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An upper bound for the multidimensional dimer problem

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Abstract. A recently proved upper bound for the permanents of (0, 1) matrices is used to improve the Fowler–Rushbrooke upper bound for the constant λ_d occurring in the d-dimensional dimer problem, $d \geqslant 3$.

The three-dimensional dimer problem is one of the classical unsolved problems of solid-state chemistry. The general d-dimensional dimer problem can be formulated (4) as follows. An *n-brick* is a d-dimensional parallelopiped of volume n with sides whose lengths $\alpha_1, \ldots, \alpha_d$ are integers. A *dimer* is a 2-brick. The problem is to determine the number of ways of dissecting an n-brick into dimers; denote this number by f_α, where $\alpha = (\alpha_1, \ldots, \alpha_d)$.

It is known (3) that if $\alpha_i \to \infty$ $(i = 1, \ldots, d)$, then $n^{-1} \log f_\alpha$ tends to a finite limit λ_d. Several writers proved that

$$\lambda_2 = 0{\cdot}291\,560\,90\ldots.$$

For $d \geqslant 3$ the exact value of λ_d is not known. However, it was shown by Fowler and Rushbrooke (2) that

$$0 \leqslant \lambda_d \leqslant \tfrac{1}{2} \log d. \tag{1}$$

Hammersley (4) obtained the lower bound

$$\lambda_3 \geqslant 0{\cdot}418347 \tag{2}$$

as a particular case of a general lower bound for λ_d. Thus from (1) and (2) we have

$$0{\cdot}418347 \leqslant \lambda_3 \leqslant 0{\cdot}54931. \tag{3}$$

In this paper we improve the upper bound in (1) and (3).

Consider a d-dimensional n-brick. It consists of n unit cubes; let them be labelled $e_1, \ldots, e_n$. Now, let S_i be the set of cubes facially adjacent to the cube e_i, $i = 1, \ldots, n$. The set S_i contains $2d$ cubes if the cube e_i lies in the interior of the brick, and fewer than $2d$ cubes if e_i lies on its surface. Let $V = (v_{ij})$ be the incidence matrix of the configuration: $v_{ij} = 1$ or 0 according as the cube e_j belongs to S_i or not, $i, j = 1, \ldots, n$. Hammersley (4) showed that

$$f_\alpha^2 = \operatorname{per}(V). \tag{4}$$

Unfortunately, no efficient method is known for computing permanents of large matrices. This precludes the evaluation of λ_d by means of the formula (4).

In 1963 Minc (5) conjectured that if A is an n-square (0, 1) matrix with row sums $r_1, \ldots, r_n$, then

$$\operatorname{per}(A) \leqslant \prod_{i=1}^{n} (r_i!)^{1/r_i}. \tag{5}$$

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The conjecture was proved in 1973 by Brégman (1). We shall use the upper bound in (5) in conjunction with (4) to obtain upper bounds for f_α and for λ_d.

To simplify matters we shall consider the given n-brick as a toroidal brick by identifying and joining opposite faces of the original brick. In this toroidal brick each set S_i contains exactly $2d$ cubes, and thus every row sum of the corresponding incidence matrix $\tilde{V}$ equals $2d$. Clearly for a given n-brick

$$\operatorname{per}(V) \leqslant \operatorname{per}(\tilde{V}).$$

Hence, by (4) and (5)

$$\begin{aligned} f_\alpha^2 &\leqslant \operatorname{per}(\tilde{V}) \\ &\leqslant ((2d)!)^{n/2d}. \end{aligned}$$

It follows that

$$\lambda_d \leqslant \frac{1}{4d} \log (2d)!. \tag{6}$$

The bound in (6) is sharper than that in (1), since

$$((2d)!)^{1/2d} < d$$

for $d \geqslant 3$. In particular, we have

$$\lambda_3 \leqslant 0{\cdot}548\,270\,9\ldots.$$

It should be noted that the use of the matrix V instead of $\tilde{V}$ would have led to the same bound for λ_d.

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REFERENCES

(1) BRÉGMAN, L. M. Certain properties of nonnegative matrices and their permanents. *Dokl. Akad. Nauk SSSR* **211** (1973), 27–30 (*Soviet Math. Dokl.* **14** (1973), 945–949).

(2) FOWLER, R. H. and RUSHBROOKE, G. S. Statistical theory of perfect solutions. *Trans. Faraday Soc.* **33** (1937), 1272–1294.

(3) HAMMERSLEY, J. M. Existence theorems and Monte Carlo methods for the monomer-dimer problem. *Research papers in statistics: Festschrift for J. Neyman* (1966), 125–146.

(4) HAMMERSLEY, J. M. An improved lower bound for the multidimensional dimer problem. *Proc. Cambridge Philos. Soc.* **64** (1968), 455–463.

(5) MINC, H. Upper bounds for permanents of (0, 1)-matrices. *Bull. Amer. Math. Soc.* **69** (1963), 789–791.

https://doi.org/10.1017/S0305004100054748 Published online by Cambridge University Press