Math. Proc. Camb. Phil. Soc. (1988), **103**, 207
Printed in Great Britain

On fixed-point-free elements

BY ALBERTO ESPUELAS
Departamento de Algebra, Universidad de Zaragoza, Zaragoza, Spain

(*Received* 31 *July* 1987)

In Theorem 1·2 of [3] the following extension of the classical theorem of Shult is proved.

THEOREM. *Let G be a p-solvable group and let V be a faithful KG-module, where* $\text{char}(K) \nmid |O_{p'}(G)|$. *Assume that G contains an element x of order p^n acting fixed-point-freely on V. If p is a Fermat prime suppose further that the Sylow 2-subgroups of G are abelian. If $p = 2$ assume that the Sylow q-subgroups of G for each Mersenne prime q less than 2^n are abelian. Then $x^{p^{n-1}} \in O_p(G)$.*

This result provides the motivation for the restriction on the characteristic of the field in the following

Definition. Let G be a group and let p be an arbitrary prime. An element x of p-power order of G is said to be a *fixed-point-free element* of G provided that there exists a faithful KG-module, where $\text{char}(K) \nmid |O_{p'}(G)|$, on which x acts fixed-point freely.

If G has no fixed-point-free element of p-power order we define $F(p, G) = 1$. Otherwise define $F(p, G)$ to be the group generated by the elements $x^{o(x)/p}$ where $x \in G$ is fixed-point-free of p-power order.

Thus Theorem 1·2 of [3] ensures that if p is not exceptional (in the Hall-Higman sense) and x is a fixed-point-free element of order p^n of a p-solvable group G then $x^{p^{n-1}} \in O_p(G)$ and hence $F(p, G) \leqslant O_p(G)$. It is also shown in [3] that the element $x^{p^{n-1}}$ may be non-central in G. We try to repair this in Proposition 1 below. We recall that a *Fitting chain* of a solvable group G is a sequence $\{A_i \mid i = 1, \ldots, f\}$ of sections $A_i = C_i/D_i$ of G satisfying (8·3a)–(8·3i) of Lemma 8·2 of [1] for $H = 1$. In Fitting length problems one may replace the group G by $\Pi_{i=1}^{f} C_i$. Thus an element centralizing a Fitting chain plays the role of a central element in many cases. In Proposition 1 we prove that, in the above situation, the element $x^{p^{n-1}}$ centralizes a Fitting chain of G.

We show that if G is p-solvable and p is not exceptional then $F(p, G)$ is a characteristic elementary abelian p-subgroup of G (see Proposition 2). Using this we obtain a complete characterization of fixed-point-free elements of prime order for non-exceptional primes (see Corollary 2).

A well-known result ensures that a Frobenius complement of order pq is cyclic. This is considerably generalized in Proposition 3. We prove that if $G = \langle x, y \rangle$ is a solvable group having a faithful complex representation on which x and y act fixed-point-freely and $o(x) = p^n$, $o(y) = q^m$, p, q being distinct odd primes, then $x^{p^{n-1}}$ and $y^{q^{m-1}}$ commute. We do not know if the stronger assertion $[F(p, G), F(q, G)] = 1$ holds in this situation.

https://doi.org/10.1017/S0305004100064781 Published online by Cambridge University Press

PROPOSITION 1. *Let G be a p-solvable group and let x be a fixed-point-free element of order p^n, where p is an odd prime but not a Fermat prime. Then $x^{p^{n-1}}$ centralizes a Fitting chain of G.*

Proof. Take a Fitting chain like the one constructed in Proposition 2·2 of [2] where $x = 1$. It is easy to see that if $p_f = p$, then, dropping the assumption (2·2g) for $i = f$, it is possible to assume that C_f is a characteristic subgroup of exponent p of $O_p(G)$ and $[O_p(G), C_f] \leqslant Z(C_f)$. By theorem 1·2 of [3] we have $x^{p^{n-1}} \in O_p(G)$. Let V be a faithful KG-module on which x acts fixed-point-freely. If $p_f \neq p$ then clearly $x^{p^{n-1}}$ centralizes C_f. If $p_f = p$ then (1·1) and (1·3) of [2] applied to the action of $\langle x \rangle C_f$ on V yield that $[x^{p^{n-1}}, C_f] = 1$. As C_{f-1}/D_{f-1} is a p_{f-1}-group and $p_{f-1} \neq p_f$, we have $C_{f-1} \cap F(G) \leqslant D_{f-1}$. Now it is easy to see by descending induction on i that $C_i \cap F(G) \leqslant D_i$ for $i \leqslant f-1$. Define $C_i^* = C_i[x^{p^{n-1}}, G]$ and $D_i^* = D_i[x^{p^{n-1}}, G]$ for $i < f$ and $C_f^* = C_f$. As $C_i \cap F(G) \leqslant D_i$ then $C_i^*/D_i^* \simeq C_i/D_i$. Furthermore D_{f-1}^* centralizes C_f^* since $x^{p^{n-1}}$ centralizes C_f. Thus the quotients $A_i^* = C_i^*/D_i^*$ with $i = 1, \ldots, f$ form a Fitting chain of G centralized by $x^{p^{n-1}}$.

PROPOSITION 2. *If G is p-solvable and p is an odd prime but not a Fermat prime, then $F(p, G)$ is a characteristic elementary abelian p-subgroup of G. In particular, if x is a fixed-point-free element of order p^n of G, then $\langle x^{p^{n-1}} \rangle^G$ is elementary abelian.*

Proof. If $\alpha \in \mathrm{Aut}(G)$ and ρ is a faithful K-representation of G, then ρ^α given by $\rho^\alpha(g) = \rho(g^\alpha)$ is also a faithful K-representation of G. If x acts fixed-point-freely under ρ, then $x^{\alpha^{-1}}$ acts fixed-point-freely under ρ^α. Thus $F(p, G)$ is a characteristic subgroup of G. By theorem 1·2 of [3] we have $F(p, G) \leqslant O_p(G)$.

Now let V be a faithful KG-module, where $\mathrm{char}(K) \nmid |O_{p'}(G)|$, and assume that x and y are p-elements acting fixed-point-freely on V. Put $o(x) = p^n$ and $o(y) = p^m$. Our claim is that $[x^{p^{n-1}}, y^{p^{m-1}}] = 1$. We argue by induction on $|G| + \dim_K(V)$. Clearly $\mathrm{char}(K) \neq p$. Thus we may choose a G-composition factor W of V such that if our claim is false, then it is false in $G/\ker(G \text{ on } W)$ acting on W. Hence we may assume that V is an irreducible KG-module. Also $G = \langle x, y \rangle$.

We show next that V is a quasi-primitive KG-module, i.e. V is an homogeneous N-module for every normal subgroup N of G. To prove this we consider the decomposition of G into homogeneous components:

$$V = V_1 \oplus \ldots \oplus V_k.$$

We fix an index i and write $N_{\langle x \rangle}(V_i) = \langle x^{p^t} \rangle$ and $N_{\langle y \rangle}(V_i) = \langle y^{p^r} \rangle$. Now x^{p^t} and y^{p^r} act fixed-point-freely on V_i. Hence $t < n$ and $r < m$. Thus, if $k > 1$, then the inductive hypothesis yields that $[x^{p^{n-1}}, y^{p^{m-1}}]$ acts trivially on V_i for each i, and $x^{p^{n-1}}$ and $y^{p^{m-1}}$ commute. Hence we may assume that V is a homogeneous N-module.

Now every characteristic abelian subgroup of $O_p(G)$ is cyclic. Thus $O_p(G)$ is the central product of $Z(O_p(G))$ and an extra-special group E. Put $P = \Omega_1(O_p(G))$. By theorem 1·2 of [3] we have $x^{p^{n-1}} \in O_p(G)$ and therefore $[x^{p^{n-1}}, P] \leqslant Z(P)$. Now lemma 1·1 and lemma 1·3 of [2] applied to the action of $\langle x \rangle P$ on V yield that $x^{p^{n-1}}$ centralizes P. As $y^{p^{m-1}} \in P$, our claim is established.

If x acts fixed-point-freely on V then so does any conjugate of x. Thus we deduce from the preceding paragraph that $X = \langle x^{p^{n-1}} \rangle^G$ is abelian. Now let F be a field and

https://doi.org/10.1017/S0305004100064781 Published online by Cambridge University Press

let W be a faithful FG-module. Suppose that y is a p-element of order p^m acting fixed-point-freely on W. We apply lemma 1·1 of [2] to the action of $\langle y\rangle X$ on W to conclude that $y^{p^{m-1}}$ centralizes X. Hence $[x^{p^{n-1}}, y^{p^{m-1}}] = 1$. Thus $F(p, G)$ $\leqslant O_p(G)$, the Proposition is now obvious.

NOTATION. *Let G be a group and p a prime. Put $m_p(G) = m$ if G contains a characteristic elementary abelian p-subgroup of order p^m and it has none of order p^{m+1}.*

COROLLARY 1. *Let G be a p-solvable group, p being an odd prime but not a Fermat prime. Then the number of fixed-point-free elements of order p of G is at most $p^{m_p(G)} - 1$. The bound is best possible.*

Proof. The first assertion is obvious since $|F(p, G)| \leqslant p^{m_p(G)}$. To prove the second, let A be an elementary abelian p-group of order p^m and $1 \neq x \in A$. Let ρ be a faithful irreducible representation of $\langle x\rangle$ over the complex field. Now ρ^A is faithful and x acts fixed-point-freely under it. Thus any non-trivial element of A is a fixed-point-free element.

We may now obtain a complete characterization of fixed-point-free elements of prime order for non-exceptional primes. Observe that it is purely group-theoretic.

COROLLARY 2. *Let G be a p-solvable group, p being an odd prime but not a Fermat prime. Consider $x \in G$ of order p. Then the following assertions are equivalent:*

(1) *x is a fixed-point-free element of G.*

(2) (*a*) *$A = \langle x\rangle^G$ is abelian.*

(*b*) *There exist maximal subgroups $A_1, \ldots, A_k$ of A with*

$$\bigcap_{i=1}^{k} \operatorname{core}_G(A_i) = 1 \quad \text{and} \quad x \notin \bigcup_{i=1}^{k} \bigcup_{g \in G} A_i^g.$$

Remark. We observe that condition (*b*) is equivalent to the following: Take $\Omega_i = \{xA_i \mid x \in G\}$ and let Ω be the disjoint union of the Ω_i. Then (*b*) asserts that G acts faithfully on Ω and x acts fixed-point-freely on Ω.

Proof. We show first that (1) implies (2). Statement (*a*) is clear from Proposition 2. Let V be a faithful KG-module, where $\operatorname{char}(K) \nmid |O_{p'}(G)|$, on which x acts fixed-point-freely. Let

$$0 = V_0 \subset V_1 \subset \ldots \subset V_t = V$$

be a G-composition series of V. Put $W_i = V_i/V_{i-1}$ for $i = 1, \ldots, t$, and let ρ_i be the representation of G afforded by W_i. Let A_i be the kernel of an irreducible constituent of $\rho_{i|A}$. As A is normal in G, we have $\ker \rho_i = \operatorname{core}_G(A_i)$. As $\operatorname{char}(K) \neq p$ and A is a p-group, statement (*b*) is clear.

To prove the converse, let ρ_i be an irreducible complex representation of A with kernel A_i. Put $\rho = \rho_1^G + \ldots + \rho_k^G$. Then ρ is faithful and x acts fixed-point-freely under ρ.

Remark. By theorem 1·2 of [3] we have $F(p, G) \leqslant O_{p'p}(G)$ if $p \geqslant 5$. The result is false if $p = 3$. It is only known to be true for $p = 2$ if the Sylow 3-subgroups of G are abelian.

PROPOSITION 3. *Let $G = \langle x, y\rangle$ be a solvable group with $o(x) = p^n$, $o(y) = q^m$, where*

https://doi.org/10.1017/S0305004100064781 Published online by Cambridge University Press

p, q are distinct odd primes. Let V be a faithful KG-module, where $\mathrm{char}(K) = 0$, *such that x and y act fixed-point-freely on V. Then* $[x^{p^{n-1}}, y^{q^{m-1}}] = 1$.

Proof. We argue by induction on $|G| + \dim_K(V)$.

We may suppose that V is G-irreducible and K is algebraically closed. First we show that V is a quasi-primitive KG-module. Let A be a normal subgroup of G. By Clifford's Theorem $V = V_1 \oplus \ldots \oplus V_t$, where the V_i are homogeneous A-modules. Fix an index i and put $N_{\langle x\rangle}(V_i) = \langle x^{p^r}\rangle$ and $N_{\langle y\rangle}(V_i) = \langle y^{q^k}\rangle$. Now x^{p^r} and y^{q^k} act fixed-point-freely on V_i. If $t > 1$ then the inductive hypothesis ensures that $[x^{p^{n-1}}, y^{q^{m-1}}]$ acts trivially on each V_i. Hence we may suppose that $t = 1$ and V is a homogeneous A-module.

We recall the following definition (see [**6**], p. 67, taking $A = 1$): A subgroup P of a solvable group G is called a *generating support subgroup* of G if:

(1) P is a normal p-subgroup of G for a prime p;

(2) there exist normal subgroups P_1 and H of G such that

(a) $P_1 \leqslant Z(P)$, P/P_1 is elementary abelian and a completely reducible G-module;

(b) $H \leqslant C_G(P_1)$;

(c) $H/C_H(P/P_1)$ is elementary abelian for some prime r and

(d) H acts non-trivially on each H-chief factor of P/P_1.

We call the *support* of G (denoted $\mathrm{supp}(G)$) the subgroup generated by the normal subgroups S of G such that S is abelian or a generating support subgroup of G.

Next we show that $[x^{p^{n-1}}, y^{q^{m-1}}]$ centralizes $\mathrm{supp}(G)$. As every normal abelian subgroup of G is central, it clearly suffices to show that it centralizes every non-abelian generating support subgroup P of G. If this is false, then we deduce from Shult's Theorem (see [**5**]) that P is a 2-subgroup. Observe that every characteristic abelian subgroup of P is cyclic and central and hence P is the central product of a cyclic group and an extra-special group. Let H, P_1 be the subgroups given by the definition of generating support subgroup. Now $H/C_H(P/P_1)$ is an r-group, $r \neq 2$. Clearly $r \neq p$ or $r \neq q$. Suppose that $r \neq p$ for example. We will prove that $V|_{\langle x\rangle}$ contains a regular $\langle x\rangle$-module, a contradiction. Let T be a Hall $\{p, r\}$-subgroup of $\langle x\rangle H$ containing x and consider the action of TP on V. As $P/Z(P)$ is elementary abelian, then $|P'| = 2$. Now $P/P' = C_{P/P'}(T) \times [P/P', T]$. As $P_1 \leqslant Z(P) \leqslant Z(G)$ and a Sylow r-subgroup R of T acts non-trivially on each R-chief factor of P/P_1, we deduce that $P_1/P' = Z(P)/P' = C_{P/P'}(T)$. Now $[P, T]$ is extra-special and it is a generating support subgroup of G. Replacing P by $[P, T]$ we may suppose that P is extra-special.

Let V_0 be an irreducible P-submodule of V. As $(|T|, |P|) = 1$, we may extend V_0 to a TP-module by Gallagher's Theorem (see [**4**], corollary 6·28). We may consider the canonical extension W_0 of V_0 to a TP-module, i.e. $\det_{W_0}(x) = 1$ if $x \in T$. Then $C_T(P/Z(P))$ acts trivially on W_0. As $V|_{\langle x\rangle} \simeq W_0|_{\langle x\rangle} \otimes U$ for a certain $\langle x\rangle$-module U and we are trying to prove that V contains the regular $\langle x\rangle$-module, we may replace V by W_0 and thus suppose that $C_T(P/Z(P))$ acts trivially on V. Hence $T = \langle x\rangle R$ where R is elementary abelian and it acts non-trivially on each R-chief factor of $P/Z(P)$. Furthermore $2 \neq r \neq p$. Now theorem 3·5 of [**6**] may be applied to yield

https://doi.org/10.1017/S0305004100064781 Published online by Cambridge University Press

that $V|_{\langle x\rangle}$ contains the regular module, a contradiction. Therefore the element $[x^{p^{n-1}}, y^{q^{m-1}}]$ centralizes supp (G).

Put $a = x^{p^{n-1}}$ and $b = y^{q^{m-1}}$. By proposition 4·3 (1) of [6] we have $[a,b] \in F(G)$. As $o(a^b) = o(a)$ and $o(b^a) = o(b)$ and every Sylow s-subgroup of $F(G)$ for s odd is centralized by either a or b, we deduce that $[a,b] \in O_2(G)$. Put $W = O_2(G)/Z(O_2(G))$ and let $0 = W_0 \subset W_1 \subset \ldots \subset W_t = W$ be a composition series of the G-module W. Put $\overline{W}_i = W_i/W_{i-1}$ and $Z = Z(O_2(G))$. Clearly $Z \leqslant Z(G)$. We prove that each $\overline{W}_i$ is centralized by either a or b. If this is false, let i be an index contradicting the assertion. Take a minimal normal subgroup $H_1 = H/C_G(\overline{W}_i)$ of $G/C_G(\overline{W}_i)$. Suppose that H_1 is an r-group, where $r \neq 2$, and, for example, $r \neq p$. Let X be a Hall $\{p,r\}$-subgroup of G containing x. Now $W_i = W_{i-1} \oplus W^*$, where W^* is X-invariant. Put $P_o/Z = W^*$. Now let P/Z be an irreducible X-submodule of W^* not centralized by a. As a centralizes every x-invariant abelian subgroup of G we have $Z(P) = Z$. Also $|P'| = 2$ and $P/P' = C_{P/P'}(X) \times [P/P', X]$. Now $[P,X]$ is extra-special since $C_P(X) \leqslant Z$. Clearly $[P,X]$ is a generating support subgroup for $X[P,X]$. The argument of the preceding paragraph yields that $V|_{\langle x\rangle}$ contains the regular module, a contradiction.

Observe that W is a completely reducible $\langle a,b\rangle$-module because a and b commute modulo $C_G(W)$. Consider G/Z and the images $\bar{a}$ and $\bar{b}$ of a and b in G/Z. As $o(\bar{a}^{\bar{b}}) = o(\bar{a})$ and $o(\bar{b}^{\bar{a}}) = o(\bar{b})$, we have $[\bar{a},\bar{b}] = 1$ and clearly a and b commute, which finishes the proof.

Question. Do $F(p,G)$ and $F(q,G)$ commute when p and q are different odd primes?

Let p be an odd prime but not a Fermat prime. Suppose that x is a fixed-point-free element of order p^n of a solvable group G. By Proposition 2 we have that $\langle x^{p^{n-1}}\rangle^G$ is abelian. Now if y is a fixed-point-free element of G of order 2^m, then it is easy to see that $y^{2^{m-1}}$ and $x^{p^{n-1}}$ commute. Hence $[F(p,G), F(2,G)] = 1$ when p is an odd prime which is not a Fermat prime.

The same argument proves that $[F(p,G), F(q,G)] = 1$ when p and q are different odd primes and one of them is not a Fermat prime.

REFERENCES

[1] E. C. Dade. Carter subgroups and Fitting heights of finite solvable groups. *Illinois J. Math.* **13** (1969), 449–514.
[2] A. Espuelas. The Fitting length of the Hughes subgroup. *J. Algebra* **105** (1987), 365–371.
[3] A. Espuelas. A noncoprime Shult type theorem. *Math. Z.*, **196** (1987), 323–329.
[4] M. Isaacs. *Character Theory of Finite Groups* (Academic Press, 1976).
[5] E. Shult. On groups admitting fixed-point-free abelian operator groups. *Illinois J. Math.* **9** (1965), 701–720.
[6] A. Turull. Supersolvable automorphism groups of solvable groups. *Math. Z.* **183** (1983), 47–73.

https://doi.org/10.1017/S0305004100064781 Published online by Cambridge University Press