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Non-singular boundary integral methods for fluid mechanics applications

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A formulation of the boundary integral method for solving partial differential equations has been developed whereby the usual weakly singular integral and the Cauchy principal value integral can be removed analytically. The broad applicability of the approach is illustrated with a number of problems of practical interest to fluid and continuum mechanics including the solution of the Laplace equation for potential flow, the Helmholtz equation as well as the equations for Stokes flow and linear elasticity.

Key words: boundary integral methods

1. Introduction

The boundary integral formulation is an efficient method of representing the solutions of certain linear partial differential equations by reducing the dimensionality of the problem by one. The solution in a volume or area domain is represented in terms of an integral over surface(s) or line(s) that enclose the domain. Solutions to fluid dynamics problems that can be modelled by the Laplace equation for potential flow or the Stokes equation for low-Reynolds-number flow as well as continuum mechanics problems such as the Helmholtz equation in scattering problems or problems in linear elasticity can all be represented in terms of such boundary integrals (Becker 1992). Symm (1963) provided a practical way to solve the integral equations by treating the surfaces or lines as discrete elements. Since the 1970s the boundary integral method (BIM) has gained increasing prominence (see Cheng & Cheng 2005, for a historical overview). The advantage of the BIM is self-evident. The reduction in the dimensionality of the problem from a volume (surface) mesh to a surface (line) mesh provides a substantial gain in computational efficiency. However, this gain is offset by the fact that the numerical implementation of the BIM is not straightforward because the approach is plagued by '*a mathematical monster that leaps out of every page*' (Becker 1992). In essence, the boundary element formulation uses the Green's function that has a $1/r$ divergence and a $1/r^2$ divergence in its derivative around the source point. The integral over the $1/r$ divergence gives rise to a weak singularity that

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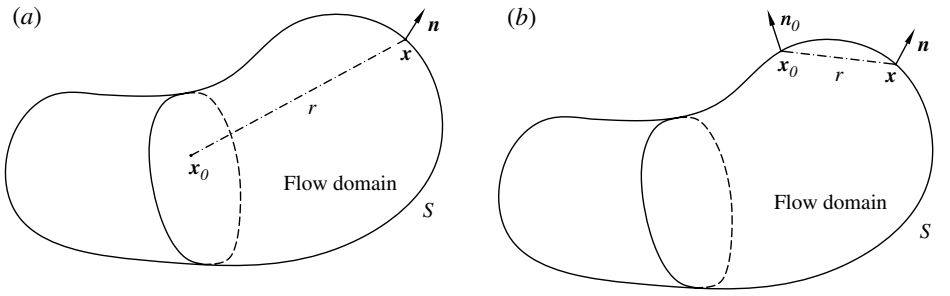


FIGURE 1. Illustration of the BIM applied to flow problems in an arbitrary three-dimensional flow domain with closed surface S: (*a*) $\boldsymbol{x}_0$ is inside the flow domain; (*b*) $\boldsymbol{x}_0$ is on the flow domain surface S.

can be evaluated using semi-analytical techniques. The term from the $1/r^2$ divergence gives rise to a Cauchy principal value (PV) integral that requires careful numerical treatment.

In this communication, we develop a general non-singular boundary integral formulation that is applicable to the Laplace equation for the potential problem, the Helmholtz equation, and equations associated with Stokes flow and linear elastic deformations. The approach is based on removing the singularities in the BIM formulation by subtracting the solution of a special related problem. We demonstrate the details of our approach using the potential problem from which it is easy to see how the method can be extended to the more complicated cases of Stokes flow and linearly elastic deformations. Validation of the approach is obtained by comparing numerical results for problems in Stokes flow for which analytic results are known. This non-singular boundary integral formulation simplifies numerical solutions based on this popular technique.

2. Potential problem: non-singular boundary integral formulation

In fluid dynamics, the potential problem arises in incompressible, inviscid or high-Reynolds-number flows. The velocity field $\boldsymbol{u} = \nabla\phi$ can be expressed in terms of a scalar potential that satisfies the Laplace equation $\nabla^2\phi = 0$. The corresponding free space Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0) = 1/r$, $r = |\boldsymbol{x} - \boldsymbol{x}_0|$ satisfies $\nabla^2 G = -4\pi\delta(\boldsymbol{x} - \boldsymbol{x}_0)$, where $\delta(\boldsymbol{x} - \boldsymbol{x}_0)$ is the Dirac δ-function. Consider the solution in the fluid domain enclosed by the surface S. With the help of Green's second identity (Becker 1992), the solution at $\boldsymbol{x}_0$ *inside* the domain can be written as the surface integral (see figure 1*a*)

$$4\pi\phi(\boldsymbol{x}_0) + \int_S \phi(\boldsymbol{x})\nabla G(\boldsymbol{x}, \boldsymbol{x}_0)\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x}) = \int_S G(\boldsymbol{x}, \boldsymbol{x}_0)\nabla\phi(\boldsymbol{x})\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x}). \tag{2.1}$$

This integral relates the potential ϕ at $\boldsymbol{x}_0$ inside the domain to integrals over ϕ and its normal derivative $\nabla\phi\cdot\boldsymbol{n} = \partial\phi/\partial n$, on the surface S with $\boldsymbol{n}$ being the outward unit normal. The vector $\boldsymbol{x}$ points to the integration position on the surface S. By letting $\boldsymbol{x}_0$ onto the surface S, we have an equation that can be solved for ϕ (or $\partial\phi/\partial n$) on the surface if $\partial\phi/\partial n$ (or ϕ) is specified. This corresponds to the Dirichlet (or Neumann) problem. This is the boundary integral formulation. However, with $\boldsymbol{x}_0$ on the surface, then as $\boldsymbol{x} \to \boldsymbol{x}_0$, the integral involving G with a $1/r$ singularity (the single layer term) has a weak singularity that can be handled numerically by changing to local polar coordinates on the surface whereas the integral involving ∇G with a $1/r^2$ singularity

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(the double layer term) gives rise to a Cauchy PV integral and a Dirac δ-function contribution. Thus, with $\boldsymbol{x}_0$ on the surface S as illustrated in figure 1(*b*), the boundary integral equation that needs to be solved is

$$(4\pi - c)\phi(\boldsymbol{x}_0) + \int_{S,\mathrm{PV}} \phi(\boldsymbol{x})\nabla G(\boldsymbol{x},\boldsymbol{x}_0)\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x}) = \int_S G(\boldsymbol{x},\boldsymbol{x}_0)\nabla\phi(\boldsymbol{x})\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x}), \quad (2.2)$$

where c is the solid angle subtended at $\boldsymbol{x}_0$ with $c = 2\pi$ if the surface has a defined curvature at $\boldsymbol{x}_0$. The numerical evaluation of the weak singularity associated with G and the Cauchy principal value integral associated with ∇G in (2.2) requires special considerations as ordinary integration methods such as Gaussian quadrature can no longer be used (Becker 1992). Different numerical methods have been developed to handle these singularities (see for example Lean & Wexler 1985, Bazhlekov, Anderson & Meijer 2004). Thus, if either the potential or the normal velocity, $\partial\phi/\partial n$, is known on the surface S, the other unknown quantity can be calculated (see for example Blake, Taib & Gibson 1986, Wang 1998, Zhang *et al.* 2001, Wrobel 2002, Fong *et al.* 2009 and Gonzalez-Avila *et al.* 2011). The aim therefore is to avoid the numerical effort needed when having to deal with these singularities.

We recapitulate the earlier work of Klaseboer, Rosales-Fernandez & Khoo (2009) and show that both singular terms associated with G and ∇G in (2.2) can be removed by considering the linear potential function

$$\psi(\boldsymbol{x}) = \phi(\boldsymbol{x}_0) + \boldsymbol{a}(\boldsymbol{x}_0)\cdot(\boldsymbol{x} - \boldsymbol{x}_0), \quad (2.3)$$

where the vector $\boldsymbol{a}(\boldsymbol{x}_0)$ will be chosen to eliminate the singularities in (2.2). Clearly $\psi(\boldsymbol{x})$ satisfies $\nabla^2\psi = 0$ and the Green's identity, (2.1):

$$4\pi\psi(\boldsymbol{x}_0) + \int_S \psi(\boldsymbol{x})\nabla G(\boldsymbol{x},\boldsymbol{x}_0)\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x}) = \int_S G(\boldsymbol{x},\boldsymbol{x}_0)\nabla\psi(\boldsymbol{x})\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x}), \quad (2.4)$$

with $\boldsymbol{x}_0$ inside the domain. Thus, subtracting (2.4) from (2.1) and using (2.3) gives

$$\int_S [\phi(\boldsymbol{x}) - \psi(\boldsymbol{x})]\nabla G(\boldsymbol{x},\boldsymbol{x}_0)\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x}) = \int_S G(\boldsymbol{x},\boldsymbol{x}_0)\nabla[\phi(\boldsymbol{x}) - \psi(\boldsymbol{x})]\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x}). \quad (2.5)$$

When $\boldsymbol{x}_0$ is located on the surface S, the singularities in (2.5) can be eliminated with the following choice of the vector

$$\boldsymbol{a}(\boldsymbol{x}_0) = [\nabla\phi(\boldsymbol{x}_0)\cdot\boldsymbol{n}_0]\boldsymbol{n}_0 \equiv \left(\frac{\partial\phi}{\partial n}\right)_0 \boldsymbol{n}_0, \quad (2.6)$$

that depends on the point $\boldsymbol{x}_0$, where the outward unit normal is $\boldsymbol{n}_0 \equiv \boldsymbol{n}(\boldsymbol{x}_0)$ (see figure 1*b*). Finally the required result of the non-singular formulation of the boundary integral equation, with $\boldsymbol{x}_0$ *now located on the surface* S, takes the form

$$\int_S \left[\phi(\boldsymbol{x}) - \phi(\boldsymbol{x}_0) - \left(\frac{\partial\phi}{\partial n}\right)_0 \boldsymbol{n}_0\cdot(\boldsymbol{x} - \boldsymbol{x}_0)\right]\nabla G(\boldsymbol{x},\boldsymbol{x}_0)\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x})$$
$$= \int_S G(\boldsymbol{x},\boldsymbol{x}_0)\left[\frac{\partial\phi}{\partial n} - \left(\frac{\partial\phi}{\partial n}\right)_0 \boldsymbol{n}_0\cdot\boldsymbol{n}\right]\mathrm{d}S(\boldsymbol{x}). \quad (2.7)$$

This result supersedes the traditional form of the boundary integral formulation given in (2.2) because all singularities have now been removed. In particular, as $\boldsymbol{x} \to \boldsymbol{x}_0$ which is now on the surface S, the weak singularity associated with the integral over G has been eliminated because $[(\partial\phi/\partial n) - (\partial\phi/\partial n)_0\boldsymbol{n}_0\cdot\boldsymbol{n}] \to 0$. In the integral over ∇G,

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there will no longer be a Cauchy PV integral or Dirac δ-function contribution as in (2.2). The approach in (2.7) that we present here will give a relationship between ϕ and $\partial\phi/\partial n$ as in the original problem. The numerical algorithm for solving (2.7) using the BIM is given by Klaseboer *et al.* (2009). A proof of the convergence of the integrals in (2.7) is given in the Appendix. Liu & Rudolphi (1999) have suggested a similar method of removing the singularities via a Taylor series expansion. Unfortunately, when their approach is implemented, it would give a relationship between ϕ, $\partial\phi/\partial n$ and $\nabla\phi(\boldsymbol{x}_0)\cdot\boldsymbol{n}$, which requires knowledge of $\nabla\phi(\boldsymbol{x}_0)\cdot\boldsymbol{n}$.

The surface integral in (2.7) is taken over all surfaces that enclose the domain. In particular, for problems in an infinite domain outside the surface S, one must also take into account the 'surface at infinity' which will give an additional term $4\pi\phi(\boldsymbol{x}_0)$ on the left-hand side of (2.7), see Liu & Rudolphi (1991), Liu & Rudolphi (1999) and Klaseboer *et al.* (2009).

3. Helmholtz problem: non-singular boundary integral formulation

For the solution of the Helmholtz equation $\nabla^2\phi + k^2\phi = 0$, in which k is the wavenumber, we have the free space Green's function $H = \cos(kr)/r$, $r = |\boldsymbol{x} - \boldsymbol{x}_0|$ that satisfies $\nabla^2 H + k^2 H = -4\pi\delta(\boldsymbol{x} - \boldsymbol{x}_0)$. In the same way as we obtained (2.5) for the potential problem, we find, for $\boldsymbol{x}_0$ in the domain

$$\int_S [\phi(\boldsymbol{x}) - \psi(\boldsymbol{x})]\nabla H(\boldsymbol{x}, \boldsymbol{x}_0)\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x}) = \int_S H(\boldsymbol{x}, \boldsymbol{x}_0)\nabla[\phi(\boldsymbol{x}) - \psi(\boldsymbol{x})]\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x}), \quad (3.1)$$

where

$$\psi(\boldsymbol{x}) = \phi(\boldsymbol{x}_0)\cos[k\boldsymbol{n}_0\cdot(\boldsymbol{x} - \boldsymbol{x}_0)] + \frac{b(\boldsymbol{x}_0)}{k}\sin[k\boldsymbol{n}_0\cdot(\boldsymbol{x} - \boldsymbol{x}_0)]. \quad (3.2)$$

Upon putting $\boldsymbol{x}_0$ onto the surface S in (3.1), we can eliminate all singular terms with the choice

$$b(\boldsymbol{x}_0) = \nabla\phi(\boldsymbol{x}_0)\cdot\boldsymbol{n}_0 \equiv \left(\frac{\partial\phi}{\partial n}\right)_0. \quad (3.3)$$

Thus, the non-singular formulation of boundary integral equation for the Helmholtz problem when $\boldsymbol{x}_0$ *is now located on the surface* S, with $\chi \equiv k\boldsymbol{n}_0\cdot(\boldsymbol{x} - \boldsymbol{x}_0)$, is

$$\int_S \left\{\phi(\boldsymbol{x}) - \phi(\boldsymbol{x}_0)\cos\chi - \frac{1}{k}\left(\frac{\partial\phi}{\partial n}\right)_0 \sin\chi\right\}\nabla H(\boldsymbol{x}, \boldsymbol{x}_0)\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x})$$
$$= \int_S H(\boldsymbol{x}, \boldsymbol{x}_0)\left\{\frac{\partial\phi}{\partial n} + k\phi(\boldsymbol{x}_0)\boldsymbol{n}_0\cdot\boldsymbol{n}\sin\chi - \left(\frac{\partial\phi}{\partial n}\right)_0 \boldsymbol{n}_0\cdot\boldsymbol{n}\cos\chi\right\}\mathrm{d}S(\boldsymbol{x}). \quad (3.4)$$

This is the key result in which all singularities associated with the boundary integral formulation of the Helmholtz problem have been removed. As $\boldsymbol{x} \to \boldsymbol{x}_0$, the analytic structure of the integrands (3.4) is essentially the same as that in (2.7) where the integrands do not diverge. In the limit $k \to 0$, this reduces to result (2.7) for the potential problem.

4. Stokes problem: non-singular boundary integral formulation

The governing equations for the pressure, p, and velocity field, $\boldsymbol{u}$, for incompressible Stokes flow in a Newtonian fluid of dynamic viscosity μ are

$$-\nabla p + \mu\nabla^2\boldsymbol{u} = 0 \quad \text{and} \quad \nabla\cdot\boldsymbol{u} = 0. \quad (4.1)$$

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The stress tensor $\boldsymbol{\sigma}_{ik}$ is given by

$$\boldsymbol{\sigma}_{ik} = -p\delta_{ik} + \mu\left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right], \tag{4.2}$$

where δ_{ik} is the Kronecker delta function. Using the Lorentz reciprocal theorem (Lorentz 1907), the velocity component in the jth direction, $u_j^0 \equiv u_j(\boldsymbol{x}_0)$, at position $\boldsymbol{x}_0$ in the fluid domain can be written as a boundary integral over the enclosing surface S (Pozrikidis 1992) as

$$\begin{aligned} 8\pi u_j^0 + \int_S u_i \boldsymbol{T}_{ijk} n_k \,\mathrm{d}S &= \frac{1}{\mu}\int_S \boldsymbol{\sigma}_{ik} n_k \boldsymbol{U}_{ij}\,\mathrm{d}S \\ &= \frac{1}{\mu}\int_S f_i \boldsymbol{U}_{ij}\,\mathrm{d}S. \end{aligned} \tag{4.3}$$

The ith component of the traction vector $\boldsymbol{f}$, is defined as $f_i = \boldsymbol{\sigma}_{ik} n_k$. Equation (4.3) for the Stokes problem is the analogue of (2.1) for the potential problem. The fundamental solutions for Stokes flow $\boldsymbol{U}_{ij}$ and $\boldsymbol{T}_{ijk}$ are given by (Pozrikidis 1992):

$$\boldsymbol{U}_{ij}(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{\delta_{ij}}{r} + \frac{\hat{x}_i \hat{x}_j}{r^3}, \tag{4.4}$$

$$\boldsymbol{T}_{ijk}(\boldsymbol{x}, \boldsymbol{x}_0) = -6\frac{\hat{x}_i \hat{x}_j \hat{x}_k}{r^5}, \tag{4.5}$$

where $\hat{x}_i$ etc., are the components of $\hat{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_0$, $r = |\hat{\boldsymbol{x}}|$ and n_k is the kth component of the unit normal of the surface pointing out of the flow domain. The functions $\boldsymbol{U}_{ij}$ and $\boldsymbol{T}_{ijk}$ diverge as $1/r$ and $1/r^2$, respectively, with the same behaviour as G and ∇G for the potential problem and give rise to singular behaviour in (4.3) when $\boldsymbol{x}_0$ is on the surface S. The traditional boundary integral formulation is obtained by putting $\boldsymbol{x}_0$ onto the surface S in (4.3), and as in (2.2), this will give rise to Cauchy PV integrals and Dirac δ-function contributions on the left-hand side and weakly singular integrands on the right-hand side of (2.2). To remove such singularities, consider a zero-pressure linear velocity field

$$\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}_0) + \frac{1}{\mu}\boldsymbol{M}(\boldsymbol{x}_0)\cdot(\boldsymbol{x} - \boldsymbol{x}_0), \tag{4.6}$$

where the matrix $\boldsymbol{M}(\boldsymbol{x}_0)$ will be chosen to cancel the arising singularities in (4.3) when $\boldsymbol{x}_0$ is on the surface S. The symmetric stress tensor corresponding to this linear flow field is

$$\boldsymbol{\Sigma}(\boldsymbol{x}_0) = \boldsymbol{M}(\boldsymbol{x}_0) + \boldsymbol{M}^{\mathrm{T}}(\boldsymbol{x}_0) \quad \text{or} \quad \Sigma_{ik}^0 = \boldsymbol{M}_{ik}^0 + \boldsymbol{M}_{ki}^0. \tag{4.7}$$

For the velocity field $\boldsymbol{w}$ to meet the incompressibility condition: $\nabla \cdot \boldsymbol{w}(\boldsymbol{x}) = 0$,

$$\mathrm{Tr}[\boldsymbol{M}(\boldsymbol{x}_0)] = \mathrm{Tr}[\boldsymbol{\Sigma}(\boldsymbol{x}_0)]/2 = 0 \tag{4.8}$$

must hold. Since $\boldsymbol{w}$ satisfies the equations for Stokes flow, the difference $(\boldsymbol{u} - \boldsymbol{w})$ also satisfies (4.3). In component form the difference becomes ($x_j^0 = j$th component of $\boldsymbol{x}_0$)

$$\int_S \left[u_i - u_i^0 - \frac{1}{\mu}\boldsymbol{M}_{il}^0 (x_l - x_l^0)\right] \boldsymbol{T}_{ijk} n_k \,\mathrm{d}S = \frac{1}{\mu}\int_S \left(f_i - \Sigma_{il}^0 n_l\right) \boldsymbol{U}_{ij}\,\mathrm{d}S. \tag{4.9}$$

This integral equation, with $\boldsymbol{x}_0$ *located on the surface* S, will have no singular behaviour if we choose (adopting the convention of implicit summation over repeated

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indices)

$$\boldsymbol{M}(\boldsymbol{x}_0) = \boldsymbol{f}(\boldsymbol{x}_0)\boldsymbol{n}(\boldsymbol{x}_0) - \tfrac{1}{4}\boldsymbol{f}(\boldsymbol{x}_0)\cdot\boldsymbol{n}(\boldsymbol{x}_0)\,[\boldsymbol{I} + \boldsymbol{n}(\boldsymbol{x}_0)\boldsymbol{n}(\boldsymbol{x}_0)]\,, \tag{4.10}$$

$$\boldsymbol{M}^0_{il} = f^0_i n^0_l - \tfrac{1}{4}(f^0_k n^0_k)(\delta_{il} + n^0_i n^0_l), \tag{4.11}$$

and

$$\boldsymbol{\Sigma}(\boldsymbol{x}_0) = \boldsymbol{f}(\boldsymbol{x}_0)\boldsymbol{n}(\boldsymbol{x}_0) + \boldsymbol{n}(\boldsymbol{x}_0)\boldsymbol{f}(\boldsymbol{x}_0) - \tfrac{1}{2}\boldsymbol{f}(\boldsymbol{x}_0)\cdot\boldsymbol{n}(\boldsymbol{x}_0)\,[\boldsymbol{I} + \boldsymbol{n}(\boldsymbol{x}_0)\boldsymbol{n}(\boldsymbol{x}_0)]\,, \tag{4.12}$$

$$\Sigma^0_{il} = \boldsymbol{M}^0_{il} + \boldsymbol{M}^0_{li} = (f^0_i n^0_l + f^0_l n^0_i) - \tfrac{1}{2}(f^0_k n^0_k)(\delta_{il} + n^0_i n^0_l). \tag{4.13}$$

The relation between the stress tensor $\boldsymbol{\Sigma}(\boldsymbol{x}_0)$ and the matrix $\boldsymbol{M}(\boldsymbol{x}_0)$ in terms of the traction $\boldsymbol{f}(\boldsymbol{x}_0)$ and the surface normal $\boldsymbol{n}(\boldsymbol{x}_0)$ needed to ensure (4.9) is non-singular is in fact not unique. It is easy to verify that the expressions in (4.10)–(4.13) obey those constraints in (4.8) and, as a result, the integrands in (4.9) are not singular as $\boldsymbol{x} \to \boldsymbol{x}_0$ on the surface (see the proof in Appendix). Thus, (4.9)–(4.13) form the non-singular boundary integral formulation of the Stokes problem. Analogous to the potential problem, the elements of the matrix $\boldsymbol{M}(\boldsymbol{x}_0)$ vary for each $\boldsymbol{x}_0$ on the surface S.

5. Linear elasticity problem: non-singular boundary integral formulation

The regularization method described thus far is quite general. The non-singular boundary integral formulation of the Stokes problem can be adapted to the linear elastic problem in solid mechanics as follows. The strain tensor $\boldsymbol{\varepsilon}$ in the elastic problem is defined in terms of the position vector field $\boldsymbol{u}$

$$\boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{5.1}$$

For a linear elastic material in equilibrium and in the absence of body forces, the stress tensor satisfies $\partial\sigma_{ij}/\partial x_j = 0$ and is given by

$$\sigma_{ij} = \frac{2\mu\nu}{1-2\nu}\delta_{ij}\boldsymbol{\varepsilon}_{ll} + 2\mu\boldsymbol{\varepsilon}_{ij}, \tag{5.2}$$

where the Poisson ratio ν, and the shear modulus μ are related to the Young's modulus, $E = 2\mu(1+\nu)$. The displacement field $\boldsymbol{u}(\boldsymbol{x}_0)$ at an interior point of the elastic material can be expressed in terms of integrals over the displacement field and the surface traction on the enclosing surface S by the same equation as (4.3), except the fundamental solutions $\boldsymbol{U}_{ij}$ and $\boldsymbol{T}_{ijk}$ for the linear elastic problem are now given by Becker (1992):

$$\boldsymbol{U}_{ij} = \frac{1}{2(1-\nu)}\left[(3-4\nu)\frac{\delta_{ij}}{r} + \frac{\hat{x}_i\hat{x}_j}{r^3}\right], \tag{5.3}$$

and

$$\boldsymbol{T}_{ijk} = -\frac{1}{1-\nu}\left[(1-2\nu)\delta_{ij}\frac{\hat{x}_k}{r^3} + 3\frac{\hat{x}_i\hat{x}_j\hat{x}_k}{r^5}\right] - \frac{1-2\nu}{1-\nu}\left[\frac{\hat{x}_j}{r^3}n_i n_k - \frac{\hat{x}_i}{r^3}n_j n_k\right]. \tag{5.4}$$

With these replacements, (4.9)–(4.13) are also the non-singular boundary integral formulation of the linear elastic problem for the displacement field $\boldsymbol{u}$ with the traction vector $\boldsymbol{f}$ defined by $f_i = \boldsymbol{\sigma}_{ik}n_k$. For an incompressible material: $\nu = 1/2$, then $\boldsymbol{U}_{ij}$ and $\boldsymbol{T}_{ijk}$ for the linear elastic problem and the Stokes problem become identical.

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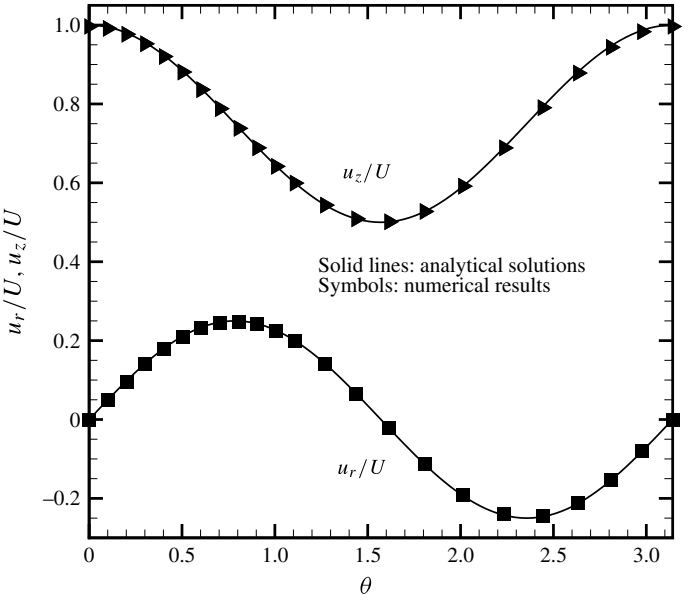


FIGURE 2. Distributions of the normalized velocities u_r/U and u_z/U along θ in the second test case for the Stokes problem of a spherical bubble rising under buoyancy force.

6. Discussion and implementation

In this communication we have developed a non-singular boundary integral formulation for solving four common and related problems in hydrodynamics and solid mechanics. The common theme in the formulation is the removal of the singularities associated with the traditional boundary integral formulation by subtracting a simpler solution of a related problem with an appropriate choice of the free parameter in the solution.

The numerical implementation of our non-singular formulation for the potential problem, (2.7), has been described in Klaseboer *et al.* (2009). The Stokes problem is very similar, except that the matrix elements appear in blocks of sub-matrices. When the surface is discretized in nodes and elements, the usual Gaussian–quadrature integration procedure can be applied for all elements, including the singular ones. This will result in a system of equations relating the potential and its normal derivative through two influence matrices, which is the discretized equivalent of (2.7). The previously singular contributions can be found on the diagonals of the influence matrices, one corresponding to $G(\boldsymbol{x}, \boldsymbol{x}_0)$ and one to its normal derivative. All of the terms corresponding to $\boldsymbol{x}_0$ now correspond to those contributions and can be obtained by simple summation. Several examples for which analytical solutions exist were tested (see Klaseboer *et al.* 2009 for more details).

For the Stokes problem, the singularities appear in blocks of 3×3 around the diagonals of the influence matrices. A procedure very similar to that followed for the potential flow can be followed to get those values by summation once more, based on (4.9).

Two examples are provided for the Stokes flow implementation. Both use flat three-node linear elements in which the surface representation and the shape functions are linear. The first example is that of a sphere moving with a constant velocity $\boldsymbol{U}$.

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The mesh is similar to that used in Klaseboer *et al.* (2009). Thus, the velocity vectors, $\boldsymbol{u}$, are given at all nodes as $\boldsymbol{u} = \boldsymbol{U}$. The traction, $\boldsymbol{f}$, is then calculated and, within the expected discretization error, agrees with the analytic solution $\boldsymbol{f} = 3\mu\boldsymbol{U}/(2R)$, where R is the radius of the sphere.

In the second test case, the traction $\boldsymbol{f}$ is given instead and the velocity $\boldsymbol{u}$ is calculated. We use the same test case as presented in Pigeonneau & Sellier (2011), and take $\boldsymbol{f} = -\rho g z \boldsymbol{n}$ that corresponds to a spherical bubble rising under buoyancy force, where g is the magnitude of gravity, and ρ is the fluid density. The exact solution in cylindrical coordinates is given by (34) and (35) in Pigeonneau & Sellier (2011): $u_r = U\sin(2\theta)/4$ and $u_z = U(1 - \sin^2\theta/2)$, where $U = \rho g R^2/(3\mu)$, and θ is the angle between the unit vector in the z-direction and the radial direction. The results are shown in figure 2. Even for a mesh consisting of only 252 nodes (500 elements) the accuracy is within 2 %.

The present non-singular formulation therefore offers all the advantages associated with the reduction of dimension afforded by the boundary integral technique without the extra numerical effort needed to handle the singularities that arise with the traditional boundary integrals formulation. Although the size of the numerical problem remains the same, the absence of singularities means that there will be a significant reduction in coding effort which will minimize the opportunity for coding error.

We believe the present contribution is a novel advance that will have both pedagogical and practical implications.

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Appendix. Non-singular proof

We show that the integrands in (2.7) for the potential flow problems and (4.9) for the Stokes flow problems are non-singular by analysing the analytic behaviour of the integrand in the neighbourhood of $\boldsymbol{x}_0$. Define a Cartesian system (ξ, η, ζ) with $\boldsymbol{x}_0 = (0, 0, 0)$ as the origin and $\boldsymbol{n}_0 = (0, 0, 1)$. In the neighbourhood of $\boldsymbol{x}_0$, a point $\boldsymbol{x} = (\xi, \eta, \zeta)$ that lies on the surface S with a suitable choice of the local coordinates, ξ, η and ζ, must satisfy

$$S = \zeta + \tfrac{1}{2}a_s\xi^2 + \tfrac{1}{2}b_s\eta^2 = 0 \tag{A 1}$$

where higher-order terms of $O(\xi^3, \eta^3)$ have been omitted. The constants a_s and b_s are related to the principal curvatures of S at $\boldsymbol{x}_0$ and ζ is quadratic in ξ and η. The unit normal vector at $\boldsymbol{x}$ is $\boldsymbol{n} = \nabla S/|\nabla S| = \cos\gamma\ (a_s\xi, b_s\eta, 1)$, where $\cos\gamma = [1 + (a_s\xi)^2 + (b_s\eta)^2]^{-1/2}$ is the direction cosine. The differential surface is $\mathrm{d}S(\boldsymbol{x}) = \mathrm{d}\xi\,\mathrm{d}\eta/\cos\gamma$. The Green function has the form: $G(\boldsymbol{x}, \boldsymbol{x}_0) = (\xi^2 + \eta^2 + \zeta^2)^{-1/2}$.

With these preliminary results, the integral on the left-hand side of (2.7), has the following limiting form obtained by using a Taylor expansion $\sim\boldsymbol{x}_0$ in terms of the

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local coordinates

$$\int_{x\to x_0}\left[\phi(\boldsymbol{x})-\phi(\boldsymbol{x}_0)-\left(\frac{\partial\phi}{\partial n}\right)_0\boldsymbol{n}_0\cdot(\boldsymbol{x}-\boldsymbol{x}_0)\right]\nabla G(\boldsymbol{x},\boldsymbol{x}_0)\cdot\boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x})$$
$$\sim\int_{x\to x_0}\left[\xi\left(\frac{\partial\phi}{\partial\xi}\right)_0+\eta\left(\frac{\partial\phi}{\partial\eta}\right)_0\right]\left[\frac{1}{2}a_s\xi^2+\frac{1}{2}b_s\eta^2\right]\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{(\xi^2+\eta^2+\zeta^2)^{3/2}}. \quad (A\,2)$$

We see that both the numerator and the denominator of the integrand are of $O(\xi^3,\eta^3)$, thus the integrand remains finite, as $\xi\to 0$ and $\eta\to 0$, that is $\boldsymbol{x}\to\boldsymbol{x}_0$. Furthermore, if the surface around $\boldsymbol{x}_0$ is a planar element, the constants a_s and b_s will be zero and the integrand vanishes. For non-planar elements, the point-wise discontinuity at $(\xi,\eta)=(0,0)$ is a set of measure zero and therefore does not contribute to the value of the integral.

Similarly, as $\boldsymbol{x}\to\boldsymbol{x}_0$, the integral on the right-hand side of (2.7), has the form

$$\int_{x\to x_0}G(\boldsymbol{x},\boldsymbol{x}_0)\left[\frac{\partial\phi}{\partial n}-\left(\frac{\partial\phi}{\partial n}\right)_0\boldsymbol{n}_0\cdot\boldsymbol{n}\right]\mathrm{d}S(\boldsymbol{x})$$
$$\sim\int_{x\to x_0}\left[(a_s\xi)\left(\frac{\partial\phi}{\partial\xi}\right)_0+(b_s\eta)\left(\frac{\partial\phi}{\partial\eta}\right)_0+\xi\left(\frac{\partial^2\phi}{\partial\xi\partial\zeta}\right)_0+\eta\left(\frac{\partial^2\phi}{\partial\eta\partial\zeta}\right)_0\right]$$
$$\times\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{(\xi^2+\eta^2+\zeta^2)^{1/2}}. \quad (A\,3)$$

We see that both the numerator and the denominator of the integrand vanish linearly with ξ and η, as $\xi\to 0$ and $\eta\to 0$. Thus, the integrand has no divergences. Again, if the surface around $\boldsymbol{x}_0$ is a planar element and the normal derivative $(\partial\phi/\partial\zeta)$ is constant over that element, then the integrand vanishes. For non-planar elements, the point-wise discontinuity at $(\xi,\eta)=(0,0)$ is a set of measure zero and therefore does not contribute to the value of the integral. This has been demonstrated for quadratic elements (Klaseboer *et al.* 2009).

This completes the proof that (2.7) is non-singular. In the same way, (3.4) for the Helmholtz problem can also be shown to be non-singular.

Using the same local coordinate system, we can also show that (4.9) for the Stokes problem is not singular. First we consider the integral on the left-hand side of (4.9). As $\boldsymbol{x}\to\boldsymbol{x}_0$, it is straightforward to show using a Taylor expansion that the two terms in the integrand have the limiting form using the notation $\hat{\boldsymbol{x}}\equiv\boldsymbol{x}-\boldsymbol{x}_0=(\xi,\eta,\zeta)$, $\boldsymbol{f}(\boldsymbol{x})=(f_1,f_2,f_3)$ and $\boldsymbol{f}(\boldsymbol{x}_0)=(f_1^0,f_2^0,f_3^0)$

$$\int_{x\to x_0}(u_i-u_i^0)\boldsymbol{T}_{ijk}n_k\,\mathrm{d}S$$
$$\sim\int_{x\to x_0}-\frac{6(a_s\xi^2+b_s\eta^2+\zeta)}{(\xi^2+\eta^2+\zeta^2)^{5/2}}\left\{\left[\xi\left(\frac{\partial u_i}{\partial\xi}\right)_0+\eta\left(\frac{\partial u_i}{\partial\eta}\right)_0+\zeta\left(\frac{\partial u_i}{\partial\zeta}\right)_0\right]\hat{x}_i\right\}\hat{x}_j\,\mathrm{d}\xi\,\mathrm{d}\eta, \quad (A\,4)$$

$$\int_{x\to x_0}\boldsymbol{M}_{il}^0\hat{x}_l\boldsymbol{T}_{ijk}n_k\,\mathrm{d}S$$
$$\sim\int_{x\to x_0}-\frac{6(a_s\xi^2+b_s\eta^2+\zeta)[\xi(f_1^0\zeta-f_3^0\xi/4)+\eta(f_2^0\zeta-f_3^0\eta/4)+f_3^0\zeta^2/2]}{(\xi^2+\eta^2+\zeta^2)^{5/2}}\hat{x}_j\,\mathrm{d}\xi\,\mathrm{d}\eta. \quad (A\,5)$$

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The numerator and the denominator of both terms are of $O(\xi^5, \eta^5)$ and, thus, they approach constant values as $\boldsymbol{x} \to \boldsymbol{x}_0$, and as a consequence, the integral on the left-hand side of (4.9) is non-singular.

Turning now to the integral on the right-hand side of (4.9) where in the limit $\boldsymbol{x} \to \boldsymbol{x}_0$, the integrand has the limiting form

$$\begin{aligned}\int_{\boldsymbol{x}\to\boldsymbol{x}_0} \left(f_i - \Sigma^0_{il} n_l\right) \boldsymbol{U}_{ij}\, \mathrm{d}S \sim{}& \int_{\boldsymbol{x}\to\boldsymbol{x}_0} \frac{(1-\cos\gamma)(f^0_1\xi + f^0_2\eta + f^0_3\zeta) + L_i\hat{x}_i}{\cos\gamma\ \ (\xi^2+\eta^2+\zeta^2)^{3/2}}\hat{x}_j\, \mathrm{d}\xi\, \mathrm{d}\eta \\ &+ \int_{\boldsymbol{x}\to\boldsymbol{x}_0} \frac{(1-\cos\gamma)f^0_j + L_j - [\cos\gamma(f^0_1 a_s\xi + f^0_2 b_s\eta)]n^0_j + (f^0_3/2)(n_j - \cos\gamma\ n^0_j)}{\cos\gamma\ (\xi^2+\eta^2+\zeta^2)^{1/2}}\, \mathrm{d}\xi\, \mathrm{d}\eta \\ &+ \int_{\boldsymbol{x}\to\boldsymbol{x}_0} \frac{(f^0_3/2)(a_s\xi^2 + b_s\eta^2) - (f^0_1 a_s\xi + f^0_2 b_s\eta)\zeta}{(\xi^2+\eta^2+\zeta^2)^{3/2}}\hat{x}_j\, \mathrm{d}\xi\, \mathrm{d}\eta, \end{aligned} \tag{A 6}$$

where $L_i \equiv \xi\,(\partial f_i/\partial\xi)_0 + \eta\,(\partial f_i/\partial\eta)_0 + \zeta\,(\partial f_i/\partial\zeta)_0$. The numerator and the denominator of the first term are of $O(\xi, \eta)$ and those of the second and third terms are of $O(\xi^3, \eta^3)$ and thus all terms approach constant values as $\boldsymbol{x} \to \boldsymbol{x}_0$, and so it follows that the integral on the right-hand side of (4.9) is non-singular.

For the case in which the surface S is a planar element for which the curvatures a_s and b_s are zero, $\cos\gamma = 1$ and the traction $\boldsymbol{f}(\boldsymbol{x}_0)$ is constant within the plane, the integrands of both integrals in (4.9) will vanish. For non-planar elements, the point-wise discontinuity at $(\xi, \eta) = (0, 0)$ is a set of measure zero and therefore does not contribute to the value of the integral.

This completes the proof that (4.9) is non-singular. In the same way, the linear elasticity problem can also shown to be non-singular.

REFERENCES

BAZHLEKOV, I. B., ANDERSON, P. D. & MEIJER, H. E. H. 2004 Nonsingular boundary integral method for deformable drops in viscous flows. *Phys. Fluids* **16**, 106–1081.

BECKER, A. A. 1992 *The Boundary Element Method in Engineering: A Complete Course*. McGraw-Hill International (UK) Limited.

BLAKE, J. R., TAIB, B. B. & GIBSON, D. C. 1986 Transient cavities near boundaries. Part 1. Rigid boundary. *J. Fluid Mech.* **170**, 479–497.

CHENG, A. H. D. & CHENG, D. T. 2005 Heritage and early history of the boundary element method. *Engng Anal. Bound. Elem.* **29**, 268–302.

FONG, S. W., ADHIKARI, D., KLASEBOER, E. & KHOO, B. C. 2009 Interactions of multiple spark-generated bubbles with phase differences. *Exp. Fluids* **46**, 705–724.

GONZALEZ-AVILA, S. R., KLASEBOER, E., KHOO, B. C. & OHL, C. D. 2011 Cavitation bubble dynamics in a liquid gap of variable height. *J. Fluid Mech.* **682**, 241–260.

KLASEBOER, E., ROSALES-FERNANDEZ, C. & KHOO, B. C. 2009 A note on true desingularization of boundary element methods for three-dimensional potential problems. *Engng Anal. Bound. Elem.* **33**, 796–801.

LEAN, M. H. & WEXLER, A. 1985 Accurate numerical integration of singular boundary element kernels over boundaries with curvature. *Intl J. Numer. Meth. Engng* **21**, 211–228.

LIU, Y. J. & RUDOLPHI, T. J. 1991 Some identities for fundamental solutions and their applications to weakly-singular boundary element formulations. *Engng Anal. Bound. Elem.* **8**, 301–311.

LIU, Y. J. & RUDOLPHI, T. J. 1999 New identities for fundamental solutions and their applications to non-singular boundary element formulations. *Comput. Mech.* **24**, 286–292.

LORENTZ, H. A. 1907 *Abhandlungen Über Theoretische Physik*. B. G. Teubner.

PIGEONNEAU, F. & SELLIER, A. 2011 Low-Reynolds-number gravity-driven migration and deformation of bubbles near a free surface. *Phys. Fluids* **23**, 2102–2116.

https://doi.org/10.1017/jfm.2012.71 Published online by Cambridge University Press

POZRIKIDIS, C. 1992 *Boundary Integral and Singularity Methods for Linearized Viscous Flow*. Cambridge University Press.

SYMM, G. T. 1963 Integral equation methods in potential theory. ii. *Proc. R. Soc. Lond. Ser.* A *Math. Phys. Sci.* **275**, 33–46.

WANG, Q. X. 1998 The evolution of a gas bubble near an inclined wall. *Theor. Comput. Fluid Dyn.* **12**, 29–51.

WROBEL, L. C. 2002 *The Boundary Element Method, Vol. 1, Applications in Thermo-Fluids and Acoustics*. Wiley.

ZHANG, Y. L., YEO, K. S., KHOO, B. C. & WANG, C. 2001 3D jet impact and toroidal bubbles. *J. Comput. Phys.* **166**, 336–360.

https://doi.org/10.1017/jfm.2012.71 Published online by Cambridge University Press