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Prime Poisson suspensions

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Abstract. We establish a necessary and sufficient condition for a Poisson suspension to be prime. The proof is based on the Fock space structure of the L^2-space of the Poisson suspension. We give examples of explicit infinite measure-preserving systems, in particular of non-singular compact group rotations that give rise to prime Poisson suspensions. We also compare some properties of so far known prime transformations with those of our examples, showing that these examples are new.

1. *Introduction*

The aim of this paper is to build new examples of prime dynamical systems, that is, systems which have a trivial factor structure: their only factors are (up to isomorphism) the original system and the one point system.

The systems we construct are particular cases of Poisson suspensions which are probability-preserving systems canonically built from infinite measure-preserving systems.

The paper is organized as follows: we first recall basic notions on factors and give an overview of prime systems that exist in the literature. Then in §2 we introduce Poisson measures and all the specific tools that we shall need to derive structural results on the σ-algebra of a Poisson measure. Section 3 is devoted to Poisson suspensions, which are Poisson measures endowed with particular measure-preserving transformations, and in §4, we give the main result that will be used to exhibit prime Poisson suspensions. We also give a collection of ergodic consequences of primeness for Poisson suspensions, in particular disjointness properties from other families of dynamical systems.

The last section gives concrete examples of infinite measure systems that satisfy the hypothesis required to obtain prime suspensions and prove that these systems actually exist.

1.1. *On factors of a dynamical system with an invariant probability measure.*

We first recall what a factor of a dynamical system is. Let a system $(X, \mathcal{A}, \mu, T)$ be given, where

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$(X, \mathcal{A}, \mu)$ is a standard Borel probability space and T is an invertible measure-preserving transformation on X. There are two equivalent points of view, as below.

(1) We say that a sub-σ-algebra $\mathcal{C} \subset \mathcal{A}$ is a *factor* if $T^{-1}\mathcal{C} = \mathcal{C}$.

(2) Another system $(Y, \mathcal{B}, \nu, S)$ is said to be a *factor* of $(X, \mathcal{A}, \mu, T)$ if there is a measurable *factor map* $\varphi : X \to Y$ such that $\nu = \mu \circ \varphi^{-1}$ and the following diagram is commutative:

$$\begin{array}{ccc} (X, \mathcal{A}, \mu) & \xrightarrow{\quad T \quad} & (X, \mathcal{A}, \mu) \\ \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \varphi} \\ (Y, \mathcal{B}, \nu) & \xrightarrow{\quad S \quad} & (Y, \mathcal{B}, \nu). \end{array}$$

These two definitions can be unified; indeed, with the latter we obtain a factor in the first sense by observing that $\varphi^{-1}\mathcal{B}$ satisfies $\varphi^{-1}\mathcal{B} \subset \mathcal{A}$ and $T^{-1}(\varphi^{-1}\mathcal{B}) = \varphi^{-1}\mathcal{B}$. For the other direction, if $\mathcal{B} \subset \mathcal{A}$ is a factor, we can form the quotient system $(X_{/\mathcal{B}}, \mathcal{A}_{/\mathcal{B}}, \mu_{/\mathcal{B}}, T_{/\mathcal{B}})$ and observe that the natural projection $\pi_{\mathcal{B}}$ is a factor map.

We give the definition of a prime system with the first, internal, point of view.

Definition 1. A system $(X, \mathcal{A}, \mu, T)$ is said to be *prime* if $\mathcal{A}$ and $\{X, \emptyset\}$ are the only factors of the system.

In other words, prime systems are those systems with the simplest factor structure.

1.2. *An overview of previously known prime systems.* The first examples are Ornstein's mixing rank one constructions [**21**], proved to be prime by Polit in [**23**]. Indeed, those systems are part of the larger class of *simple systems* [**6**, **29**] which possess their own theory: they are those systems $(X, \mathcal{A}, \mu, T)$ whose ergodic selfjoinings are either the product joining or graph joinings Δ_S with $S \in C(T)$, the centralizer of T. In particular, factors $\mathcal{K}$ of simple systems correspond to compact subgroups $K \subset C(T)$ as follows:

$$\mathcal{K} = \{A \in \mathcal{A},\ \forall S \in K,\ SA = A\}.$$

Therefore, if K is a maximal compact subgroup of the centralizer of a simple system T, then it induces a prime system $T_{/\mathcal{K}}$. The most drastic situation occurs when the centralizer of a simple system is reduced to the powers of the transformation, and thus the system itself is prime. It is then said to have minimal self-joinings (MSJ(2)). Mixing rank one transformations are MSJ(2) [**13**], and so is the Chacon transformation [**5**] and many others (see for example [**3**]).

There also exist examples of rigid, and therefore not MSJ(2), simple prime transformations (see [**7**]). In [**9**] (see also [**4**]), examples are given of simple systems with a centralizer possessing a non-normal maximal compact subgroup K, giving way to non-simple prime systems since, if T is simple, the factor system $T_{/\mathcal{K}}$ is simple if and only if K is normal in $C(T)$.

We recall that there exist horocycle flows whose non-zero times maps are prime. However, it is proved in [**28**] that they can always be seen as factors of simple systems, and they are thus part of the above theory.

Many examples of prime maps are also of rank one. Indeed, mildly mixing rank one maps are prime, as King showed in [**12**] that a strict factor of a rank one map is rigid. It

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is yet unknown whether prime rank one maps are always factors of simple systems. To be complete, we mention another situation which does not fit into those already described, where prime systems occur: if T is MSJ(2), then the symmetric factor of $T \times T$ is prime and so is the map $(x, y) \mapsto (y, Tx)$ (see [**25**, p. 128]).

We believe that we have listed, if not every example, at least all families of prime probability measure-preserving transformations so far known.

2. *Technology*

2.1. *Poisson measure.* Let $(X, \mathcal{A}, \mu)$ be a σ-finite, infinite measure space, μ being continuous, and $(X^*, \mathcal{A}^*, \mu^*)$ be the corresponding *Poisson measure space*. We recall the definition below.

X^* is the space of counting measures on $(X, \mathcal{A})$, i.e. measures of the form $\nu = \sum_{i \in I} \delta_{x_i}$, where the x_i are in X and I is countable.

We define the maps $N(A)$, $A \in \mathcal{A}$, on X^* by $N(A)(\nu) = \nu(A)$. The map $N(A)$ 'counts' the number of points that fall into the set A. We set $\mathcal{A}^* := \sigma\{N(A),\ A \in \mathcal{A}\}$, the smallest σ-algebra on X^* that makes the operation of measuring sets in $\mathcal{A}$ measurable.

The measure μ^* is now defined as the only probability measure on $(X^*, \mathcal{A}^*)$ such that, for any $k \in \mathbb{N}$ and arbitrary disjoint sets $A_1, \ldots, A_k$ in $\mathcal{A}$ so that $0 < \mu(A_i) < +\infty$, the random variables $N(A_1), \ldots, N(A_k)$ are independent and distributed according to Poisson laws with parameters $\mu(A_1), \ldots, \mu(A_k)$ respectively.

The underlying measure μ is often called the *intensity* of the Poisson measure.

It is common to denote the identity on X^* by N, that is $N(\nu) = \nu$ where ν is a counting measure on X. Under the distribution μ^*, N is therefore a random measure.

2.2. *Fock space.* For $n \geq 1$, let $L^2_{\mathrm{sym}}(\mu^{\otimes n})$ denote the subspace of $L^2(\mu^{\otimes n})$ of functions invariant by permutations of coordinates. It is convenient to endow it with the normalized scalar product $\langle \cdot, \cdot \rangle_n := (1/n!)\langle \cdot, \cdot \rangle_{L^2(\mu^{\otimes n})}$.

We recall that $L^2(\mu^*)$ has a Fock-space structure based on $L^2(\mu)$. Namely:

$$L^2(\mu^*) \simeq \mathbb{C} \oplus L^2(\mu) \oplus L^2_{\mathrm{sym}}(\mu^{\otimes 2}) \oplus \cdots \oplus L^2_{\mathrm{sym}}(\mu^{\otimes n}) \oplus \cdots .$$

That is, $L^2(\mu^*)$ can be seen as an orthogonal sum of subspaces H^n, $n \in \mathbb{N}$, where H^n (called *chaos of order n*) is naturally identified to $L^2_{\mathrm{sym}}(\mu^{\otimes n})$ (and H^0, the subspace of constant functions, is identified to $\mathbb{C}$). The identification is done through the (normalized) multiple stochastic integrals $(1/n!)I^{(n)}$ from $L^2_{\mathrm{sym}}(\mu^{\otimes n})$ to H^n whose constructions are detailed in [**19**] for example.

In this paper, we only need to know $I^{(n)}$ explicitly in the case $n = 1$: for $f \in L^1(\mu) \cap L^2(\mu)$, define

$$I^{(1)}(f) := \int_X f(x)(N(dx) - \mu(dx)).$$

2.3. *Difference operators.* In this section, we collect some information about *difference operators* we shall need in the next section. It is taken from the important paper of Last and Penrose [**15**]. Lemmas 1 and 2 represent a convenient reformulation of the

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various properties of difference operators we need to obtain Theorem 1. They also appear, somewhat implicitly, in [**15**].

Let F be a measurable function on $(X^*, \mathcal{A}^*)$. Define the difference operator $D^1_y F$ by

$$D^1_y F(\nu) := F(\nu + \delta_y) - F(\nu).$$

This consists of adding a particle at the position $y \in X$ and evaluating F with and without this particle and taking the difference.

By induction, we define $D^n_{y_1,\dots,y_n} F$ as

$$D^n_{y_1,\dots,y_n} F := D^{n-1}_{y_2,\dots,y_n}(D^1_{y_1} F).$$

It can be observed that this operator is symmetric in y's for any $n \in \mathbb{N}$.

Closely related to these operators is the following formula (known as *Mecke's formula* [**20**]):

$$\int_{X^*}\int_X h(\nu, x)\nu(dx)\mu^*(d\nu) = \int_{X^*}\int_X h(\nu + \delta_x, x)\mu(dx)\mu^*(d\nu), \tag{2.1}$$

which is valid for all positive measurable functions h defined on $X^* \times X$.

As a first consequence of this formula, we recall the following result which is an immediate extension of Lemma 2.4 in [**15**].

LEMMA 1. *If F and G are two measurable functions on (X^*, μ^*) that coincide with μ^*-almost surely, then for any $n \in \mathbb{N}^*$,*

$$D^n_{y_1,\dots,y_n} F(\nu) = D^n_{y_1,\dots,y_n} G(\nu),$$

for $\mu^* \otimes \mu^{\otimes n}$-almost every $(\nu, y_1, \dots, y_n) \in X^* \times X^n$.

So these operators are well defined for $F \in L^2(\mu^*)$. It turns out that they establish a remarkable link with the Fock space structure (see [**15**]):

- For $\mu^{\otimes n}$-almost every $(y_1, \dots, y_n) \in X^n$, $\nu \mapsto D^n_{y_1,\dots,y_n} F(\nu)$ is μ^*-integrable.
- $(y_1, \dots, y_n) \mapsto \mathbb{E}[D^n_{y_1,\dots,y_n} F]$ is in $L^2_{\mathrm{sym}}(\mu^{\otimes n})$.
- If we set $P_n F(y_1, \dots, y_n) := \mathbb{E}[D^n_{y_1,\dots,y_n} F]$, and $P_0 F := \mathbb{E}[F]$, F decomposes in the Fock space as:
$$F \simeq P_0 F + \cdots + P_n F + \cdots .$$

In particular, for $f \in L^2_{\mathrm{sym}}(\mu^{\otimes n})$, $P_n[(1/n!)I^{(n)}(f)] = f$, $\mu^{\otimes n}$-almost everywhere, and in this case we can even remove the expectation:

$$D^n_{y_1,\dots,y_n}\left[\frac{1}{n!} I^{(n)}(f)\right](\nu) = f(y_1, \dots, y_n)$$

for $\mu^* \otimes \mu^{\otimes n}$-almost all $\nu, y_1, \dots, y_n \in X^* \times X^n$.

LEMMA 2. *If $F \in H^n$, then, for μ-almost all $y \in X$, $D^1_y F \in H^{n-1}$.*

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Proof. We first claim that $D^1_y F$ is in $L^2(\mu^*)$ for μ-almost all $y \in X$ when $F \in H^n$. Then equation (3.9) in [**15**] reduces to $F = I^{(n)}(f_n)$, where $f_n = (1/n!)P_n F \in L^2_{\mathrm{sym}}(\mu^{\otimes n})$, and condition (3.11) is satisfied. Therefore, by Theorem 3.3 in [**15**], $D_y F(\nu) = D'_y F(\nu)$ $\mu^* \otimes \mu$-almost every (ν, y), where D'_y is given by formula (3.10), that is in our case $D'_y F = nI^{(n-1)} f_n(\cdot, \dots, \cdot, y)$. The claim follows.

Now, let $k \neq n-1$; we get

$$\begin{aligned} P_k(D^1_y F)(y_1, \dots, y_k) &= \mathbb{E}[D^k_{y_1,\dots,y_k}(D^1_y F)] \\ &= \mathbb{E}[D^{k+1}_{y_1,\dots,y_k,y} F] \\ &= P_{k+1} F(y_1, \dots, y_k, y). \end{aligned}$$

But as $F \in H^n$, $P_{k+1}F$ is zero $\mu^{\otimes k+1}$-almost everywhere and we deduce that, for μ-almost all $y \in X$, $P_k(D^1_y F)$ is zero $\mu^{\otimes k}$-almost everywhere. This proves the lemma. □

2.4. *A result on σ-algebras.* Let us present the key tool that will be applied to prove the main result of this paper (Theorem 2).

THEOREM 1. *Let $(X^*, \mathcal{A}^*, \mu^*)$ be a Poisson measure and let Φ be a conditional expectation on a σ-algebra $\mathcal{C} \subset \mathcal{A}^*$ that preserves the nth chaos H^n for every $n \geq 1$. If Φ is zero on H^1, then Φ is zero on H^n for every $n \geq 1$. In other words, Φ is the conditional expectation on the trivial σ-algebra $\{X^*, \emptyset\}$.*

Proof. Let F be in H^n, $n \geq 2$; then we want to show that $\Phi F = 0$. Without loss of generality we can assume that F is real. As $\Phi F \in H^n$, we have $P_k(\Phi F) = 0$ for all $k \neq n$, and therefore it is enough to show that $P_n(\Phi F) = 0$.

We shall prove that $D^1_a \Phi F(\nu) = 0$ for $\mu^* \otimes \mu$-almost every (ν, a). It will follow from Lemma 1 that $D^n_{y_1,\dots,y_n} \Phi F(\nu) = 0$ for $\mu^* \otimes \mu^{\otimes n}$-almost every $(\nu, y_1, \dots, y_n)$, so $P_n(\Phi F) = 0$, and the proof will be complete.

Let $a \in X$; then

$$\begin{aligned} \mathbb{E}[(D^1_a \Phi F)^2] &= \mathbb{E}[(\Phi F(\cdot + \delta_a) - \Phi F)^2] \\ &= \mathbb{E}[\Phi F(\cdot + \delta_a)^2] + \mathbb{E}[(\Phi F)^2] - 2\mathbb{E}[\Phi F(\cdot + \delta_a)\Phi F]. \end{aligned}$$

We have

$$\begin{aligned} \mathbb{E}[\Phi F(\cdot + \delta_a)\Phi F] &= \mathbb{E}[(\Phi F(\cdot + \delta_a) - \Phi F)\Phi F] + \mathbb{E}[(\Phi F)^2] \\ &= \mathbb{E}[D^1_a \Phi F \cdot \Phi F] + \mathbb{E}[(\Phi F)^2]. \end{aligned}$$

But, as ΦF is in H^n, $D^1_a \Phi F$ is in H^{n-1} for μ-almost all $a \in X$, thanks to Lemma 2. These two vectors are therefore orthogonal, which means that $\mathbb{E}[\Phi F(\cdot + \delta_a)\Phi F] = \mathbb{E}[(\Phi F)^2]$. So

$$\mathbb{E}[(D^1_a \Phi F)^2] = \mathbb{E}[\Phi F(\cdot + \delta_a)^2] - \mathbb{E}[(\Phi F)^2].$$

Now we apply Mecke's formula (2.1) with $h(\nu, x) = (\Phi F)^2(\nu) f(x)$, where f is a non-negative function in $L^1(\mu) \cap L^2(\mu)$. We get

$$\int_{X^*} \int_X \Phi F(\nu)^2 f(x) \nu(dx) \mu^*(d\nu) = \int_{X^*} \int_X \Phi F(\nu + \delta_x)^2 f(x) \mu(dx) \mu^*(d\nu),$$

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which can be rewritten as

$$\mathbb{E}\left[(\Phi F)^2 \int_X f(x)N(dx)\right] = \int_X \mathbb{E}[\Phi F(\cdot + \delta_x)^2] f(x)\mu(dx).$$

Since $I^{(1)}(f) = \int_X f(x)N(dx) - \int_X f(x)dx$, we also have

$$\mathbb{E}\left[(\Phi F)^2 \int_X f(x)N(dx)\right] = \mathbb{E}[(\Phi F)^2 I^{(1)}(f)] + \int_X \mathbb{E}[(\Phi F)^2] f(x)\mu(dx).$$

As Φ is the conditional expectation on $\mathcal{C}$, ΦF is $\mathcal{C}$-measurable and so is $(\Phi F)^2$. But, by assumption, Φ vanishes on H^1, which implies that the conditional expectation of $I^{(1)}(f)$ on $\mathcal{C}$ is zero, and therefore $\mathbb{E}[(\Phi F)^2 I^{(1)}(f)] = 0$. Hence

$$\mathbb{E}\left[(\Phi F)^2 \int_X f(x)N(dx)\right] = \int_X \mathbb{E}[(\Phi F)^2] f(x)\mu(dx)$$

and so we get

$$\int_X \mathbb{E}[(\Phi F)^2] f(x)\mu(dx) = \int_X \mathbb{E}[(\Phi F)^2(\cdot + \delta_x)] f(x)\mu(dx).$$

As this equality holds for any non-negative $f \in L^1(\mu) \cap L^2(\mu)$, we obtain

$$\mathbb{E}[(\Phi F)^2] = \mathbb{E}[(\Phi F)^2(\cdot + \delta_x)]$$

for μ-almost all $x \in X$.

Summing up, for μ-almost all $a \in X$,

$$\mathbb{E}[(D_a^1 \Phi F)^2] = 0$$

and thus $D_a^1 \Phi F(\nu) = 0$ $\mu^* \otimes \mu$-almost everywhere. □

3. *Poisson suspensions*

If T is a measure-preserving automorphism of $(X, \mathcal{A}, \mu)$, then $T_* : \nu \mapsto \nu \circ T^{-1}$ is a measure-preserving automorphism of $(X^*, \mathcal{A}^*, \mu^*)$. $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is the *Poisson suspension* over the *base* $(X, \mathcal{A}, \mu, T)$.

Let us recall the most basic ergodic result about Poisson suspensions (see [**24**] for a proof).

$(X^*, \mathcal{A}^*, \mu^*, T_*)$ is ergodic (and then weakly mixing) if and only if $(X, \mathcal{A}, \mu, T)$ has no T-invariant set A with $0 < \mu(A) < +\infty$.

In particular, if T is ergodic and μ is infinite, then T_* is ergodic.

The ergodic theory of Poisson suspension deals with the interplay between a dynamical system with a σ-finite measure $(X, \mathcal{A}, \mu, T)$ and the canonically built probability measure-preserving system $(X^*, \mathcal{A}^*, \mu^*, T_*)$.

Two points of view are to be considered: Poisson suspensions can be seen as a probabilistic tool to study infinite ergodic theory and also as a family of probabilistic systems indexed by infinite measure-preserving ones in order to see what kind of properties we get. This paper belongs to the latter category.

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3.1. *Poissonian factors.* There are two main ways to obtain natural factors of a Poisson suspension $(X^*, \mathcal{A}^*, \mu^*, T_*)$.

First assume you can find a T-invariant measurable set $A \subset X$, of positive measure. Then the Poisson measure restricted to A is such a factor. Indeed, the map

$$X^* \to A^*$$
$$\nu \mapsto \nu_{|A}$$

realizes a factor map between $(X^*, \mathcal{A}^*, \mu^*, T_*)$ and $(A^*, (\mathcal{A}_{|A})^*, (\mu_{|A})^*, (T_{|A})^*)$.

In terms of σ-algebra, the above factor corresponds to

$$\sigma\{N(C), C \in \mathcal{A}, C \subset A\} \subset \mathcal{A}^*.$$

The second way consists of considering σ-finite factors of the base (we mean T-invariant σ-algebras $\mathcal{B} \subset \mathcal{A}$ such that $\mu_{|\mathcal{B}}$ remains σ-finite). If $\mathcal{B}$ is such a σ-finite factor, we have the factor map

$$(X, \mathcal{A}, \mu, T) \xrightarrow{\psi} (X_{/\mathcal{B}}, \mathcal{A}_{/\mathcal{B}}, \mu_{/\mathcal{B}}, T_{/\mathcal{B}})$$

and if we define ψ_* by $\nu \mapsto \nu \circ \psi^{-1}$ we obtain the factor relationship at the level of the Poisson suspensions:

$$(X^*, \mathcal{A}^*, \mu^*, T_*) \xrightarrow{\psi_*} ((X_{/\mathcal{B}})^*, (\mathcal{A}_{/\mathcal{B}})^*, (\mu_{/\mathcal{B}})^*, (T_{/\mathcal{B}})^*).$$

In terms of σ-algebra, this corresponds to $\mathcal{B}^* := \sigma\{N(C), C \in \mathcal{B}\} \subset \mathcal{A}^*$.

A Poissonian factor is a combination of both situations which is obtained by first considering a T-invariant subset $A \subset X$ and then considering a σ-finite factor $\mathcal{B}$ of the restricted system $(A, \mathcal{A}_{|A}, \mu_{|A}, T_{|A})$. We also consider the trivial factor $\{\emptyset, X^*\}$ a Poissonian factor.

In particular, if $(X, \mathcal{A}, \mu, T)$ is ergodic, then the only Poissonian factors of $(X^*, \mathcal{A}^*, \mu^*, T_*)$ are:

- the trivial factor $\{\emptyset, X^*\}$;
- $\mathcal{B}^*$, for a σ-finite factor $\mathcal{B} \subset \mathcal{A}$.

We shall need a result which can be read from Proposition 4.4 and the proof of Lemma 5.5 in [**24**]; for the sake of clarity we give some elements of the proof. We recall that a *sub-Markov* operator on $L^2(\mu)$ is a positive operator Φ such that $\Phi f \leq 1$ and $\Phi^* f \leq 1$, for $0 \leq f \leq 1$.

PROPOSITION 1. *Let $\mathcal{C} \subset \mathcal{A}^*$ be a factor of $(X^*, \mathcal{A}^*, \mu^*, T_*)$ and Φ the corresponding conditional expectation. Assume moreover that Φ preserves the first chaos H^1 and does not vanish on H^1. Then we have the following results.*

- *Φ induces on $L^2(\mu)$ a sub-Markov operator Ψ.*
- *There exists a T-invariant set $A \subset X$ such that Ψ restricted to $L^2(\mu_{|A})$ is a conditional expectation on a σ-finite factor $\mathcal{G} \subset \mathcal{A}_{|A}$ and vanishes on $L^2(\mu_{|A^c})$.*

Proof. The fact that Φ induces a sub-Markov operator Ψ on $L^2(\mu)$ is a direct application of Proposition 4.4 in [**24**].

We can extend Ψ in a natural way such that it sends positive bounded functions to positive bounded functions. Then $\Psi(1)$ is well defined and $0 \leq \Psi(1) \leq 1$. Observe next

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that, as Φ is an orthogonal projection, so is Ψ, and in particular it is self-adjoint. Thus

$$\int_X \Psi(1_F)\,d\mu = \int_X \Psi(\Psi(1_F))\,d\mu = \int_X \Psi(1_F)\Psi(1)\,d\mu,$$

for any set F of finite measure; thus it follows that $\int_X \Psi(1)(1-\Psi(1))\,d\mu = 0$ which implies that $\Psi(1) = 1_A$ where A is the T-invariant set $A := \{\Psi(1) = 1\}$.

On the restricted dynamical system $(A, \mathcal{A}_{|A}, \mu_{|A}, T_{|A})$, Ψ now acts as a positive operator and an orthogonal projection such that $\Psi(1) = 1$, which implies that Ψ is the conditional expectation with respect to a σ-finite factor $\mathcal{G} \subset \mathcal{A}_{|K}$.

Indeed, a proof of this latter fact in the probability context can be found in [**16**] (see Lemma 1 and Proposition 1) and readily extends to the σ-finite case once we observe that Ψ allows to define a *self-joining* η (i.e. a $T_{|A} \times T_{|A}$-invariant measure on $A \times A$ which projects on both sides as $\mu_{|A}$) thanks to the formula

$$\eta(B \times C) := \int_B \Psi(1_C)\,d\mu_{|A}. \qquad \square$$

3.2. *Superposition of Poisson suspensions.* We shall need later (for Propositions 2 and 9) an easy and classical fact about Poisson measures which says that if we consider two independent Poisson measures on the same base, with intensities μ_1 and μ_2, the superposition of Poisson suspensions, that is the full collection of particles coming from both systems, gives birth to a Poisson measure with intensity $\mu_1 + \mu_2$. More precisely, we have the following map:

$$\begin{array}{c} (X^* \times X^*, \mathcal{A}^* \otimes \mathcal{A}^*, \mu_1^* \otimes \mu_2^*) \\ \Big\downarrow \Psi \\ (X^*, \mathcal{A}^*, (\mu_1 + \mu_2)^*) \end{array}$$

defined by

$$\Psi(\nu_1, \nu_2) = \nu_1 + \nu_2.$$

From an ergodic point of view, as $\Psi \circ (T_* \times T_*) = T_* \circ \Psi$, the Poisson suspension $(X^*, \mathcal{A}^*, (\mu_1 + \mu_2)^*, T_*)$ can be seen as a factor of the direct product $(X^* \times X^*, \mathcal{A}^* \otimes \mathcal{A}^*, \mu_1^* \otimes \mu_2^*, T_* \times T_*)$. If $\mu_1 = \mu_2 = \mu$, as Ψ is symmetric, it is a factor of the symmetric factor of this direct product (the σ-algebra $\mathcal{A}^* \odot \mathcal{A}^* \subset \mathcal{A}^* \otimes \mathcal{A}^*$ generated by symmetric functions on $(X^* \times X^*, \mathcal{A}^* \otimes \mathcal{A}^*, \mu^* \otimes \mu^*)$).

3.3. *Unitary operators U_T and U_{T_*}.* T acts unitarily on $L^2(\mu)$ by $U_T : f \mapsto f \circ T$ and so does T_* on $L^2(\mu^*)$ by $U_{T_*} : F \mapsto F \circ T_*$. Each chaos is preserved by U_{T_*} and, through the above identification, it is easy to see that it corresponds to U_T on $H^1 \simeq L^2(\mu)$, and more generally to $U_T^{\odot n}$ (the nth symmetric tensor power of U_T) on $H^n \simeq L^2_{\mathrm{sym}}(\mu^{\otimes n})$.

If σ is the maximal spectral type of U_T on $L^2(\mu)$ then σ^{*n} is the maximal spectral type of $U_T^{\odot n}$ on $L^2_{\mathrm{sym}}(\mu^{\otimes n})$.

We shall need another definition.

Definition 2. A Poisson suspension is said to have the *property CP* (for 'chaos-preserving') if any conditional expectation with respect to a factor preserves each chaos H^n.

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Below is the main situation where we obtain this property.

Example. If the maximal spectral type σ of $(X, \mathcal{A}, \mu, T)$ satisfies $\sigma^{*n} \perp \sigma^{*m}$, for all distinct $n, m \in \mathbb{N}^*$, then $(X^*, \mathcal{A}^*, \mu^*, T_*)$ has property CP (see [**18**] for the proof in the Gaussian case; the Poissonian one is completely analogous).

4. *Prime Poisson suspensions*

4.1. *Main result.* We have to extend the definition of prime systems to the case of an infinite measure. The difference with the probability measure case resides in the fact that the trivial factor $\{X, \emptyset\}$ is no longer σ-finite factor when the measure is infinite.

Definition 3. An ergodic system $(X, \mathcal{A}, \mu, T)$ with an infinite measure is said to be *prime* if $\mathcal{A}$ is its only σ-finite factor.

We are now able to prove the main result of the paper. We shall give examples of such systems in the last section.

THEOREM 2. *Let $(X, \mathcal{A}, \mu, T)$ be an ergodic infinite measure-preserving system such that $(X^*, \mathcal{A}^*, \mu^*, T_*)$ has property CP. If $\mathcal{C} \subset \mathcal{A}^*$ is a non-trivial factor, then it contains a non-trivial Poissonian factor. In particular, if we assume moreover that $(X, \mathcal{A}, \mu, T)$ is prime, then $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is prime.*

Proof. Let $\mathcal{C}$ be a non-trivial T_*-invariant σ-algebra included in $\mathcal{A}^*$ and Φ the corresponding conditional expectation; then it preserves H^1 thanks to property CP.

Assume firstly that Ψ vanishes on H^1. Then we can apply Theorem 1 to conclude that $\mathcal{C} = \{X^*, \emptyset\}$, which is impossible as we have assumed $\mathcal{C}$ to be a non-trivial factor.

Now if Ψ doesn't vanish on H^1 we can apply Proposition 1, combined with the ergodicity of T to deduce that Φ induces on $L^2(\mu)$ a sub-Markov operator Ψ which is also a conditional expectation on a σ-finite factor $\mathcal{T}$. The image of Ψ contains all the indicator functions of finite measure sets contained in $\mathcal{T}$. Coming back to $L^2(\mu^*)$, the image of Φ contains all the vectors $I^{(1)}(1_A) = N(A) - \mu(A)$, for $A \in \mathcal{T}$ of finite measure, which are therefore $\mathcal{C}$-measurable. This proves that $\mathcal{C}$ contains the Poissonian factor $\mathcal{T}^*$. □

Remark 1. The conditions of Theorem 2 are also necessary in order that the Poisson suspension be prime: then, we have no non-trivial conditional expectation with respect to a factor, so property CP holds obviously, and there is no proper non-trivial Poissonian factor so $(X, \mathcal{A}, \mu, T)$ must be prime.

4.2. *Some consequences.* We mentioned in the Introduction that when T is MSJ(2), then $T \odot T$, the symmetric factor of the direct product $T \times T$, is prime and so is the map $(x, y) \mapsto (y, Tx)$ with respect to the product measure. It is therefore natural to ask if this is the case in our context.

PROPOSITION 2. *$T_* \odot T_*$ is never prime.*

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Proof. With the result on the superposition of two independent Poisson measures recalled in §3.2, we obtain the scheme

$$\begin{array}{c}(X^*\times X^*,\ \mathcal{A}^*\otimes\mathcal{A}^*,\ \mu^*\otimes\mu^*,\ T_*\times T_*)\\ \downarrow\\ (X^*\times X^*,\ \mathcal{A}^*\odot\mathcal{A}^*,\ \mu^*\otimes\mu^*,\ T_*\odot T_*)\\ \downarrow\\ (X^*,\ \mathcal{A}^*,\ (2\mu)^*,\ T_*).\end{array}$$

The direct product $(X^*\times X^*,\ \mathcal{A}^*\otimes\mathcal{A}^*,\ \mu^*\otimes\mu^*,\ T_*\times T_*)$ can be thought as the Poisson suspension

$$((X\times X)^*,\ (\mathcal{A}\otimes\mathcal{A})^*,\ (\mu\otimes\delta_\infty+\delta_\infty\otimes\mu)^*,\ (T\times T)_*)$$

where ∞ is an artificially added point in X, fixed by T. In this way, $(X^*,\ \mathcal{A}^*,\ (2\mu)^*,\ T_*)$ appears as a Poissonian factor (corresponding to the symmetric factor of $X\times X$), and we know that the corresponding relatively independent joining is ergodic (see [**24**]). It follows that

$$\begin{array}{c}(X^*\times X^*,\ \mathcal{A}^*\otimes\mathcal{A}^*,\ \mu^*\otimes\mu^*,\ T_*\times T_*)\\ \downarrow\\ (X^*,\ \mathcal{A}^*,\ (2\mu)^*,\ T_*)\end{array}$$

is a relatively weakly mixing extension.

However, we know that

$$\begin{array}{c}(X^*\times X^*,\ \mathcal{A}^*\otimes\mathcal{A}^*,\ \mu^*\otimes\mu^*,\ T_*\times T_*)\\ \downarrow\\ (X^*\times X^*,\ \mathcal{A}^*\odot\mathcal{A}^*,\ \mu^*\otimes\mu^*,\ T_*\odot T_*)\end{array}$$

is a compact extension. This proves that $(X^*,\ \mathcal{A}^*,\ (2\mu)^*,\ T_*)$ is a strict and non trivial factor of $(X^*\times X^*,\ \mathcal{A}^*\odot\mathcal{A}^*,\ \mu^*\otimes\mu^*,\ T_*\odot T_*)$. □

PROPOSITION 3. *If T_* is prime and $T_*\times T_*$ has property CP, then $(\nu_1,\ \nu_2)\mapsto(\nu_2,\ T_*\nu_1)$ is prime.*

Proof. We use the representation of $(X^*\times X^*,\ \mathcal{A}^*\otimes\mathcal{A}^*,\ \mu^*\otimes\mu^*,\ T_*\times T_*)$ introduced in the above proof, that is

$$((X\times X)^*,\ (\mathcal{A}\otimes\mathcal{A})^*,\ (\mu\otimes\delta_\infty+\delta_\infty\otimes\mu)^*,\ (T\times T)_*)$$

In this representation, $(\nu_1,\ \nu_2)\mapsto(\nu_2,\ T_*\nu_1)$ becomes $(\nu_1\otimes\delta_\infty+\delta_\infty\otimes\nu_2)\mapsto(\nu_2\otimes\delta_\infty+\delta_\infty\otimes T_*\nu_1)$ and is indeed a Poisson suspension over

$$(X\times X,\ \mathcal{A}\otimes\mathcal{A},\ \mu\otimes\delta_\infty+\delta_\infty\otimes\mu,\ R),$$

where R maps $(x,\ y)$ to $(y,\ Tx)$.

Observe that $R^2=T\times T$ and that R is ergodic. Indeed, if A is an R-invariant set, then it is also a $T\times T$-invariant set. But, as T is ergodic, it is easy to see that, modulo null sets with respect to the measure $\mu\otimes\delta_\infty+\delta_\infty\otimes\mu$, the only $T\times T$-invariant sets are $\emptyset$,

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$X \times X$, $X \times \{\infty\}$ and $\{\infty\} \times X$. Since the last two sets are obviously not R-invariant, R is ergodic.

In the same vein, a σ-finite factor of R is also a σ-finite factor of $T \times T$ and, with respect to the measure $\mu \otimes \delta_\infty + \delta_\infty \otimes \mu$, the only σ-finite factor of $T \times T$ is the symmetric factor which is not a factor of R, so R is prime.

It remains to check that $((X \times X)^*, (\mathcal{A} \otimes \mathcal{A})^*, (\mu \otimes \delta_\infty + \delta_\infty \otimes \mu)^*, R_*)$ has property CP. It follows from the assumption that $T_* \times T_*$ has property CP; hence so does $(T \times T)_*$ under the measure $(\mu \otimes \delta_\infty + \delta_\infty \otimes \mu)^*$, and from the fact that $R_*^2 = (T \times T)_*$. □

In particular, if the maximal spectral type σ of T satisfies $\sigma^{*n} \perp \sigma^{*m}$, then $T_* \times T_*$ also has property CP, as σ is still the maximal spectral type of $T \times T$ with respect to the measure $\mu \otimes \delta_\infty + \delta_\infty \otimes \mu$.

4.3. *Disjointness.* The following disjointness results come from [**17**] where the notion of Joining Primeness of order n (JP(n)) was introduced. Simple maps and their factors are JP(1) and direct products of such maps are JP(2).

THEOREM 3. [**17**] *A Poisson suspension is disjoint from every JP(n) map for any* $n \geq 1$.

Therefore our prime Poisson suspensions are disjoint from prime maps that are simple or factors of simple maps.

PROPOSITION 4. *If a transformation S is distally simple, then $S \odot S$ and $K := (x, y) \mapsto (y, Sx)$ are disjoint from Poisson suspensions.*

Proof. The first point follows from the fact that $S \times S$ is JP(2) and so is $S \odot S$.

For the second point, a non-trivial joining between K and a Poisson suspension T_* would yield a non-trivial joining between $K^2 = S \times S$ and $(T_*)^2 = (T^2)_*$ which is impossible by the above arguments. □

5. *Examples*

5.1. *Non-singular compact group rotations.* We introduce a family of examples that was studied, among other sources, in [**1**] and [**11**]. We use the notation of the latter paper.

Consider the 2-adding machine $\Omega := \{0, 1\}^{\mathbb{N}}$ equipped with the uniform Bernoulli probability measure $\nu := (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)^{\otimes \mathbb{N}}$, and the transformation $\omega \mapsto \omega + \overline{1}$, where $\overline{1} = (1, 0, 0, \dots)$ and addition is modulo 2 with 'carry to the right'. Next, consider a measurable positive integer-valued function h on ω.

We build the Kakutani tower over Ω with height function h. Define $X \subset \Omega \times \mathbb{N}$ as the set of points (ω, n) such that $1 \leq n \leq h(\omega)$ and let T be the transformation on X given by

$$T(\omega, n) = \begin{cases} (\omega, n+1) & \text{if } 1 \leq n < h(\omega), \\ (\omega + \overline{1}, 1) & \text{if } n = h(\omega). \end{cases}$$

We endow X with the σ-algebra $\mathcal{A}$ naturally inherited from the Borel σ-algebra of Ω and the T-invariant measure on $(X, \mathcal{A})$ defined by

$$\int_X f(\omega, n)\mu(d(\omega, n)) = \int_\Omega \left[\sum_{n=1}^{h(\omega)} f(\omega, n)\right] \nu(d\omega)$$

for every measurable positive function f on X.

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Let us now give a more precise specification of h. Consider a sequence of integers $\{m_i\}_{i\geq 0}$ where $m_i \geq 3$ and set $n_{i+1} = m_i n_i$ where $n_0 = 1$. For $\omega \in \Omega$, let us denote $k(\omega)$ as the smallest integer k such that $\omega_k = 0$ and

$$h(\omega) = n_{k(\omega)} - \sum_{j<k(\omega)} n_j.$$

Then it is easy to see that the measure μ is infinite. It can be noted (see [**11**]) that this system encodes an ergodic infinite measure-preserving compact group rotation, namely the adding machine on $\prod_{j\geq 0}\{0, \ldots, m_j - 1\}$ endowed with a measure singular with respect to the Haar measure on this group.

For the sequel of this section, $(X, \mathcal{A}, \mu, T)$ denotes the above defined system.

5.2. *Properties.* In [**1**], the following proposition is proved.

PROPOSITION 5. *Joinings between $(X, \mathcal{A}, c_1\mu, T)$ and $(X, \mathcal{A}, c_2\mu, T)$ exist only for $c_1 = c_2$ and are graph joinings Δ_{T^n}, $n \in \mathbb{Z}$. In particular, the system is prime.*

In [**11**], the spectrum of T is determined as below.

PROPOSITION 6. *The spectrum of T is simple and its maximal spectral type is the Riesz product*

$$\sigma := \prod_{j=0}^{+\infty}(1 + \cos 2\pi n_j t).$$

Those Riesz products are the most classical ones and we have the following proposition [**2**, **22**].

PROPOSITION 7. *All the convolution powers σ^{*n} are continuous and singular. Moreover, $\sigma^{*n} \perp \sigma^{*m}$ for all $n \neq m$.*

5.3. *Poisson suspensions over $(X, \mathcal{A}, \mu, T)$.* As a direct application, we obtain our first examples of prime Poisson suspensions.

PROPOSITION 8. *The Poisson suspension $(X^*, \mathcal{A}^*, \mu^*, T_*)$ is prime. Moreover, it is mildly mixing, non-mixing, has trivial centralizer and singular spectrum with infinite multiplicity.*

Proof. The requirements of Theorem 2 are satisfied since T is ergodic, prime, preserves an infinite measure, and has property CP as $\sigma^{*n} \perp \sigma^{*m}$ for all $n \neq m$. This latter property also implies (see [**24**]) that each transformation S that commutes with T_* is of the form R_* for some transformation R of the base that commutes with T. Therefore, as the centralizer of T is trivial, so is the centralizer of T_*, and it follows that T_* is not rigid. As T_* is prime, the only rigid factor is the trivial one, consequently T_* is mildly mixing. To see that it is not mixing, it is sufficient to notice that $\widehat{\sigma}(n_j) = \frac{1}{2}$ for all $j \geq 0$.

By Proposition 7, T_* has singular spectrum. It remains to prove that its multiplicity is infinite. Since T has simple spectrum, T_* is spectrally isomorphic to the dynamical system generated by the Gaussian process with spectral measure σ. By Girsanov's theorem, it has either infinite multiplicity or simple spectrum. In the latter case, for all $n \geq 1$ the map

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$\pi_n : \mathbb{T}^n \to \mathbb{T}, t_1, \ldots, t_n \mapsto t_1 + \cdots + t_n$ should be $n!$ to 1 on some Borel set $F \subset \mathbb{T}^n$ with $\sigma^{\otimes n}(F) = 1$ (for details, see e.g. [**14**]). However, this is impossible even for $n = 2$ by a result of [**10**].

Indeed it is shown there (Ch. IV, §2.3) that for any positive Borel measure $\tau \ll \sigma$ the measure $\tau * \sigma$ is actually equivalent to σ^{*2}. Given a σ-compact set $F \subset \mathbb{T}^2$ with $\sigma^{\otimes 2}(F) = 1$, it follows that for every compact set $A \subset \mathbb{T}$ with $\sigma(A) > 0$ we have $\sigma^{*2}(\pi_2(F \cap (A \times \mathbb{T})) = 1$. Hence, given any positive integer k, if we choose k disjoint compact sets $A_1, \ldots, A_k$ with $\sigma(A_j) > 0$ for all j, we obtain that $\bigcap_{j=1}^{k} \pi_2(F \cap (A_j \times \mathbb{T}) \neq \emptyset$, and thus there are at least k points in F with the same image under π_2 (this proves actually that U_{T_*} has infinite multiplicity on H^2). □

The fact that these systems possess a singular spectrum of infinite multiplicity makes them, as far as we know, new examples of prime systems. The corollary below also follows.

COROLLARY 1. *The Poisson suspension* $(X^*, \mathcal{A}^*, \mu^*, T_*)$ *is not a rank one system.*

Besides, it is mildly mixing and, in Appendix A, we give a short proof of the fact of independent interest that if a Poisson suspension is of rank one, then it is necessarily rigid.

With the above examples we obtain a Poisson suspension with a continuum array of non-disjoint, non-isomorphic prime factors.

PROPOSITION 9. *The Poisson suspension*

$$((X \times [0, 1])^*, (\mathcal{A} \otimes \mathcal{B})^*, (\mu \otimes \lambda_{[0,1]})^*, (T \times \mathrm{Id})_*)$$

possesses the Poisson suspensions $(X^*, \mathcal{A}^*, (c\mu)^*, T_*)$, $0 < c \leq 1$ *as factors. These factors are prime, non-disjoint, unitarily isomorphic and non metrically isomorphic for different values of* c.

Proof. The factor relationship is implemented by the map $\nu \mapsto \nu(\cdot \times [0, c])$.

The systems $(X, \mathcal{A}, c\mu, T)$ have the same properties as c spans $[0, 1]$. In particular $(X^*, \mathcal{A}^*, (c\mu)^*, T_*)$ are prime and have the same spectrum; hence they all are unitarily isomorphic.

It is recalled in §3.2 that the superposition of two independent Poisson measures with intensity μ_1 and μ_2 leads to a Poisson measure with intensity $\mu_1 + \mu_2$. Therefore, if $c_1 < c_2$ then $(X^*, \mathcal{A}^*, (c_2\mu)^*, T_*)$ is a factor of the direct product of $(X^*, \mathcal{A}^*, (c_1\mu)^*, T_*)$ with $(X^*, \mathcal{A}^*, ((c_2 - c_1)\mu)^*, T_*)$. This yields a joining between $(X^*, \mathcal{A}^*, (c_2\mu)^*, T_*)$ and $(X^*, \mathcal{A}^*, (c_1\mu)^*, T_*)$; it is not independent for obvious reasons.

Now assume there exists an isomorphism S between both systems. As $\sigma \perp \sigma^{*n}$, $n \geq 2$, this implies, thanks to Proposition 5.2 in [**24**], that $S = R^*$ for an isomorphism R between $(X, \mathcal{A}, c_1\mu, T)$ and $(X, \mathcal{A}, c_2\mu, T)$, but such an isomorphism does not exist by Proposition 5. □

5.4. *A mixing example.* Another source of examples is furnished by recent Ryzhikov's infinite measure-preserving 'mixing' rank one transformations (see [**27**]), whose construction would be too long to be given here.

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He proved, in particular, that all those systems have the *minimal self-joining* property in infinite measure (the only ergodic self-joinings are off-diagonal joinings), which implies that they are prime as in previous examples (see Proposition 5). Moreover he proved, with some extra assumptions, that Poisson suspensions over such systems have simple (and singular) spectrum, which in turn implies that they have the property CP (indeed, a necessary condition for a Poisson suspension to have simple spectrum is that $\sigma^{*n} \perp \sigma^{*m}$ for all $n \neq m$, where σ is the maximal spectral type of the base). If we sum up and apply Theorem 2, we get the following proposition.

PROPOSITION 10. *There exist prime Poisson suspensions which are mixing, with simple singular spectrum and trivial centralizer.*

Observe that, as any mixing rank one is MSJ(2), those prime mixing Poisson suspensions are disjoint from any previously known prime systems, thanks to Theorem 3 and Proposition 4.

A. *Appendix*

PROPOSITION A.1. *If a Poisson suspension is of rank one, then it is rigid.*

Proof. We need the following property of rank one systems, established by Ryzhikov in [**26**]: If Φ is a Markov operator corresponding to an ergodic selfjoining of a rank one transformation T, then there exists $a > 0$, a Markov operator Ψ and a sequence n_k such that $U_{T^{n_k}}$ converges weakly to $a\Phi + (1-a)\Psi$.

We recall that every Poisson suspension $(X^*, \mathcal{A}^*, \mu^*, T_*)$ has the so-called ELF property (see [**8**]); that is every limit of off-diagonal joinings is ergodic. Therefore, in the above situation, $a = 1$ and $U_{T_*^{n_k}} \to \Phi$.

However, we cannot apply this result directly to $\Phi = \mathrm{Id}$, as we have to rule out a sequence (n_k) that would be eventually equal to zero.

We recall moreover that we build a *Poissonian joining* (see [**8**] and [**24**]) of a Poisson suspension T_* by considering a sub-Markov operator φ that commutes with the base transformation T and forming the exponential $\exp(\varphi)$ that acts on each chaos H^n of $L^2(\mu^*)$ as $\varphi^{\odot n}$. Moreover, Poissonian joinings of an ergodic Poisson suspension are ergodic. Therefore we can apply Ryzhikov's result to the Markov operators $\exp((1-(1/n))\mathrm{Id}_{L^2(\mu)})$ for each $n > 1$. As $(1-(1/n))\mathrm{Id}_{L^2(\mu)}$ tends to $\mathrm{Id}_{L^2(\mu^*)}$, we have that $\exp((1-(1/n))\mathrm{Id}_{L^2(\mu)})$ tends to $\exp(\mathrm{Id}_{L^2(\mu)}) = \mathrm{Id}_{L^2(\mu^*)}$. It now follows that $\mathrm{Id}_{L^2(\mu^*)}$ is a limit point of $(U_{T_*^n})$ $(n \neq 0)$ and thus T_* is rigid. □

Remark A.1. Valery Ryzhikov informed us that the same proof shows even more, namely that non-rigid Poisson suspensions are of local rank zero (and therefore of infinite rank).

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