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A note on restricted X-ray transforms

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Abstract

We show how the techniques introduced in [**1**] and [**2**] can be employed to derive endpoint $L^p \to L^q$ bounds for the X-ray transform associated to the line complex generated by the curve $t \to (t, t^2, \dots, t^{d-1})$. Almost-sharp Lorentz space estimates are produced as well.

1. *Introduction*

The purpose of this paper is to give yet another application of the far-reaching techniques of Christ (see [**1**]), and in particular to utilise the refinement provided in [**2**] to establish strong type (p, q) bounds for the X-ray transform first studied in [**4**] and [**3**] (see also [**8**] for some related results). Work on averages along curves using these techniques is currently being undertaken by Stovall ([**9**]); however, the simple behaviour of the X-ray transform makes it a very natural object of study. Thus, we shall be concerned with the operator

$$Xf(x) = \int_I f\big(s, x_2 + sx_1, x_3 + sx_1^2, \dots, x_d + sx_1^{d-1}\big)ds, \tag{1·1}$$

where $I \subset \mathbb{R}$ is a closed interval.

To characterise the set of (p, q) such that $X\colon L^p \to L^q$, we let $\Delta_d \subset [0, 1]^2$ be the convex hull of the points $(1, 1)$, $(0, 0)$ and (p_d^{-1}, q_d^{-1}), where $p_d = d(d+1)/(d^2 - d + 2)$, $q_d = (d+1)/(d-1)$. We have the following.

THEOREM 1.

$$X : L^p(\mathbb{R}^d) \longrightarrow L^q(\mathbb{R}^d) \iff (p^{-1}, q^{-1}) \in \Delta_d.$$

The necessity of this condition can be found in [**6**]. We remark that this result was established in [**5**] for $d = 3$, whilst [**6**] and [**7**] provide estimates in dimension $d = 4$, [**4**] contains estimates in dimension $d = 4, 5$ and [**3**] contains estimates for general d; further mixed-norm estimates for the operator (1·1) have been obtained as well, and we refer the reader to [**3**] and [**4**] and the references therein for these results. However the work of the aforementioned authors establishes only restricted weak-type bounds at (p_d, q_d) when $d \geqslant 4$. We are able to establish the following estimate, from which Theorem 1 follows by simple interpolation.

THEOREM 2. *For every* $\epsilon > 0$,

$$X : L^{p_d}(\mathbb{R}^d) \longrightarrow L^{q_d, p_d + \epsilon}.$$

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Here $L^{s,r}(\mathbb{R}^d)$ denote the familiar Lorentz spaces; since $p_d < q_d$, it is clear that the strong (p_d, q_d) bound for X follows from Theorem 2.

The following example shows that X maps L^{p_d} to $L^{q_d,r}$ only if $r \geqslant p_d$. Hence, with the possible exception of ϵ, Theorem 2 is sharp in the scale of Lorentz spaces.

For simplicity we shall suppose that $0 \in I$ in the definition of (1·1), otherwise we may translate things accordingly. Let $\chi = \chi_{[-1,1]^d}$ be the characteristic function of the cube centred at the origin of sidelength 2. Now consider the nonisotropic dilations given by

$$\delta \circ y = (\delta y_1, \ldots, \delta^d y_d),$$

and let $\chi_k = \chi_k(y) = (k^{-1} \circ y)$. We define the function

$$f(x) = \sum_{k \geqslant N} \chi_k(x_1, x_2 - k^2, \ldots, x_d - k^d), \quad N \gg 1,$$

where the single elements in the sum have clearly disjoint supports. Thus, if $A_k = supp(\chi_k(\cdot - (0, k^2, \ldots, k^d))$

$$\begin{aligned} \|f\|_{L^{p_d}} &= \left(\sum_{k \geqslant N} |A_k| \right)^{\frac{d^2-d+2}{d(d+1)}} = \left(\sum_{k \geqslant N} k^{-d(d+1)/2} \right)^{\frac{d^2-d+2}{d(d+1)}} \\ &\approx N^{(-d(d+1)/2+1)\frac{d^2-d+2}{d(d+1)}} = N^{(-1/2+1/d(d+1))(d^2-d+2)}. \end{aligned}$$

However

$$\begin{aligned} Xf(x) &= \sum_{k \geqslant N} \int_I \chi_k(s, x_2 + sx_1 - k, x_3 + sx_1^2 - k^2, \ldots, x_d + sx_1^{d-1} - k^d) ds \\ &\geqslant \sum_{k \geqslant N} \psi_{2k}(x_1) \int_{s \ll k^{-1}} \chi_k(s, x_2 + sx_1 - k, x_3 + sx_1^2 - k^2, \ldots, x_d + sx_1^{d-1} - k^d) ds \\ &\geqslant \sum_{k \geqslant N} k^{-1} \psi_{2k}(x_1) \psi_{2k^2}(x_2 - k^2) \cdots \psi_{2k^d}(x_d - k^d), \end{aligned}$$

where $\psi_j(t) = \chi_{[-1/j,1/j]}(t)$. Note that, again, all the functions in the sum are characteristic functions of disjoint sets. Thus,

$$\begin{aligned} \|Xf\|_{L^{q_d,r}} \gtrsim \left[\sum_{k \geqslant N} \left(k^{-1} |B_k|^{\frac{d-1}{d+1}} \right)^r \right]^{1/r} &= \left[\sum_{k \geqslant N} \left(k^{-1} k^{-\frac{d(d+1)}{2}\frac{d-1}{d+1}} \right)^r \right]^{1/r} \\ &= \left[\sum_{k \geqslant N} k^{-\frac{d^2-d+2}{2} r} \right]^{1/r} \approx N^{(-\frac{d^2-d+2}{2}+1/r)}. \end{aligned}$$

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Hence, in order for boundedness to hold, we must have

$$\begin{aligned} N^{(-\frac{d^2-d+2}{2}+1/r)} &\lesssim N^{(-1/2+1/d(d+1))(d^2-d+2)} \\ &\Longrightarrow -\frac{d^2-d+2}{2}+\frac{1}{r} \leqslant -\frac{d^2-d+2}{2}+\frac{d^2-d+2}{d(d+1)} \\ &\Longleftrightarrow r \geqslant \frac{d(d+1)}{d^2-d+2}, \end{aligned}$$

as $N \to \infty$.

Notation. Whenever we write (or we have written) $A \lesssim B$ for any two nonnegative quantities A and B, we shall mean that there exists a striclty positive constant c such that $A \leqslant cB$; such constant is subject to change from line to line and even from step to step.

2. *Preliminary statements*

We first summarise some of the results contained in [**3**] that are necessary to our arguments. Let $E \subset \mathbb{R}^d_1$, $F \subset \mathbb{R}^d_2$ be any two measurable sets[1] and let

$$\mathcal{T}(E,F) = \langle X\chi_E, \chi_F\rangle = \langle \chi_E, X^*\chi_F\rangle, \tag{2·1}$$

where the dual operator X^* is given by

$$X^*g(x) = \int_{\mathbb{R}} g(t, x_2 - x_1 t, x_3 - x_1 t^2, \dots, x_d - x_1 t^{d-1})dt.$$

The restricted weak-type (p_d, q_d) estimate for (1·1) then amounts to prove

$$\langle X\chi_E, \chi_F\rangle \lesssim |E|^{1/p_d}|F|^{1/q_d'}.$$

If one lets

$$\alpha = \mathcal{T}(E,F)/|F|, \quad \beta = \mathcal{T}(E,F)/|E|,$$

then it suffices to show that either

$$|E| \gtrsim \alpha^d \beta^{d(d-1)/2} \quad \text{or} \quad |F| \gtrsim \alpha^{d-1}\beta^{(d^2-d+2)/2.} \tag{2·2}$$

THEOREM 3. (Christ-Erdogan [**3**]). *Estimates* (2·2) *hold.*

It is important to keep in mind the manner by which estimates (2·2) were established. Define, for fixed x, the maps

$$\begin{aligned} \gamma(x,s) &= \big(s, x_2 + sx_1, x_3 + sx_1^2, \dots, x_d + sx_1^{d-1}\big), \\ \gamma^*(x,t) &= (t, x_2 - x_1 t, x_3 - x_1 t^2, \dots, x_d - x_1 t^{d-1}), \end{aligned}$$

and the maps $\Phi_j \equiv \Phi_{j,x} : \mathbb{R}^j \to \mathbb{R}^d_1$ if j is even, $\Phi_j \equiv \Phi_{j,x} : \mathbb{R}^j \to \mathbb{R}^d_2$ if j is odd, by letting $\Phi_1(t_1) = \gamma^*(x,t_1)$, $\Phi_2(t_1,s_1) = \gamma(\Phi_1(t_1), s_1)$ and further

$$\begin{aligned} \Phi_{2k+1}(t_1,s_1,t_2,s_2,\dots,t_{k+1}) &= \gamma^*(\Phi_{2k}(t_1,s_1,t_2,s_2,\dots,t_k,s_k), t_{k+1}), \\ \Phi_{2k+2}(t_1,s_1,t_2,s_2,\dots,t_{k+1},s_{k+1}) &= \gamma(\Phi_{2k+1}(t_1,s_1,t_2,s_2,\dots,t_{k+1}), s_{k+1}). \end{aligned}$$

[1] We use the notation $\mathbb{R}^d_i$, $i = 1, 2$ to stress the fact that the sets E and F lie in different ambient spaces, albeit of equal dimension d.

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We also define maps $\Psi_j \equiv \Psi_{j,x}$ by setting $\Psi_1(s_1) = \gamma(x, s_1)$, $\Psi_2(s_1, t_2) = \gamma^*(\Psi_1(s_1), t_2)$ and

$$\Psi_{2k+1}(s_1, t_2, \dots, s_k, t_{k+1}, s_{k+1}) = \gamma(\Psi_{2k}(s_1, t_2, \dots, s_k, t_{k+1}), s_{k+1})$$
$$\Psi_{2k+2}(s_1, t_2, \dots, t_{k+1}, s_{k+1}, t_{k+2}) = \gamma^*(\Psi_{2k+1}(s_1, t_2, \dots, s_k, t_{k+1}, s_{k+1}), t_{k+2}),$$

where $\Psi_j : \mathbb{R}^j \to \mathbb{R}^d_2$ if j is even, and $\Psi_j \colon \mathbb{R}^j \to \mathbb{R}^d_1$ if j is odd.

LEMMA 1. *Consider the maps Φ_d and Ψ_d and let J_Φ and J_Ψ be the determinants of the associated Jacobian matrices. Then*

$$J_\Phi = c_d \prod_{j=1}^{k}(s_j - s_{j-1}) \prod_{1\leqslant j<\ell\leqslant k}(t_j - t_\ell)^4 \quad \textit{for } d = 2k,$$
$$J_\Phi = c_d \prod_{j=1}^{k}(s_j - s_{j-1}) \prod_{1\leqslant j<\ell\leqslant k}(t_j - t_l)^4 \prod_{j=1}^{k}(t_j - t_{k+1})^2 \quad \textit{for } d = 2k+1,$$

where we have set $x_1 = s_0$ for notational convenience, and

$$J_\Psi = c_d(t_{k+1} - t_1) \prod_{j=1}^{k-1}(s_{j+1} - s_j) \prod_{2\leqslant j<\ell\leqslant k}(t_j - t_\ell)^4 \prod_{j=2}^{k}(t_j - t_{k+1})^2 \prod_{j=2}^{k}(t_j - t_1)^2$$

when $d = 2k$, whilst

$$J_\Psi = c_d \prod_{j=1}^{k}(s_{j+1} - s_j) \prod_{2\leqslant j<\ell\leqslant k+1}(t_j - t_\ell)^4 \prod_{j=2}^{k+1}(t_j - t_1)^2$$

when $d = 2k+1$. We set $x_1 = t_1$ in the formulas characterising J_Ψ.

Proof. The formulas concerning J_Φ have been proven in [**3**]. To compute J_Ψ first observe that for even d

$$\begin{aligned}\Psi_d(s_1, t_2, \dots, s_k, t_{k+1}) &= \left(t_{k+1}, x_2 + \sum_{j=1}^{k}(t_j - t_{j+1})\right) s_j, x_3 + \sum_{j=1}^{k}\left(t_j^2 - t_{j+1}^2\right) s_j, \dots, x_d \\ &\quad + \sum_{j=1}^{k}\left(t_j^{d-1} - t_{j+1}^{d-1} s_j\right)\end{aligned}$$

and for odd d

$$\begin{aligned}\Psi_d(s_1, t_2, \dots, s_k, t_{k+1}, s_{k+1}) &= \Bigg(s_{k+1}, x_2 + \sum_{j=1}^{k}(t_j - t_{j+1})s_j + s_{k+1}t_{k+1}, \dots, x_d \\ &\quad + \sum_{j=1}^{k}\left(t_j^{d-1} - t_{j+1}^{d-1}\right)s_j + s_{k+1}t_{k+1}^{d-1}\Bigg).\end{aligned}$$

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The corresponding Jacobian matrices $\partial\Psi_d/\partial(s,t)$[2] are then given by

$$\begin{pmatrix} 1 & -s_k & -2t_{k+1}s_k & \dots & -(d-1)t_{k+1}^{d-2}s_k \\ 0 & t_k - t_{k+1} & t_k^2 - t_{k+1}^2 & \dots & t_k^{d-1} - t_{k+1}^{d-1} \\ 0 & s_k - s_{k-1} & 2t_k(s_k - s_{k-1}) & \dots & (d-1)t_k^{d-2}(s_k - s_{k-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & t_2 - t_3 & t_2^2 - t_3^2 & \dots & t_2^{d-1} - t_3^{d-1} \\ 0 & s_2 - s_1 & 2t_2(s_2 - s_1) & \dots & (d-1)t_2^{d-1}(s_2 - s_1) \\ 0 & t_1 - t_2 & t_1^2 - t_2^2 & \dots & t_1^{d-1} - t_2^{d-1} \end{pmatrix}$$

for even d, and by

$$\begin{pmatrix} 1 & t_{k+1} & t_{k+1}^2 & \dots & t_{k+1}^{d-1} \\ 0 & s_{k+1} - s_k & 2t_{k+1}(s_{k+1} - s_k) & \dots & (d-1)t_{k+1}^{d-2}(s_{k+1} - s_k) \\ 0 & t_k - t_{k+1} & t_k^2 - t_{k+1}^2 & \dots & t_k^{d-1} - t_{k+1}^{d-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & t_2 - t_3 & t_2^2 - t_3^2 & \dots & t_2^{d-1} - t_3^{d-1} \\ 0 & s_2 - s_1 & 2t_2(s_2 - s_1) & \dots & (d-1)t_2^{d-1}(s_2 - s_1) \\ 0 & t_1 - t_2 & t_1^2 - t_2^2 & \dots & t_1^{d-1} - t_2^{d-1} \end{pmatrix}$$

for odd d.

Let us first examine the case where $d = 2k$ is even. Observe, just as in [**3**], that J_Ψ must be a polynomial of degree $d(d-1)/2$, and that the polynomial considered in the statement of the lemma has the same degree. Now, it suffices to factor out of the determinant all the terms of the form $(s_{j+1} - s_j)$, $j = 1, \dots, k-1$, to obtain an expression involving only the t variables. One can then prove that the determinant is divisible by the quadratic and quartic terms just as in [**3**], whilst the presence of the linear term can be seen by observing that by adding every other row starting from the bottom one, we obtain a matrix with a row of the form

$$\left(t_1 - t_{k+1}\, t_1^2 - t_{k+1}^2 \cdots t_1^{d-1} - t_{k+1}^{d-1}\right),$$

where all entries have the common factor $(t_1 - t_{k+1})$. One may then prove that the constant $c_d \neq 0$, again as in [**3**].

The case of odd $d = 2k+1$ is even simpler; again we may factor all the terms of the form $(s_{j+1} - s_j)$, $j = 1, \dots, k$ to obtain an expression involving only the t variables. However, this is completely analogous to the case treated in [**3**], and the same can be said about the constant c_d.

We conclude this section with a lemma that can be seen as the natural analogue of [2, lemma 8·1].

[2] It is important to keep in mind that t_1 is just a dummy variable.

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LEMMA 2. *Let $E, E' \subset \mathbb{R}_1^d$, $G \subset \mathbb{R}_2^d$ be measurable sets of finite measure, and suppose that $X\chi_{E'}(x) \geqslant \delta_1$ for all $x \in G$. Then*

$$|E'| \gtrsim \delta_1^2 (\mathcal{T}(E,G)/|G|)^{d-2} (\mathcal{T}(E,G)/|E|)^{d(d-1)/2}. \tag{2·3}$$

Further, let $F, F' \subset \mathbb{R}_2^d$, $H \subset \mathbb{R}_1^d$ be measurable sets of finite measure, and suppose $X^*\chi_{F'}(y) \geqslant \delta_2$ for all $y \in H$. Then

$$|F'| \gtrsim \delta_2^d (\mathcal{T}(H,F)/|F|)^{d-1} (\mathcal{T}(H,F)/|F|)^{(d^2-d+2)/2-d}. \tag{2·4}$$

Proof. We first prove (2·3), by splitting the argument in the two cases of even d and odd d.[3] To simplify notation, we shall write $z = (z_1, \dots, z_{m-1}, z_m) = (\hat{z}, z_m) \in \mathbb{R}^m$ for any variable z and appropriate $m \in \mathbb{Z}_+$.

Case $d = 2k$. By using the method of refinements developed in [**1**], we may find a point $x_0 \in E$ and a sequence of sets $\Omega_j \subset \mathbb{R}^j$, $j = 1, \dots, d$ satisfying:

(i) for each j, $\Omega_{j+1} \subset \Omega_j \times \mathbb{R}$;
(ii) $|\Omega_1| \gtrsim \mathcal{T}(E,G)/|E|$;
(iii) for even j, for each point $\omega \in \Omega_j$, $|\{t \in \mathbb{R} : (\omega, t) \in \Omega_{j+1}\}| \gtrsim \mathcal{T}(E,G)/|E|$;
(iv) for odd $j \neq d-1$, for each point $\omega \in \Omega_j$, $|\{t \in \mathbb{R}: (\omega, t) \in \Omega_{j+1}\}| \gtrsim \mathcal{T}(E,G)/|G|$;
(v) for $j = d-1$, for each point $\omega \in \Omega_{d-1}$, $|\{s \in \mathbb{R}: (\omega, s) \in \Omega_d\}| \gtrsim \delta_1$;
(vi) $\Phi_{j,x_0}(\Omega_j) \subset E$ for even j, $\Phi_{j,x_0}(\Omega_j) \subset F$ for odd j, and $\Phi_{d,x_0}(\Omega_d) \subset E'$.

Thus, by Bezout's theorem (see [1, 3]) we have the lower bound

$$\begin{aligned}
|E'| \gtrsim |\Phi_{d,x_0}(\Omega_d)| &\gtrsim \int_{\Omega_d} |J_\Phi(s,t)| ds dt \\
&= \int_{\Omega_d} \prod_{j=1}^{k} |s_j - s_{j-1}| \prod_{1 \leqslant j < \ell \leqslant k} |t_j - t_\ell|^4 ds dt \\
&\gtrsim \delta_1^2 \int_{\Omega_{d-1}} \prod_{j=1}^{k-1} |s_j - s_{j-1}| \prod_{1 \leqslant j < \ell \leqslant k} |t_j - t_\ell|^4 d\hat{s} dt \\
&\gtrsim \delta_1^2 (\mathcal{T}(E,G)/|G|)^{d-2} (\mathcal{T}(E,G)/|E|)^{d(d-1)/2}.
\end{aligned}$$

Case $d = 2k+1$. Here the method of refinements gives us a point $y_0 \in F$ and a sequence of sets $\Omega_j \subset \mathbb{R}^{j-1}$, $j = 2, \dots, d+1$ satisfying;

(i) for each j, $\Omega_{j+1} \subset \Omega_j \times \mathbb{R}$;
(ii) $|\Omega_2| \gtrsim \mathcal{T}(E,G)/|G|$;
(iii) for odd $j \neq d$, for each point $\omega \in \Omega_j$, $|\{s \in \mathbb{R}: (\omega, s) \in \Omega_{j+1}\}| \gtrsim \mathcal{T}(E,G)/|G|$;
(iv) for even j, for each point $\omega \in \Omega_j$, $|\{t \in \mathbb{R}: (\omega, t) \in \Omega_{j+1}\}| \gtrsim \mathcal{T}(E,G)/|E|$;
(v) for $j = d$ for each point $\omega \in \Omega_d$, $|\{s \in \mathbb{R}: (\omega, s) \in \Omega_{d+1}\}| \gtrsim \delta_1$;
(vi) $\Psi_{j,y_0}(\Omega_{j+1}) \subset E$ for odd j, $\Psi_{j,y_0}(\Omega_{j+1}) \subset F$ for even j, and $\Psi_{d,y_0}(\Omega_{d+1}) \subset E'$.

[3] This is why we shall need to utilise the formulae we derived for the maps Ψ_d, as well as the formulae for Φ_d.

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Again, by Bezout's theorem,

$$\begin{aligned}|E'| \gtrsim |\Psi_{d,y_0}(\Omega_{d+1})| &\gtrsim \int_{\Omega_{d+1}} |J_\Phi(s,t)|dsdt \\ &= \int_{\Omega_{d+1}} \prod_{j=1}^{k} |s_{j+1}-s_j| \prod_{2\leqslant j<\ell\leqslant k+1} |t_j-t_\ell|^4 \prod_{j=2}^{k+1} |t_j-t_1|^2 dsdt \\ &\gtrsim \delta_1^2 \int_{\Omega_d} \prod_{j=1}^{k-1} |s_{j+1}-s_j| \prod_{2\leqslant j<\ell\leqslant k+1} |t_j-t_\ell|^4 \prod_{j=2}^{k+1} |t_j-t_1|^2 d\hat{s}dt \\ &\gtrsim \delta_1^2 (\mathcal{T}(E,G)/|G|)^{d-2} (\mathcal{T}(E,G)/|E|)^{d(d-1)/2}.\end{aligned}$$

We now turn to the proof of (2·4).

Case $d = 2k+1$. Here we may apply the previous method as in the case of even d for (2·3); however, now properties (5) and (6) should now be:

(5) for $j = d-1$, for each point $\omega \in \Omega_{d-1}$, $|\{t \in \mathbb{R}: (\omega,t) \in \Omega_d\}| \gtrsim \delta_2$;

(6) $\Phi_{j,x_0}(\Omega_j) \subset E$ for even j, $\Phi_{j,x_0}(\Omega_j) \subset F$ for odd j, and $\Phi_{d,x_0}(\Omega_d) \subset F'$.

The lower bound one gets thanks to Bezout's theorem is now

$$\begin{aligned}|F'| \gtrsim |\Phi_{d,x_0}(\Omega_d)| &\gtrsim \int_{\Omega_d} |J_\Phi(s,t)|dsdt \\ &= \int_{\Omega_d} \prod_{j=1}^{k} |s_j-s_{j-1}| \prod_{1\leqslant j<\ell\leqslant k} |t_j-t_l|^4 \prod_{j=1}^{k} |t_j-t_{k+1}|^2 dsdt \\ &\gtrsim \delta_2^{2k+1} \int_{\Omega_{d-1}} \prod_{j=1}^{k} |s_j-s_{j-1}| \prod_{1\leqslant j<\ell\leqslant k} |t_j-t_l|^4 dsd\hat{t} \\ &\gtrsim \delta_2^d (\mathcal{T}(H,F)/|F|)^{d-1} (\mathcal{T}(H,F)/|H|)^{(d^2-d+2)/2-d}.\end{aligned}$$

Case $d = 2k$. Here we may apply the method of refinements as in the case of odd d for (2·3); now conditions (5) and (6) are

(5) for $j = d$ for each point $\omega \in \Omega_d$, $|\{t \in \mathbb{R}: (\omega,t) \in \Omega_{d+1}\}| \gtrsim \delta_2$;

(6) $\Psi_{j,y_0}(\Omega_{j+1}) \subset E$ for odd j, $\Psi_{j,y_0}(\Omega_{j+1}) \subset F$ for even j, and $\Psi_{d,y_0}(\Omega_{d+1}) \subset F'$. Thus, using Bezout's theorem once more, we have

$$\begin{aligned}|F'| \gtrsim |\Psi_{d,y_0}(\Omega_{d+1}| &\gtrsim \int_{\Omega_{d+1}} |J_\Psi(s,t)|dsdt \\ &\gtrsim \int_{\Omega_{d+1}} |t_{k+1}-t_1| \prod_{j=1}^{k-1} |s_{j+1}-s_j| \prod_{2\leqslant j<\ell\leqslant k} |t_j-t_\ell|^4 \prod_{j=2}^{k} |t_j-t_{k+1}|^2 \prod_{j=2}^{k} |t_j-t_1|^2 dsdt \\ &\gtrsim \delta_2^{2k} \int_{\Omega_d} \prod_{j=1}^{k-1} |s_{j+1}-s_j| \prod_{2\leqslant j<\ell\leqslant k} |t_j-t_\ell|^4 \prod_{j=2}^{k} |t_j-t_{k+1}|^2 dsd\hat{t} \\ &\gtrsim \delta_2^d (\mathcal{T}(H,F)/|F|)^{d-1} (\mathcal{T}(H,F)/|H|)^{(d^2-d+2)/2-d}.\end{aligned}$$

3. *Strong type estimates*

The purpose of this section is to show how the arguments in [**2**] can be utilised to obtain the statement of Theorem 2; naturally, we shall have to make suitable modifications, the

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main one being the use of Lemma 2. We are aiming to show that

$$|\langle Xf, g\rangle| \lesssim \|f\|_{L^{p_d}} \|g\|_{L^{q_d', r'}}, \quad r' < p_d', \tag{3·1}$$

which naturally implies $X: L^{p_d}(\mathbb{R}^d) \to L^{q_d, r}(\mathbb{R}^d)$ for $r > p_d$. For the sake of notational simplicity, from now on we shall relabel $p \equiv p_d$, $q \equiv q_d$, as we shall only deal with inequality (3·1) in this section.

As pointed out in [**2**], it suffices to consider f, g of the form $f = \sum_{k\in\mathbb{Z}} 2^k \chi_{E_k}$, $g = \sum_{j\in\mathbb{Z}} 2^j \chi_{F_j}$ were the sets E_k's are pairwise disjoint and so are the F_j's; the indices k, j are completely independent of each other. The key step is to show that

$$|\langle Xf, g\rangle| \lesssim \|f\|_p \|g\|_{q'} \quad \text{if } f = \sum_{k\in\mathbb{Z}} 2^k \chi_{E_k} \text{ and } g \equiv \chi_F \quad \text{for a single set } F, \tag{3·2}$$

and its counterpart[4]

$$|\langle Xf, g\rangle| \lesssim \|f\|_p \|g\|_{q'} \quad \text{if } f \equiv \chi_E \quad \text{for a single set } E, \quad \text{and } g = \sum_{j\in\mathbb{Z}} 2^j \chi_{F_j}. \tag{3·3}$$

We follow the scheme of [**2**] to prove (3·2). Let $\epsilon_1, \eta_1 \in (0, 1/2]$ be arbitrary and normalise the p norm of f by setting $\sum_k 2^{kp}|E_k| = 1$. Suppose

$$|E_k| \approx \eta_1 2^{-kp} \quad \text{for all } k, \quad \mathcal{T}(E_k F) \approx \epsilon_1 |E_k|^{1/p} |F|^{1/q'} \quad \text{for all } k.$$

Then the number M_1 of indices k is finite and $M_1\eta_1 \lesssim 1$. Further, assume that any two indices k_1, k_2 in the sum satisfy $|k_1 - k_2| \geqslant A \log(1/\epsilon_1)$,[5] and define the sets

$$G_k = \{x \in F : X\chi_{E_k}(x) \geqslant c_0 \epsilon_1 |E_k|^{1/p} |F|^{1/q'-1}\},$$

where the constant $c_0 > 0$ is chosen sufficiently small to have $\mathcal{T}(E_k, F \setminus G_k) \leqslant \frac{1}{2}\mathcal{T}(E_k F)$, so that $\mathcal{T}(E_k, G_k) \approx \mathcal{T}(E_k, F)$. Since $\mathcal{T}(E_k, G_k) \lesssim |E_k|^{1/p} |G_k|^{1/q'}$, this implies

$$|G_k| \gtrsim \epsilon_1^{q'} |F|. \tag{3·4}$$

A simple observation[6] then shows that one has the dichotomy

$$\text{either } \sum_{k\in\mathbb{Z}} |G_k| \lesssim |F|, \quad \text{or} \tag{3·5}$$

there exists indices k_1, k_2, $k_1 \neq k_2$ so that

$$|G_{k_1} \cap G_{k_2}| \gtrsim \epsilon_1^{2q'} |F|. \tag{3·6}$$

We first wish to show that (3·6) cannot hold; we shall then complete the proof as in [**2**].

Arguing by contradiction, assume that (3·6) does hold; we start by applying (2·3) of Lemma 2 with $E = E_{k_1}, E' = E_{k_2}, G = G_{k_1} \cap G_{k_2}$ and $\delta_1 \approx \epsilon_1 |E'|^{1/p} |F|^{-1/q}$. We also have that $X\chi_E \geqslant c_0 \epsilon_1 |E|^{1/p} |F|^{-1/q}$ at each point of G and thus

$$\mathcal{T}(E, G) \gtrsim \epsilon_1 |E|^{1/p} |F|^{-1/q} |G|.$$

[4] This is utterly redundant when the operator in question is (essentially) self-adjoint, but the X-ray transform does not have this property.

[5] This is done by simply splitting the sum into $O(A \log(1/\epsilon_1))$ sums; the logarithmic factor that is lost will not affect the estimates in a crucial way.

[6] So far we have only described the arguments in [**2**], which we have included for the sake of completeness; all the details can be found in that paper.

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By Lemma 2 one may conclude

$$\begin{aligned}|E'| &\gtrsim (\epsilon_1|E'|^{1/p}|F|^{-1/q})^2(\epsilon_1|E|^{1/p}|F|^{-1/q})^{d-2}\\ &\quad\times(\epsilon_1|E|^{1/p-1}|F|^{-1/q}|G|)^{d(d-1)/2}\\ &\gtrsim \epsilon_1^{d+q'd(d-1)/2}|E'|^{2/p}|E|^{(d-2)/p}|F|^{-d/q}|F|^{d(d-1)/2q'}\\ &\gtrsim \epsilon_1^{d+q'd(d-1)/2}|E'|^{2/p}|E|^{(d-2)/p},\end{aligned}$$

where we have used (3·6) and the actual expressions for (p, q). After a bit of algebra one then reaches the conclusion

$$|E'| \lesssim \epsilon_1^{-\varphi}|E|, \quad \text{for some } \varphi > 0.$$

From here, since $|E| = |E_{k_1}| \approx \eta_1 2^{-k_1 p}$ and $|E'| = |E_{k_2}| \approx \eta_1 2^{-k_2 p}$ and the fact that the roles of E and E' can be interchanged, one obtains that $|k_1 - k_2| \lesssim \log(1/\epsilon_1)$, a contradicion to the assumption $|k_1 - k_2| \geqslant A\log(1/\epsilon_1)$ if A is chosen sufficiently large.

Hence (3·5) holds and we may now conclude the argument. We have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} 2^k \mathcal{T}(E_k, F) &\approx \sum_{k\in\mathbb{Z}} 2^k \mathcal{T}(E_k, G_k)\\ &\lesssim \left(\sum_{k\in\mathbb{Z}} 2^{kq}|E_k|^{q/p}\right)^{1/q}\left(\sum_{k\in\mathbb{Z}}|G_k|\right)^{1/q'}\\ &\lesssim \left(\sum_{k\in\mathbb{Z}} 2^{kp}|E_k|2^{k(q-p)}|E_k|^{q/p-1}\right)^{1/q}|F|^{1/q'}\\ &\leqslant \max_k(2^{kp}|E_k|)^{(1/p-1/q)}|F|^{1/q'}\\ &\lesssim \eta^{1/p-1/q}|F|^{1/q'},\end{aligned}$$

where $1/p - 1/q > 0$ and we used $\sum_{k\in\mathbb{Z}} 2^{kp}|E_k| = 1$. However, since the number of indices k in the sum is $M_1 \lesssim \eta_1^{-1}$, one may also argue that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} 2^k \mathcal{T}(E_k, F) &\approx \sum_{k\in\mathbb{Z}} 2^k \eta_1 |E_k|^{1/p}|F|^{1/q'}\\ &\lesssim \epsilon_1 M_1 \eta_1^{1/p}|F|^{1/q'} = \epsilon_1 \eta_1^{-1/p'}|F|^{1/q'}.\end{aligned}$$

If we now recall the assumption $|k_1 - k_2| \geqslant A\log(1/\epsilon_1)$ and retain the normalisations in ϵ_1, η_1 we have $\langle Tf, \chi_F\rangle \lesssim \log(1/\epsilon_1)\min(\eta_1^{1/p-1/q}, \epsilon_1\eta_1^{-1/p'})|F|^{1/q'}$. Thus,

$$\langle Tf, \chi_F\rangle \lesssim \min(\epsilon_1^a, \eta_1^b)\|f\|_p|F|^{1/q'} \tag{3·7}$$

for positive a, b and all f, F subject to the normalisations in ϵ_1, η_1. Summing over dyadic values of η_1 we have

$$\langle Tf, \chi_F\rangle \lesssim \epsilon_1^a\|f\|_p|F|^{1/q'}, \tag{3·8}$$

where now f, F are only subject to ϵ_1 normalisations. Summing again over dyadic values of ϵ_1 gives (3·2), although it is equation (3·8) we shall use to prove the strong type bounds.

We now give an outline of the argument needed to prove (3·3). Again, let $\epsilon_2, \eta_2 \in (0, 1/2]$ be arbitrary, and normalise the q' norm of g by setting $\sum_{j\in\mathbb{Z}} 2^{jq'}|F_j| = 1$. Suppose

$$|F_j| \approx \eta_2 2^{-kq'} \quad \text{for all } j, \quad \mathcal{T}(E, F_j) \approx \epsilon_2|E|^{1/p}|F_j|^{1/q'} \quad \text{for all } j.$$

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We define M_2 as the number of indices j in the sum, and again $M_2\eta_2 \lesssim 1$; further, we shall split the sum in $O(\log(1/\eta_2))$ sums. If we define

$$H_j = \{x \in E : X^*\chi_{F_j} \gtrsim d_0\epsilon_2|F_j|^{1/q'}|E|^{1/p-1}\}$$

where d_0 is to be chosen sufficiently small so that

$$\mathcal{T}(H_j, F_j) \approx \mathcal{T}(E, F_j).$$

Proceeding as in the proof of (3·2) one deduces that $|H_j| \gtrsim \epsilon_2^p|E|$, and the new dichotomy becomes that either $\sum_{j\in\mathbb{Z}}|H_j| \lesssim |E|$, or there exist j_1, j_2 with $j_1 \neq j_2$ so that $|H_{j_1} \bigcap H_{j_2}| \gtrsim \epsilon_2^{2p}|E|$. Again, the key step is now to show that the latter can't happen by applying (2·4) of Lemma 2 in the following manner; set $F = F_{j_1}$, $F' = F_{j_2}$, $H = H_{j_1} \bigcap H_{j_2}$, and $\delta_2 \approx \epsilon_2|F'|^{1/q'}|E|^{-1/p'}$. Further, we have $X^*\chi_F \gtrsim \epsilon_2|E|^{-1/p'}|F|^{1/q'}$ at every point of H, hence

$$\mathcal{T}(H, F) \gtrsim \epsilon_2|E|^{-1/p'}|F|^{1/q'}|H|.$$

By Lemma 2 we can now conclude

$$\begin{aligned}|F'| &\gtrsim (\epsilon_2|F'|^{1/q'}|E|^{-1/p'})^d(\eta_2|F|^{1/q'-1}|E|^{-1/p'}|H|)^{d-1}\\ &\quad\times(\eta_2|F|^{1/q'}|E|^{-1/p'})^{(d^2-d+2)/2-d}\\ &\gtrsim \epsilon_2^{\psi}|F'|^{d/q'}|F|^{-(d-1)/q+(d^2-d+2)/2q'},\end{aligned}$$

for some $\psi > 0$, where we used that $|H| \gtrsim \epsilon_2^{2p}|E|$. The same rearrengement as before then provides the desired contradiction and shows that $\sum_{j\in\mathbb{Z}}|H_j| \lesssim |E|$. Inequality (3·3) can then be proven just like inequality (3·2).

Conclusion of the proof. Now let $f = \sum_{k\in\mathbb{Z}} 2^k\chi_{E_k}$, $g = \sum_{j\in\mathbb{Z}} 2^j\chi_{F_j}$, assume $\|f\|_p = \|g\|_{q'} = 1$, and let $\epsilon_2, \eta_2 \in (0, 1/2]$. We shall suppose $|F_j| \approx \eta_2 2^{-kq'}$ for all j with $|F_j| > 0$. Then we consider the sum $\sum^*_{j,k}\mathcal{T}(E_k, F_j)$ where the $*$ indicates that the sum is taken only with respect to j, k or pairs (j, k) with $\mathcal{T}(E_k, F_j) \approx \epsilon_2|E_k||F_j|^{1/q'}$. Again, one assumes $|j_1 - j_2| \geqslant B\log(1/\epsilon_2)$. The proof of inequality (3·3) gives us, for each pair (j, k) sets $H_{j,k} \subset E_k$ so that $\mathcal{T}(E_k, F_j) \approx \mathcal{T}(H_{j,k}, F_j)$ and $\sum^*_j|H_{j,k}| \lesssim |E_k|$. Hence

$$\begin{aligned}\sum_{j,k}^* 2^j2^k\mathcal{T}(E_k, F_j) &\lesssim \sum_{j,k} 2^j2^k\mathcal{T}(H_{j,k}, F_j)\\ &= \sum_j 2^j\left\langle X\left(\sum_k^* 2^k\chi_{H_{j,k}}\right), \chi_{F_j}\right\rangle \lesssim 2^j|F_j|^{1/q'}\left(\sum_k^* 2^{kp}|H_{j,k}|\right)^{1/p},\end{aligned}$$

where in the last step one uses inequality (3·2). By Hölder's inequality this last quantity is controlled by

$$\left(\sum_j 2^{jp'}|F_j|^{p'/q'}\right)^{1/p'}\left(\sum_j\sum_k^* 2^{kp}|H_{j,k}|\right)^{1/p} \lesssim \eta_2^{1/q'-1/p'}\left(\sum_k^* 2^{kp}|E_k|\right)^{1/p}. \qquad (3·9)$$

On the other hand, one may use the alternative bound

$$\sum_{j,k}^*\mathcal{T}(E_k, F_j) \lesssim \epsilon_1^a\sum_j 2^j|F_j|^{1/q'}\left(\sum_k 2^{kp}|E_k|\right)^{1/p} \leqslant \epsilon_1^a M_2\eta_2^{1/q'} \lesssim \epsilon_1^a\eta_2^{-1/q},$$

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where in the first step inequality (3·8) has been used. Now, summing over dyadic values of ϵ_1 and η_2 gives the strong (p, q) bound; the Lorentz space bound may be obtained by observing that the first term of (3·9) may be controlled by $\sum_j 2^{jr'} |F_j|^{r'/q'}$ if $r' < p'$. This implies that $r > p$, giving the conclusion ot Theorem 2.

4. *Final remarks*

The material presented in this paper is an interesting application of the techniques first introduced in [**1**], and then further developed in [**2**]. Whilst a number of results have been proven by utilising these ideas, it is not yet clear to which extent these techniques can be applied, although it is perhaps fair to say that the $L^p \to L^q$ regularity of many interesting operators may be studied this way. In [**2**] Christ has already shown that one need not be restricted to studying averages along curves, but may consider submanifolds of $\mathbb{R}^d$ of higher dimension, specifically the paraboloid. Further, the fact that strong type estimates may be established by exploiting the Lorentz "smoothing" that these objects present at the endpoints suggests that endpoint estimates may be established as well, at least in the case of translation-invariant operators. The work of Stovall in [**9**], as well as the simple application we gave in this article certainly raise hope that this may indeed be possible.

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