DIFFRACTION BY A SCREEN CAUSING LARGE RANDOM PHASE FLUCTUATIONS

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ABSTRACT. The propagation of a scalar wave is studied, after it has suffered changes of phase in its passage through an irregular refracting medium, such as the ionosphere. The rms phase fluctuations may be large; i.e. greater than one radian. The paper is in three parts. In the first, the receiver is at a great distance from the screen (Fraunhofer diffraction), and a formula is found for all even order moments of the amplitude, and for the correlation function of the even powers of the amplitude. The probability distribution of the amplitude is proved to be a Rice distribution. The multiple integrals expressing the moments of the amplitude reduce to sums of correlations of the original phase fluctuations amongst points over the screen, which are evaluated using combinatorial arguments.

In the second part the variance of the square of the amplitude is calculated numerically for points near the screen (Fresnel diffraction). When the rms phase fluctuation is greater than one the variance is maximum near the point where the irregularities in the screen would bring the rays to a focus.

In the third part radio star scintillation and scatter propagation are discussed in the light of the theory.

1. *Introduction.* The passage of a radio wave through the ionosphere is sometimes accompanied by large random changes of phase across the wavefront. It is well known that there are problems encountered beyond those involved in diffraction by an amplitude or a shallow phase screen. For example, there appears to be a certain amount of spatial fine structure in the diffraction pattern at some distance beyond the screen —smaller than the original irregularity size (6), (11). This is thought to occur in the scintillation of radio stars.

In the present paper a statistical method is used in studying such propagation beyond the screen. It is a continuation of the author's previous work on amplitude and thin phase screens (10). No attempt is made to describe the passage of the wave through the ionosphere itself; for this, see (2), (3), (4), (8).

In the first part of this paper, §§ 3–6, the signal amplitude at a great distance from the screen is studied. We calculate the $(2n)$th moment of the amplitude and also the auto-correlation of the $(2n)$th power of the amplitude. The results apply when one is so far that there are many irregularities in a Fresnel zone. This is the case for ordinary fading in reflexion from the E region.

In the second part, §§ 7–8, the variance of the amplitude squared is calculated numerically for a small distance from the screen, where the Fresnel zone size is less than or equal to the irregularity size. The amplitude fluctuations increase from zero to the steady value found in the first part. When the phase fluctuations in the screen are large

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enough the amplitude fluctuations pass through a maximum as the distance is increased. Thus the random phase screen is still able to focus the waves to some extent. This part of the work would apply to radio star scintillations, bursts of amplitude in reflexion from the E and F regions (14), and possibly to scatter propagation via the troposphere or the ionosphere.

In the third part, § 9, experimental results concerning scintillation and scatter propagation are discussed.

Some of the mathematical work is perhaps excessively general, but it is hoped that it will shed some light on the difficulties of the calculation and the nature of propagation from a randomly varying screen.

2. *Diffraction theory.* The complex amplitude of the signal at the plane $z = 0$ (the 'screen') is $F(x, y)\exp(i\omega t)$. The Helmholtz formula gives without approximation the complex amplitude E at the point (x_0, y_0, z_0) in front of the screen $(z > 0)$,

$$E(x_0, y_0, z_0) = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{e^{-ikr}}{r}\frac{\partial F}{\partial z}(x, y, 0) - F(x, y, 0)\frac{\partial}{\partial z}\left(\frac{e^{-ikr}}{r}\right)\right\}dx\,dy,$$

where

$$r^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2,$$

$$k = 2\pi/\lambda,$$

λ = wavelength,

ω = angular frequency.

We now assume (1) $\partial F/\partial z = -ikF$ at $z = 0$; (2) $z_0 \gg \lambda$. Under the latter assumption it is valid to use the stationary phase approximation, putting

$$r = z_0 + \frac{1}{2z_0}\{(x-x_0)^2 + (y-y_0)^2\}.$$

Outside the range of validity of this formula for r the oscillations in the integrand essentially cancel out leaving errors of the order of one part in z_0/λ. We may also replace r by z_0 in the other parts of the integrand, incurring errors of the same order. Finally we have,

$$E(x_0, y_0, z_0) = \frac{i}{\lambda z_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x, y)\exp\left\{-\frac{\pi i}{\lambda z_0}[(x-x_0)^2 + (y-y_0)^2]\right\}dx\,dy \qquad (1)$$

dropping a factor $\exp(-2\pi i z_0/\lambda)$.

The function $F(x, y)$ is subject to no restriction, except that it must not behave like $\exp\{(\pi i/\lambda z_0)[(x-x_0)^2 + (y-y_0)^2]\}$ outside the range of validity of the binomial expansion; i.e. when $(x-x_0)^2 + (y-y_0)^2 > z_0$; otherwise the oscillations will not cancel out and some error will be introduced. This leaves $F(x, y)$ unrestricted within an area surrounding the point $(x_0, y_0, 0)$ many Fresnel zones across.

The amplitude R is defined to be

$$R = (EE^*)^{\frac{1}{2}}. \qquad (2)$$

We shall be able to calculate averages of multiple integrals arising from even powers of R; however, odd powers of R cannot be averaged unless we know the actual probability

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distribution of R, which we do not. We shall therefore give the following results (denoting an average over assemblies by a bar):

$$\lim_{z_0\to\infty} \overline{R^{2n}(x_0, y_0, z_0)}, \tag{3}$$

$$\lim_{z_0\to\infty} \overline{R^{2n}(x_0, y_0, z_0)\, R^{2n}(x_0+\xi, y_0+\eta, z_0)} \tag{4}$$

(where n is any positive integer), and

$$\overline{R^4(x_0, y_0, z_0)} \tag{5}$$

for certain values of z_0. This last result is largely a matter for numerical computation. This has been carried out by Dr M. L. V. Pitteway, of the Mathematical Laboratory, using EDSAC II, and will be discussed in the Appendix.

3. *Calculation of $\overline{R^{2n}}$ for large z.* We shall assume F to be the complex amplitude due to a phase screen, so that F varies only in phase with x, y.

$$F(x, y) = A \exp(i\phi(x, y)), \tag{6}$$

where A is some complex constant, and $\phi(x, y)$ is a real stationary stochastic function of x, y.

The moments of R are calculated at the point $(0, 0, z)$; ($x_0 = y_0 = 0$ in (1)). Because the properties of the screen are statistically stationary the moments do not depend on x_0, y_0.

From (1) and (2) we have

$$\begin{aligned} R^{2n} &= (EE^*)^n \\ &= \frac{1}{(\lambda z_0)^{2n}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} F(x_1, y_1) F(x_2, y_2) \dots F(x_n, y_n) F^*(x_{n+1}, y_{n+1}) \dots F^*(x_{2n}, y_{2n}) \\ &\quad \times \exp\left[-\frac{\pi i}{\lambda z_0} (x_1^2 + x_2^2 + \dots + x_n^2 - x_{n+1}^2 - \dots - x_{2n}^2 + y_1^2 + y_2^2 + \dots + y_n^2 - y_{n+1}^2 - \dots - y_{2n}^2) \right] \\ &\qquad \times dx_1 dx_2 \dots dy_{2n}. \end{aligned} \tag{7}$$

It is convenient to make the change of variable

$$x_i = (\lambda z_0)^{\frac{1}{2}} u_i, \quad y_i = (\lambda z_0)^{\frac{1}{2}} v_i, \quad (\lambda z_0)^{\frac{1}{2}} = t. \tag{8}$$

Then taking an average over assemblies,

$$\begin{aligned} \overline{R^{2n}} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{F(u_1 t, v_1 t) \dots F(u_n t, v_n t) F^*(u_{n+1} t, v_{n+1} t) \dots F^*(u_{2n} t, v_{2n} t)} \\ &\quad \times \exp[-\pi i (u_1^2 + \dots + u_n^2 - u_{n+1}^2 - \dots - u_{2n}^2 + v_1^2 + \dots + v_n^2 - v_{n+1}^2 - \dots - v_{2n}^2)] \\ &\qquad \times du_1 du_2 \dots dv_{2n}. \end{aligned} \tag{9}$$

The average in the integrand may be considered with the aid of the following theorem: If $\alpha_1, \dots, \alpha_n$ be n complex quantities whose $2n$ real and imaginary parts have a joint Gaussian distribution, then

$$\overline{\exp(\alpha_1 + \dots + \alpha_n)} = \exp[\tfrac{1}{2}\overline{(\alpha_1 + \dots + \alpha_n)^2}]. \tag{10}$$

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This follows from the formula for the characteristic function of $2n$ variables, say $x_1, \ldots, x_{2n}$, having a joint Gaussian distribution,

$$\overline{\exp\left(i\sum_k z_k x_k\right)} = \exp\left(-\tfrac{1}{2}\sum_{l,k}\overline{x_l x_k} z_l z_k\right),$$

where z_k is a complex number (see (12), § 2·9). The average of a complex number $x+iy$ is $\bar{x}+i\bar{y}$. We therefore have

$$\overline{F(u_1 t, v_1 t)\ldots F(u_n t, v_n t)\, F^*(u_{n+1} t, v_{n+1} t)\ldots F^*(u_{2n} t, v_{2n} t)}$$
$$= (AA^*)^n \exp\left[-\tfrac{1}{2}\overline{\{\phi(u_1 t, v_1 t)+\ldots+\phi(u_n t, v_n t)-\phi(u_{n+1} t, v_{n+1} t)-\ldots-\phi(u_{2n} t, v_{2n} t)\}^2}\right]. \tag{11}$$

The correlation function ρ of ϕ between two points (x_i, y_i), (x_j, y_j) is defined as follows

$$\overline{\phi(x_i y_i)\,\phi(x_j y_j)} = \phi_0^2 \rho(x_i - x_j, y_i - y_j), \tag{12}$$

where $\phi_0^2 = \overline{\phi^2}$. Denote $\rho(x_i - x_j, y_i - y_j)$ by ρ_{ij}.

When the irregularities are isotropic ρ_{ij} is a function only of the distance between the points, d_{ij}; i.e. $[(x_i - x_j)^2 + (y_i - y_j)^2]^{\frac{1}{2}}$.

The right-hand side of (11) is now

$$(AA^*)^n \exp\left[-\phi_0^2\{n+\rho_{12}+\rho_{13}+\ldots+\rho_{1n}-\rho_{1\,n+1}-\ldots-\rho_{1\,2n}+\rho_{23}+\ldots+\rho_{2n-1\,2n}\}\right]. \tag{13}$$

The exponential in (13) being expanded in a power series, (9) becomes

$$\overline{R^{2n}} = (AA^*)^n \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\left[1+\sum_{m=1}^{\infty}\frac{(-\phi_0^2)^m}{m!}(n+\rho_{12}+\ldots+\rho_{2n-1\,2n})^m\right]$$
$$\times \exp\left[-\pi i(u_1^2+\ldots-v_{2n}^2)\right] du_1 \ldots dv_{2n} \tag{14}$$

$$= (AA^*)^n\left\{1+\sum_{m=1}^{\infty}\frac{(-\phi_0^2)^m}{m!}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}(n+\rho_{12}+\ldots+\rho_{2n-1\,2n})^m\right.$$
$$\left.\times \exp\left[-\pi i(u_1^2+\ldots-v_{2n}^2)\right] du_1 \ldots dv_{2n}\right\}. \tag{15}$$

In going from (14) to (15) the integration of the first term, which is just the exponential, has been done. This is simply a product of Fresnel integrals of the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{-\pi i(u^2-v^2)\}\,du\,dv = 1.$$

The factor $(n+\rho_{12}+\ldots+\rho_{2n-1\,2n})^m$ may be expanded using multinomial theory. We then have to find integrals of products of ρ's with the exponential factor. Certain of these integrals will be independent of t, and others will depend on it. The products whose integrals are independent of t contain the factor $\rho_{ij}\rho_{kl}\ldots$, where no two of the ρ's have a suffix in common (i, j, k, l all different) and the first suffix of ρ comes from the sequence 1 to n and the second from the sequence $n+1$ to $2n$. The integral in this case is of the form

$$\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\rho[t(u_i-u_j), t(v_i-v_j)]\,\rho[t(u_k-u_l), t(v_k-v_l)]\ldots$$
$$\times \exp\left[-\pi i(u_i^2-u_j^2+v_i^2-v_j^2+u_k^2-u_l^2+v_k^2-v_l^2)\right] du_i \ldots dv_l \ldots. \tag{16}$$

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This may be factorized, a typical factor being

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho[t(u_i-u_j),t(v_i-v_j)]\exp\left[-\pi i(u_i^2-u_j^2+v_i^2-v_j^2)\right]du_i\,du_j\,dv_i\,dv_j. \tag{17}$$

If we make the change of variable

$$\sqrt{2}\,u_i=u_i'-u_j',\quad \sqrt{2}\,u_j=u_i'+u_j',\quad \sqrt{2}\,v_i=v_i'-v_j',\quad \sqrt{2}\,v_j=v_i'+v_j', \tag{18}$$

the integral becomes

$$\begin{aligned}&\iiiint\rho(2tu_j',2tv_j')\exp\{2\pi i(u_i'u_j'+v_i'v_j')\}\,du_i'du_j'dv_i'dv_j'\\&\quad=\iint\rho(2tu_j',2tv_j')\,\delta(u_i')\,\delta(v_j')\,du_j'dv_j'\\&\quad=\rho(0,0)=1.\end{aligned} \tag{19}$$

Therefore the integral of products of ρ's not depending on t is always one.

When the integral depends on t, for other combinations of the factors ρ_{ij}, we may proceed as follows. Divide the volume of integration into two parts:

$$|u_j|,|v_j|<L;\quad |u_j|,|v_j|\geqslant L.$$

In the first part, as $t\to\infty$ the integral tends to zero because $\rho_{ij}\to 0$ except at $u_i=u_j$, $v_i=v_j$, where $\rho_{ij}=1$. In the second region of integration we suppose that ρ_{ij} may be approximated by $\exp(-d_{ij}^2/r_0^2)$. Now we see from the Appendix that the integral over the whole space can then be evaluated and tends to zero like t^{-4}. Therefore the integral over the whole of the space tends to zero, since the first part is bounded. We may note that it is a reasonable assumption that the correlation tends to zero exponentially so that all the moments of the corresponding power spectrum exist. Although it is probably too strong an assumption to say that $\rho\sim\exp(-d_{ij}^2/r_0^2)$, we have not found a weaker one, from which we can deduce the desired result.

We may think of the $2n$ points having $2n(2n-1)/2$ correlations amongst them. As $t\to\infty$ only those points remain correlated (in fact perfectly correlated), which contribute opposite phases in the integral for $(EE^*)^n$.

Next we must find the total contribution of these highly correlated points. Since ρ_{ij} contributes nothing if $0\leqslant i,j\leqslant n$, or $n+1\leqslant i,j\leqslant 2n$, we may drop such terms in the multinomial expansion in (15). With this in mind we may write

$$(n+\rho_{12}+\ldots+\rho_{2n-1\,2n})^m=n^m\left[1+\sum_{j=1}^{m}(\rho_{1\,n+1}+\ldots+\rho_{n\,2n})^j\left(-\frac{1}{n}\right)^j\binom{m}{j}\right]. \tag{20}$$

The combinatorial problem is done in two stages.

(A) How many products are there of the form ρ_{ij}, $\rho_{ij}\rho_{kl}$, $\rho_{ij}\rho_{kl}\rho_{pq}$, ...; i.e. how many different combinations of r ρ's? ρ_{ij} may be chosen in n^2 ways, since i is any number from 1 to n, and j is any number from $n+1$ to $2n$. ρ_{kl} may then be chosen in $(n-1)^2$ ways, and so r ρ's may be chosen in

$$\frac{n^2(n-1)^2\ldots(n-r+1)^2}{r!}\quad\text{ways.} \tag{21}$$

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(B) Since there are j factors we have more than one way of forming the product of r ρ's. A given term such as $\rho_{1\,n+1}^{i_1}\rho_{2\,n+2}^{i_2}\dots\rho_{r\,n+r}^{i_r}$ may be chosen in

$$\sum_{i_l=1}^{i_l=j-1}\frac{j!}{i_1!\,i_2!\dots i_r!},\quad\left(\sum_{l=1}^{r}i_l=j\right) \tag{22}$$

ways. This, as shown in the Appendix, is equal to

$$r^j-\binom{r}{1}(r-1)^j+\binom{r}{2}(r-2)^j-\binom{r}{3}(r-3)^j+\dots+(-1)^r\binom{r}{r-1}. \tag{23}$$

The contribution of $(\rho_{1\,n+1}+\dots+\rho_{n\,2n})^j$ to the integral is therefore

$$\sum_{r=1}^{n}\sum_{l=0}^{r-1}\frac{n^2(n-1)^2\dots(n-r+1)^2}{r!}(-1)^l\binom{r}{l}(r-l)^j. \tag{24}$$

Therefore the integral of the term containing $(n+\rho_{12}+\dots+\rho_{2n-1\,2n})^m$ is

$$\begin{aligned}
&n^m\left[1+\sum_{j=1}^{m}\sum_{r=1}^{n}\sum_{l=0}^{r-1}\frac{n^2(n-1)^2\dots(n-r+1)^2}{r!}(-1)^l\binom{r}{l}(r-l)^j\left(-\frac{1}{n}\right)^j\binom{m}{j}\right]\\
&\quad=n^m\left[1+\sum_{r=1}^{m}\sum_{l=0}^{r-1}\left\{\left(1-\frac{r-l}{n}\right)^m-1\right\}(-1)^l\binom{r}{l}\frac{n^2(n-1)^2\dots(n-r+1)^2}{r!}\right]\\
&\quad=n^m+\sum_{r=0}^{n}\sum_{l=0}^{r-1}\{(n-r+l)^m-n^m\}(-1)^l\binom{r}{l}\frac{n^2(n-1)^2\dots(n-r+1)^2}{r!}.
\end{aligned} \tag{25}$$

Substituting this in (15) we have finally

$$\begin{aligned}
\frac{\overline{R^{2n}}}{(AA^*)^n}&=1+\sum_{m=1}^{\infty}\frac{(-\phi_0^2)^m}{m!}n^m\\
&\quad+\sum_{m=1}^{\infty}\sum_{r=1}^{n}\sum_{l=0}^{r-1}\left[\frac{(-\phi_0^2)^m}{m!}\{(n-r+l)^m-n^m\}(-1)^l\binom{r}{l}\frac{n^2(n-1)^2\dots(n-r+1)^2}{r!}\right]\\
&=\exp(-n\phi_0^2)\left[1+\sum_{r=1}^{n}\sum_{l=0}^{r-1}(\exp\{(r-l)\phi_0^2\}-1)(-1)^l\binom{r}{l}\frac{n^2(n-1)^2\dots(n-r+1)^2}{r!}\right]\\
&=\exp(-n\phi_0^2)\left[1+\sum_{r=1}^{n}(\exp(\phi_0^2)-1)^r\frac{n^2(n-1)^2\dots(n-r+1)^2}{r!}\right].
\end{aligned} \tag{26}$$

Now one might have expected that as $t\to\infty$ the fluctuations in the signal would have become normally distributed, because then the signal is made up of a large number of independent scattered waves. Although one often appeals to the Central Limit Theorem, it is not, of course, directly applicable, as it applies to only a sum of independent vectors. However, we have shown in the above analysis how the integral also behaves as a sum, if $\rho(d)\to 0$ sufficiently rapidly for large d. We shall now prove that R has a Rice distribution ((12), § 3·10).

Rice has considered the distribution of R when $E_1=\Re E$, $E_2=\Im E$, have a normal distribution about means $\bar{E}_1$, $\bar{E}_2$. He finds

$$\overline{R^n}=(2\psi_0)^{\frac{1}{2}n}\,\Gamma\left(\frac{n}{2}+1\right){}_1F_1\left(-\frac{n}{2};\,1;\,-\frac{P^2}{2\psi_0}\right),$$

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where ψ_0 is the variance of E_1 or E_2, $P = |\bar{E}_1 + i\bar{E}_2|$ and

$$ {}_1F_1(a; b; x) = 1 + \frac{ax}{b1!} + \frac{a(a+1)}{b(b+1)}\frac{x^2}{2!} + \ldots. $$

In our case $$2\psi_0 = \overline{(E-\bar{E})(E-\bar{E})^*} = \overline{EE^*} - \bar{E}\bar{E}^*.$$

We have already shown (10) that

$$\bar{E} = \bar{F}, \quad \overline{EE^*} = \overline{FF^*} = AA^*. \tag{27}$$

But $$\bar{F} = A\,\overline{\exp(i\phi)} = A\exp(-\tfrac{1}{2}\phi_0^2). \tag{28}$$

Thus $$2\psi_0 = AA^*(1-\exp(-\phi_0^2)), \tag{29}$$

$$\frac{\overline{R^{2n}}}{(AA^*)^n} = (1-\exp(-\phi_0^2))^n\, n!\, {}_1F_1\left(-n;\ 1;\ \frac{\exp(-\phi_0^2)}{1-\exp(-\phi_0^2)}\right). \tag{30}$$

The series ${}_1F_1$ terminates. It is no problem to show that (26) and (30) are identical, and that therefore R has a Rice distribution.

It does not follow immediately that E_1, E_2 are normally distributed; we must make some assumption about the phase. We have seen in the case of shallow phase (10) that the phase of the fluctuations becomes uniformly distributed from 0 to 2π as $z \to \infty$. If we accept this conclusion here, the desired result follows, by working backwards from amplitude and phase E_1 to E_2.

It is interesting to consider the formal basis of the joint normal distribution, summarized in the following Theorem, due to Maxwell ((9), p. 296) and found independently by the author:

THEOREM. *If E_1 and E_2 have a joint distribution which depends only on $E_1^2 + E_2^2$ (i.e. the phase is uniformly distributed from 0 to 2π) and E_1, E_2 are independent, then they have a joint normal distribution.*

Proof.

$$p(E_1, E_2) = f(E_1^2 + E_2^2),$$

$$p(E_1, E_2) = p(E_1)\,p(E_2) = g(E_1^2)\,g(E_2^2).$$

Clearly f and g differ only by a constant factor and hence f is an exponential. Q.E.D.

We may regard the results leading to (26) as showing the independence of E_1, E_2.

It must be stressed, although it is obvious mathematically, that only when $\phi_0 \gg 1$ can P vanish: only a *deep*-phase screen can produce a signal with a Rayleigh distribution.

4. *Calculation of* $\lim_{z_0 \to \infty} \overline{R^{2n}(x_0, y_0, z_0)\, R^{2n}(x_0+\xi, y_0+\eta, z_0)}$. In this section we calculate the correlation of intensity R^2 and its moments between two points very far from the screen, lying in a plane parallel to the screen.

Now, from (1), making a translation of the receiver to $(x_0+\xi, y_0+\eta)$,

$$E(x_0+\xi, y_0+\eta) = -\frac{i}{\lambda z_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x+x_0+\xi, y+y_0+\eta)\exp\left[-\frac{\pi i}{\lambda z_0}(x^2+y^2)\right]dx\,dy. \tag{31}$$

Let $$\phi(x_i y_i) \equiv \phi_i, \quad \phi(x_i+\xi, y_i+\eta) \equiv \phi_i', \quad R(x_1+\xi, y_1+\eta) = R'. \tag{32}$$

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Then we have, using the u, v notation (8),

$$\frac{\overline{R^{2n}R'^{2n}}}{\overline{(AA^*)^{2n}}} = \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} \overline{\exp\{i(\phi_1+\dots+\phi_n-\phi_{n+1}-\dots-\phi_{2n}+\phi'_{2n+1}+\dots+\phi'_{3n}-\phi'_{3n+1}-\dots-\phi'_{4n})\}} \times \exp[-\pi i(u_1^2+\dots-v_{4n}^2)]\,du_1\dots dv_{4n} \quad (33)$$

$$= \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} \exp\{-\phi_0^2(2n+\rho_{12}+\dots+\rho_{1n}-\rho_{1n+1}-\dots-\rho_{12n}+\rho'_{12n+1}+\dots+\rho_{4n-14n})\} \times \exp[-\pi i(u_1^2+\dots-v_{4n}^2)]\,du_1\dots dv_{4n}, \quad (34)$$

where ρ' denotes correlation between ϕ and ϕ',

$$= 1+\sum_{m=1}^{\infty}\left\{\frac{(-\phi_0^2)^m}{m!}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}(2n+\rho_{12}+\dots+\rho_{4n-14n})^m \times \exp[-\pi i(u_1^2+\dots-v_{4n}^2)]\,du_1\dots dv_{4n}\right\}. \quad (35)$$

Before continuing with the general case, we consider the case $n = 1$. The integrand in (35) is

$$(2-\rho_{12}+\rho'_{13}-\rho'_{14}-\rho'_{23}+\rho'_{24}-\rho_{34})^m. \quad (36)$$

Dropping ρ'_{13}, ρ'_{24} we have

$$\sum_{j=0}^{m} 2^{m-j}\binom{m}{j}(-1)^j[(\rho_{12}+\rho_{34})+(\rho'_{14}+\rho'_{23})]^j = 2^m+\sum_{j=1}^{\infty}2^{m-j}\binom{m}{j}(-1)^j[(\rho_{12}+\rho_{34})+(\rho'_{14}+\rho'_{23})]^j. \quad (37)$$

Contributions from $\rho_{12}\rho_{14}$, etc., are zero, since the ρ's have a common suffix. Contributions from ρ_{12} and $\rho_{34} = 1$. Contributions from ρ'_{14} and $\rho'_{23} = \rho(\xi,\eta)$. The last result follows from a suitable modification of (16)–(19). We have, as the contribution from ρ_{14}, for example,

$$\iiiint \rho\{t(u_1-u_4)+\xi, t(v_1-v_4)+\eta\}\exp\{-\pi i(u_1^2-u_4^2+v_1^2-v_4^2)\}\,du_1\dots dv_4 = \iint\rho\{tu'_1+\xi, tv'_1+\eta\}\,\delta(u'_1)\,\delta(v'_1)\,du'_1dv'_1 = \rho(\xi,\eta). \quad (38)$$

The contribution from (36) is therefore

$$2^m+\sum_1^m 2^{m-j}\binom{m}{j}(-1)^j[2^j+(2\rho)^j] = 2^m+2^m\sum_1^m\binom{m}{j}(-1)^j(1+\rho^j) = 2^m+2^m(-2+(1-\rho)^m) = -2^m+2^m(1-\rho)^m. \quad (39)$$

Substituting in (35) we have

$$\frac{\overline{R^2R'^2}}{(AA^*)^2} = 1+\sum_1^{\infty}\frac{(-\phi_0)^2}{m!}(-2^m+2^m(1-\rho)^m) = 1+\exp[-2\phi_0^2(1-\rho)]-\exp(-2\phi_0^2). \quad (40)$$

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In the general case, in order to facilitate the combinatorial analysis, we suppose that all the ρ's are arranged in a triangular array, using $n = 2$ as an example.

$$\begin{array}{ccc|cc:cc}
12 & 13 & 14 & 15 & 16 & 17 & 18 \\
 & 23 & 24 & 25 & 26 & 27 & 28 \\
\hdashline
 & & 34 & 35 & 36 & 37 & 38 \\
 & & & 45 & 46 & 47 & 48 \\
\hline
 & & & & 56 & 57 & 58 \\
 & & & & & 67 & 68 \\
 & & & & & & 78
\end{array}
=
\begin{array}{c:c}
\text{I} & \text{II} \\
\hdashline
\text{III} & \text{IV}
\end{array}$$

The ρ's which are the correlations between points contributing opposite phases are in the large square. All the ρ's outside this contribute zero as $t \to \infty$. Of those inside the square, those in quarters I, IV give 1 in the integral; those in II, III give $\rho(\xi, \eta)$. Now the number of times a term of the form $\rho(0,0)^{r+t}\rho(\xi,\eta)^{s+u}$ occurs, where r different ρ's are from I, s from II, u from III, t from IV is

$$\begin{aligned}
&\frac{n^2(n-1)^2 \dots (n-r+1)^2}{r!} \times \frac{n^2(n-1)^2 \dots (n-t+1)^2}{t!} \\
&\qquad \times \frac{(n-r)(n-t)\dots(n-r-s+1)(n-t-s+1)}{s!} \\
&\qquad \times \frac{(n-r)(n-t)\dots(n-r-u+1)(n-t-u+1)}{u!} \\
&= \frac{n!^4}{(n-r-s)!\,(n-r-u)!\,(n-t-s)!\,(n-t-u)!\,r!\,t!\,s!\,u!}. \qquad (41)
\end{aligned}$$

We write as before (equation (20))

$$(2n+\rho_{12}+\dots+\rho_{4n-1\,4n})^m = (2n)^m\left[1+\sum_{j=1}^{m}\left\{\left(-\frac{1}{2n}\right)^j\binom{m}{j}(\rho_{12}+\dots+\rho_{4m-1\,4n})^j\right\}\right]. \qquad (42)$$

Now, as in (22), we have the contribution of all the terms in the multinomial expansion, giving rise to the same term $\rho(0,0)^{r+t}\rho(\xi,\eta)^{s+u}$ as,

$$\sum_{1\leqslant i\leqslant j-1} \frac{j!\,\rho(\xi,\eta)^{i_{r+t+1}+\dots+i_{r+t+s+u}}}{i_1!\,i_2!\dots i_{r+s+t+u}!} \quad (\Sigma i = j). \qquad (43)$$

This is shown in the Appendix to be

$$\sum_{l=0}^{r+t+s+u-1}\ \sum_{p+q=l} (-1)^l\binom{r+t}{p}\binom{s+u}{q}[r+t-p+(s+u-q)\rho(\xi,\eta)]^j. \qquad (44)$$

Multiplying by (41) and summing over r, t, s, u, gives the total contribution of

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$(\rho_{12}+\ldots+\rho_{4n-14n})^j$. Substituting the result into (42) we have the contribution of $(2n+\rho_{12}+\ldots+\rho_{4n-14n})^m$:

$$(2n)^m+\sum_{r+t+s+u=1}^{2n}\sum_{l=0}^{r+t+s+u-1}\sum_{p+q=l}(-1)^l\binom{r+t}{p}\binom{s+u}{q}$$
$$\times\{(2n-[r+t-p+(s+u-q)\rho(\xi,\eta)])^m-(2n)^m\}$$
$$\times\frac{n!^4}{(n-r-s)!\,(n-r-u)!\,(n-t-s)!\,(n-t-u)!\,r!\,t!\,s!\,u!}. \tag{45}$$

Substituting into (35) and summing over m, and then over p, q, and l we have finally,

$$\frac{\overline{R^{2n}R'^{2n}}}{(AA^*)^{2n}}=\exp(-2n\phi_0^2)+\sum_{r+t+s+u}\sum_{l}\sum_{p+q}(-1)^l\binom{r+t}{p}\binom{s+u}{q}$$
$$\times\exp(-2n\phi_0^2)\,(\exp\{[r+t-p+(s+u-q)\rho]\phi_0^2\}-1)$$
$$\times\frac{n!^4}{(n-r-s)!\,(n-r-u)!\,(n-t-s)!\,(n-t-u)!\,r!\,t!\,s!\,u!}$$
$$=\exp(-2n\phi_0^2)\sum_{r+s+t+u=0}^{2n}\{\exp(\phi_0^2-1)\}^{r+t}[\exp\{\phi_0^2\rho(\xi,\eta)\}-1]^{s+u}$$
$$\times\frac{n!^4}{(n-r-s)!\,(n-r-u)!\,(n-t-s)!\,(n-t-u)!\,r!\,t!\,s!\,u!}. \tag{46}$$

5. *Discussion.* When $n=1$, equation (26) gives

$$\overline{R^2}=AA^* \tag{47}$$

which is a form of the result obtained in the previous paper (10), $\overline{EE^*}=\overline{FF^*}$. Although equation (26) is true only when $z\to\infty$, this special case is true for all values of z. It expresses conservation of energy.

When $n=2$,
$$\overline{R^4}=(AA^*)^2\,(2-\exp(-2\phi_0^2)).$$

The variance of R^2 is therefore

$$\frac{\overline{(R^2-\overline{R^2})^2}}{\overline{R^2}^2}=1-\exp(-2\phi_0^2), \tag{48}$$
$$\doteqdot 2\phi_0^2 \quad \text{if} \quad \sigma_0^2\ll 1.$$

From equation (40),

$$\frac{\overline{R^2R'^2}}{(AA^*)^2}=1+\exp(-2\phi_0^2(1-\rho))-\exp(2-\phi_0^2).$$

Therefore the correlation function of R^2 in a plane parallel to the screen is, as $z\to\infty$,

$$\rho_{R^2}(\xi,\eta)=\frac{\overline{R^2R'^2}-\overline{R^2}^2}{\overline{R^4}-\overline{R^2}^2}=\frac{\exp(2\phi_0^2\rho)-1}{\exp(2\phi_0^2)-1}$$
$$\doteqdot\rho(\xi,\eta) \quad \text{if} \quad \phi_0^2\ll 1, \tag{49}$$
$$\doteqdot\exp(-2\phi_0^2[1-\rho(\xi,\eta)]) \quad \text{if} \quad \phi_0^2>1. \tag{50}$$

In the case of E region fading it is believed that one is effectively at a very large distance from the screen, except for bursts of amplitude. Hence one may apply

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equations (48) and (50) to find ϕ_0^2 and then $\rho(\xi, \eta)$ from measurements of the variance of R^2 and its correlation function (using spaced receivers). Some calculations of R^2 by the author using experimental fading records indicate that ϕ_0^2 is about one.

When $\phi_0^2 > 1$, and if we assume $\rho = \exp(-d^2/r_0^2)$ we have

$$\rho_{R^2}(d) = \exp(2\phi_0^2(\rho - 1)) \doteqdot \exp\left(-\frac{2\phi_0^2 d^2}{r_0^2}\right). \tag{51}$$

The distance d at which the correlation has fallen to e^{-1} is thus reduced by $\sqrt{2}\phi_0$ compared to the original phase fluctuations. Hewish found that at a large distance the correlation of R fell to e^{-1} in a distance r_0/ϕ_0. Although his methods were different it is likely that his result is different mainly because he considered R instead of R^2.

Bramley (3), has studied the correlation of the complex amplitude E immediately in front of the screen. He deduced the result

$$\rho_E(d) \equiv \frac{\overline{EE'^*} - \bar{E}\bar{E}'^*}{\overline{EE^*} - \bar{E}\bar{E}^*} = \exp(-\phi_0^2[1 - \rho(d)]),$$

(in our notation). This function is actually independent of the distance from the screen, since it may be shown that $\overline{EE'^*} = \overline{FF'^*}$ (1). Therefore the correlation of the complex amplitude is the same, except for a scale factor $\sqrt{2}$, as the correlation of the amplitude squared at a great distance and when $\phi_0^2 > 1$.

6. *Amplitude—phase screens.* If we had started with a complex amplitude

$$F = A\, e^{\mu} = A\, e^{\lambda + i\nu}, \tag{52}$$

where λ and ν are real stochastic functions with a joint normal distribution, we would have obtained results similar in many respects to those listed above. We shall indicate the modifications required in the formulae. Equation (11) becomes

$$\overline{F(u_1 t, v_1 t) \dots F^*(u_{2n} t, v_{2n} t)} = (AA^*)^n \exp\{\tfrac{1}{2}\overline{(\mu_1 + \dots + \mu_n + \mu_{n+1}^* + \dots + \mu_{2n}^*)^2}\}. \tag{53}$$

Now we know we can disregard correlation between

$$\mu_i \text{ and } \mu_j \quad \text{if} \quad 1 \leqslant i, j \leqslant n; \qquad \mu_i^* \text{ and } \mu_j^* \quad \text{if} \quad n+1 \leqslant i, j \leqslant 2n,$$

and so keeping only the other terms, (53) is

$$(AA^*)^n \exp\{n(\overline{\lambda^2} - \overline{\nu^2}) + (\overline{\lambda^2} + \overline{\nu^2})(\rho_{1\,n+1} + \dots + \rho_{n\,2n})\}, \tag{54}$$

where ρ is now the correlation function of μ,

$$\overline{\mu(x, y)\, \mu^*(x + \xi, y + \eta)} = \overline{\mu\mu^*}\, \rho(\xi, \eta). \tag{55}$$

Thus instead of (26), we have

$$\frac{\overline{R^{2n}}}{\overline{(AA^*)^n}} = \exp\{n(\lambda^2 - \nu^2)\} \sum_{r=0}^{n} (\exp(\overline{\lambda^2} + \overline{\nu^2}) - 1)^r \frac{n^2(n-1)^2 \dots (n-r+1)^2}{r!}, \tag{56}$$

and instead of (46),

$$\frac{\overline{R^{2n} R'^{2n}}}{\overline{(AA^*)^{2n}}} = \exp\{2n(\overline{\lambda^2} - \overline{\nu^2})\} \sum_{r,s,t,u}^{n} (\exp(\overline{\lambda^2} + \overline{\nu^2}) - 1)^{r+t} (\exp[(\lambda^2 + \nu^2)\rho(\xi, \eta)] - 1)^{s+u} \times \frac{n!^4}{(n-r-s)!\,(n-r-u)!\,(n-t-s)!\,(n-t-u)!\,r!\,s!\,t!\,u!}. \tag{57}$$

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The variance and correlation of R^2 become

$$\frac{\overline{(R^2-\bar{R}^2)^2}}{\bar{R}^{2^2}} = 1-\exp\{-2(\overline{\lambda^2}+\overline{\nu^2})\}, \tag{58}$$

$$\rho_{R^2}(\xi,\eta) = \frac{\exp\{2(\overline{\lambda^2}+\overline{\nu^2})\rho(\xi,\eta)\}-1}{\exp\{2(\overline{\lambda^2}+\overline{\nu^2})\}-1}. \tag{59}$$

A medium causing both absorption and refraction may be characterized by a complex refractive index μ, whose real and imaginary parts are proportional to the density of the medium. If the variation in the wave in passing through the medium is negligible—i.e. a thin screen—then the complex amplitude of the outgoing wave will be given by (52). Equations (58), (59) show that the variations of refraction and absorption are indistinguishable in terms of the variance and correlation.

7. *Variation of* $\overline{R^4}$ *with* z_0 *for* $\phi_0^2 \ll 1$. We may start with (15), keeping only the term in ϕ_0^2 and putting $n = 2$.

$$\frac{\overline{R^4}}{(AA^*)^2} = 1-\phi_0^2\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}(2+\rho_{12}+\rho_{13}+\dots+\rho_{34}) \times \exp[-\pi i(u_1^2+u_2^2-u_3^2-u_4^2+v_1^2+v_2^2-v_3^2-v_4^2)]\,du_1\dots dv_4. \tag{60}$$

Let us now assume that

$$\rho_{ij} = \exp\left\{-\frac{t^2}{r_0^2}[(u_i-u_j)^2+(v_i-v_j)^2]\right\}. \tag{61}$$

The integrals may be evaluated without too much difficulty (a general formula is given in the Appendix) and we finally have

$$\overline{R^4} = (AA^*)^2\left(1+\phi_0^2\frac{8\lambda^2z_0^2/\pi^2r_0^4}{1+4\lambda^2z_0^2/\pi^2r_0^4}\right). \tag{62}$$

Hewish (7) obtained a similar result according to his theory of a shallow phase screen. He defined a scintillation index F_{rms} as follows,

$$F_{\text{rms}}^2 = \frac{\overline{(\Delta R)^2}}{\bar{R}^2}, \tag{63}$$

and found that, for a one-dimensional screen,

$$\frac{2F_{\text{rms}}^2}{\phi_0^2} = 1-\frac{1}{s_0}\int_0^{s_0}\cos\frac{\pi s^2}{2}\,ds, \tag{64}$$

where

$$s_0 = \frac{2\sqrt{(\lambda z_0)}}{r_0}.$$

For small z_0, (64) becomes

$$\frac{2F_{\text{rms}}^2}{\phi_0^2} = \frac{2\pi^2\lambda^2z_0^2}{5r_0^2}. \tag{65}$$

The variance of R^2 (from 62) is

$$\frac{\overline{R^4}-\overline{R^2}^2}{\bar{R}^{2^2}} = \phi_0^2\frac{1+4\lambda^2z_0^2/\pi^2r_0^4}{8\lambda^2z_0^2/\pi^2r_0^4} \doteq \frac{8\phi_0^2\lambda^2z_0^2}{\pi^2r_0^4}, \tag{66}$$

which is approximately the same as (65). It must be remembered that Hewish's result was for a one-dimensional screen and applied to variation in R rather than R^2.

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8. *The numerical calculation of* $\overline{R^4}$ *for* $\phi_0^2 \simeq 1$. In the Appendix the calculation by Dr M. L. V. Pitteway is given of $\overline{R^4}$ for different values of ϕ_0 up to $\phi_0^2 = 2$. The method, described in more detail there, was essentially a continuation of that begun in the previous section, equation (60). The expansion in powers of ϕ_0^2 was continued up to ϕ_0^{32}. The numerical results are shown in Fig. 1. The limit $2 - \exp(-2\phi_0^2)$ is approached very closely. The accumulation of rounding-off error made the calculation impossible when $\phi_0^2 > 2$. No approximation, alternative to the power series expansion, could be found suitable for large ϕ_0.

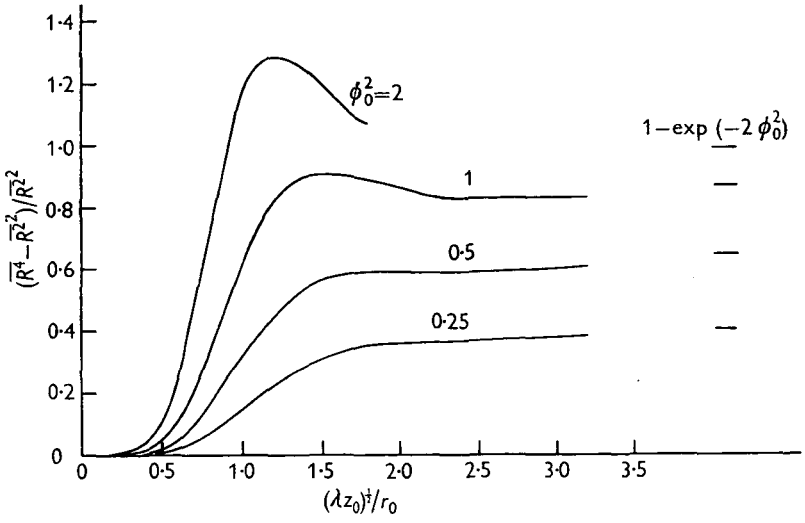


Fig. 1. This shows the dependence of the variance of R^2 on $\sqrt{(\lambda z_0)}/r_0$ for a deep phase screen with isotropic irregularities and Gaussian correlogram. Curves are given for different values of ϕ_0, the rms phase fluctuation at the screen in radians. R = amplitude, λ = wavelength, z_0 = distance from the screen, r_0 = irregularity size at the screen.

The appearance of maxima in the curves, at distances $z_0 \simeq 2r_0^2/\lambda$ and for $\phi_0 \gtrsim 1$, is a hitherto unnoticed phenomenon. If we think of the irregularities as refracting objects then we might expect the rays coming from the screen to converge together. Near the screen the phase fluctuation is not deep enough, and far from the screen all the focusing is smeared out. But at some particular distance, the rays converge together more than elsewhere. The larger ϕ_0 the closer is the maximum to the screen.

A fading signal is known to have a Rayleigh distribution if all the power is in the fluctuations. It seems odd therefore to find that $(\overline{R^4} - \overline{R^2}^2)/\overline{R^2}^2$ actually exceeds 1, its value for a Rayleigh distribution. The explanation is that the fluctuations are not randomly phased, and the in-phase and quadrature components E_1 and E_2 are not statistically independent. This allows R to be more peaked than in a Rayleigh distribution. The contours of constant probability of E_1, E_2 are difficult to conjecture. However, the original arc at $z = 0$ is probably twisted in such a way as to give a large range of amplitude. As $z \to \infty$ the contour becomes circular. Note that maxima still occur when $(\overline{R^4} - \overline{R^2}^2)/\overline{R^2}^2$ is less than one.

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The more the signal is randomly phased the more difficult it is to get focusing, because of the lack of coherence. Therefore, although the rays converge together for any ϕ_0, when the point of convergence is so far that the signal is randomly phased, the convergence cannot cause a maximum. The focal distance from a lens of radius r_0 is $\pi r_0^2/\phi_0\lambda$. On the other hand the signal is randomly phased if $\pi r_0^2/\lambda z_0 < 1$. Thus $\phi_0 > 1$ is the condition for focusing. This is in agreement with the calculations.

9. *Experimental applications.* (i) *Radio star scintillation.* In the case of radio star scintillation we have the following values for λ, z and r_0:

$$\lambda \simeq 8\,\text{m.}, \quad z_0 \simeq 300\,\text{km.}, \quad r_0 \simeq 1\text{–}5\,\text{km.}$$

Therefore
$$\sqrt{(\lambda z_0)}/r_0 \simeq 0{\cdot}3\text{–}1{\cdot}5.$$

We may expect to find focusing effects, if the phase fluctuations produced by the ionosphere are large enough.

If $(\overline{R^4} - \overline{R^2}^2)/\overline{R^2}^2$, which we shall call σ^2 in the rest of the paper, exceeds one, we may be sure there is focusing. The author has examined three records made at Cambridge using 38 Mc/s. The receiver records R^2 (13), but the mean amplitude of the signal varies sinusoidally as the radio star moves through the lobes of the interference pattern. When the latter modification is taken into account, it is possible to find σ^2, and hence to apply the theory. The three values of σ^2 obtained were 0·097, 0·3, 0·31. These are all definitely less than one, so there is no evidence of focusing. The empirical 'scintillation index' F, was 2–3, 4–4½, 4½–5, respectively, on a scale of one to five. This makes $F \propto \sigma$, nearly. $F = 5$ includes, however, all records in which $F > 4$. There may be some, then, in which σ is much nearer one.

We must conclude that ϕ_0 is probably less than one. The minimum values of ϕ_0 may be found from equation (48), and for $\sigma^2 = 0{\cdot}1$, 0·3, $\phi_0 = 0{\cdot}23$, 0·42, respectively. Therefore, the ionosphere acts as a thin phase screen for radio star scintillations.

These results may be compared with those of Hewish (7), who made a similar analysis. He found that some records had a Rayleigh distribution, but that $\phi_0 = 0{\cdot}2$ represented more usual ionospheric conditions. This is not entirely consistent, as a Rayleigh distribution requires $\phi_0 > 1$ but agrees more or less with what we find.

If ϕ_0 could be estimated from the received signal, and σ measured, then $\sqrt{z_0}/r_0$ would be determined. It was by this means that Hewish (7) found the height of the irregularities (r_0 was measured using spaced receiver correlation).

By making measurements of different frequencies and finding how σ depends on $\sqrt{(\lambda z_0)}/r_0$ it may be possible to find how ϕ_0 depends on λ. This would tell us something about the way the irregularities cause scattering. For in a dispersive medium like the ionosphere the refractive index fluctuation $1 - n \propto \lambda^2$. If the wave is unaltered in the medium (Born approximation) then $\phi_0 \propto (1-n)/\lambda \propto \lambda$. However, if there is multiple scattering, ϕ_0 will not be proportional to λ.

If $\phi_0^2 \ll 1$, then $\sigma \propto \phi_0\lambda$, according to equation (66). Now Hewish (7) found that $F_{\text{rms}} \propto \lambda^2$, by comparing scintillations on two frequencies. Therefore $\sigma \propto \lambda^2$, and $\phi_0 \propto \lambda$. We may conclude, on rather slender evidence, that there is no multiple scattering.

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It is necessary, strictly speaking, to modify the calculations to take into account the fact that the irregularities are not isotropic, but are elongated in a ratio of 5 : 1 to 10 : 1. This modification will not be very difficult since the integrals in the power series in ϕ_0^2 factor into two parts, depending respectively on u and v.

(ii) *Scatter propagation.* This term refers to beyond the horizon propagation at frequencies from about 200 Mc/s to the micro-wave region. The propagation takes place by means of irregularities in the troposphere at micro-wave frequencies, or in the ionosphere at the lower frequencies. The signal is of a highly irregular nature in all cases, and it is of interest to inquire whether the present study of irregular phase screens has any application.

I shall only draw some general conclusions from the statistical distribution of R, based on a volume of experimental papers which appeared a few years ago (15). It seems that the fading of the signal can be considered as a combination of short and long period fading, and that these probably refer to different mechanisms. The short period fading (less than 5 min. period) seems to have a Rayleigh distribution, but a closer examination reveals departures from this (5). The long period fading (longer than 15 min. period) shows the so-called log-normal distribution. Unfortunately, not one of the many authors specifically defined this term. It would seem to mean that the frequency function of R is given by βR^a, where β and a are some constants. For the case of microwave propagation $a = -2{\cdot}34 \pm 0{\cdot}06$ (average of 4 cases), and for the tropospheric case $a \doteq -1{\cdot}2$. Naturally β cannot be determined, because of the divergence at $R = 0$.

This 'log-normal' distribution means that the signal has a greater proportion of large values of R than with a Rayleigh distribution. It we could estimate σ it would surely be greater than one, since it is one for the Rayleigh distribution. Thus the long-period fading seems to be an example of focusing. It may be due to the motion of large clouds across the path of propagation.

Herbstreit and Thompson (5) measured phase as well as amplitude fluctuations in the short period fading (1046 Mc/s). They found that the signal could be described as a steady component plus fluctuations, the latter not being uniformly distributed in phase from 0 to 2π. The fluctuation in quadrature was larger than the in-phase fluctuation. This would indicate that the receiver was near a phase screen (10). The rms phase fluctuation was about 5°, so that the fading is due to a shallow phase screen.

10. *General conclusions.* We have determined a number of statistical properties of a wave propagated from a deep-phase screen. When there are many irregularities in a Fresnel zone; i.e. at a great distance from the screen, the calculation of all even moments of R shows that R has a Rice distribution. This is true for any correlation function of the phase fluctuations in the screen, provided it falls to zero sufficiently rapidly (like $\exp(-d^2/r_0^2)$ for large d). This is, in a sense, a generalization of the Central Limit Theorem to integrals instead of sums. The correlation function of R^{2n} is also calculated.

When we consider the case where the Fresnel zone is comparable with the irregularity size, the integrals are more difficult and were programmed for numerical calculation

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on EDSAC II (Cambridge Mathematical Laboratory) by Dr Pitteway. It is found that the irregularities are still able to bring the rays to a focus provided $\phi_0 \gtrsim 1$.

Although radio star scintillations are sometimes thought to be due to a deep-phase screen, there is in fact no evidence that it is usually deep, in agreement with Hewish's conclusions. It would also appear that there is no multiple scattering.

Scatter propagation shows two types of fading. The long-period fading appears to be due to a deep-phase screen and the short period fading to shallow-phase screen. The receiver is in the region of Fresnel diffraction in both cases, since there are significant departures from the Rice distribution of amplitude.

APPENDIX

The equations (23) and (44) were found by the author using heuristic arguments and consideration of special cases. He owes the following proof using a generating function to Mr J. Fowler, of Trinity College, Cambridge.

Consider $(e^x-1)^r$. Using the binomial theorem and then expanding the exponential it may be written

$$\sum_{l=0}^{r} e^{lx}\binom{r}{l}(-1)^{r-l} = \sum_{j=0}^{\infty}\sum_{l=0}^{r}\binom{r}{l}\frac{e^j x^j}{j!}(-1)^{r-l}. \tag{67}$$

Using the exponential expansion first and then the multi-nominal theorem,

$$(e^x-1)^r = \left(\sum_{i=0}^{\infty}\frac{x^{i+1}}{(i+1)!}\right)^r. \tag{68}$$

The coefficient of x^j is

$$\sum_{i\geqslant 0}\frac{x^{i_1+\ldots+i_r+r}}{(i_1+1)!\ldots(i_r+1)!} = \sum_{i\geqslant 1}\frac{x^j}{i_1!\,i_2!\ldots i_r!}.$$

Therefore,

$$\sum_{l=0}^{r}(-1)^{r-l}\binom{r}{l}e^j = \sum_{i\geqslant 1}\frac{j!}{i_1!\ldots i_r!},$$

which is the result desired.

Next consider $(e^x-1)^v(e^{x\rho}-1)^w$. Using the exponential expansion, and the binomial theorem, this becomes,

$$\left(\sum_{m=0}^{v}(-1)^m\binom{v}{m}e^{x(v-m)}\right)\left(\sum_{n=0}^{w}(-1)^n\binom{w}{n}e^{x\rho(w-n)}\right)$$
$$= \sum_{j=0}^{\infty}\sum_{m=0}^{v}\sum_{n=0}^{w}(-1)^{m+n}\binom{v}{m}\binom{w}{n}\frac{[v-m+(w-m)\rho]^j x^j}{j!}. \tag{69}$$

The coefficient of x^j is

$$\sum_{m=0}^{v}\sum_{n=0}^{w}(-1)^{m+n}\binom{v}{m}\binom{w}{n}\frac{[v-m+(w-m)\rho]^j}{j!}. \tag{70}$$

Otherwise, using the multinomial expansion,

$$(e^x-1)^v(e^{x\rho}-1)^w = \left(\sum_{i=0}^{\infty}\frac{x^{i+1}}{(i+1)!}\right)^v\left(\sum_{k=0}^{\infty}\frac{x^{k+1}\rho^{k+1}}{(k+1)!}\right)^w$$

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The coefficient of x^j is

$$\sum_{i=0}\sum_{k=0}\frac{\rho^{k_1+k_2+\ldots+k_w+w}}{(i_1+1)!\ldots(i_v+1)!\,(k_1+1)!\ldots(k_w+1)!}$$

$$=\sum_{i\geqslant 0}\frac{\rho^{i_{v+1}+i_{v+2}+\ldots+i_{n+w}+w}}{(i_1+1)!\,(i_2+1)!\ldots(i_{v+w}+1)!}$$

$$=\sum_{i\geqslant 1}\frac{\rho^{i_{v+1}+i_{v+2}+\ldots+i_{v+w}}}{i_1!\,i_2!\ldots i_{v+w}!}. \tag{71}$$

Equation (44) is obtained by putting $v=r+t$, $w=s+u$, and equating coefficients of x^j in (70), (71).

The numerical evaluation of $\overline{R^4}$ was done by Dr M. L. V. Pitteway, of the Mathematical Laboratory, who has communicated the following summary.

The problem is to evaluate, as a function of ϕ_0^2 and t the integral I given by

$$I=\int\int\int\int\int\int\int\int_{-\infty}^{\infty}\exp(-\phi_0^2A+\pi iB)\,du_1du_2du_3du_4dv_1dv_2dv_3dv_4, \tag{72}$$

where

$$\begin{aligned}A=2&+\exp\{-[(u_2-u_1)^2+(v_2-v_1)^2]t^2\}\\&-\exp\{-[(u_3-u_1)^2+(v_3-v_1)^2]t^2\}\\&-\exp\{-[(u_4-u_1)^2+(v_4-v_1)^2]t^2\}\\&-\exp\{-[(u_3-u_2)^2+(v_3-v_2)^2]t^2\}\\&-\exp\{-[(u_4-u_2)^2+(v_4-v_2)^2]t^2\}\\&-\exp\{-[(u_4-u_3)^2+(v_4-v_3)^2]t^2\}\end{aligned} \tag{73}$$

and

$$B=(u_1^2+u_2^2-u_3^2-u_4^2)+(v_1^2+v_2^2-v_3^2-v_4^2). \tag{74}$$

The eightfold integral is difficult to evaluate numerically. The following method was devised for evaluation on EDSAC II, the digital computer of the Cambridge University Mathematical Laboratory.

Expand I as a power series in ϕ_0^2. There are seven terms in A, and we expand A^m by a multinomial process. Symmetry between u and v allows the integral to be separated, and gives

$$I=\sum_{m=0}^{\infty}(-1)^m\phi_0^{2m}\sum_{n_1\ldots n_7}\pm\frac{2^{n_1}}{\prod_{i=1}^{7}n_i!}U^2, \tag{75}$$

where $\sum_{i=1}^{7}n_i=m$, the sign is $+$ or $-$ according as $\sum_{i=3}^{6}n_i$ is even or odd, and

$$U=\int\int\int\int_{-\infty}^{\infty}\exp(-\mathbf{uAu'})\,du_1du_2du_3du_4. \tag{76}$$

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u is the four-vector $(u_1 u_2 u_3 u_4)$, and **A** is a 4×4 matrix involving t^2 and the n_i. By using an orthogonal transformation to diagonalize **A**, so that the four integrals in (76) separate, it can be shown (subject to conditions of convergence) that

$$U^2 = \frac{\pi^4}{|\mathbf{A}|}. \tag{77}$$

This yields at length

$$I = -\sum_{m=0}^{\infty} \phi_0^{2m} \times \sum_{n_1 \ldots n_6} \frac{(-1)^{n_1+n_2+n_7} 2^{n_1}}{\prod_{j=1}^{7} n_j! \left[\left\{\frac{t^2}{\pi}(n_7 - n_2) + i\right\}^2 - \frac{t^4}{\pi^2}\{(n_4 - n_1)(n_2 + n_7) + (n_4 + n_5)(n_3 + n_6)\}\right]}.$$

The coefficients of the powers ϕ_0^{2m}, for a given value of t, have been evaluated on EDSAC II. The limits of the sixfold sum in the multinomial expansion are

$$\left.\begin{aligned} &0 \leqslant n_1 \leqslant m, \\ &0 \leqslant n_2 \leqslant m - n_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &0 \leqslant n_6 \leqslant m - n_1 - n_2 - n_3 - n_4 - n_5, \\ &n_7 \text{ fixed.} \end{aligned}\right\} \tag{79}$$

The imaginary parts of the coefficients vanish, but were evaluated as a measure of the error. The calculation is slow for values of m greater than 15, and errors accumulate. The results of the calculations are shown in Fig. 1, where, for different values of ϕ_0^2, I is plotted as function of t. Agreement with the asymptotic limit is indicated for large t.

This method of integration is not suitable for evaluation of I beyond the limits of Fig. 1; for further work, a different expression or approximation is required.

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