = \mathrm{Quad}(S)/\mathrm{Diffeo}_0(S)$ and call it the Teichmüller space of quadratic differentials. This space can be identified with the cotangent bundle of Teichmüller space, and the projection map of the cotangent bundle can be identified with the map which takes a marked quadratic differential to its underlying marked Riemann surface. The collection of unit-area quadratic differentials can be identified with the unit sphere bundle in this cotangent bundle with respect to an appropriate metric [**E**]. Since the Teichmüller space of quadratic differentials is the total space of a fiber bundle with smooth fiber and smooth base, it is a smooth manifold.

The group $\mathrm{Mod}(S)$ acts smoothly and properly discontinuously on $\widetilde{\mathcal{Q}}_g$. We define the moduli space of quadratic differentials,

$$\mathcal{Q}_g = \widetilde{\mathcal{Q}}_g/\mathrm{Mod}(S).$$

This space has an orbifold structure by Proposition 9.

Following the same paradigm, we can define versions of these spaces for surfaces with finite sets of marked points. Note that we are not assuming any relation between the marked points and the quadratic differential. In particular, the collection of marked points need not contain the collection of zeros and poles of the quadratic differential. We define $\widetilde{\mathcal{Q}}_{g,b} = \widetilde{\mathcal{Q}}(S, \Sigma)$ to be the quotient $\mathrm{Quad}(S, \Sigma)/\mathrm{Diffeo}_0(S, \Sigma)$; this space has a manifold structure. We can see this by looking at the projection p_b from $\widetilde{\mathcal{Q}}(S, \Sigma)$ to $\widetilde{\mathcal{T}}(S, \Sigma)$ which takes a quadratic differential on a marked surface to the underlying marked surface. This map is the projection map of a fiber bundle, where the fiber is the (topological) sphere of unit-area quadratic differentials on the surface.

The space

$$\mathcal{Q}_{g,b} = \widetilde{\mathcal{Q}}(S, \Sigma)/\mathrm{Mod}(S, \Sigma)$$

has an orbifold structure. We define the forgetful map $\bar{\Phi}_{g,b} : \mathcal{Q}_{g,b} \to \mathcal{Q}_g$ using the inclusion $\mathrm{Mod}(S, \Sigma) \subset \mathrm{Mod}(S)$.

PROPOSITION 12. *The natural maps $p_0 : \mathcal{Q}_g \to \mathcal{M}_g$ and $p_b : \mathcal{Q}_{g,b} \to \mathcal{M}_{g,b}$ are orbifold fibrations with the sphere as the generic fiber. They satisfy $p_0\bar{\Phi}_{g,b} = \Phi_{g,b}p_b$.*

Proof. The projection p_0 is mapping class group equivariant. The quotient spaces are the moduli space of unit-area quadratic differentials and the moduli space of Riemann surfaces. There is an induced map between the quotient spaces, which is just the map that projects a quadratic differential to its underlying Riemann surface. According to Proposition 10, the induced projection map on the quotient spaces is an orbifold fibration with the sphere as generic fiber (in the terminology of Proposition 10 the group Γ_0 is trivial). The argument for p_b is the same. The second assertion is clear from the definitions. □

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Orbifold fibrations are useful for us because of the following result, whose proof is immediate from the definitions and therefore omitted.

PROPOSITION 13. *An orbifold fibration with compact generic fiber is a proper map.*

We now have the tools we need for the following.

Proof of Proposition 8.

$$\begin{array}{ccc} \mathcal{Q}_{g,1} & \xrightarrow{\bar{\Phi}_{g,1}} & \mathcal{Q}_g \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_0} \\ \mathcal{M}_{g,1} & \xrightarrow{\Phi_{g,1}} & \mathcal{M}_g \end{array}$$

The map p_1 mapping quadratic differentials to their underlying Riemann surfaces is proper because it is an orbifold fibration with the sphere as generic fiber. By Proposition 11, the fiber of the forgetful map $\Phi_{g,1}$ is the surface of genus g. Since the fiber is compact, the map $\Phi_{g,1}$ is proper. Thus $\Phi_{g,1}p_1$ is proper and, because the diagram commutes, so is $p_0\bar{\Phi}_{g,1}$. In particular, $\bar{\Phi}_{g,1}$ is proper. □

3. *The moduli space of a branched cover*

Let $\pi : \widetilde{S} \to S$ be a branched covering map, where $\widetilde{S}$ is a surface of genus $\widetilde{g}$, and let $\Sigma \subset S$ be the set of branch points, i.e. the points over which π is not a covering map. We want to follow the same paradigm to construct a moduli space of branched covering maps of Riemann surfaces or holomorphic quadratic differentials topologically equivalent to π. We shall focus on the case of quadratic differentials because that is what we will need, but the same construction would also give a moduli space of branched covers of Riemann surfaces, as in [**Nat**].

Consider quintuples $(p, \widetilde{X}, X, \widetilde{h}, h)$, where $\widetilde{X}$ and X are Riemann surfaces with holomorphic quadratic differentials, $p : \widetilde{X} \to X$ is a holomorphic branched covering map which pulls back the quadratic differential on X to that on $\widetilde{X}$, and $\widetilde{h} : \widetilde{S} \to \widetilde{X}$ and $h : S \to X$ are diffeomorphisms such that $h\pi = p\widetilde{h}$.

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\widetilde{h}} & \widetilde{X} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ S & \xrightarrow{h} & X \end{array}$$

We say $(p_1, \widetilde{X}_1, X_1, \widetilde{h}_1, h_1)$ and $(p_2, \widetilde{X}_2, X_2, \widetilde{h}_2, h_2)$ are equivalent if there are holomorphic equivalences $\widetilde{w} : \widetilde{X}_1 \to \widetilde{X}_2$ and $w : X_1 \to X_2$ which preserve quadratic differentials so that $p_2\widetilde{w} = wp_1$, $\widetilde{w}\widetilde{h}_1 = \widetilde{h}_2$ and $wh_1 = h_2$. Let $\mathrm{Quad}(\pi)$ denote the set of equivalence classes.

$$\begin{array}{ccccccc} \widetilde{S} & \xrightarrow{\widetilde{h}_1} & \widetilde{X}_1 & \xrightarrow{\widetilde{w}} & \widetilde{X}_2 & \xleftarrow{\widetilde{h}_2} & \widetilde{S} \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{h_1} & X_1 & \xrightarrow{w} & X_2 & \xleftarrow{h_2} & S \end{array}$$

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We define $\mathrm{Diffeo}(\pi)$ to be the set of pairs of diffeomorphisms $(\widetilde{\gamma}, \gamma)$, where $\gamma : S \to S$ and $\widetilde{\gamma} : \widetilde{S} \to \widetilde{S}$ are such that $\gamma\pi = \pi\widetilde{\gamma}$ and γ fixes Σ. We obtain a group structure by defining multiplication in $\mathrm{Diffeo}(\pi)$ to be coordinate-wise composition. We define $\mathrm{Diffeo}_0(\pi)$ to be the set of pairs $(\widetilde{\gamma}, \gamma)$ that are isotopic to (id, id) via an *isotopy respecting* π, i.e. parametrized families H_t and $\widetilde{H}_t$ of diffeomorphisms of S and $\widetilde{S}$, respectively, such that H_t fixes Σ and $H_0 = \mathrm{id}$, $H_1 = \gamma$, $\widetilde{H}_0 = \mathrm{id}$, $\widetilde{H}_1 = \widetilde{\gamma}$ and $\pi\widetilde{H}_t = H_t\pi$ for each t. We define $\mathrm{Mod}(\pi) = \mathrm{Diffeo}(\pi)/\mathrm{Diffeo}_0(\pi)$ and $\widetilde{\mathcal{Q}}(\pi) = \mathrm{Quad}(\pi)/\mathrm{Diffeo}_0(\pi)$. There is a natural action of $\mathrm{Mod}(\pi)$ on $\widetilde{\mathcal{Q}}(\pi)$, and we denote the quotient by $\mathcal{Q}(\pi)$. Below we will equip $\mathcal{Q}(\pi)$ with a natural topology.

Let b be the cardinality of Σ and let d be the degree of the cover π. Observe that the definition of $\mathrm{Diffeo}_0(\pi)$ ensures that if $(\widetilde{\gamma}, \gamma) \in \mathrm{Diffeo}(\pi)$, then $\gamma \in \mathrm{Diffeo}(S, \Sigma)$, and if $(\widetilde{\gamma}, \gamma) \in \mathrm{Diffeo}_0(\pi)$, then $\gamma \in \mathrm{Diffeo}_0(S, \Sigma)$. Let $\widetilde{P} : \mathrm{Quad}(\pi) \to \mathrm{Quad}(S, \Sigma)$ be the map which takes the quintuple $(p, \widetilde{X}, X, \widetilde{h}, h)$ to the pair (X, h). It follows that $\widetilde{P}$ induces a map on equivalence classes $\widetilde{\mathcal{Q}}(\pi) \to \widetilde{\mathcal{Q}}_{g,b}$ which we will also denote by $\widetilde{P}$. It also follows that the homomorphism

$$H : \mathrm{Mod}(\pi) \to \mathrm{Mod}(S, \Sigma), \quad H(\widetilde{\gamma}, \gamma) = \gamma$$

is well-defined. H gives us a $\mathrm{Mod}(\pi)$-action on $\widetilde{\mathcal{Q}}_{g,b}$, and $\widetilde{P}$ is $\mathrm{Mod}(\pi)$-equivariant.

Let $z \in S \setminus \Sigma$, $\widetilde{z} \in \pi^{-1}(z)$ and $\widetilde{\Sigma} = \pi^{-1}(\Sigma)$. Also take $\Gamma \subset \pi_1(S \setminus \Sigma, z)$ to be the image of $\pi_1(\widetilde{S} \setminus \widetilde{\Sigma}, \widetilde{z})$ under the projection π, and let $N(\Gamma)$ be the normalizer of Γ in $\pi_1(S \setminus \Sigma, z)$.

LEMMA 14. *$\ker H$ is finite and can be identified with $N(\Gamma)/\Gamma$.*

Proof. Suppose $(\widetilde{\gamma}, \gamma)$ represents an element of $\mathrm{Mod}(\pi)$ which is in $\ker H$, so that γ is isotopic to the identity via an isotopy fixing Σ. The map $\pi|_{\widetilde{S} \setminus \widetilde{\Sigma}}$ is a covering map, so by lifting the isotopy to $\widetilde{S} \setminus \widetilde{\Sigma}$, we obtain that $\widetilde{\gamma}$ is isotopic to a diffeomorphism $\widetilde{\gamma}_1 : \widetilde{S} \setminus \widetilde{\Sigma} \to \widetilde{S} \setminus \widetilde{\Sigma}$ covering the identity. The collection of such diffeomorphisms forms a group F, which we can identify with the deck group $N(\Gamma)/\Gamma$ (see [**Ha**, Proposition 1.39]). This is a finite group, since Γ is of finite index (its index is d, the degree of the cover). Moreover, all elements of the deck group can arise. By the unique lifting property [**Ha**, Proposition 1.34], $F \cap \mathrm{Diffeo}_0(\widetilde{S}) = \{\mathrm{id}\}$, i.e. the inclusion of F in $\mathrm{Mod}(\widetilde{S})$ induces an inclusion of groups. □

Let $\widetilde{\mathcal{Q}}'(\pi)$ denote the quotient of $\widetilde{\mathcal{Q}}(\pi)$ by $\ker H$ (considered at this point only as a set, with no additional structure). By equivariance, two quintuples which differ by an element of $\ker H$ will map to the same point under $\widetilde{P}$, so that $\widetilde{P}$ induces a map $\widetilde{P}' : \widetilde{\mathcal{Q}}'(\pi) \to \widetilde{\mathcal{Q}}(S, \Sigma)$.

LEMMA 15. *The map $\widetilde{P}'$ is bijective.*

Proof. We begin by proving surjectivity. Given (X, h), we need to produce $\widetilde{X}$, p and $\widetilde{h}$ so that the quintuple $(p, \widetilde{X}, X, \widetilde{h}, h)$ represents an element of $\widetilde{\mathcal{Q}}(\pi)$. We define $\widetilde{X}$ to be the surface $\widetilde{S}$ with the Riemann surface structure and quadratic differential structure pulled back via the branched cover $h\pi$. We take p to be the map $h\pi$ and take $\widetilde{h}$ to be the identity.

We now prove injectivity. Suppose we are given $(p_1, \widetilde{X}_1, X_2, \widetilde{h}_1, h_1)$ and $(p_2, \widetilde{X}_2, X_2, \widetilde{h}_2, h_2)$ for which (X_1, h_1) and (X_2, h_2) are equivalent in $\widetilde{\mathcal{Q}}(S, \Sigma)$, so there is a holomorphic $f : X_1 \to X_2$ preserving the quadratic differentials, with $fh_1 = h_2$. Pick

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a basepoint z in $S \smallsetminus \Sigma$, and let $\widetilde{z} \in \pi^{-1}(z)$. Let $\widetilde{x}_1 = \widetilde{h}_1(\widetilde{z})$ and $\widetilde{x}_2 = \widetilde{h}_2(\widetilde{z})$; also let $\widetilde{\Sigma} = \pi^{-1}(\Sigma)$ and $\widetilde{\Sigma}_i = p_i^{-1}(h(\Sigma))$ for $i = 1, 2$. By chasing the diagram below, we see that

$$(fp_1)_*\pi_1(\widetilde{X}_1 \smallsetminus \widetilde{\Sigma}_1, \widetilde{x}_1) = (h_2\pi)_*\pi_1(\widetilde{S} \smallsetminus \widetilde{\Sigma}, \widetilde{z}) = p_{2*}\pi_1(\widetilde{X}_2 \smallsetminus \widetilde{\Sigma}_2, \widetilde{x}_2).$$

The Galois correspondence for covering spaces shows that these two covering spaces are equivalent, so there is $\widetilde{f} : \widetilde{X}_1 \smallsetminus \widetilde{\Sigma}_1 \to \widetilde{X}_2 \smallsetminus \widetilde{\Sigma}_2$ taking $\widetilde{x}_1$ to $\widetilde{x}_2$ such that $p_1 = p_2\widetilde{f}$. We can extend $\widetilde{f}$ to a map $\widetilde{X}_1 \to \widetilde{X}_2$, which we continue to denote by $\widetilde{f}$. Since both $\widetilde{X}_1$ and $\widetilde{X}_2$ have the quadratic differential structure obtained by pulling back the quadratic differential from X, the map $\widetilde{f}$ preserves the quadratic differential structure and thus $(p_1, \widetilde{X}_1, X_1, \widetilde{h}_1, h_1)$, $(p_2, \widetilde{X}_2, X_2, \widetilde{h}_2, h_2)$ are equivalent.

$$\begin{array}{ccccccc}
\widetilde{S} & \xrightarrow{\widetilde{h}_1} & \widetilde{X}_1 & \xrightarrow{\widetilde{f}} & \widetilde{X}_2 & \xleftarrow{\widetilde{h}_2} & \widetilde{S} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi} \\
S & \xrightarrow{h_1} & X_1 & \xrightarrow{f} & X_2 & \xleftarrow{h_2} & S
\end{array}$$

□

Since $\widetilde{\mathcal{Q}}_{g,b}$ is a manifold, we can use $\widetilde{P}$ to induce a manifold topology on $\widetilde{\mathcal{Q}}'(\pi)$ as well. Taking quotients by the action, the map $\widetilde{P}'$ induces a map $P : \mathcal{Q}(\pi) \to \mathcal{Q}(S, \Sigma)$. Our next objective is to analyze this map.

We begin with some general observations on diffeomorphisms and fundamental groups. Let Y be a surface, and let $z \in Y$. Let $\mathrm{Aut}(\pi_1(Y, z))$ be the group of automorphisms and $\mathrm{Inn}(\pi_1(Y, z))$ the group of inner automorphisms of $\pi_1(Y, z)$. Define $\mathrm{Out}(Y) = \mathrm{Out}(\pi_1(Y, z))$ to be

$$\mathrm{Aut}(\pi_1(Y, z))/\mathrm{Inn}(\pi_1(Y, z)).$$

There is a well-defined homomorphism from $\mathrm{Mod}(Y)$ to $\mathrm{Out}(Y)$ which we now describe. Given $f : Y \to Y$, set $w = f(z)$ and consider the isomorphism $f_* : \pi_1(Y, z) \to \pi_1(Y, w)$. Given a path ρ from w to z, there is a canonical map $\rho_* : \pi_1(Y, w) \to \pi_1(Y, z)$. If we change the path ρ to a path σ, then the two identifications differ by the inner automorphism corresponding to conjugation by $\rho \cdot \sigma^{-1}$. In particular, the composition $\rho_* f_*$ determines an element of Out which is independent of the path ρ and depends only on the isotopy class of f.

LEMMA 16. *The image of H is of finite index in* $\mathrm{Mod}(S, \Sigma)$.

Proof. By definition, the image of H consists of those elements $\gamma \in \mathrm{Diffeo}(S, \Sigma)$ for which there is a lift $\widetilde{\gamma} \in \mathrm{Diffeo}(\widetilde{S})$ with $\gamma\pi = \pi\widetilde{\gamma}$. We claim that it suffices to consider this lifting problem for $Y = S \smallsetminus \Sigma$ and $\widetilde{Y} = \widetilde{S} \smallsetminus \widetilde{\Sigma}$ in place of S and $\widetilde{S}$. Indeed, if $\gamma \in \mathrm{Diffeo}(S, \Sigma)$ and $f = \gamma|_Y$ has a lift $\widetilde{f} : \widetilde{Y} \to \widetilde{Y}$, then since $\widetilde{S}$ is the completion of $\widetilde{Y}$ with respect to a complex structure for which $\widetilde{f}$ is holomorphic, we can continuously extend $\widetilde{f}$ to obtain a lift of $\widetilde{\gamma}$ defined on $\widetilde{S}$.

Let $\widetilde{z}$ be a point in $\widetilde{Y}$ that maps to z and let $\widetilde{w}$ be a point that maps to w. Let $\Gamma_{\widetilde{z}}$ be the group $p_*(\pi_1(\widetilde{Y}, \widetilde{z})) \subset \pi_1(Y, z)$ and let $\Gamma_{\widetilde{w}}$ be the group $p_*(\pi_1(\widetilde{Y}, \widetilde{w})) \subset \pi_1(Y, w)$. The Galois theory of covering spaces tells us that the diffeomorphism $f : Y \to Y$ lifts to

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a diffeomorphism $\widetilde{f}:\widetilde{Y}\to\widetilde{Y}$ taking $\widetilde{z}$ to $\widetilde{w}$ if and only if $f_*(\Gamma_{\widetilde{z}})=\Gamma_{\widetilde{w}}$. If we want to know whether f has any lift to a map $\widetilde{f}$, then we need to consider all possible values for $\widetilde{w}'=f(\widetilde{z})$. As $\widetilde{w}'$ ranges over points in $p^{-1}(w)$, the groups $\Gamma_{\widetilde{w}'}$ range over all conjugates of $\Gamma_{\widetilde{w}}$. Thus f lifts if and only if f_* maps $\Gamma_{\widetilde{z}}$ to a conjugate of $\Gamma_{\widetilde{w}}$.

As before, let d denote the degree of the covering, which is also the index of $\Gamma_{\widetilde{z}}$. Let $\mathcal{F}_d$ be the collection of subgroups of $\pi_1(Y,z)$ of index d, and let $\mathcal{F}'_d$ be the set of conjugacy classes in $\mathcal{F}_d$. The group Aut acts on $\mathcal{F}_d$ and on $\mathcal{F}'_d$, and its subgroup Inn acts trivially on $\mathcal{F}'_d$; so Out acts on $\mathcal{F}'_d$.

If we choose a path ρ from w to z, then $\rho_*(\Gamma_{\widetilde{w}})$ is conjugate to $\Gamma_{\widetilde{z}}$, and so f lifts to a $\widetilde{f}$ if and only if the element of Aut corresponding to f fixes the conjugacy class of $\Gamma_{\widetilde{z}}$ in $\mathcal{F}'_d$. This shows that the image of H corresponds to the stabilizer of the conjugacy class of $\Gamma_{\widetilde{z}}$ in $\mathcal{F}'_d$. Since the set $\mathcal{F}'_d$ is finite, the image of this homomorphism has finite index. □

COROLLARY 17. *The moduli space $\mathcal{Q}(\pi)$ has a natural orbifold structure, and $P:\mathcal{Q}(\pi)\to\mathcal{Q}(S,\Sigma)$ is a finite covering of orbifolds.*

Proof. Since $\mathrm{Mod}(S,\Sigma)$ acts properly discontinuously on $\widetilde{\mathcal{Q}}(S,\Sigma)$, by Lemma 15 and equivariance we have that $\mathrm{Mod}(\pi)$ acts properly discontinuously on the manifold $\widetilde{\mathcal{Q}}'(\pi)$. Proposition 9 then implies that $\mathcal{Q}(\pi)$ is an orbifold. The degree of P is the index of the image of H in $\mathrm{Mod}(S,\Sigma)$, which, by Lemma 16, is finite. □

Proof of Proposition 7. The map P is an orbifold fibration with finite generic fiber. In particular, the generic fiber is compact. By Proposition 13, the map P is proper. □

Proof of Proposition 6. Let $\widetilde{R}:\mathrm{Quad}(\pi)\to\mathcal{Q}(\widetilde{S})$ be the map which sends the class of the quintuple $(p,\widetilde{X},X,\widetilde{h},h)$ to the class of the pair $(\widetilde{X},\widetilde{h})$. As explained earlier, the group $\mathrm{Diffeo}(\pi)$ acts on $\mathrm{Quad}(\pi)$. The group $\mathrm{Diffeo}(\pi)$ also acts on $\mathrm{Quad}(\widetilde{S})$ via the natural map from $\mathrm{Diffeo}(\pi)$ to $\mathrm{Diffeo}(\widetilde{S})$. It follows that $\widetilde{R}$ descends to a well-defined map $R:\mathcal{Q}(\pi)\to\mathcal{Q}(\widetilde{S})$, which is continuous since $\widetilde{R}$ factors through $\widetilde{\mathcal{Q}}'(\pi)$, and the topology on $\mathcal{Q}(\pi)$ is defined via Lemma 15.

The required maps have been constructed. The only thing that remains is to discuss the G-action. In the case of the space of quadratic differentials, there is a natural action of G. We begin by describing this action on the set of pairs (X,h). We think of a quadratic differential as being given by a set of charts mapping open subsets of X to open subsets of $\mathbb{C}$. Then the action of $g\in G$ is obtained by identifying $\mathbb{C}$ with $\mathbb{R}^2$ and post-composing the charts with the map $g:\mathbb{R}^2\to\mathbb{R}^2$. The action of $\mathrm{Diffeo}(S)$ is by pre-composition on the charts; hence these actions commute, and therefore we have a well-defined action of G on the quotient space. The same reasoning implies that there is a well-defined G-action on $\mathcal{Q}(\pi)$, and since P and R both descend from maps defined on $\mathrm{Quad}(\pi)$, they are G-equivariant. □

Proof of Proposition 3. Let $x\in S$, let U be a small disk around x, and let y be a point of S outside U. If we are given a finite-index subgroup Γ of $\pi_1(S\smallsetminus\{x\},y)$, we can construct a branched cover of S by constructing the corresponding cover of $S\smallsetminus\{x\}$ and then taking the metric completion. This branched cover will have non-trivial branching if a loop obtained by travelling from y to the boundary of U, traversing the boundary of U

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once, and then returning to y along the same path is not in Γ. The fundamental group of $S \setminus \{x\}$ is a free group on $2g$ generators. Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be loops on S so that a loop γ corresponding to the boundary of U is given by the product of commutators $\alpha_1\beta_1{\alpha_1}^{-1}{\beta_1}^{-1} \ldots \alpha_g\beta_g{\alpha_g}^{-1}{\beta_g}^{-1}$. Let D_d be the dihedral group of order $2d$ generated by a and b, where $a^2 = b^2 = (ab)^d = 1$. Define a homomorphism $h : \pi_1(S \setminus \{x\}, y) \to D_d$ so that $h(\alpha_1) = a$, $h(\beta_1) = b$ and $h(\alpha_j) = h(\beta_j) = 1$ for $j > 1$. This homomorphism has the property that $h(\gamma) = aba^{-1}b^{-1} = (ab)^2$; this element is not the identity since $d > 2$. Let $F \subset D_d$ be the subgroup containing a and 1. Let $\Gamma \subset \pi_1(S \setminus \{x\}, y)$ be $h^{-1}(F)$. The group Γ has index d, hence the covering space of $S \setminus \{x\}$ corresponding to Γ has degree d. Let $\widetilde{S}$ be the branched cover obtained by completing this covering space, and let π be the projection. The branching at x is non-trivial because the loop γ corresponds to an element of $\pi_1(S \setminus \{x\}, y)$ which is not in Γ. □

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