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A SLICE CROMWELL INEQUALITY FOR HOMOGENEOUS LINKS

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Abstract

Cromwell ['Homogeneous links', *J. London Math. Soc. (2)* **39**(3) (1989), 535–552] proved that the minimum v-degree of the HOMFLY polynomial of a homogeneous link L is bounded above by $1-\chi(L)$, where $\chi(L)$ is the maximum Euler characteristic of Seifert surfaces of L. We prove its slice version, stating that the minimum v-degree of the HOMFLY polynomial of a homogeneous link L is bounded above by $1-\chi_4(L)$, where $\chi_4(L)$ is the maximum four-dimensional Euler characteristic of L. As a byproduct, we prove a conjecture of Stoimenow ['Some inequalities between knot invariants', *Internat. J. Math.* **13**(4) (2002), 373–393] that for an alternating link, the minimum v-degree of the HOMFLY polynomial is smaller than or equal to its signature.

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1. Introduction

A *homogeneous link* is a link represented by a homogeneous diagram, a diagram whose Seifert graph is homogeneous. It is a common generalisation of alternating links and positive or negative links and has various nice properties.

Let $\chi(L)$ be the maximum Euler characteristic of Seifert surfaces of L and let $P_L(v,z)$ be the HOMFLY polynomial of a link L, given by the skein relation

$$v^{-1}P_+(v,z)-vP_-(v,z)=zP_0(v,z), \quad P_{\text{Unknot}}(v,z)=1.$$

In [4, Theorem 4(b)], Cromwell proved the following result which we call the *Cromwell inequality*.

THEOREM 1.1 (Cromwell inequality). *For a homogeneous link L,*

$$\min\deg_v P_L(v,z)\le 1-\chi(L). \tag{1.1}$$

Furthermore, (1.1) is an equality if and only if L is positive.

In [4], the link is always assumed to be nonsplit and the theorem is proved for nonsplit links, but one can check that the theorem also applies for split links.

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For a link L with homogeneous diagram D, Seifert's algorithm gives a maximum Euler characteristic Seifert surface of L; hence, $\chi(L) = s(D) - c(D)$, where $s(D)$ is the number of Seifert circles and $c(D)$ is the number of crossings of D [4, Corollary 4.1]. Thus, we may understand Theorem 1.1 as the inequality

$$\min\deg_v P_L(v,z) \le -s(D) + c(D) + 1 \tag{1.2}$$

for a homogeneous diagram D of L.

The aim of this note is to give a strengthened version of Cromwell's inequality. We point out that Cromwell's argument, as we will state in Theorem 2.2, actually proves a much stronger result than stated in Theorem 1.1. Based on this observation, we prove the following generalisation of the diagram version of the Cromwell inequality (1.2).

THEOREM 1.2. *For a homogeneous link L with homogeneous diagram D,*

$$\min\deg_v P_L(v,z) \le -s(D) + w(D) + 2s_+(D) + 1 - 2\#_{\mathrm{sp}}L.$$

Here, $w(D)$ is the writhe of D, $s_+(D)$ is the number of connected components of the diagram obtained from D by resolving all the negative crossings of D and $\#_{\mathrm{sp}}L$ is the number of split components of L.

The right-hand side in Theorem 1.2 appears in the following so-called sharper Bennequin-type inequality. Let $\chi_4(L)$ be the maximum Euler characteristic of a smoothly embedded surface in B^4 whose boundary is L. A *slice-torus invariant* of a knot is a homomorphism $\phi : C \to \mathbb{R}$ from the smooth knot concordance group C to $\mathbb{R}$, such that $\phi(T_{p,q}) = g_4(T_{p,q})$ for the (p,q)-torus knot $T_{p,q}$ [7]. For the definition of the slice-torus invariant of links, we refer to [2].

THEOREM 1.3 [2, Theorem 1.4]. *Let ϕ be a slice-torus invariant. Then, for a link L and its nonsplittable diagram D,*

$$-s(D) + w(D) + 2s_+(D) + 1 - 2\#_{\mathrm{sp}}L \le 2\left(\phi(L) - \frac{\#L - 1}{2}\right) \le 1 - \chi_4(L).$$

Here, $\#L$ denotes the number of components of L, and a *nonsplittable* diagram D is one for which the number of connected components of D is equal to $\#_{\mathrm{sp}}L$, the number of split components of L.

Since a homogeneous diagram is nonsplittable [4, Corollary 3], Theorems 1.2 and 1.3 upgrade the Cromwell inequality (1.1) to the *slice Cromwell inequality*.

COROLLARY 1.4 (Slice Cromwell inequality). *For a homogeneous link L,*

$$\min\deg_v P_L(v,z) \le 1 - \chi_4(L).$$

Remark that $\min\deg_v P_L(v,z)$ appears in the Morton–Franks–Williams inequality [5, 8]

$$\overline{sl}(L) \le \min\deg_v P_L(v,z) - 1$$

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for the maximum self-linking number

$$\begin{aligned}\overline{sl}(L) &= \max\{-s(D) + w(D) \mid D \text{ is a diagram of } L\} \\ &= \max\{sl(\mathcal{T}) \mid \mathcal{T} \text{ is a transverse link topologically isotopic to } L\}.\end{aligned}$$

Theorem 1.2 says that for homogeneous links, the Morton–Franks–Williams inequality subsumes all the Bennequin-type inequalities

$$\overline{sl}(L) \leq 2\left(\phi(L) - \frac{\#L - 1}{2}\right) - 1$$

of the slice-torus invariants ϕ.

As a generalisation of the condition for equality in the Cromwell inequality, we pose the following conjectures.

CONJECTURE 1.5. Let L be a homogeneous link.

(i) The slice Cromwell inequality is an equality if and only if L is positive.
(ii) For a homogeneous diagram D of L, the inequality in Theorem 1.2 is an equality if and only if D is a positive diagram.

Conjecture 1.5 is related to the following problem asked by Baader [1].

QUESTION 1.6. Is a homogeneous quasipositive link positive?

Here, a link L is called *quasipositive* if it is a closure of a quasipositive braid, a product of conjugates of the positive generators $\sigma_1, \ldots, \sigma_{n-1}$ of the braid group.

Although Question 1.6 remains open even for alternating links, in [6], extending the argument in the alternating link case [10], we gave a partial affirmative answer. We showed that if a homogeneous link L admits a homogeneous diagram that attains the maximum self-linking number (for example, when the number of Seifert circles of D is equal to the braid index of L), then L is quasipositive if and only if the diagram is positive.

Conjecture 1.5 leads to an affirmative answer to Question 1.6.

THEOREM 1.7. *If Conjecture 1.5(i) or (ii) is true, quasipositive homogeneous links are positive.*

PROOF. For a homogeneous link L with homogeneous diagram D,

$$\overline{sl}(L) \leq \min\deg_v P_L(v,z) - 1 \leq -s(D) + w(D) + 2s_+(D) - 2\#_{\mathrm{sp}}L \leq -\chi_4(L).$$

Since $-\chi_4(L) = \overline{sl}(L)$ holds for a quasipositive link L [11], it follows that a quasipositive homogeneous link attains the equality for both the slice Cromwell inequality and Theorem 1.2. Thus, assuming Conjecture 1.5(i) or (ii), we conclude that L is positive. □

Finally, we point out that Theorem 2.2, which contains the actual content of Cromwell's proof, can be used to solve [9, Problem 1.16] and [13, Conjecture 4.1]

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concerning the signature $\sigma(L)$ and the minimum degree of the HOMFLY polynomial for alternating links.

We remark that [9, Problem 1.16] and [13, Conjecture 4.1] also ask for the inequality $\min\deg_a F_L(a^{-1},z) \le \min\deg_v P_L(v,z)$ for the Kauffman polynomial $F_L(a,z)$, but this inequality was already proven in [12, Corollary 5.4]. Also, we remark that for the signature σ, we are using the convention that $\sigma(\text{Positive trefoil}) = 2$.

THEOREM 1.8. *For an alternating link L,*

$$\min\deg_v P_L(v,z) + \#_{\mathrm{sp}}(L) - 1 \le \sigma(L).$$

The anonymous referee pointed out that Theorem 1.8 also follows from Theorems 1.2, 1.3 and the equality $\sigma(L) = 2\phi(L) - \#L + 1$ for a nonsplit alternating link L and a slice-torus invariant ϕ [3, (2.10)]. (Note the different convention on the signature σ in [3].)

2. Proof of Theorem 1.2

Let G be a signed graph which is not necessarily connected, but is not allowed to have a self-loop. A signed graph means that the sign + or − is assigned to each edge. A signed graph is *positive* (respectively *negative*) if all the signs of the edges are positive (respectively negative).

A vertex v of a graph G is a *cut vertex* if removing v increases the number of connected components. By splitting the graph along its cut vertices, each graph is decomposed as a union of *blocks*, connected maximum subgraphs without cut vertices. The *rank* of a graph G is defined by $\mathrm{rank}(G) = -\#V(G) + \#E(G) + \#G$, where $V(G)$ and $E(G)$ denote the set of vertices and edges of G, and $\#G$ denotes the number of connected components of G.

The *Seifert graph* G_D of a link diagram D is a signed graph whose vertices are Seifert circles and whose edges are crossings (see Figure 1).

DEFINITION 2.1 (Homogeneous graphs and links). A signed graph is *homogeneous* if each block is either positive or negative. A link diagram D is *homogeneous* if its Seifert graph G_D is homogeneous.

To prove Theorem 1.2, we pin down what Cromwell actually proved in his proof of the Cromwell inequality.

THEOREM 2.2 (Cromwell [4]). *Let D be a connected homogeneous diagram of a link L. Let B_i $(i = 1,\dots,d)$ be the blocks of its Seifert graph G_D. Then, $P_L(v,z)$ contains a monomial $v^a z^b$ with $a = \sum_{i=1}^{d} \varepsilon(B_i)\,\mathrm{rank}(B_i)$ and $b = \mathrm{rank}\, G_D$ $(= 1 - \chi(L))$. Here, $\varepsilon(B_i)$ is the sign of the block B_i.*

For the reader's convenience, we give a brief outline of the proof, emphasising the key properties derived from the proof.

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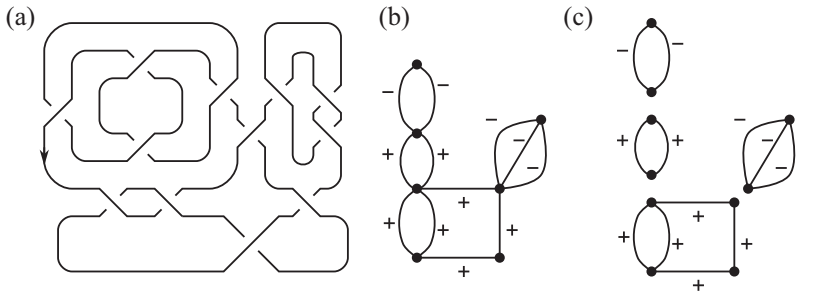


FIGURE 1. (a) Homogeneous diagram; (b) Seifert graph; (c) the blocks of the Seifert graph.

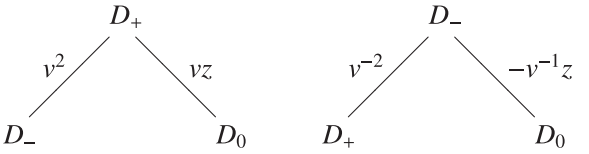


FIGURE 2. Skein resolution tree.

SKETCH OF PROOF OF THEOREM 2.2. Starting from a given diagram D of a link L, the HOMFLY polynomial can be computed by applying the skein relation repeatedly. Such a computation is summarised by a so-called *skein (resolution) tree*, a rooted binary tree having the following properties.

- Each node is labelled by a link diagram and each edge is labelled by a monomial.
- The root is labelled by the diagram D.
- The triple $(D_{\text{parent}}, D_{\text{child1}}, D_{\text{child2}})$ consisting of the diagram D_{parent} associated to a nonterminal node and the diagrams $D_{\text{child1}}, D_{\text{child2}}$ associated to its child nodes forms a skein triple (D_+, D_-, D_0) or (D_-, D_+, D_0). In the former case, the edge connecting D_{parent} to D_{child1} (respectively D_{parent} to D_{child2}) is labelled by the monomial v^2 (respectively vz). In the latter case, the edge connecting D_{parent} to D_{child1} (respectively D_{parent} to D_{child2}) is labelled by the monomial v^{-2} (respectively $-v^{-1}z$) (see Figure 2).
- A diagram associated with a terminal node represents the unlink.

By the skein relation of the HOMFLY polynomial, the HOMFLY polynomial $P_L(v, z)$ is given by

$$P_L(v,z) = \sum_n \pi(n)\left(\frac{v^{-1}-v}{z}\right)^{\#n-1},$$

where the summation runs over all the terminal nodes n, $\pi(n)$ is the product of labels of edges that appear in the path from the terminal node n to the root, and $\#n$ is the number of components of the unlink associated to the terminal node n.

Cromwell showed that in an appropriate skein tree coming from an ascending diagram construction, if a terminal node n satisfies $\max \deg_z \pi(n)((v^{-1}-v)/z)^{\#n-1} =$

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rank G_D (that is, if a terminal node n contributes the highest z-degree terms of $P_L(v,z)$), it has the following properties.

(a) The Seifert graph of a diagram associated to n is a tree. Hence, it represents the unknot and $\pi(n)((v^{-1}-v)/z)^{\#n-1} = \pi(n)$ is a monomial.
(b) The sign of the monomial $\pi(n)((v^{-1}-v)/z)^{\#n-1}$ does not depend on the choice of such a terminal node n. The sign is given by $\prod_{i=1}^{d} \varepsilon(B_i)^{\operatorname{rank} B_i}$. In particular, contributions of the terminal nodes never cancel.

Then, the assertion comes from an observation that for the terminal node n_0 obtained by repeatedly resolving the crossings (that is, the rightmost terminal node), its contribution is given by

$$\pi(n_0)\left(\frac{v^{-1}-v}{z}\right)^{\#n_0-1} = v^{\sum_{i=1}^{d} \varepsilon(B_i)\operatorname{rank}(B_i)} z^{\operatorname{rank} G_D}.$$

We rewrite $\sum_{i=1}^{d} \varepsilon(B_i)\operatorname{rank}(B_i)$ in terms of the quantities from the diagram, rather than its Seifert graph.

PROPOSITION 2.3. *For a connected homogeneous diagram D,*

$$\sum_{i=1} \varepsilon(B_i)\operatorname{rank}(B_i) = -s(D) + w(D) + 2s_+(D) - 1.$$

PROOF. Let P and N be the number of positive and negative blocks, and let $B_1^+, \dots, B_P^+$ (respectively $B_1^-, \dots, B_N^-$) be the positive (respectively negative) blocks of G_D.

In the procedure for the decomposition of the Seifert graph G_D into its blocks, cutting a graph at a cut vertex increases the number of vertices by one. Therefore,

$$s(D) + (P + N - 1) = \sum_{i=1}^{P} \#V(B_i^+) + \sum_{i=1}^{N} \#V(B_i^-).$$

Similarly, $s_+(D)$ is the number of connected components of the graph obtained by deleting all the negative-signed edges, Hence,

$$s_+(D) + (N - 1) = \sum_{i=1}^{N} \#V(B_i^-).$$

Thus,

$$\begin{aligned}\sum_{i=1}^{d} \varepsilon(B_i)\operatorname{rank}(B_i) &= \sum_{i=1}^{P} \operatorname{rank}(B_i^+) - \sum_{i=1}^{N} \operatorname{rank}(B_i^-) \\ &= \sum_{i=1}^{P} (-\#V(B_i^+) + \#E(B_i^+) + 1) - \sum_{i=1}^{N} (-\#V(B_i^-) + \#E(B_i^-) + 1)\end{aligned}$$

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$$= -\sum_{i=1}^{P} \#V(B_i^+) + \sum_{i=1}^{N} \#V(B_i^-) + w(D) + P - N$$
$$= -s(D) - (P + N - 1) + 2\sum_{i=1}^{N} \#V(B_i^-) + w(D) + P - N$$
$$= -s(D) + w(D) + 2s_+(D) - 1. \qquad \square$$

Next, we prove the following assertion which is a bit stronger than Theorem 1.2.

THEOREM 2.4. *For a homogeneous link L with homogeneous diagram D, let $h_L(v) \in \mathbb{Z}[v, v^{-1}]$ be the highest z-degree term of the HOMFLY polynomial $P_L(v, z)$ of L, the coefficient of $z^{1-\chi(L)}$. Then, $h_L(v)$ contains a monomial $v^{-s(D)+w(D)+2s_+(D)+1-2\#_{sp}L}$. In particular,*

$$\min\deg_v h_L(v) \le -s(D) + w(D) + 2s_+(D) + 1 - 2\#_{\mathrm{sp}}L.$$

PROOF. The case for L nonsplit follows from Theorem 2.2 and Proposition 2.3.

For a split link $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_{\#_{sp}L}$, its homogeneous diagram D is a disjoint union of the homogeneous diagrams D_i of L_i. Hence,

$$\min\deg_v h_L(v) = \sum_{i=1}^{\#_{\mathrm{sp}}L} \min\deg_v h_{L_i}(v) - (\#_{\mathrm{sp}}L - 1)$$
$$\le \sum_{i=1}^{\#_{\mathrm{sp}}L} (-s(D_i) + w(D_i) + 2s_+(D_i) - 1) - (\#_{\mathrm{sp}}L - 1)$$
$$= -s(D) + w(D) + 2s_+(D) + 1 - 2\#_{\mathrm{sp}}L. \qquad \square$$

REMARK 2.5. The inequality

$$\min\deg_v h_L(v) \le -s(D) + w(D) + 2s_+(D) + 1 - 2\#_{\mathrm{sp}}L \le 1 - \chi_4(L)$$

in Theorem 2.4 often attains the equality for nonpositive diagrams, even for alternating links. For example, a nonpositive alternating knot 6_1 attains the equality. Thus, we cannot prove Conjecture 1.5(i) and (ii) just by looking at the highest z-degree part $h_L(v)$ of the HOMFLY polynomial.

The proof of Theorem 1.8 is similar to the proof of Theorem 1.2. We observe that $\sum_{i=1}^{d} \varepsilon(B_i)\,\mathrm{rank}(B_i)$ is nothing but the signature for alternating diagrams.

PROOF OF THEOREM 1.8. For a connected, reduced alternating diagram D of a link L, the signature is given by

$$\sigma(L) = w(D) - (d_+ - d_-),$$

where $d_\pm$ is the number of positive and negative edges of a spanning tree of the Seifert graph G_D [14, Theorem 2(1)]. Let $B_1^+, \ldots, B_P^+$ (respectively $B_1^-, \ldots, B_N^-$) be the positive

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(respectively negative) blocks of the Seifert graph G_D. We remark that the formula in [14, Theorem 2(1)] contains an erroneous factor $\frac{1}{2}$.

Since $d_+ = \sum_{i=1}^{P}(\#V(B_i^+) - 1)$ and $d_- = \sum_{i=1}^{N}(\#V(B_i^-) - 1)$, we conclude

$$\begin{aligned}\sigma(L) &= w(D) - (d_+ - d_-)\\ &= \sum_{i=1}^{P} \#E(B_i^+) - \sum_{i=1}^{N} \#E(B_i^-) - \sum_{i=1}^{P}(\#V(B_i^+) - 1) + \sum_{i=1}^{N}(\#V(B_i^-) - 1)\\ &= \sum_{i=1}^{P}(-\#V(B_i^+) + \#E(B_i^+) + 1) - \sum_{i=1}^{N}(-\#V(B_i^-) + \#E(B_i^-) + 1)\\ &= \sum_{i=1}^{d} \varepsilon(B_i)\,\mathrm{rank}(B_i).\end{aligned}$$

By Theorem 2.2, this shows the theorem for the nonsplit case. The split case follows easily from the nonsplit case. □

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