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101.28 The versatile exponential inequality $e^x \geqslant 1 + x$

Introduction

In this Note, we survey and gather together a variety of applications of the inequality $e^x \geqslant 1 + x$. Many of them are scattered throughout the existing literature, but *en masse* they exemplify Bromwich's premise in [1] that "These simple inequalities are often sufficient to obtain useful properties of the exponential function.". The inequality

$$e^x \geqslant 1 + x \text{ for all real } x, \text{ with equality if, and only if, } x = 0 \qquad (1)$$

expresses the fact that the graph of $y = e^x$ lies above its tangent line at $x = 0$ (indeed, at any point $x = a$, since $e^{x-a} \geqslant 1 + x - a$ leads to $e^x \geqslant e^a + e^a(x - a)$). There are many different proofs of (1): we give two, probably less familiar ones, adapted from David Bradley's article [2] on the corresponding inequality for $\ln x$.

The first is the 'one-liner'

$$e^x - x - 1 = \begin{cases} \int_0^x \int_0^y e^t \, dt \, dy, & x \geqslant 0 \\ \int_x^0 \int_y^0 e^t \, dt \, dy, & x < 0 \end{cases}$$

where the integrands are manifestly non-negative.

For the second,

$$\begin{aligned} 0 \leqslant \sum_{n=1}^{N} 2^{n-1}\left(e^{x2^{-n}} - 1\right)^2 &= \sum_{n=1}^{N} \left(2^{n-1}e^{x2^{1-n}} - 2^n e^{x2^{-n}} + 2^{n-1}\right) \\ &= e^x - 2^N e^{x2^{-N}} + 2^N - 1 \text{ (by telescoping)} \\ &= e^x - 1 - \frac{x\left(e^{x2^{-N}} - 1\right)}{x2^{-N}} \end{aligned}$$

where the right-hand side tends to $e^x - 1 - x$ as $N \to \infty$ because e^x has derivative 1 at $x = 0$.

We will also use two immediate consequences of (1). First,

$$e^x \leqslant \frac{1}{1 - x} \text{ for } x < 1, \text{ with equality if, and only if, } x = 0. \qquad (2)$$

For (1) gives $e^{-x} \geqslant 1 - x$ which then gives (2) for the range of values quoted. We also have

$$e^x \geqslant ex \text{ for all real } x, \text{ with equality if, and only if, } x = 1. \qquad (3)$$

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For (1) gives $e^{x-1} \geqslant 1 + x - 1 = x$ which rearranges to give (3).

Finally, we will also have need of the following version of the sandwich principle for sequences: we will simply refer to it as 'the Lemma' when we use it.

Lemma: Let the sequences (a_n) and (b_n) satisfy the conditions

- $0 < a_n \leqslant l \leqslant b_n$ for some fixed l and all n,
- $\lim_{n \to \infty} \dfrac{b_n}{a_n} = 1$.

Then $\lim_{n \to \infty} a_n = l = \lim_{n \to \infty} b_n$.

Proof: We have $0 \leqslant l - a_n \leqslant b_n - a_n = a_n\left(\dfrac{b_n}{a_n} - 1\right) \leqslant l\left(\dfrac{b_n}{a_n} - 1\right) \to 0$, with an almost identical argument for (b_n).

Applications

1. *The exponential limit*: $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n+1}$

By (1) and (2), $1 + \frac{1}{n} < e^{1/n}$ and $e^{1/(n+1)} < \dfrac{1}{1 - \frac{1}{n+1}} = 1 + \frac{1}{n}$ so that $\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$. Also $\dfrac{\left(1 + \frac{1}{n}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n} = 1 + \frac{1}{n} \to 1$ as $n \to \infty$ so, by the Lemma, the exponential limit holds.

Actually, this is the special case $(u_n = 1)$ of a more general result.

Let $u_n > 0$ and $s_N = u_1 + \ldots + u_N$ and suppose that

$$\frac{u_1^2 + \ldots + u_N^2}{(u_1 + \ldots + u_N)^2} \to 0 \text{ as } N \to \infty. \tag{4}$$

Then, if $a_N = \prod_{n=1}^{N} \left(1 + \dfrac{u_n}{s_N}\right)$ and $b_N = \prod_{n=1}^{N} \left(1 - \dfrac{u_n}{s_N}\right)^{-1}$, we have $e = \lim_{N \to \infty} a_N = \lim_{N \to \infty} b_N$. For, by (1), $1 + \dfrac{u_n}{s_N} < \exp\left(\dfrac{u_n}{s_N}\right)$ so that $a_N < e$ and $1 - \dfrac{u_n}{s_N} < \exp\left(-\dfrac{u_n}{s_N}\right)$ so that $e < b_N$. Also

$$1 > \frac{a_N}{b_N} = \prod_{n=1}^{N} \left(1 - \frac{u_n^2}{s_N^2}\right) \geqslant 1 - \frac{u_1^2 + \ldots + u_N^2}{(u_1 + \ldots + u_N)^2}$$

on using the standard (Weierstrass) inequality

$$\prod_{n=1}^{N} (1 - x_n) \geqslant 1 - (x_1 + \ldots + x_N) \text{ for } 0 < x_n < 1. \tag{5}$$

By (4) and the Lemma, we are done.

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It is also worth noting that (1) and (2) are sufficient to establish that e^x has derivative 1 at $x = 0$, by letting $x \to 0$ in the following chain of inequalities:

$$1 < \frac{e^x - 1}{x} < \frac{1}{x}\left(\frac{1}{1-x} - 1\right) = \frac{1}{1-x}.$$

2. *Divergence of* $\sum \frac{1}{n}$

In [3], Honsberger gives the following lovely proof of the divergence of $\sum \frac{1}{n}$. By (1), $e^{1/n} > 1 + \frac{1}{n} = \frac{n+1}{n}$ so, multiplying for $1 \leqslant n \leqslant N$, we have

$$e^{1+\frac{1}{2}+\ldots+\frac{1}{N}} > \frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{N+1}{N} = N + 1$$

and so

$$1 + \frac{1}{2} + \ldots + \frac{1}{N} > \ln(N+1) \to \infty \text{ as } N \to \infty.$$

3. *Which is larger*: e^π *or* π^e?

The following argument may be found in many places, such as [4].

By (3) applied to $\frac{x}{e}$ we have $e^{x/e} \geqslant \frac{ex}{e} = x$ or $e^x \geqslant x^e$ with equality if, and only if, $x = e$. In particular, $e^\pi > \pi^e$.

4. *For* $\alpha, \beta > 0$ *what is the maximum value of* $x^\alpha e^{-\beta x}$ *when* $x > 0$?

By (3) applied to $\frac{\beta x}{\alpha}$ we have $e^{\beta x/\alpha} \geqslant \frac{e\beta x}{\alpha}$ from which $xe^{-\beta x/\alpha} \leqslant \frac{\alpha}{e\beta}$ and $x^\alpha e^{-\beta x} \leqslant \left(\frac{\alpha}{e\beta}\right)^\alpha$. The maximum value of $x^\alpha e^{-\beta x}$ when $x > 0$ is thus $\left(\frac{\alpha}{e\beta}\right)^\alpha$, attained when $x = \frac{\alpha}{\beta}$. Concrete examples of functions of this form include the Maxwell-Boltzmann distribution and, more generally, the probability density function of the χ^2-distribution.

5. *What is the maximum value of* $x^{1/x}$ *for* $x > 0$?

Dörrie singles this out as Problem 89 ('Steiner's Problem') in his famous collection [5] and uses (1) in his solution. We can also use (3) applied to $\frac{x}{e}$:

$e^{x/e} \geqslant \frac{ex}{e} = x$ from which $x^{1/x} \leqslant e^{1/e}$ with equality if, and only if, $x = e$.

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A nice use of this result is in the greatest product problem: given a positive number N, what is the largest product possible from positive numbers that add up to N? If $a_1 + \ldots + a_n = N$ then, by the AM-GM inequality,

$$a_1 \ldots a_n \leqslant \left(\frac{N}{n}\right)^n = \left(x^{\frac{1}{x}}\right)^N$$

where $x = \frac{N}{n}$. From the above, we then have $a_1 \ldots a_n \leqslant \left(e^{\frac{1}{e}}\right)^N \approx 1.445^N$. This provides a tight upper bound for the problem: a common restriction is to insist that N and the a_i are positive integers, [6].

6. *What is the minimum value of x^x for $x > 0$?*

Here, we just replace x with $\frac{1}{x}$ in Application 5 to obtain $\left(\frac{1}{x}\right)^x \leqslant e^{1/e}$ which rearranges to $x^x \geqslant e^{-1/e}$ with equality if, and only if, $x = \frac{1}{e}$.

7. *The AM-GM inequality*

Let $x_1, \ldots, x_n$ be positive real numbers with $A = \frac{1}{n}\left(x_1 + \ldots + x_n\right)$ their arithmetic mean and $G = \left(x_1 \ldots x_n\right)^{1/n}$ their geometric mean. Then applying (1) to $\dfrac{x_i}{A} - 1$ gives $e^{\frac{x_i}{A} - 1} \geqslant \frac{x_i}{A}$ so, on multiplying for $1 \leqslant i \leqslant n$, $e^0 = e^{\frac{nA}{A} - n} \geqslant \dfrac{G^n}{A^n}$. It follows that $A \geqslant G$ with equality if, and only if, $\frac{x_i}{A} = 1$ or $x_i = A$ for all $1 \leqslant i \leqslant n$. In the context of the survey article [7], this constitutes a direct proof that (d) $\Rightarrow$ (a) in Proposition 1 of that article.

8. *A problem from Pólya and Szegő*

Problem II.58 from [8] asks for the value of $\lim\limits_{n\to\infty} \dfrac{(n^2+1)(n^2+2)\ldots(n^2+n)}{(n^2-1)(n^2-2)\ldots(n^2-n)}$. By (1) and (2) we have, for $1 \leqslant k \leqslant n$, $1 - \dfrac{k}{n^2} < e^{-\frac{k}{n^2}} < \dfrac{1}{1 + \frac{k}{n^2}}$ so, multiplying these inequalities and rewriting $e^{-(1+2+\ldots+n)/n^2}$ as $e^{-1/2}\cdot e^{-1/(2n)}$, $a_n = e^{\frac{1}{2n}} \prod\limits_{k=1}^{n}\left(1 - \dfrac{k}{n^2}\right) < e^{-\frac{1}{2}} < b_n = \dfrac{e^{\frac{1}{2n}}}{\prod\limits_{k=1}^{n}\left(1 + \frac{k}{n^2}\right)}$.

But $\dfrac{a_n}{b_n} = \prod\limits_{k=1}^{n}\left(1 - \dfrac{k^2}{n^4}\right) \geqslant 1 - \dfrac{1}{n^4}\sum\limits_{k=1}^{n} k^2 \to 1$ as $n \to \infty$ on using (5) and the fact that $\sum\limits_{k=1}^{n} k^2 = O\left(n^3\right)$. It then follows from the Lemma that, as $n \to \infty$, $a_n, b_n \to e^{-1/2}$ and

$$\frac{e^{\frac{1}{n}}}{a_n b_n} = \frac{\prod\limits_{k=1}^{n}\left(1 + \frac{k}{n^2}\right)}{\prod\limits_{k=1}^{n}\left(1 - \frac{k}{n^2}\right)} = \frac{(n^2+1)(n^2+2)\ldots(n^2+n)}{(n^2-1)(n^2-2)\ldots(n^2-n)} \to e.$$

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9. *The probability integral*, $\int_0^\infty e^{-\frac{1}{2}x^2}dx = \sqrt{\frac{\pi}{2}}$

The following is a tweaking of the argument presented in [9, 10]. From (1) and (2) we have $1 - x < e^{-x} < \frac{1}{1+x}$ for $x > 0$ so that

$$\left(1 - \frac{x^2}{n}\right)^{\frac{n}{2}} < e^{-\frac{1}{2}x^2} < \frac{1}{\left(1 + \frac{x^2}{n}\right)^{\frac{n}{2}}}$$

and $$\int_0^{\sqrt{n}}\left(1 - \frac{x^2}{n}\right)^{\frac{n}{2}}dx < \int_0^{\sqrt{n}} e^{-\frac{1}{2}x^2}dx < \int_0^\infty e^{-\frac{1}{2}x^2}dx < \int_0^\infty \frac{1}{\left(1 + \frac{x^2}{n}\right)^{\frac{n}{2}}}dx.$$

Substituting $x = \sqrt{n}\sin\theta$ in the left-hand integral and $x = \sqrt{n}\tan\theta$ in the right-hand integral then shows that

$$\sqrt{n}C_{n+1} < \int_0^\infty e^{-\frac{1}{2}x^2}dx < \sqrt{n}C_{n-2} \tag{6}$$

where $C_k = \int_0^{\frac{\pi}{2}} \cos^k\theta\, d\theta$. But integration by parts gives $kC_k = (k-1)C_{k-2}$ so that $kC_kC_{k-1} = (k-1)C_{k-1}C_{k-2} = \ldots = 1.C_1C_0 = \frac{\pi}{2}$ and

$$kC_k^2 < kC_kC_{k-1} = \frac{\pi}{2} = (k+1)C_{k+1}C_k < (k+1)C_k^2.$$

By the Lemma, it follows that $\sqrt{k}C_k \to \sqrt{\frac{\pi}{2}}$ and using this in (6) shows that $\int_0^\infty e^{-\frac{1}{2}x^2}dx = \sqrt{\frac{\pi}{2}}$.

10. *The birthday problem*

This famous problem asks for the least number, k, of people for which the probability that any two share the same birthday exceeds 0.5. If there are N days in the year and birthdays are uniformly distributed, this is equivalent to seeking the smallest k for which the inequality

$$0.5 > p_k = \left(1 - \frac{1}{N}\right)\left(1 - \frac{2}{N}\right)\ldots\left(1 - \frac{k-1}{N}\right)$$

holds.

I gave one approach to this in [11], but we can also solve it using our exponential inequalities. From (1) and (2), $1 - \frac{i}{N} < e^{-\frac{i}{N}} < \frac{1}{1+\frac{i}{N}} = \frac{1-\frac{i}{N}}{1-\frac{i^2}{N^2}}$ so, multiplying for $1 \leqslant i \leqslant k-1$,

$$p_k < \exp\left[-\frac{1+2+\ldots+(k-1)}{N}\right] < \frac{p_k}{\prod_{i=1}^{k-1}\left(1-\frac{i^2}{N^2}\right)} < \frac{p_k}{1-\left[\frac{1^2+\ldots+(k-1)^2}{N^2}\right]},$$

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on using (5). Summing the two series then gives the bounds

$$p_k < \exp\left[-\frac{k(k-1)}{2N}\right] < \frac{p_k}{1 - \dfrac{k(k-1)(2k-1)}{6N^2}}.$$

With $k = 23, N = 365$ the left-hand inequality gives $p_{23} < 0.499999$, and with $k = 22, N = 365$ the right-hand inequality gives $p_{22} > 0.517863$ which verifies the familiar answer, $k = 23$.

Finally, my recent article [12] contains one further application of (1) and (2) used in tandem.

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