REPULSIVE FIXPOINTS OF ANALYTIC FUNCTIONS WITH APPLICATIONS TO COMPLEX DYNAMICS

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ABSTRACT

Let $\mathscr{G}$ be a family of functions analytic in a domain D in the complex plane. It is proved that $\mathscr{G}$ is a normal family, provided that for each $f\in\mathscr{G}$, there exists $k=k(f)>1$ such that the kth iterate f^k has no repulsive fixpoint in D. A new proof of a result of Bergweiler and Terglane concerning the dynamics of entire functions is also given.

1. *Introduction*

In the theory of complex dynamical systems, it is well known that the Julia set is the closure of the set of all repulsive periodic points. Thus if there is no repulsive periodic point in a domain, the domain must be contained in the Fatou set.

Lo Yang posed the following problem related to this kind of situation in [**11**, Problem 8, p. 164].

PROBLEM 1. Let D be a domain in the complex plane, let $\mathscr{G}$ be a family of entire functions and let k be a positive integer. If, for every $f\in\mathscr{G}$, both f and its iteration f^k of order k have no fixpoint in D, is $\mathscr{G}$ normal in D?

Here $f^2=f(f)$ and f^k is defined inductively via $f^k=f^{k-1}(f)$, $k=3,4,\ldots$.

Our main result concerns the more general case when $\mathscr{G}$ is a family of functions analytic in a domain D. It is possible that for some $k\geqslant 1$, f^k is not defined in some subset of D, or that although f^k is defined in some point $z\in D$, we might have $f^j(z)\notin D$ for some j, $1\leqslant j\leqslant k-1$ (for example, f may be the restriction of a rational map of the sphere). In either case we say that f^k has no fixpoint in D.

DEFINITION 1. Let D be a domain in the complex plane and let f be analytic in D. We say that f^k has a *fixpoint* $a\in D$ if the following conditions hold:

$$f^j(a)\in D,\quad j=1,2,\ldots,k,$$

and $f^k(a)=a$.

If a is a fixpoint of f^k and $|(f^k)'(a)|>1$, a is called a *repulsive fixpoint* of f^k or a *repulsive periodic point* of f. If a is a fixpoint of f^k and $|(f^k)'(a)|>1$ or $(f^k)'(a)=1$, then we say that a is *weakly repulsive* (see [**4**] and the references therein for the definitions).

In [**7**], we solved Yang's problem and proved the following more general result.

THEOREM A. *Let D be a domain in the finite complex plane; let $\mathscr{G}$ be a family of analytic functions in D. If for every $f\in\mathscr{G}$ there exists $k=k(f)>1$ such that f^k has no fixpoint in D, then $\mathscr{G}$ is a normal family.*

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Is Theorem A true if 'fixpoint' is replaced by 'repulsive fixpoint'? The answer is affirmative and given in the following theorem.

THEOREM 1. *Let D be a domain in the finite complex plane, and let $\mathcal{G}$ be a family of analytic functions in D. If for every $f \in \mathcal{G}$ there exists $k = k(f) > 1$ such that f^k has no repulsive fixpoint in D, then $\mathcal{G}$ is a normal family.*

The method of proof of Theorem 1 can be used to give a new proof of a result of Bergweiler and Terglane concerning the dynamics of entire functions.

Let us recall some notations and results in complex dynamics (see [**5**, **9**]).

Let f be a rational function in the extended plane $\bar{\mathcal{C}}$ or a transcendental entire function in the plane $\mathcal{C}$. The Fatou set $\mathcal{F}(f)$ of f is the subset of $\bar{\mathcal{C}}$ (or $\mathcal{C}$) where the set of iterations $\{f^n\}$ is normal in the sense of Montel. The complement of $\mathcal{F}(f)$ is the Julia set $\mathcal{J}(f)$. It is easy to see that Theorem 1 implies the deep result due to Baker (cf. [**1**]) that, for an entire function f, the repulsive periodic points of f are dense in $\mathcal{J}(f)$.

Now suppose that $\mathcal{F}(f) \neq \varnothing$ and D is a component of $\mathcal{F}(f)$. Since $\mathcal{F}(f)$ and $\mathcal{J}(f)$ are completely invariant under f, for each positive integer n, $f^n(D) = D_n$ is also a component of $\mathcal{F}(f)$. If there exist no integers $n > m > 0$ such that $D_m \cap D_n \neq \varnothing$, D is called a wandering domain.

A fundamental result due to Sullivan [**10**] is that a rational function f of degree bigger than 1 has no wandering domain in $\mathcal{F}(f)$. For transcendental entire functions, however, $\mathcal{F}(f)$ may have wandering domains, and even multiply-connected wandering domains (see for example [**2**, **3**]).

Let f be a transcendental entire function. In [**4**], the authors have got a sufficient result for $\mathcal{F}(f)$ to have no multiply-connected wandering domains.

THEOREM 2 [**4**]. *Let f be a transcendental entire function. If f has only finitely many weakly repulsive fixpoints, then $\mathcal{F}(f)$ has no multiply-connected wandering domains.*

In Section 3, we study repulsive fixpoints of polynomial-like mappings to prove Lemma 3 which is our basic tool in the proof of Theorems 1 and 2. On the way, we prove two results on polynomial mappings which have independent interest (cf. Theorems 3 and 4). In Section 5 we give a new proof of Theorem 2 which is completely different from that used in [**4**]. Besides this, our proof also gives some information concerning the location of the fixpoints. In Section 6 we give an example which shows that Theorem 2 is no longer true if the condition of weakly repulsive is replaced by that of repulsive.

2. *Lemmas*

One of the main tools in the proof of Theorem 1 is a consequence of Ahlfors' theory of covering surfaces which was first used in the field of complex dynamics by I. N. Baker. We give the last part of a lemma of Baker (cf. [**1**, Lemma 2], and also cf. [**8**, p. 170]).

LEMMA 1. *Let $f(z)$ be an analytic function in $|z| < r \leqslant \infty$ and let D_j $(j = 1, 2, 3)$ be three discs in the finite complex plane with disjoint closures. Then there is a constant*

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$C > 0$ depending only on r and D_j $(j = 1, 2, 3)$ (and not depending on f) such that if the spherical derivative $f^{\#}$ is such that

$$f^{\#}(0) = \frac{|f'(0)|}{1+|f(0)|^2} > C,$$

then there exists a region $\Omega \subset \{|z| < r\}$ such that f maps Ω univalently onto one of the D_j $(j = 1, 2, 3)$.

REMARK 1. If Ω is as above, we say that Ω is a simple (or univalent) island with respect to f over $f(\Omega)$ (cf. [**8**, p. 145]).

For $z \in C$ and $0 < \rho < +\infty$, let

$$D(z, \rho) = \{w \in C; |w - z| < \rho\}, \quad C(z, \rho) = \partial D(z, \rho).$$

LEMMA 2. *Let r be a positive number such that $0 < r < 1$, let f be analytic in $D(0, 1)$ and assume that $\Omega \subset D(0, r)$ is a univalent island over $D(0, 1)$ with respect to f, that is, $f(\Omega) = D(0, 1)$ and $f: \Omega \longrightarrow D(0, 1)$ is univalent. Then f has a repulsive fixpoint in Ω.*

This lemma is well known. For completeness we include the simple proof.

Proof of Lemma 2. We know that $f^{-1}(C(0, 1)) = \partial\Omega$ and that the correspondence is univalent. When w describes $C(0, 1)$, $f^{-1}(w)$ describes the closed curve $\partial\Omega$; hence f has one zero in Ω. Since

$$|f(z) - z - f(z)| = |z| \leqslant r < 1 = |f(z)|, \quad z \in \partial\Omega,$$

$f(z) - z$ has one zero in Ω. It follows from Schwarz's lemma that this fixpoint must be repulsive. This proves the lemma. □

3. *Repulsive fixpoints of polynomial-like mappings*

Another main tool in our proofs is the theory of polynomial-like mappings due to Douady and Hubbard (see [**6**]).

Let U, V contained in the finite complex plane be a pair of topological discs, that is, Jordan domains. A proper mapping between discs $f: U \longrightarrow V$ is a holomorphic mapping such that $f^{-1}(K) \subset U$ is compact for every compact set $K \subset V$. A necessary and sufficient condition for f to be proper is that $f(\partial U) = \partial V$ (see for example [**9**, p. 5]). The cardinality of the inverse image (counted with multiplicity) in U of a point is the degree of f.

A polynomial-like mapping $f: U \longrightarrow V$ is a proper mapping of degree greater than 1 between discs such that $\bar{U} \subset V$.

We need a fundamental result of Douady and Hubbard (see [**6**] or [**5**, Theorem VI.1.1]).

THEOREM B. *Let $f: U \longrightarrow V$ be a polynomial-like mapping of degree d. Then there is a quasiconformal mapping ϕ with $\phi(z) = z + o(1)$ near ∞ and such that $f = \phi \circ P \circ \phi^{-1}$ on U, where P is a polynomial of degree d.*

Let $f: U \longrightarrow V$ be a proper mapping such that $\bar{U} \subset V$. If the degree of f is 1, Lemma 2 tells us that f has a repulsive fixpoint in U. In this case $f: U \longrightarrow V$ is called a regular mapping. If the degree d of f is greater than 1, we need to prove that, for

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all $n > 1$, f^n has at least one repulsive fixpoint in U. In order to prove that, we first consider polynomial mappings.

Let P be a polynomial of degree $d > 1$ and assume that z_0 is a periodic point of P with period q ($\geqslant 1$). Then we have a cycle $\{z_0, P(z_0), \ldots, P^{q-1}(z_0)\}$. We define $(P^q)'(z_0)$ to be the multiplier of the cycle. The cycle is said to be nonrepulsive if $|(P^q)'(z_0)| \leqslant 1$.

Now assume that $(P^q)'(z_0)$ is a primitive mth root of unity for some integer m, $1 \leqslant m < \infty$. In this case we say that the cycle is rational neutral. We have the following formula:

$$P^{qm}(z) = z + b_0(z - z_0)^v + \ldots, \tag{1}$$

(cf. [**5**, p. 41]), where $b_0 \neq 0$ and $v = km + 1$ for some positive integer k. We say that the numbers v and k are the multiplicity and the dynamical multiplicity of the q-cycle, respectively. According to [**5**, p. 60], there are k immediate basins of attraction associated with z_0. If the q-cycle is not rational neutral, that is, either the modulus of the multiplier of the cycle is not 1 or $(P^q)'(z_0) = e^{2\pi i\theta_0}$, where θ_0 is irrational, we define its multiplicity and dynamical multiplicity both to be 1.

Recall that a point $w_0 \in D$ is said to be a critical point of P if $f'(w_0) = 0$. We claim that every immediate basin of attraction of z_0 contains at least one critical point of P (see [**5**] for the definition of the immediate attracting basins). We use the argument in [**5**, proof of Theorem III.2.2] which immediately gives us the attracting case. Assume now that the cycle is rational neutral. Let A^* be the immediate basin of attraction containing a petal Q of P^{qm} at z_0. Also in the case when the dynamical multiplicity k is larger than 1, there exists a Fatou coordinate function defined on Q (cf. [**5**, p. 39]). Let $R = P^{qm}$. Arguing as in [**5**, proof of Theorem III.2.3], we find $w_0 \in A^*$ such that $(R^N)'(w_0) = 0$ for some $N \geqslant 1$. Hence we can also find $w_1 \in A^*$ such that $P'(w_1) = 0$. In each one of the k immediate basins of attraction, we can in this way find a critical point of P.

The following result, which will be used to prove Theorem 4, gives us a more accurate bound for the number of nonrepulsive cycles. As it has some independent interest, we include it as a theorem.

THEOREM 3. *If P is a polynomial of degree d, then P has at most $d-1$ nonrepulsive cycles counting dynamical multiplicities in the finite complex plane.*

Proof. Let $\mathscr{M}$ be the set of points occurring in nonrepulsive cycles which are not rational neutral and let $\mathscr{M}_1$ be the set of points occurring in nonrepulsive cycles which are rational neutral. As there are at most finitely many nonrepulsive cycles (cf. [**5**, Theorem III.2.7, p. 62]), there exists an integer $K > 1$ such that

$$K \geqslant \max\{\text{multiplicities of cycles in } \mathscr{M}_1\}.$$

Following the argument in [**5**, pp. 100–101], we find a polynomial Q such that

(i) $Q(z) = 0$ for all $z \in \mathscr{M}$;

(ii) $\sum \mathrm{Re}(Q'(z_j)/P'(z_j)) < 0$ for all neutral cycles $\{z_1, \ldots, z_m\} \subset \mathscr{M}$;

and another polynomial g such that

(iii) every $w \in \mathscr{M}_1$ is a zero of g with multiplicity $K+1$;

(iv) $g(z) = 1$ for every $z \in \mathscr{M}$.

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To find such a polynomial g we let

$$g_1(z) = \prod_{w_j \in \mathscr{M}_1} (z-w_j)^{K+1}$$

and

$$g_2(z) = \prod_{z_j \in \mathscr{M}} (z-z_j).$$

Then $g_1(z_k) := a_k \neq 0$ for every $z_k \in \mathscr{M}$. Define

$$g_3(z) = \sum_{z_j \in \mathscr{M}} \frac{g_2(z)}{a_j(z-z_j)\, g_2'(z_j)}.$$

Then $g_3(z)$ is a polynomial with $g_3(z_j) = 1/a_j$. Finally we define $g(z) = g_1(z)\, g_3(z)$.

Now for small $\varepsilon > 0$, set $f = P + \varepsilon Qg$. We can check that all $z \in \mathscr{M}$ give us attracting cycles of f and that all $z \in \mathscr{M}_1$ will be rational neutral periodic points of f with the same (dynamical) multiplicities as they are periodic points of P. Arguing as in [**5**, p. 100] and using [**5**, Theorem 2.4, p. 61], we deduce the theorem. □

In order to prove Theorem 4, which is one of the main tools in our proofs, we have to say more about different types of multiplicities.

Let P be a polynomial of degree $d > 1$ and let $n > 1$ be an integer. Assume that z_0 is a fixpoint of P^n. Then z_0 is a periodic point of P with period $1 \leqslant q \leqslant n$ and we have a cycle $\{z_0, P(z_0), \ldots, P^{q-1}(z_0)\}$. It is easy to see that q is a factor of n. We claim that the multiplicity of any point in the cycle as a root of the equation

$$P^n(z) - z = 0 \tag{2}$$

is constant over the cycle. To see this, we note that as long as $(P^n)'(z_0) \neq 1$, there will be no problem since $(P^n)'(z_j)$ is indeed the same for all points of the cycle. If

$$P^n(z) = z + a_\mu(z-z_0)^\mu + \ldots, \quad a_\mu \neq 0,\ \mu \geqslant 2,$$

and

$$t = h(z) = P(z) = z_1 + b(z-z_0) + \ldots$$

has inverse $z = h^{-1}(t) = z_0 + b^{-1}(t-z_1) + \ldots$ (we have necessarily $b \neq 0$), then near z_1 we have

$$P^n(t) = h \circ P^n \circ h^{-1}(t) = t + a'_\mu(t-z_1)^\mu + \ldots, \quad a'_\mu \neq 0,$$

and our claim is proved.

We need some facts connecting the multiplicity μ of z_0 as a root of equation (2) and the multiplicity ν of the q-cycle. Suppose that

$$P^n(z) = z + a_0(z-z_0)^\mu + \ldots,$$

where $a_0 \neq 0$. In the rational neutral case, there are two possibilities.

(i) $\mu > 1$. Then $\nu = \mu$.

(ii) $\mu = 1$. Then $\nu > \mu$.

In the remaining cases, we know by definition that

(iii) $\nu = \mu = 1$.

To prove (i), we assume that (1) holds for some $m \geqslant 1$. Since $\mu > 1$, we have $qm \leqslant n$ and there exists an integer q' such that $n = mqq'$. From (1), it is easy to see that $P^n(z) = z + q' b_0(z-z_0)^\nu + \ldots$. It follows that $\nu = \mu$.

In case (ii), it is clear that $\nu > 1 = \mu$.

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Now we are able to prove the following result, which we also include as a theorem.

THEOREM 4. *Let P be a polynomial of degree $d > 1$. Then*
(a) *P has at least one weakly repulsive fixpoint;*
(b) *for any $n > 1$, P^n has at least one repulsive fixpoint.*

Proof. Without loss of generality we suppose that $P(z)$ is monic. If $P(z)-z$ has no multiple zeros, we have d fixpoints of P. According to Douady's theorem (cf. [**5**, Theorem VI.1.2]) or Theorem 3, at least one of these fixpoints is repulsive and (a) and (b) are true. If P has no repulsive fixpoint, there must be at least one fixpoint z_0 such that $P'(z_0) = 1$ and (a) holds.

To prove (b), we assume that P^n has no repulsive fixpoint and obtain a contradiction by the following argument.

Let the cycles defined by the fixpoints of P^n start at $\{z_j\}$ and have periods $\{q_j\}$. We define

$$\nu_j^* = \nu_j, \quad \mu_j > 1; \quad \nu_j^* = 1, \quad \mu_j = 1,$$

where μ_j is the multiplicity of z_j as a root of equation (2) and ν_j is the multiplicity of the jth cycle. From our discussion above, we know that $\nu_j^* = \mu_j$. Hence

$$P^n(z)-z = \prod_{j=1}^{M}\prod_{l=1}^{q_j}(z-P^l(z_j))^{\nu_j^*}$$

with $\sum_{j=1}^{M}\nu_j^* q_j = d^n$. Again by Douady's theorem (cf. [**5**, Theorem VI.1.2]) or Theorem 3, we know that $1 \leqslant M \leqslant d-1$.

Denote the dynamical multiplicity of the jth cycle by N_j. If $\nu_j^* > 1$, then there exists m_j $(1 \leqslant m_j \leqslant n)$ such that $\nu_j^* = m_j N_j + 1$. *If* $\nu_j^* = 1$, we write $\nu_j^* = m_j N_j + 1$, with $m_j = 0$.

We claim that

$$m_j q_j \leqslant n^2/2, \quad 1 \leqslant j \leqslant n.$$

If $\nu_j^* = 1$, we have $m_j = 0$ and there is nothing to prove. It remains to discuss the case $\nu_j^* > 1$; then we have $(P^n)'(z_j) = 1$. If $q_j = n$, the period of the jth cycle is n. Then we have either $\mu_j > 1$ which implies that $m_j = 1$ or $\mu_j = 1$ which implies that $m_j = 0$. In both cases, the inequality holds. We now consider the case when $q_j < n$. Since q_j is a factor of n, it is clear that $q_j \leqslant n/2$. Thus the claim is proved in all cases.

Since P has at least one fixpoint, there exists at least one index j_0 such that $q_{j_0} = 1$. We conclude that $\sum_{j=1}^{M} q_j < nM$.

It follows from Theorem 3 that

$$\sum_{j=1}^{M} N_j \leqslant d-1.$$

Using this inequality and the fact that $M \leqslant d-1$, we deduce that

$$\begin{aligned} d^n &= \sum_{j=1}^{M} q_j \nu_j^* = \sum_{j=1}^{M} q_j(m_j N_j + 1) \\ &< n^2/2 \sum_{j=1}^{M} N_j + nM \leqslant (d-1)(n^2/2+n). \end{aligned}$$

If $d > 1$ and $n > 1$, this inequality is false. This completes the proof of Theorem 4. □

Returning to our two topological discs U and V in the finite plane with $\bar{U} \subset V$, we can now prove the following lemma.

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LEMMA 3. *Let $f: U \longrightarrow V$ be a regular mapping or a polynomial-like mapping of degree d. Then*

(1) *f has at least one weakly repulsive fixpoint in U;*

(2) *for any integer $n > 1$, f^n has at least one repulsive fixpoint in U.*

Proof. Applying Theorem B, we deduce that it suffices to treat the case when f is a polynomial P of degree $d\ (>1)$ defining a proper mapping of degree d between discs U' and V'. It is clear that $P^{-1}(U') = \bigcup_{n\geqslant 1} P^{-n}(U')$ is a relatively compact subset of U'. We know that the Julia set $\mathscr{J}(P)$ is adherent to this inverse orbit (cf. [**5**, p. 57]). We conclude that $\mathscr{J}(P^n) = \mathscr{J}(P) \subset U'$. From Theorem 4, we know that P^n has at least one repulsive fixpoint if $n > 1$. It is easy to see that this fixpoint belongs to the Julia set $\mathscr{J}(P^n)$ and is thus in U' (cf. [**5**, Theorem III.1.1]) in the sense of Definition 1. The same argument works for weakly repulsive fixpoints in the case $n = 1$. We have proved Lemma 3. □

Alternatively, we could use the filled-in Julia set $\mathscr{K}(P) = \bigcap_{n\geqslant 1} P^{-n}(U')$ which is a relatively compact subset of U'. We know that $\mathscr{J}(P)$ is the boundary of $\mathscr{K}(P)$ (cf. [**6**, p. 295]). Arguing as above, we see again that Lemma 3 holds.

4. *Proof of Theorem* 1

In order to prove Theorem 1, we first prove the following lemma.

LEMMA 4. *Let $z_0 \in D$ and let $\{f_n\}$ be a sequence of functions analytic in D which is normal in $D\backslash\{z_0\} = D^*$ but is such that no subsequence is normal at z_0. Then for each $r > 0$ such that $D(z_0, r) \subset D$, there exists $n_0 = n_0(r)$ such that for each integer $k > 1$, f_n^k has a repulsive fixpoint in $D(z_0, r)$ for all $n \geqslant n_0$.*

Proof. Without loss of generality, we suppose that D is the unit disc and $z_0 = 0$. Let $r \in (0, 1)$ be given. Since $\{f_n\}$ is normal in D^*, we can find a subsequence of $\{f_n\}$ which we also call $\{f_n\}$, such that if $f_n \to F$ uniformly in $\{r_1 < |z| < r_2\}$ as $n \to \infty$, where $0 < r_1 < r < r_2$, then either $F = \infty$ or F is analytic in $\{r_1 < |z| < r_2\}$. If F is analytic, it follows from the maximum principle that $|f_n(z)|$ is uniformly bounded on $|z| \leqslant r$. This is impossible since $\{f_n\}$ is not normal at the origin. Thus we must have $F = \infty$.

We claim that f_n must have at least one zero in $\{|z| < r\}$ for all large n. Otherwise, the family $\{1/f_n\}$ is normal at $z = 0$ which contradicts our assumptions. Since $\{f_n\}$ is normal in D^*, for any given $0 < r_1 < r$, we deduce that $f_n \to \infty$ uniformly on $r_1 \leqslant |z| \leqslant r$. Therefore there exists n_0 such that $|f_n(z)| > 1$ on $r_1 \leqslant |z| \leqslant r$ for all $n \geqslant n_0$. Furthermore, we note that

$$|f_n(z) - z - f_n(z)| = |z| = r_1 < |f_n(z)|, \quad |z| = r_1.$$

Since $f_n(z)$ has at least one zero in $|z| < r_1$, it follows from Rouché's theorem that $f_n(z) - z$ must have at least one zero in $|z| < r_1$, that is, f_n has at least one fixpoint in $|z| < r_1$.

We next show, for each $k > 1$, that f_n^k has at least one repulsive fixpoint in D for all $n \geqslant n_0$. Let us assume that $n_0 \geqslant 2$. We recall that if w_0 is a critical point of f, then $f(w_0)$ is called a critical value of f.

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We now fix $r_1 < r$ and choose r_n such that $r_1 \leqslant r_n \leqslant r$ and there is no critical value of f_n on $C(0, r_n)$. We can do this, because f_n has only finitely many critical points in $D(0, r)$. Since $|f_n| > 1$ on $C(0, r_1)$ and we have a fixpoint inside $C(0, r_1)$, it is clear from the argument principle that $C(0, r_n)$ is contained in the range of f_n. Consider $f_n^{-1}(C(0, r_n))$, which is a set of disjoint Jordan curves. We note that $C(0, r_n) \cap f_n^{-1}(C(0, r_n)) = \varnothing$. Using again our fixpoint in $C(0, r_1)$, we deduce that there is a component $C_n \subset f_n^{-1}(C(0, r_n))$ such that C_n is inside $C(0, r_n)$ and there is one fixpoint inside C_n. Denote the topological disc bounded by C_n by D_n. Then f_n maps C_n onto $C(0, r_n)$. This implies that $f_n : D_n \longrightarrow C(0, r_n)$ is a regular mapping or a polynomial-like mapping. It follows from Lemma 3 that f_n^k has at least one repulsive fixpoint in $D_n \subset C(0, r)$ for all $k > 1$. The lemma is proved. □

The proof of Theorem 1 is almost the same as the proof of the main result in [7]. We give an outline here for the convenience of the reader; see [7] for the details.

Suppose that $\mathscr{G}$ satisfies the assumptions of Theorem 1 but is not normal in D. Then we may select from $\mathscr{G}$ a sequence which we may without loss of generality take to be the whole of $\mathscr{G}$, such that there is some $z_1 \in D$ with the property that no subsequence of $\mathscr{G}$ is normal in any neighbourhood of z_1. If there exists a sequence in $\mathscr{G}$ which is normal in a punctured neighbourhood of z_1, we use Lemma 4 and find repulsive fixpoints of f_n^k for all large n and all $k > 1$ which contradicts our assumptions. We can now find distinct points $\{z_j\}_1^5$ in D and a sequence in $\mathscr{G}$ which is not normal at these points. Let $\{D_j\}$ be discs centred at these points, all with radius ε, chosen so small that the closures of the discs are disjoint and contained in D. We can now find $f \in \mathscr{G}$ such that, for each $k > 1$, there exists a j such that f^k defines a univalent map from an island in D_j onto D_j. It follows from Lemma 2 that f^k has a repulsive fixpoint in this island. This is a contradiction and Theorem 1 is proved. □

5. *Proof of Theorem* 2

Suppose that f is a transcendental entire function in the plane and that there is a multiply-connected wandering domain D of $\mathscr{F}(f)$. We shall prove that f has infinitely many weakly repulsive fixpoints.

It is known that $f^n \to \infty$ in D (cf. [2]). Moreover if ℓ is a simple closed curve in D which is not null-homotopic in D, then the winding number of $f^n(\ell)$ about 0 is non-zero for sufficiently large n. In fact, if $D_n = f^n(D)$ and α_n, β_n denote the boundaries of the components A_n, B_n of D_n^c which contain 0 and ∞, respectively, then $f(\alpha_n) = \alpha_{n+1}$, $f(\beta_n) = \beta_{n+1}$. Further, β_n is in the component of D_{n+1}^c whose boundary is α_{n+1}. This is clear since α_n must be a subset of $\text{int}(\alpha_{n+1})$ for all n and α_n and β_n are subsets of the boundary of the connected set D_n which does not meet α_{n+1}. We note that α_n and β_n are disjoint.

Let $B = \{\text{zeros and 1-points of } f\}$ so that by Picard's theorem B is an infinite set. Let C_n denote the region whose boundary is $\alpha_n \cup \alpha_{n+1}$. For infinitely many n, we have $C_n \cap B \neq \varnothing$. We shall prove that for each such n, there is a weakly repulsive fixpoint in C_n.

Suppose that $z_n \in C_n \cap B$. Since $f(z_n) = 0$ or 1, we have (for large n) $z_n \notin D_n$. Let us first assume that z_n belongs to a component G of D_n^c different from B_n (and of course A_n). Then $\partial f(G) = f(\partial G)$ is a component of ∂D_{n+1} which can only be α_{n+1} since $f(G)$ meets $\{0, 1\}$.

We note that f has no finite asymptotic values and that the set of critical values

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of f is countable. Since α_{n+1} and β_{n+1} are disjoint, it follows that ∂G does not meet β_n or α_{n+1}. We take a Jordan curve ℓ in $A_{n+1}\backslash\bar{G}$ with $G \subset \operatorname{Int}\ell$ which passes through no critical value of f and a component ℓ' of $f^{-1}(\ell)$ in G. Then ℓ' is a Jordan curve which bounds a topological disc $U \subset G$ and $f(U) = V = \operatorname{Int}\ell$. The map $f: U \longrightarrow V$ is either univalent or polynomial-like.

In the second case, we have $z_n \in B_n \cap A_{n+1}$. This time, we take ℓ to be a Jordan curve in D_{n+1} whose winding number around 0 is 1 and which contains no critical values of f. We may join z_n to ℓ in B_n by a path γ and there is a first point $t \in \gamma$ with $f(t) \in \ell$. The component ℓ' of $f^{-1}(\ell)$ which contains t is then a Jordan curve in $\mathscr{F}(f)$ and does not meet D_n. Either $I = \operatorname{Int}\ell'$ contains D_n or $\operatorname{Int}\ell'$ lies in $B_n \cap A_{n+1}$. In the first case, $f(I) \supset f(D_n) = D_{n+1}$ which is impossible since $\partial f(I) = f(\ell') = \ell$. Thus taking the topological discs $U = \operatorname{Int}\ell'$ and $V = \operatorname{Int}\ell$, we again have a polynomial-like or univalent map $f: U \longrightarrow V$.

Thus in both cases, it follows from Lemma 3 that f has a weakly repulsive fixed point in $U \subset C_n$. The proof is complete. □

6. *An example*

In [**3**], Baker constructed an entire function as follows. Take a constant C such that

$$0 < C < \frac{4}{e^2}.$$

Suppose that n_0 is a positive integer and r_1 is a number such that

$$r_1 > 1, \quad 2^{n_0+1}C > 2r_1^2.$$

For each integer n, denote by r_n numbers such that

$$r_{n+1} > 2r_n, \quad 1 \leqslant n \leqslant n_0,$$

and

$$r_{n+1} = C^2\left(1+\frac{r_n}{r_1}\right)^2 \ldots \left(1+\frac{r_n}{r_n}\right)^2.$$

Define the entire function

$$f(z) = C^2\prod_{j=1}^{\infty}\left(1+\frac{z}{r_j}\right)^2.$$

Then, in [**3**], Baker proved that f has multiply-connected wandering domains.

Now we set

$$g(z) = z + f(z)$$

and prove that $g(z)$ also has multiply-connected wandering domains.

In fact set

$$s_n = \left(\frac{n+1}{n+2}\right)r_n + 1$$

and

$$B_n = \{z; r_n^2 < |z| < s_n\}.$$

It is proved in [**3**] that

(i) $f(s_{n-1})/s_n = 1 - 3/4n^2 + O(1/n^3) < 1$;

(ii) $f(-s_{n-1}) > r_n^3/(n+1)^2$;

(iii) there is a positive constant $A > 0$ such that

$$\frac{f(-r_n^2)}{r_{n+1}^2} > Ar_1^2 \ldots r_{n-1}^2(r_n - 1)^2 > 1.$$

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It is easy to see that (i), (ii) and (iii) imply that $f(B_n) \subset B_{n+1}$. Therefore it follows from basic properties of complex dynamics of entire functions that f has multiply-connected wandering domains.

Next we prove that (i), (ii) and (iii) also hold with f replaced by g.

In fact, for large n, we have

$$r_n > 2^n$$

and

$$r_{n+1} > 4r_n^2.$$

We can easily deduce that

$$\frac{g(s_{n-1})}{s_n} = \frac{s_{n-1}}{s_n} + 1 - \frac{3}{4n^2} + O\left(\frac{1}{n^3}\right)$$

$$< \frac{1}{2^n} + 1 - \frac{3}{4n^2} + O\left(\frac{1}{n^3}\right) < 1,$$

$$g(s_{n-1}) > \frac{r_n^3}{(n+1)^2} - s_{n-1} > r_n^2,$$

and

$$\frac{g(-r_n^2)}{r_{n+1}^2} > Ar_1^2 \dots r_{n-1}^2 (r_n - 1)^2 - \frac{r_n^2}{r_{n+1}^2} > 1.$$

Therefore g has multiply-connected wandering domains. On the other hand, it is obvious that all fixpoints of g have multiplier 1. Thus Theorem 2 is best possible.

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