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# Comparative uncertainty: theory and automation

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*In Memory of Sauro Tulipani*

In recent decades, qualitative approaches to probabilistic uncertainty have received more and more attention. We propose a characterisation of partial preference orders through a uniform axiomatic treatment of a variety of qualitative uncertainty notions. To this end, we prove a representation result that connects qualitative notions of partial uncertainty to their numerical counterparts. We describe an executable specification, in the declarative framework of Answer Set Programming, that constitutes the core engine for qualitative management of uncertainty. Some basic reasoning tasks are also identified.

## 1. Introduction

*Where are all the numbers coming from?*
– Peter Cheeseman 1988

Numerous formalisms for dealing with uncertainty were studied in the previous century. Most of the proposed probabilistic models of decision under uncertainty rely on numerical measures and representations. All of them originate from amendments to the well-known Probability measure (Savage 1972), and usually aimed at generalising it to give a better fit to different peculiarities of specific application fields. However, several authors have criticised such use of the classical theory of probability by emphasising the fact that ordinary people, in expressing their intuitive judgments on the likelihood of events, often (deliberately) violate the postulates of such theory. It seems reasonable that the human mental process yielding a judgment is guided by a number of heuristics that unconsciously 'act behind the scenes'. Subjective, psychological and environmental factors may influence this process (Fox 1999; Fox and See 2003; Kahneman *et al.* 1982; Suppes 1974). For instance, different descriptions of the same event often give rise to substantially different judgments. A paradigmatic example of this phenomenon is the well-known *Ellsberg Paradox* (Ellsberg 1961; Fishburn 1986). A way to circumvent these problems is to drop some of the postulates characterising probability measures. Most of the proposals along these lines weaken the additivity property of probability functions. There are several ways to do this, resulting in alternative notions of uncertainty measures: Possibility and Necessity measures, Belief and Plausibility functions, 0-monotone and 0-alternating functions, and so on. For a survey, see Klir and Folger (1988), Nguyen and Walker (1997)

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and Walley (1996), among others. (Appendix A gives a brief summary of some basic notions about uncertainty measures from the quantitative point of view.)

Actually, the numerical approach itself has been criticised *in toto*, because all of the proposed numerical frameworks inevitably suffer from important drawbacks:

— the difficulty, for human beings, of producing precise numerical values;
— the difficulty of expressing a complete evaluation.

The former problem may originate from a lack of adequate knowledge or expertise, or from the inability of people to estimate numerical values correctly and to grade their preferences numerically. It seems, in fact, more plausible that assessments of uncertainty are based on comparative qualitative judgments (Brafman and Tennenholtz 1997; Cheeseman 1988; Dubois *et al.* 2002; Parsons 1994; Redelmeier *et al.* 1995; Tversky 1974). Moreover, other aspects may make numerical evaluations inappropriate from a practical point of view. Indeed, studies on the sensitivity of automated decision systems with respect to the accuracy of the given numerical assessments (see, for example, Pradhan *et al.* (1996)) suggest that there are some fields of application where there could be little advantage in forcing the user to express precise quantifications of preferences. Hence, rough assessments or even qualitative treatments could yield similar results with less complex modelling and, often, less computational effort. To avoid such weaknesses of numerical models, *qualitative approaches* have been proposed in recent decades and are receiving more and more attention, either as theoretical tools to deal directly with belief management (Bilgiç 2001; de Cooman 1997; Dubois 1986), or within the more articulated framework of decision-making theory (see, for example, Dubois *et al.* (1997a; 1997b; 2003) and Giang and Shenoy (2001)). The central idea of such methodologies is to grade uncertainty about the truth of propositions through comparisons expressing the judgment of 'less or more believed to be true'. Operationally, this translates into the use of order relations instead of numerical grades. The notions of qualitative probabilities, qualitative plausibility, and so on, are then introduced. (Often, the terms *comparative* probabilities, *comparative* plausibility, and so on, are used.)

Similar reasons are at the root of the difficulty of expressing a *complete* evaluation. In the context of qualitative models (but the same argument applies to the numerical case) it is doubtful that a set of preferences expressed by a human being constitutes a complete (that is, total) order of the domain of discernment. There could be several reasons that prevent the subject of the analysis from providing such complete information. (S)he may be incapable or reluctant to describe a total order in very rich contexts, for instance, because of substantial difficulties in comparing every possible event, or because (s)he is interested in reasoning about a restricted portion of the domain (Brafman and Tennenholtz 1997; Halpern 1997; Kaplan and Fine 1977; Lehmann 1996; Tversky and Simonson 1993). This problem can be circumvented by adapting the pioneering approach proposed by de Finetti in the context of Probability measures (de Finetti 1931; 1974) to the qualitative framework. Namely, by introducing the so called *partial models*, that is, qualitative assessments defined only on some of the situations at hand and intended to be restrictions of some complete models. (Then, we deal with partial capacities, partial probabilities, and so on.) This approach allows the analyst of the problem to focus his/her

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evaluation on the situations really judged relevant, with respect to the problem at hand. This also leaves open the possibility of enlarging the model to other scenarios that could enter the scene later.

This paper is organised as follows. In the next section we introduce basic notions and provide an axiomatic view of the most studied uncertainty orders. (Notice that we focus on the treatment of partial orders, even if total relations can easily be dealt with by exploiting the very same machinery.) Section 3 illustrates the potentialities of the declarative approach, by describing two reasoning tasks that exploit preference orders. In Section 4 we briefly describe a tool for declarative management of preference orders. Such a tool is ultimately based on a successful form of declarative programming, namely Answer Set Programming (Lifschitz 1999; Marek and Truszczyński 1999). Finally, we draw some conclusions and outline future developments. For convenience, Appendix A gives a brief summary of some basic notions about uncertainty measures from the quantitative point of view. Appendix B recalls the main features of Answer Set Programming, with particular emphasis on its application to our context.

## 2. Axiomatisation of partial preference orders

We begin by recalling some notions related to uncertainty orders. The domain of discernment is represented by a finite set of events $\mathscr{E} = \{E_1, \ldots, E_n\}$ (among them, $\phi$ and $\Omega$ denote the impossible and the certain event, respectively). The events are seen as the relevant propositions on which the subject of the analysis expresses his/her opinion. As we have already mentioned, $\mathscr{E}$ does not necessarily represent a full model, that is, it does not cover all elementary situations and all of their combinations. For this reason, a crucial component of partial assessments is the knowledge of the logical relationships (incompatibilities, implications, combinations, equivalences, and so on) holding among events. Such relationships are usually expressed by stating a collection $\mathscr{C}$ of constraints on the events (as well as, on conjunctions and disjunctions of events). By taking into account the constraints $\mathscr{C}$, the family $\mathscr{E}$ spans a minimal Boolean algebra $\mathscr{A}_\mathscr{E}$ containing $\mathscr{E}$ itself. Note that $\mathscr{A}_\mathscr{E}$ is only implicitly defined via $\mathscr{E}$ and $\mathscr{C}$ and it is not a part of the assessment. However, $\mathscr{A}_\mathscr{E}$ can be referenced as a supporting structure.

**Definition 2.1.** Let $\mathscr{E}$ be a set of events and $\mathscr{C}$ be a collection of constraints on $\mathscr{E}$. $\mathscr{A}_\mathscr{E}$ is the minimal algebra $(\mathscr{A}, \cup, \cap, \bar{}, \phi, \Omega)$ satisfying $\mathscr{C}$ and such that $\mathscr{E} \subseteq \mathscr{A}$.

Such an algebra induces a lattice structure on $\mathscr{A}$. The *atoms* of $\mathscr{A}_\mathscr{E}$ are the minimal elements of the (sub-)lattice $\mathscr{A}_\mathscr{E} \setminus \{\phi\}$. Then, each event corresponds to a set of atoms and $\mathscr{A}_\mathscr{E}$ is (partially) ordered by set inclusion.

As a useful piece of notation, given any binary relation $\mathsf{R}$, we will use $\neg(A \mathsf{R} B)$ to mean that the pair $\langle A, B \rangle$ does not belong to $\mathsf{R}$.

**Definition 2.2.** Let $\mathscr{A}_\mathscr{E}$ be an algebra of events. A binary relation $\preccurlyeq^*$ over $\mathscr{A}$ is a *(total) preference order* if it satisfies the following conditions:

(A1) $\preccurlyeq^*$ is a pre-order, that is, it is reflexive, transitive and total[†];

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(A2) $\phi \preceq^* \Omega$ and $\neg(\Omega \preceq^* \phi)$ (*non-triviality*);
(A3) for all events $A, B$, $A \subseteq B \rightarrow (A \preccurlyeq^* B)$ (*monotonicity*).

If $\preccurlyeq^*$ is a total preference order, $\sim^*$ is its *symmetric factor*, that is, $\forall E_1, E_2\,(E_1 \sim^* E_2 \leftrightarrow E_1 \preccurlyeq^* E_2 \wedge E_2 \preccurlyeq^* E_1)$. Moreover, $\prec^*$ is the *asymmetric factor* of $\preccurlyeq^*$, that is, $\forall E_1, E_2\,(E_1 \prec^* E_2 \leftrightarrow E_1 \preccurlyeq^* E_2 \wedge \neg(E_2 \sim^* E_1))$.

In order to deal with partial assessments, it is convenient to consider two distinct but correlated preference relations. Roughly speaking, the two relations model weak and strict user's preferences, respectively. The following definition formalises this concept.

**Definition 2.3.** Let $\preccurlyeq$ and $\prec$ be binary relations over a set of events $\mathscr{E}$ such that $E_1 \prec E_2 \rightarrow E_1 \preccurlyeq E_2$. The pair $\langle \preccurlyeq, \prec \rangle$ is a *weak preference structure* for $\mathscr{E}$ (*w.p.s.*, for short) if there exists a total preference order $\preccurlyeq^*$ over $\mathscr{A}_{\mathscr{E}}$ such that $\forall E_1, E_2 \in \mathscr{E}\,\big((E_1 \preccurlyeq E_2 \rightarrow E_1 \preccurlyeq^* E_2) \wedge (E_1 \prec E_2 \rightarrow E_1 \prec^* E_2)\big)$.

Note that Definition 2.3 does not require either $\preccurlyeq$ or $\prec$ to be total orders, or $\prec$ to be the asymmetric factor of $\preccurlyeq$. On the other hand, it is required that $\preccurlyeq^*$ extends $\preccurlyeq$, and that $\prec^*$ (the asymmetric factor of $\preccurlyeq^*$) extends $\prec$.

**Example 2.1.** Consider the following situation, concerning a decision making problem in gastroenterology[‡]. We consider three possible diseases that a patient might suffer: peptic ulcer, gastric cancer and biliar disease. The symptoms that might be associated to these diseases are jaundice, weight loss and dark stools. Presence of jaundice indicates biliar disease, weight loss can be associated with gastric cancer, dark stools might indicate peptic ulcer or gastric cancer. From data provided by the hospitals we know that the incidence of peptic ulcer is greater than the incidence of gastric cancer, while biliar disease affects the majority of the patients. Clearly, the age and sex of the patient is useful information for making a diagnosis: peptic ulcer and gastric cancer are more frequent in men; biliar disease is more often complained of by women. Moreover, in the male population, the incidence of ulcers is greater than the incidence of biliar disease. As for age, it is reasonable to say that older people are more subject to peptic ulcer or gastric cancer than young people. This scenario can be represented as follows:

GC ≡ *The real state of suffering from gastric cancer* JA ≡ *Jaundice symptoms*
PU ≡ *The real state of suffering from peptic ulcer* WL ≡ *Weight loss symptoms*
BD ≡ *The real state of suffering from biliar disease* DS ≡ *Dark stools.*

Let M (respectively, W) denote the event *The patient is male* (respectively, *female*), and OA (respectively, YA) denote the event *The patient is old* (respectively, *young*).
Finally, let us focus on the simplified situation in which patients suffer from at most one disease. The knowledge about diseases and symptoms can be described in terms of logical

[†] As usual, $\preccurlyeq^*$ is total over $\mathscr{A}$ if for all events $E_1, E_2 \in \mathscr{A}$, we have $E_1 \preccurlyeq^* E_2$ or $E_2 \preccurlyeq^* E_1$. Note that it could be that for $E_1 \neq E_2$ both $E_1 \preccurlyeq^* E_2$ and $E_2 \preccurlyeq^* E_1$ hold.
[‡] This example is hypothetical and for illustrative purposes only; it is not intended to express any clinical competence.

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constraints as follows:

$$\mathsf{JA} \cap \mathsf{GC} = \mathsf{JA} \cap \mathsf{PU} = \mathsf{WL} \cap \mathsf{PU} = \phi$$
$$\mathsf{WL} \cap \mathsf{BD} = \mathsf{DS} \cap \mathsf{BD} = \phi$$
$$\mathsf{GC} \cap \mathsf{PU} = \mathsf{GC} \cap \mathsf{BD} = \mathsf{PU} \cap \mathsf{BD} = \phi$$
$$\mathsf{GC} \cup \mathsf{PU} \cup \mathsf{BD} = \Omega$$
$$\mathsf{OA} \cap \mathsf{YA} = \mathsf{M} \cap \mathsf{W} = \phi$$
$$\mathsf{OA} \cup \mathsf{YA} = \mathsf{M} \cup \mathsf{W} = \Omega.$$

Moreover, from the meaning of the events, it seems reasonable to describe a w.p.s. as follows:

$$\phi \prec \mathsf{GC} \prec \mathsf{PU} \prec \mathsf{BD} \prec \Omega$$
$$\mathsf{GC} \cap \mathsf{W} \prec \mathsf{GC} \cap \mathsf{M}$$
$$\mathsf{PU} \cap \mathsf{W} \prec \mathsf{PU} \cap \mathsf{M}$$
$$\mathsf{BD} \cap \mathsf{M} \prec \mathsf{BD} \cap \mathsf{W}$$
$$\mathsf{BD} \cap \mathsf{M} \prec \mathsf{PU} \cap \mathsf{M}$$
$$\mathsf{YA} \cap (\mathsf{GC} \cup \mathsf{PU}) \preccurlyeq \mathsf{OA} \cap (\mathsf{GC} \cup \mathsf{PU}).$$

It is easy to verify the following property.

**Proposition 2.1.** For any w.p.s. $\langle \preccurlyeq, \prec \rangle$ for $\mathscr{E}$, the following properties hold:

(A1′) if there exist $E_1, \ldots, E_n \in \mathscr{E}$ such that $E_1 \preccurlyeq E_2 \preccurlyeq \ldots \preccurlyeq E_n \preccurlyeq E_1$, then $\neg(E_i \prec E_j)$ for any $i, j \in \{1, \ldots, n\}$;
(A2′) $\neg(\Omega \preccurlyeq \phi)$;
(A3′) for all $E_1, E_2 \in \mathscr{E}$, $E_1 \prec E_2 \rightarrow E_2 \not\subseteq E_1$.

Conditions (A1′)–(A3′) ensure the existence of a total preference order $\preccurlyeq^*$ that enlarges $\langle \preccurlyeq, \prec \rangle$. Considering numerical approaches to uncertainty, Capacity measures (Choquet 1954) constitute the most general framework, as they express 'common sense' behaviours. Any reasonable relation $\preccurlyeq$ must be representable by a partial Capacity (that is, a restriction of a Capacity measure to the set of events at hand). This corresponds to the satisfaction of the conditions (A1′)–(A3′) (de Cooman 1997).

The uncertainty notions are further differentiated by considering the specific way of combining distinct pieces of information (for example, as mentioned earlier, additivity is adopted for Probabilities). Within the numerical context, this yields a taxonomy of numerical measures. Following Dubois (1986), Walley and Fine (1979) and Wong *et al.* (1991), a classification of various preference notions was proposed in Capotorti and Vantaggi (2000) according to their compliance with the numerical models.

In the following, we revise the treatment presented in Capotorti *et al.* (1998) and Capotorti and Vantaggi (2000) by adapting it to the notion of a w.p.s. The correspondence between a qualitative uncertainty notion and a numerical measure is given in terms of a representability result. More formally, we have the following definition.

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**Definition 2.4.** Let $\mathscr{E}$ be a set of events. A total preference order $\preccurlyeq^*$ over $\mathscr{A}_{\mathscr{E}}$ is said to be *representable* by a numerical measure $f : \mathscr{A}_{\mathscr{E}} \to [0,1]$ if for all $E_1, E_2 \in \mathscr{A}_{\mathscr{E}}$, we have $E_1 \preccurlyeq^* E_2 \leftrightarrow f(E_1) \leqslant f(E_2)$.

A w.p.s. $\langle \preccurlyeq, \prec \rangle$ for $\mathscr{E}$ is said to be *representable* by a partial uncertainty measure $g : \mathscr{E} \to [0,1]$ if it admits an enlargement $\preccurlyeq^*$ over $\mathscr{A}_{\mathscr{E}}$ that is representable by an uncertainty measure $g^* : \mathscr{A}_{\mathscr{E}} \to [0,1]$ extension of $g$ to $\mathscr{A}_{\mathscr{E}}$.

We refer to any specific class of preference orders by the name of the corresponding numerical notion. The representability results we are going to present legitimate this choice. Note that in the following we assume that the orders are closed under monotonicity (that is, for all $E_1, E_2 \in \mathscr{E},\ E_1 \subseteq E_2 \to E_1 \preccurlyeq E_2$).

Proposition 2.2 states a necessary and sufficient condition for a w.p.s. to be representable by a belief function. Representability by other uncertainty measures is stated similarly (see below).

**Proposition 2.2 (Comparative belief).** A w.p.s. $\langle \preccurlyeq, \prec \rangle$ for $\mathscr{E}$ can be extended to a total preference order $\preccurlyeq^*$ over $\mathscr{A}_{\mathscr{E}}$ representable by a belief function if and only if for all $X, Y, Z, W \in \mathscr{E}$ such that $X \subset Y$, $Z \subset W \subset Y$, $W \setminus Z \subseteq Y \setminus X$ the following implication holds:

(B′) $$X \sim Y \to \neg(Z \prec W).$$

*Proof.*

(⇒) By hypothesis and the characterisation theorem for total comparative beliefs (Wong *et al.* 1991), there exists a total order $\preccurlyeq^*$ extending $\langle \preccurlyeq, \prec \rangle$ and satisfying the following condition (which characterises complete beliefs):

(B) $\forall A, B, C \in \mathscr{A}_{\mathscr{E}}$ such that $A \subset B, B \cap C = \phi, A \prec^* B \to A \cup C \prec^* B \cup C$.

Let us assume that (B′) does not hold. Then there exist $X, Y, Z, W \in \mathscr{E} \subseteq \mathscr{A}_{\mathscr{E}}$ such that $X \subset Y$, $Z \subset W \subset Y$, and $W \setminus Z \subseteq Y \setminus X$ with $X \sim^* Y$ and $Z \prec^* W$. Hence, by monotonicity and transitivity of $\preccurlyeq^*$, we have

$$X \sim^* X \cup Z \sim^* X \cup W \sim^* Y\,. \tag{1}$$

From $W \setminus Z \subseteq Y \setminus X$ and $Z \subset W$, we easily get that $(X \setminus Z) \cap \overline{X \setminus W} = X \cap (W \setminus Z) = \phi$ and that $\overline{X \setminus Z} \cap (X \setminus W) = X \cap (Z \setminus W) = \phi$. Then, since $(X \cup Z) \setminus Z = X \setminus Z$ and $(X \cup W) \setminus W = X \setminus W$, it follows that

$$(X \cup Z) \setminus Z = (X \cup W) \setminus W\,. \tag{2}$$

Since $(X \setminus W) \cap W = \phi$ and $Z \prec^* W$, by (2) and (B), we get

$$\begin{aligned} Z \cup (X \setminus W) &= Z \cup ((X \cup W) \setminus W) \\ &= Z \cup ((X \cup Z) \setminus Z) \\ &= X \cup Z \prec^* W \cup (X \setminus W) = X \cup W\,. \end{aligned}$$

This contradicts (1), so (B′) must hold.

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($\Leftarrow$) Observe that, since $\langle \preccurlyeq, \prec \rangle$ is a w.p.s., by Proposition 2.1, (A2′) and (A3′) hold for $\langle \preccurlyeq, \prec \rangle$. This allows us to add to $\langle \preccurlyeq, \prec \rangle$ the relationships $\phi \prec \Omega$ and $E_1 \preccurlyeq E_2$ for all $E_1, E_2 \in \mathscr{A}_{\mathscr{E}}$ such that $E_1 \subseteq E_2$, without introducing inconsistency (that is, the presence of both $E_i \prec E_j$ and $E_j \preccurlyeq E_i$, for some $E_i, E_j$). Hence, without loss of generality, we can assume from the beginning $\langle \preccurlyeq, \prec \rangle$ such that $\phi \prec \Omega$ holds and $\preccurlyeq$ is closed by monotonicity.

We will now describe a construction of the enlargement $\preccurlyeq^*$. First, we construct an enlargement $\langle \preccurlyeq', \prec' \rangle$ over $\mathscr{A}_{\mathscr{E}}$ closed under axioms (B) and (B′), that is, such that

$$\forall A, B, C \in \mathscr{A}_{\mathscr{E}}, A \subset B, B \cap C = \phi \qquad A \prec' B \rightarrow A \cup C \prec' B \cup C \tag{3}$$

$$\forall X, Y, Z, W \in \mathscr{A}_{\mathscr{E}}, X \subset Y, Z \subset W \subset Y, W \setminus Z \subseteq Y \setminus X \qquad X \sim' Y \rightarrow Z \sim' W. \tag{4}$$

Such an enlargement can be obtained by applying the following procedure.

**Set** $\langle \preccurlyeq', \prec' \rangle \equiv \langle \preccurlyeq, \prec \rangle$ and $\mathscr{E}' \equiv \mathscr{E}$

**Repeat** for as long as possible

(↑) **If** $\exists A, B \in \mathscr{E}'$ such that $A \subset B, A \prec' B$ **and** $\exists C \in \mathscr{A}_{\mathscr{E}} \setminus \{\phi\}$ such that $B \cap C = \phi$ **and** $\neg(A \cup C \prec' B \cup C)$, **then**

**Add** $A \cup C \prec' B \cup C$ to $\prec'$

**Set** $\mathscr{E}' = \mathscr{E}' \cup \{A \cup C, B \cup C\}$

(↓) **If** $\exists X, Y \in \mathscr{E}'$ such that $X \subset Y, X \sim' Y$ **and** $\exists C \in \mathscr{A}_{\mathscr{E}}$ such that $Y \cap C \neq \phi$ **and** $\neg(X \cap C \sim' Y \cap C)$, **then**

**Add** $X \cap C \preccurlyeq' Y \cap C$ and $Y \cap C \preccurlyeq' X \cap C$ to $\preccurlyeq'$

**Set** $\mathscr{E}' = \mathscr{E}' \cup \{X \cap C, Y \cap C\}$

At each iteration of the **Repeat**-loop, the step (↑) enlarges the relation $\prec'$ without affecting $\preccurlyeq'$. Similarly, each step (↓) enlarges the relation $\preccurlyeq'$ without influencing $\prec'$. Each execution of steps (↑) and (↓) preserves the validity of condition (B′) for the w.p.s. $\langle \preccurlyeq', \prec' \rangle$ over $\mathscr{E}'$, while (B) is not falsified. By construction, (B) is satisfied at the end of the procedure. Moreover, we have that steps (↑) and (↓) do not conflict, that is, it never happens that there exist a $C \in \mathscr{A}_{\mathscr{E}} \setminus \{\phi\}$ and $A, B, X, Y \in \mathscr{E}'$, with $A \subset B$, $B \cap C = \phi$, $X \subset Y$, $(B \cup C) \subset Y$, and $B \setminus A \subseteq Y \setminus X$, but such that $A \prec' B$ and $X \sim' Y$ (*cf.* Figure 1, where we depict events as nodes of a graph whose edges represent order relationships).

Things are this way because, otherwise, (B′) could be falsified in $\mathscr{E}'$ by putting $Z \equiv (A \cup C)$, $W \equiv (B \cup C)$ and $X, Y$ as they are. Finally, by construction of $\mathscr{E}'$, we have $\mathscr{A}_{\mathscr{E}'} \equiv \mathscr{A}_{\mathscr{E}}$ after each iteration. Since $\mathscr{A}_{\mathscr{E}}$ is finite, this fact guarantees the termination of the procedure. At this point, $\preccurlyeq'$ can be closed under monotonicity over $\mathscr{A}_{\mathscr{E}}$ preserving (B′). The w.p.s. $\langle \preccurlyeq', \prec' \rangle$ so obtained can be enlarged to obtain $\langle \preccurlyeq'', \prec'' \rangle$ as follows:

— first add all 'inverted' pairs $(E_j, E_i)$ such that $E_i \preccurlyeq' E_j$ and such addition does not create intransitive cycles;

— then, impose a strict order for all the remaining pairs

$$\preccurlyeq'' \quad =_{\text{Def}} \quad \preccurlyeq' \cup \{E_j \preccurlyeq'' E_i : E_i \preccurlyeq' E_j \text{ and axiom (A1′) is preserved}\}$$

$$\prec'' \quad =_{\text{Def}} \quad \prec' \cup \{E_i \prec'' E_j : E_i \preccurlyeq'' E_j \text{ and } \neg(E_j \preccurlyeq'' E_i)\}.$$

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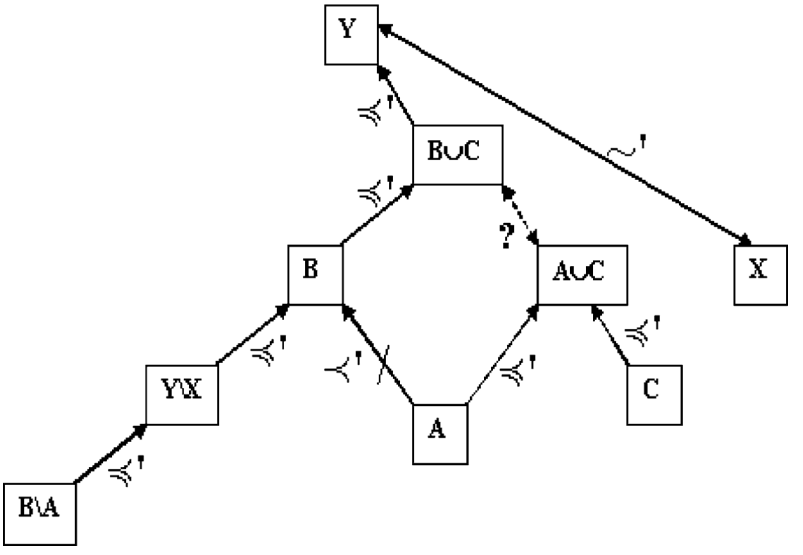



Fig. 1. A potential conflict configuration in the construction of the enlargement: considering the relationship holding between $(A \cup C)$ and $(B \cup C)$ (the edge marked by '?'), the step $(\uparrow)$ would declare that $(A \cup C) \prec' (B \cup C)$, while $(\downarrow)$ would impose $(A \cup C) \sim' (B \cup C)$.

Note that, by (3) and (4), the w.p.s. $\langle \preccurlyeq', \prec' \rangle$ already contains all the equivalences $X \sim' Y$ and all the strict preferences $A \prec' B$ over $\mathscr{A}_\mathscr{E}$ that axioms (B) and (B′) could involve, hence (B) is satisfied by the w.p.s. $\langle \preccurlyeq'', \prec'' \rangle$.
If we close $\langle \preccurlyeq'', \prec'' \rangle$ by transitivity, we obtain that (the closure of) $\preccurlyeq''$ induces a partial order relation over the quotient set $\mathscr{A}_\mathscr{E}/\sim''$. In the following, for the sake of simplicity, we will use $\preccurlyeq''$ also to denote such an induced order over $\mathscr{A}_\mathscr{E}/\sim''$. Moreover, we use $\preccurlyeq''|_S$ to denote the restriction of $\preccurlyeq''$ to the set $S$. The complete order $\preccurlyeq^*$ can be obtained by this procedure as follows:

**Set** $\mathscr{A}$ to $\mathscr{A}_\mathscr{E}/\sim''$ and $\mathscr{F}$ to the lattice $(\mathscr{A}, \preccurlyeq'')$
**Repeat** until $\mathscr{A} \neq \phi$
  **Set** $\mathscr{M}$ to be the set of minimal elements of the lattice $\mathscr{F}$
  **Set** $E_i \sim^* E_j$ for all $E_i, E_j \in \mathscr{M}$
  **Set** $E_h \prec^* E_k$ for all $E_h \in \mathscr{M}$ and $E_k \in \mathscr{A} \setminus \mathscr{M}$
  **Set** $\mathscr{A} \equiv \mathscr{A} \setminus \mathscr{M}$ and $\mathscr{F} \equiv (\mathscr{A} \setminus \mathscr{M}, \preccurlyeq''|_{\mathscr{A} \setminus \mathscr{M}})$.

This procedure preserves (B). In fact, since $\preccurlyeq'$ was closed by monotonicity, the w.p.s. $\langle \preccurlyeq'', \prec'' \rangle$ already predicates on all pairs of elements among $A, B, A \cup C$ and $B \cup C$ (with $A \subset B$ and $B \cap C = \phi$), while the construction of $\preccurlyeq^*$ does not impose further constraints among them. Hence, by the characterisation theorem for beliefs (Wong *et al.* 1991), $\preccurlyeq^*$ is representable by a belief function. □

An analogous property can be stated for comparative 0-monotonicity. Proposition 2.3 relates the qualitative notions to the corresponding quantitative measures. The proof

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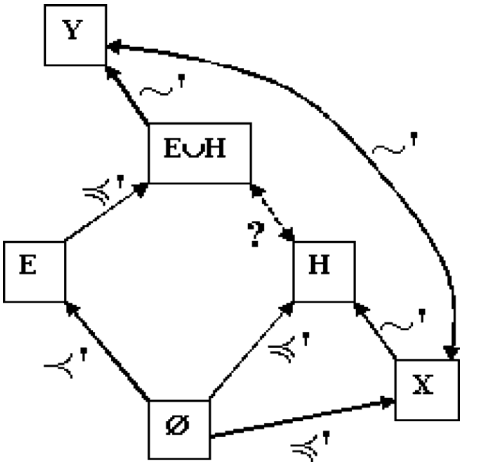



Fig. 2. A critical configuration in the enlargement procedure for comparative 0-monotonicity orders (*cf.* Figure 1 and Proposition 2.3).

proceeds along the lines of Proposition 2.2 but employing, in place of condition (B), the appropriate characterisation of total preference.

**Proposition 2.3 (Comparative 0-monotonicity).** Let $\langle \preccurlyeq, \prec \rangle$ be a w.p.s. for $\mathscr{E}$. Then $\langle \preccurlyeq, \prec \rangle$ can be extended to a total preference order $\preccurlyeq^*$ over $\mathscr{A}_\mathscr{E}$ that is representable by a 0-monotone function† if and only if for all $X, Y, Z \in \mathscr{E}$ such that $X \subset Y, Z \subseteq Y \setminus X$, we have

(0M′) $$X \sim Y \rightarrow \neg(\phi \prec Z).$$

*Sketch of proof.* Recall that complete comparative 0-monotone orders are characterised by the axiom

(0M) $$\forall E, H \in \mathscr{A}_\mathscr{E} \text{ such that } E \cap H = \phi, \quad \phi \prec^* E \rightarrow H \prec^* E \cup H.$$

So the algorithm outlined in the proof of Proposition 2.2 must be adapted to 0-monotone orders by modifying the steps (↑) and (↓) as follows:

(↑) **If** $\exists E, H \in \mathscr{E}'$ such that $E \cap H = \phi, \phi \prec' E$ **and** $\neg(H \prec' E \cup H)$, **then**
**Add** $H \prec' E \cup H$ to $\prec'$
**Set** $\mathscr{E}' = \mathscr{E}' \cup \{E \cup H\}$

(↓) **If** $\exists X, Y \in \mathscr{E}'$ such that $X \subset Y, X \sim' Y$ **and** $\exists Z \in \mathscr{A}_\mathscr{E}$ such that $Z \subseteq Y \setminus X$ **and** $\neg(\phi \sim' Z)$, **then**
**Add** $\phi \sim' Z$ to $\preccurlyeq'$
**Set** $\mathscr{E}' = \mathscr{E}' \cup \{Z\}$.

Figure 2 shows how a potential violation of (0M) (obtainable by putting $Z \equiv E$) is avoided by virtue of (0M′). □

† 0-monotonicity is often referred to as *super-additivity*, *cf.* Appendix A.

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Other notions of preference orders (Proposition 2.4) are treated uniformly and the proofs of the corresponding representability results proceed along the same lines as those sketched for Propositions 2.2 and 2.3.

**Proposition 2.4.** Let $\langle \preccurlyeq, \prec \rangle$ be a w.p.s. for $\mathscr{E}$. The following properties hold:

**Comparative plausibility.** $\langle \preccurlyeq, \prec \rangle$ can be extended to a total preference order $\preccurlyeq^*$ over $\mathscr{A}_\mathscr{E}$ *representable by a plausibility function* (equivalently, by an $n$-alternating function, with $n \geqslant 2$) if and only if for all $X, Y, Z, W \in \mathscr{E}$ such that $X \subset Y$, $Z \subset W \subset Y$, $W \setminus Z = Y \setminus X$, we have

(PL′) $$X \prec Y \rightarrow \neg(Z \sim W).$$

Compare (PL′) with the corresponding axiom (PL) characterising complete comparative plausibility orders:

(PL) $\forall A, B, C \in \mathscr{A}_\mathscr{E}$ such that $A \subseteq B, B \cap C = \phi, A \sim^* B \rightarrow A \cup C \sim^* B \cup C$.

**Comparative 0-alternation.** $\langle \preccurlyeq, \prec \rangle$ can be extended to a total preference order $\preccurlyeq^*$ over $\mathscr{A}_\mathscr{E}$ representable by a 0-alternating function† if and only if or all $X, Y \in \mathscr{E}$ such that $X \subset Y$, we have

(0A′) $$X \sim Y \rightarrow \neg(X \cup (\Omega \setminus Y) \prec \Omega).$$

(Recall that complete comparative 0-alternating orders can be characterised by

(0A) $\forall K, F \in \mathscr{A}_\mathscr{E}$ such that $K \cup F = \Omega, \quad F \prec^* \Omega \rightarrow K \cap F \prec^* K.$ )

To the best of our knowledge, there exists no purely qualitative characterisation of comparative probability. This notion seems to have an intrinsically numerical character. Among the possible characterisations proposed in the literature, the following proposition is drawn from Coletti (1990).

**Proposition 2.5 (Comparative probability).** A w.p.s. $\langle \preccurlyeq, \prec \rangle$ for $\mathscr{E}$ can be extended to a total order $\preccurlyeq^*$ over $\mathscr{A}_\mathscr{E}$ that is representable by a probability function if and only if

(CP) for any $X_1, \ldots, X_n, Y_1, \ldots, Y_n \in \mathscr{E}$, with $Y_i \preccurlyeq X_i, \forall i = 1, \ldots, n$, such that for some $r_1, \ldots, r_n > 0$ $\sup\left(\sum_{i=1}^n r_i(a_i - b_i)\right) \leqslant 0$, implies that $X_i \sim Y_i$, for all $i = 1, \ldots, n$ ($a_i, b_i$ are the indicator functions of $X_i, Y_i$, respectively).

Axiom (CP) involves quantitative notions (for example, indicator functions and summations) and its verification requires numerical elaborations. Nevertheless, it is possible to (qualitatively) state a *necessary*, though not sufficient, condition for the representability of an order through a probability function.

**Proposition 2.6 (*Weak* comparative probability).** If $\langle \preccurlyeq, \prec \rangle$ can be extended to a total preference order $\preccurlyeq^*$ over $\mathscr{A}_\mathscr{E}$ that is representable by a probability function, then for all $X, Y, Z \in \mathscr{E}$ such that $X \cap Z = Y \cap Z = \phi$, we have

(WC) $$X \preccurlyeq Y \rightarrow \neg(Y \cup Z \prec X \cup Z).$$

† 0-alternation is often referred to as *sub-additivity*, *cf.* Appendix A.

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We mentioned earlier that the introduction of different classes of orders is motivated by the presence of practical situations where a strictly probabilistic approach is not viable. Here is a simple example.

**Example 2.2.** Let $A, B$ and $C$ be three companies, each of which is a potential buyer of a firm that some other company wants to sell. Although distinct, $A$ and $C$ belong to the same holding. Hence, the following uncertainty order about which company will be the buyer could reflect specific information about the companies' strategies (by abuse of notation, let $A$ denote the event 'the company $A$ buys the firm', and similarly for $B$ and $C$): $\phi \prec A \prec B \prec B \cup C \prec A \cup C \prec \Omega$. Since $A, B$ and $C$ are incompatible, it is immediate to see that the order relation is not representable by a probability because it violates axiom (WC), while it can be managed in line with belief functions behaviours because it agrees with axiom (B′).

## 3. Reasoning tasks for preference orders

In this section we describe two reasoning tasks that exploit preference orders. Such tasks can be seen as the basic constituents of expert systems and decision-support tools that handle qualitative knowledge in the form of comparative assessments.

### *Qualitative framework detection*

This is a classification task: given a (partial) assessment (which means a description of the domain of discernment, constraints and preferences), the goal is to detect which amongst all of the compatible uncertainty frameworks is the most stringent. Notice that by proceeding in this way we actually invert the usual attitude towards the qualitative management of uncertainty, where specific axioms are usually set in advance, and only those relations satisfying them are admitted. Here, on the other hand, given a fixed preference relation, the goal is to ascertain which are the reasonable rules to work with.

Considering an assessment as the outcome of a reasoning process performed by an agent (human or not), detecting the correct uncertainty framework provides useful information about the cognitive schema of the agent. Selecting the most restrictive framework (among those appropriate) clearly corresponds to adopting a sort of 'cautious approach' in interpreting the agent's thought. This guides one in determining the agent's conceptualisation of uncertainty (that is, its way of expressing the lack of information and the variability of phenomena) and its (implicit) model of the problem at hand.

Such a detection process can be, for instance in a multi-agent system, of great help in constructing a more informed representation of (other) agents' models of reality. This translates into better strategies for agent modelling, decision making and plan recognition, that is, the attempt to infer the plans of other agents by communicating with them or by observing their behaviours.

The following is an example of framework detection.

**Example 3.1.** Suppose a physician wants to perform a preliminary assessment of the reliability of a test for SARS (Severe Acute Respiratory Syndrome). The evidence used here for the SARS diagnosis is based on moderate or severe respiratory symptoms and the

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positive or indeterminate result of an adopted clinical test for the presence of the SARS-associated antibody coronavirus SARS-CoV. The elements appearing in the analysis can be summarised as follows:

$$\begin{aligned}
\mathsf{A} &\equiv \textit{Normal respiratory symptoms}\\
\mathsf{B} &\equiv \textit{Moderate respiratory symptoms}\\
\mathsf{C} &\equiv \textit{Severe respiratory symptoms}\\
\mathsf{D} &\equiv \textit{Moderate or severe respiratory symptoms}\\
\mathsf{E} &\equiv \textit{Death from pulmonary diseases}\\
\mathsf{F} &\equiv \textit{Positive or indeterminate clinical test}
\end{aligned}$$

subject to the (logical) restrictions:

$$\begin{aligned}
\mathsf{A}\cap\mathsf{B} &= \phi\\
\mathsf{B}\cap\mathsf{C} &= \phi\\
\mathsf{A}\cap\mathsf{C} &= \phi\\
\mathsf{A}\cup\mathsf{B}\cup\mathsf{C} &= \Omega\\
\mathsf{D} &= \mathsf{B}\cup\mathsf{C}\\
\mathsf{E} &\subset \mathsf{C}\\
\mathsf{F}\cap\mathsf{A} &= \phi\,.
\end{aligned}$$

Consider the w.p.s. $\langle \preccurlyeq, \prec \rangle$ described by

$$\begin{aligned}
\phi &\prec \mathsf{C}\\
\mathsf{C} &\prec \mathsf{B}\\
\mathsf{B} &\preccurlyeq \mathsf{A}\\
\mathsf{C} &\prec \mathsf{D}\\
\mathsf{E} &\prec \mathsf{C}\\
\mathsf{E} &\prec \mathsf{D}\\
\mathsf{F} &\prec \mathsf{A}\\
\mathsf{A}\cup\mathsf{E} &\sim \mathsf{A}\cup\mathsf{C}\,.
\end{aligned}$$

The meaning of the events makes such an order seem reasonable. Considering the axioms described in Section 2, it is easy to verify that such a w.p.s. agrees with the basic axioms, however, it cannot be achieved using either a Probability or Belief function, since it does not satisfy the corresponding axioms. However, one can use comparative plausibility, 0-monotone or 0-alternating functions. (See Example 4.1 below.)

*Qualitative inference*

An interesting task, which is strictly related to the previous one, is to infer new knowledge on the basis of a partial model. This ultimately amounts to finding an extension of a preference relation so as to take into account one or more further events extraneous to the initial assessment. Clearly, this should be achieved in such a way that the extension retains

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the same character as the initial order (for example, both should satisfy the same axioms). More precisely, let an initial (partial) assessment be given, and expressed as a w.p.s. $\langle \preccurlyeq, \prec \rangle$ over a set of known events $\mathscr{E}$. Assume that $\langle \preccurlyeq, \prec \rangle$ satisfies the axioms characterising a specific class, say $\mathcal{C}$, of orders. Consider a new event $\mathsf{S}$ (not in $\mathscr{E}$), which is implicitly described by a collection $\mathscr{C}'$ of set-theoretical constraints involving the events of $\mathscr{E}$. In the spirit of Coletti (1990, Theorem 3), the problem can be formulated as: *Determine the 'minimal' extension* $\langle \preccurlyeq^+, \prec^+ \rangle$ (*over* $\mathscr{E} \cup \{\mathsf{S}\}$) *of* $\langle \preccurlyeq, \prec \rangle$ *that is induced by the new event and still belongs to the class* $\mathcal{C}$. In other words, we are interested in ascertaining how the new event $\mathsf{S}$ must relate to the members of $\mathscr{E}$ in order that $\langle \preccurlyeq^+, \prec^+ \rangle$ is still in $\mathcal{C}$. To this end, we want to determine the sub-collections $\mathscr{L}_\mathsf{S}$, $\mathscr{W}\mathscr{L}_\mathsf{S}$, $\mathscr{U}_\mathsf{S}$ and $\mathscr{W}\mathscr{U}_\mathsf{S}$ of $\mathscr{E}$ defined by

$$\begin{aligned}
E \in \mathscr{L}_\mathsf{S} &\text{ iff no extension } \preccurlyeq^* \text{ of } \preccurlyeq \text{ can infer that } \mathsf{S} \preccurlyeq^* E \\
E \in \mathscr{W}\mathscr{L}_\mathsf{S} &\text{ iff no extension } \preccurlyeq^* \text{ of } \preccurlyeq \text{ can infer that } \mathsf{S} \prec^* E \\
E \in \mathscr{U}_\mathsf{S} &\text{ iff no extension } \preccurlyeq^* \text{ of } \preccurlyeq \text{ can infer that } E \preccurlyeq^* \mathsf{S} \\
E \in \mathscr{W}\mathscr{U}_\mathsf{S} &\text{ iff no extension } \preccurlyeq^* \text{ of } \preccurlyeq \text{ can infer that } E \prec^* \mathsf{S}.
\end{aligned}$$

Consequently, in order to satisfy the axioms characterising $\mathcal{C}$, any weak preference structure $\langle \preccurlyeq^+, \prec^+ \rangle$ extending $\langle \preccurlyeq, \prec \rangle$ must, at least, impose

$$\begin{aligned}
&E \prec^+ \mathsf{S} \quad \forall E \in \mathscr{L}_\mathsf{S} \\
&E \preccurlyeq^+ \mathsf{S} \quad \forall E \in \mathscr{W}\mathscr{L}_\mathsf{S} \\
&\mathsf{S} \prec^+ E \quad \forall E \in \mathscr{U}_\mathsf{S} \\
&\mathsf{S} \preccurlyeq^+ E \quad \forall E \in \mathscr{W}\mathscr{U}_\mathsf{S}.
\end{aligned}$$

**Example 3.2.** Consider the weak preference structure of Example 3.1 and the new event

$$\mathsf{S} \equiv \textit{The real state of having SARS}$$

subject to the restrictions $\mathsf{S} \subset \mathsf{F}$ and $\mathsf{F} \cap \mathsf{E} \subset \mathsf{S}$. Since we discovered in Example 3.1 that the initial preference relation satisfies axiom (PL′), we want to impose such an axiom and compute the extension of the initial order. We will see in Example 4.2, that in order to satisfy (PL′), the following relationships among events have to be verified by any extension involving the new event $\mathsf{S}$: $\mathsf{S} \prec \mathsf{A} \cup \mathsf{C}$, $\mathsf{S} \prec \mathsf{A} \cup \mathsf{E}$, $\mathsf{S} \prec \mathsf{D}$, $\mathsf{S} \prec \mathsf{A}$, $\mathsf{S} \prec \Omega$, $\phi \preccurlyeq \mathsf{S}$, $\mathsf{S} \preccurlyeq \mathsf{F}$. In other words, we have $\phi \in \mathscr{W}\mathscr{L}_\mathsf{S}$, $\{\mathsf{A} \cup \mathsf{C}, \mathsf{A} \cup \mathsf{E}, \mathsf{D}, \mathsf{A}, \Omega\} \subseteq \mathscr{U}_\mathsf{S}$, and $\mathsf{F} \in \mathscr{W}\mathscr{U}_\mathsf{S}$. This shows that, apart from obvious relations induced by monotonicity, no significant constraint involving $\mathsf{S}$ can be inferred. Since $\mathsf{S}$ and $\mathsf{E}$ can be freely compared, this result suggests that either further investigation about the relevance of the clinical test or a revision of the initial preference relation should be performed.

The availability of automated tools for extending preference orders whenever new knowledge is acquired directly suggests applications in expert systems and decision-support tools. In automated diagnosis, planning or problem solving, to mention a few examples, one could easily imagine scenarios where knowledge is not available in its entirety from the beginning. We could outline how a rudimentary inference process could develop by identifying the basic steps an automated agent should perform:

(0) the acquisition of an initial collection of observations about the object of the analysis, together with a (qualitative) partial preference assessment;

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(1) the detection of the most adequate (that is, the most discriminant) uncertainty framework;
(2) whenever new knowledge becomes available, the refinement of the agent's description of the real world by performing order extension.

The results of step (2) could then be exploited to guide further investigations on the real world, in order to obtain new information. Then, the process will be repeated until further pieces of knowledge are obtainable or an accurate enough degree of belief is achieved.

## 4. *PreC*: working with preference orders

Apart from the qualitative probabilities, all the axioms in Section 2 have a direct declarative reading, as they involve only logical and preference relations. Hence we can almost immediately provide an executable declarative specification for them. In fact, their declarative character supports a straightforward translation within the logical framework of Answer Set Programming (ASP, for short – see Appendix B for a brief summary of some basic ASP notions). Thus, we immediately obtain an executable ASP-specification suitable for performing the tasks described in the previous section. More specifically, this is done by exploiting an ASP-solver (in our case SModels (Syrjänen 1999)).

The ASP-specification is obtained as follows. We start by defining in ASP the predicates prec(·,·), precneq(·,·) and equiv(·,·), to render the relators $\preccurlyeq$, $\prec$ and $\sim$, respectively. The characterisation of potential legal answer sets (*cf.*, Appendix B) is done by asserting properties of prec(·,·), precneq(·,·) and equiv(·,·). (Auxiliary predicates/functions set-theoretical constructs, such as event(·), subset(·,·), diffset(·,·), and so on, have an immediate reading.) For instance, (A3′) is rendered by weeding out all answer sets where $A \subseteq B \wedge B \prec A$ holds for some events $A$ and $B$:

```
:- event(A;B), subset(A,B), precneq(B,A).
```

For preference classification, we consider one of the axioms of Section 2, say (B′). The following rule also has an immediate reading:

```
failsB :- event(X;Y;Z;W), subset(X,Y), X!=Y, subset(Z,W), Z!=W, subset(W,Y),
          W!=Y, subset(diffset(W,Z),diffset(Y,X)), equiv(X,Y), precneq(Z,W).
```

Namely, the fact failsB is true (that is, belongs to the answer set) whenever there exist events falsifying (B′). All other axioms can be treated similarly.

When an ASP-solver is fed with such a program and a description of a preference relation (that is, a set of facts having the forms prec(·,·), precneq(·,·), equiv(·,·)), different outcomes may be obtained:

(a) If no answer set is produced, the input w.p.s. violates some basic requirement, such as axioms (A1′)–(A3′).
(b) Otherwise, if an answer set is generated, there exists a numerical (partial) model representing the input w.p.s. The presence in the answer set of a fact of the form failsℭ (say failsB), witnesses that the corresponding axiom ((B′) in the case) is violated. Consequently, the given order (as well as any of its extensions) is not compatible with the uncertainty framework ruled by ℭ.

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**Example 4.1.** Consider the situation described in Example 3.1. The w.p.s. $\langle \preccurlyeq, \prec \rangle$ can be described in the syntax of ASP as follows:

```
precneq(N,C) :- empty(N).
precneq(C,B).
precneq(C,D).
precneq(E,C).
precneq(E,D).
precneq(F,A).
equiv(unionset(A,E),unionset(A,C)).
prec(B,A).
```

As expected, if such an ASP specification is given as input to smodels, the answer set found includes the facts failsB1 and failsWC, testifying that the given assessment does not satisfy either of the axioms (B′) and (WC).

A similar treatment can be used to implement the order extension task. In this case, the input knowledge consists of a set of events together with a collection of logical constraints and preferences, a description of the new event, and one or more axioms to be imposed. The handling of the imposed axioms is done by ASP-rules of the form

```
:- holdsB, event(X;Y;Z;W), subset(X,Y), X!=Y, subset(Z,W), Z!=W, subset(W,Y),
   W!=Y, subset(diffset(W,Z),diffset(Y,X)), equiv(X,Y), precneq(Z,W) .
```

Compare this rule with the similar one introduced above for implementing the classification task. Rules of this kind, that is, with an empty head, are called *constraints* (in the sense described in Appendix B, see also Baral (2003)). A constraint is satisfied when at least one of the atoms in its body is not satisfied. We use such rules to declare that any extension for which an axiom is violated is 'undesirable'. Intuitively speaking, whenever the fact holdsB is true, in order to satisfy the above rule, at least one of the other facts must be false. (Notice that these facts are all true exactly when (B′) is violated.) In order to activate this constraint (that is, to impose axiom (B′)), it suffices to add the fact holdsB to the input of the solver. Since, in general, more than one extension is possible, the collections $\mathscr{L}_S$, $\mathscr{W}\mathscr{L}_S$, $\mathscr{U}_S$ and $\mathscr{W}\mathscr{U}_S$ can be obtained by computing the intersection $Cn$ of all the answer sets. (Or, equivalently, by computing the set of logical consequences of the ASP-program. Notice that, in general, $Cn$ does not need to be an answer set by itself. Some ASP-solvers offer the computation of $Cn$ as a built-in feature. This is not the case for smodels, so a post-processing of smodels' output is needed.) This allows one to detect the minimal extension of the preference relation that is mandatory for each total order.

**Example 4.2.** Consider the Examples 4.1 and 3.2. As expected, the following facts belong to each answer set and are obtained by filtering smodels' output: precneq(S,AC), precneq(S,AE), precneq(S,D), precneq(S,A), precneq(S,Omega), prec(Empty,S), prec(S,F), where AC, AE, Omega and Empty are instantiated to the events A∪C, A∪E, Ω and $\phi$, respectively.

The executable specifications we have briefly outlined in this section (together with the ASP-solver and a C-library of functions designed for efficiently handling sets and

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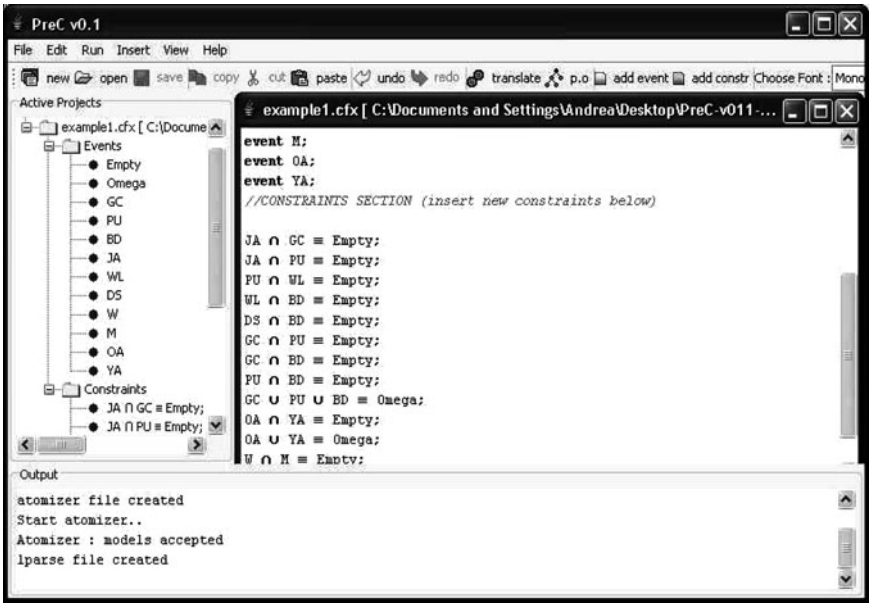



Fig. 3. Main window of the *Pre*ference *C* runcher.

operations on sets) constitute the core inference-engine of the prototype tool *PreC* (*Pre*ference *C* runcher). This tool is designed to assist the user in dealing interactively with (partial) preference orders and qualitative uncertainty. *PreC* offers a user-friendly and mouse-oriented interface for inputting, modifying and managing assessments; for activating the reasoning tasks; and for handling order extensions as described in the previous sections. In this way, the user does not interact directly with the ASP-solver and does not handle any ASP-specification. Actually, both the solver and the ASP code could be changed or improved in a transparent manner. Moreover, new uncertainty notions and new tasks can be included to extend the tool. Figure 3 depicts the main window of *PreC*, through which the user can describe, manage and modify his/her assessments. The visualisation of preference orders in the form of graphs (*cf.*, Figure 4) allows the user to modify or extend the set of events, for instance, in preparing for the execution of one of the inference tasks of Section 3.

## Conclusions

In this paper we have proposed a uniform characterisation of partial preference orders that provides a strong connection between the axiomatisations of partial preference orders and the corresponding axiomatisations for complete orders. Moreover, the uniform axiomatic treatment presented in this paper translates straightforwardly into an executable specification in a declarative programming framework. This has allowed us to explore the potential offered by Answer Set Programming for building decision support systems based on qualitative judgments. Hence, an implementation of what could be thought of as a kernel of an inference engine arose almost naturally. Moreover, the highly declarative

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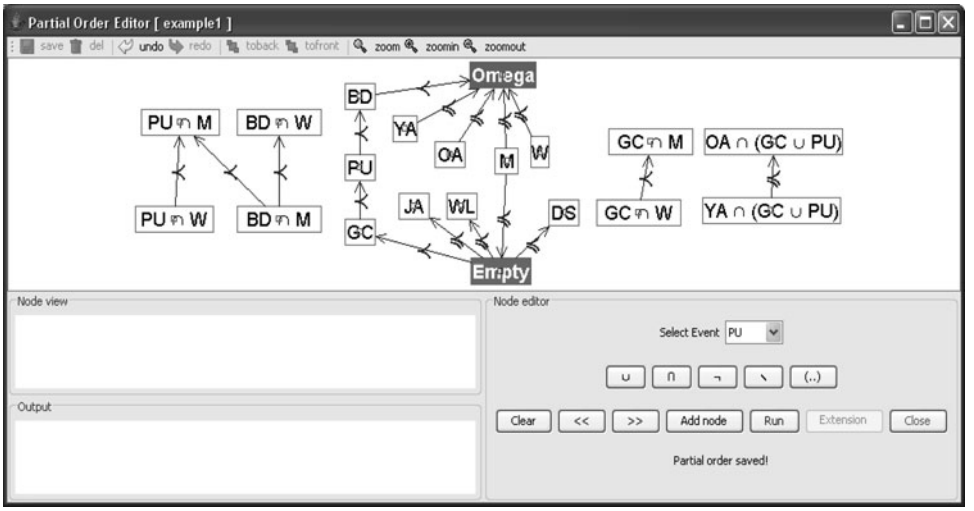



Fig. 4. Visual management of preference orders (*cf.*, Example 2.1).

character of the encoding of the axioms into executable ASP-specifications means that it is easy to modify and extend the treatment to deal with new notions of uncertainty. Similarly, the discovery of alternative axiomatisations of an uncertainty notion can be immediately 'plugged into' the framework simply by modifying or adding suitable ASP rules (consider, for instance, the case of comparative Probabilities, for which a qualitative characterisation has yet to be found). Alternatively, a challenging goal for future research is to complete our approach to handle comparative Probabilities through an integration with efficient numerical approaches such as linear optimisation tools (for example, column generation techniques, *cf.* Jaumard *et al.* (1991), among others). More generally, we envisage the design of a full-blown automated system that integrates different (in some way complementary) techniques and methods for uncertainty management: including mixed numerical/qualitative assessments and extending the range of applicability to conditional probability frameworks (Coletti and Vantaggi 2006).

## Appendix A. A gentle introduction to uncertainty measures

In this appendix we give a brief description of the various generalisations of Probability measures referred to in this article. The following material is far from an exhaustive and complete treatment – we will just give an informal introduction to the subject. The interested reader can refer to the widely available literature: an introductory treatment of the relationships between Probability measures, Belief functions and Possibility measures can be found in Klir and Folger (1988), Nguyen and Walker (1997) and Walley (1996), to mention just a few among many.

We consider, as our domain of interest, a set $\Omega$ of possibilities ($\Omega$ is often referred to as *sample space*). For our purposes it is sufficient to consider the case of a finite domain. An *event* is then defined as a subset of $\Omega$. In order to introduce uncertainty measures, we

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consider any algebra $\mathscr{A}$ (on $\Omega$) consisting of a set of subsets of $\Omega$ such that $\Omega \in \mathscr{A}$, and that is closed under union and complementation.

All of the measures we are going to introduce are (normalised) monotone real-valued functions over an algebra. Such functions are usually called (*Choquet*) *Capacities* (Choquet 1954), though they are also referred to as *fuzzy measures* or *Sugeno measures.*

**Definition A.1.** A real-valued function $F$ on $2^{\Omega}$ is a *Capacity* if

$$F(\phi) = 0, \qquad F(\Omega) = 1, \qquad \text{and for all } A, B \subseteq \Omega \;\; A \subseteq B \rightarrow F(A) \leqslant F(B).$$

Let $CAP(\Omega)$ denote the class of Capacities over $\Omega$. The notion of Capacity is often considered to be too general to be of interest by itself. In fact, by adopting it, apart from monotonicity, there is no other relationship imposed between the uncertainty of a composed event, for example, $F(A \cup B)$, and the uncertainty of its components $F(A)$ and $F(B)$. In order to reflect different rationales in managing the information, further constraints are imposed on the manner in which uncertainties of composed events are determined. In the following we describe some of the more interesting measures obtained by refining Capacities.

The most widely adopted measure of uncertainty is characterised by the additivity property used for combining events. A *Probability* $P$ over $\Omega$ is a Capacity that satisfies the following *additivity* requirement: for all $A, B \subseteq \Omega$ such that $A \cap B = \phi$ we have $P(A \cup B) = P(A) + P(B)$. We use $PROB(\Omega)$ to denote the class of all Probabilities over $\Omega$, and, clearly, $PROB(\Omega) \subseteq CAP(\Omega)$.

**Definition A.2.** Let $F_1$ and $F_2$ be two functions on $2^{\Omega}$. Then $F_1$ is the *dual* of $F_2$ if for each $A \subseteq \Omega$, we have $F_1(A) = 1 - F_2(\Omega \setminus A)$.

Note that the dual of a Capacity is a Capacity too. Moreover, the dual of a Probability is the Probability itself.

Additivity, though widely adopted in 'measurement' processes, is usually thought to be too strong a requirement. Hence, several generalisations have been proposed. In particular, the following definition characterises those Capacities satisfying only one of the weak inequalities that, taken together, give additivity.

**Definition A.3.** Let $g : \mathscr{A}_{\mathscr{E}} \rightarrow [0, 1]$ be an uncertainty measure. Then,

— $f$ is said to be a 0-*monotone* (or *super-additive*) function if and only if $\forall E_1, E_2 \in \mathscr{E}$ with $E_1 \cap E_2 = \phi$, we have $f(E_1 \cup E_2) \geqslant f(E_1) + f(E_2)$.

— $f$ is said to be a 0-*alternating* (or *sub-additive*) function if and only if $\forall E_1, E_2 \in \mathscr{E}$ with $E_1 \cap E_2 = \phi$, we have $f(E_1 \cup E_2) \leqslant f(E_1) + f(E_2)$.

Notice that the dual of a 0-monotone function is a 0-alternating function. The class of 0-monotone (respectively, 0-alternating) functions is denoted by $0MON(\Omega)$ (respectively, $0ALT(\Omega)$).

We will now introduce a pair of dual uncertainty measures that induce the slightly different notions of sub-additivity and super-additivity, respectively.

**Definition A.4.** Let $\Pi$ and $N$ be Capacities over $\Omega$.

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— $\Pi$ is a *Possibility* measure (over $\Omega$) if it satisfies the following property: for all $A, B \subseteq \Omega$ $\Pi(A \cup B) = \max\{\Pi(A), \Pi(B)\}$.
— $N$ is a *Necessity* measure (over $\Omega$) if it is the dual of a Possibility measure.

It is obvious that any Possibility measure $\Pi$ satisfies the *sub-additivity* property, that is, for all $A, B \subseteq \Omega$ $\Pi(A \cup B) \leqslant \Pi(A) + \Pi(B)$. Similarly, any Necessity measure $N$ satisfies the *super-additivity* property, that is, for all $A, B \subseteq \Omega$ $N(A \cup B) \geqslant N(A) + N(B)$.

A Possibility measure $\Pi$ is usually induced by a *possibility distribution* (that is, a fuzzy set) $\pi : \Omega \to [0, 1]$ (Dubois and Prade 1980; Zadeh 1965). The value $\pi(x)$ expresses the possibility that a singleton $x \in \Omega$ is representative of the concept being considered. Possibility is then defined by putting $\Pi(A) = \max\{\pi(x) \mid x \in A\}$ for any $A \subseteq \Omega$. The classes of Possibilities and Necessities over $\Omega$ are denoted by $POS(\Omega)$ and $NEC(\Omega)$, respectively.

Our final definition concerns Belief and Plausibility measures. By adopting the most general formulation, following (Shafer 1976), we have the following definition.

**Definition A.5.** A function $Bel : 2^{\Omega} \to [0, 1]$ is a *Belief* measure if it is a Capacity and it satisfies the following condition (known as $\infty$-*monotonicity*):

$$\text{For each } n \geqslant 1,\ Bel\Big(\bigcup_{i=1}^{n} A_i\Big) \geqslant \sum_{\phi \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} Bel\Big(\bigcap_{i \in I} A_i\Big)$$

(where $A_i \subseteq \Omega$ for each $i$).

Intuitively speaking, a Belief function $Bel$ is usually constructed through a basic assignment of uncertainty, not necessarily being a Capacity, $\mu : 2^{\Omega} \to [0, 1]$ so that, for any proposition $A \subseteq \Omega$, we have $Bel(A) = \sum_{B \subseteq A} \mu(B)$. Belief functions are also called *Capacities monotone of infinite order*. Capacities that satisfy the above condition with the restriction $n \leqslant N$ are then said to be *monotone of order N* (or $N$-monotone). Dually, if the opposite inequality ($\leqslant$) is considered, the measure is said to be an $N$-*alternating* Capacity. For $N = 2$ these properties reduce to the usual super- and sub-additivity, respectively. The dual of a Belief measure is called a *Plausibility* measure. The class of Belief (respectively, *Plausibility*) measures is denoted by $BEL(\Omega)$ (respectively, $PL(\Omega)$).

The classes of Capacities introduced above are related as follows:

$$CAP(\Omega) \supset 0MON(\Omega) \supset BEL(\Omega) \supset NEC(\Omega)$$
$$CAP(\Omega) \supset 0ALT(\Omega) \supset PL(\Omega) \supset POS(\Omega)$$
$$BEL(\Omega) \cap PL(\Omega) \supset PROB(\Omega).$$

## Appendix B. Answer set programming

In this appendix we briefly recall the basics of a successful alternative style of logic programming (Lifschitz 1999; Marek and Truszczyński 1999) known as Answer Set Programming (ASP, for short). In this logical framework, a problem can be encoded – by using a function-free logic language – as a set of properties and constraints that describe the (candidate) solutions. More specifically, an *ASP-program* is a collection of *rules* of the form

$$L_1; \ldots; L_k; not\ L_{k+1}; \ldots; not\ L_{\ell} \leftarrow L_{\ell+1}, \ldots, L_m, not\ L_{m+1}, \ldots, not\ L_n$$

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where $n \geqslant m \geqslant \ell \geqslant k \geqslant 0$ and each $L_i$ is a literal, that is, an atom $A$ or a negation of an atom $\neg A$. The symbol $\neg$ denotes classical negation, while *not* stands for negation-as-failure (Note that ',' and ';' stand for logical conjunction and disjunction, respectively.) The left- and right-hand sides of the clause are called the *head* and *body*, respectively. A rule with an empty head is a *constraint*. (The literals in the body of a constraint cannot be all true, otherwise they would imply falsity.)

The semantics of ASP is expressed in terms of *answer sets* (or equivalently *stable models* (Gelfond and Lifschitz 1988)). Consider first the case of an ASP-program $P$ that does not involve negation-as-failure (that is, $\ell = k$ and $n = m$). In this case, a set $X$ of literals is said to be closed under $P$ if for each rule in $P$, whenever $\{L_{\ell+1}, \ldots, L_m\} \subseteq X$, we have $\{L_1, \ldots, L_k\} \cap X \neq \varnothing$. If $X$ is inclusion-minimal among the sets closed under $P$, it is said to be an answer set for $P$. Such a definition is extended to any program $P$ containing negation-as-failure by considering the *reduct* $P^X$ (of $P$). $P^X$ is defined as the set of rules of the form

$$L_1; \ldots; L_k \leftarrow L_{\ell+1}, \ldots, L_m$$

for all rules of $P$ such that $X$ contains all the literals $L_{k+1}, \ldots, L_\ell$, but does not contain any of the literals $L_{m+1}, \ldots, L_n$. Clearly, $P^X$ does not involve negation-as-failure. The set $X$ is an answer set for $P$ if it is an answer set for $P^X$.

Once a problem is described as an ASP-program $P$, its solutions (if any) are represented by the answer sets of $P$. Notice that an ASP-program may have none, one or several answer sets.

Consider the program $P$ consisting of the two rules $p; q \leftarrow$ and $\neg r \leftarrow p$. Such a program has two answer sets: $\{p, \neg r\}$ and $\{q\}$. If we add the rule (actually, a constraint) $\leftarrow q$ to $P$, we rule-out the second of these answer sets, because it violates the new constraint. This simple example reveals the core of the usual approach followed in formalising/solving a problem with ASP. Intuitively speaking, the programmer adopts a 'generate-and-test' strategy. First (s)he provides a set of rules describing the collection of (all) potential solutions. Then, the addition of a group of constraints rules-out all those answer sets that are not desired real solutions. The expressive power of ASP, as well as, its computational complexity have been investigated deeply. The interested reader should refer to Dantsin *et al.* (2001), among others.

To find the solutions of an ASP-program, an ASP-solver is used. Several solvers have became available[†], each of which has its own particular strengths.

We will now give a simple example of an ASP-program (see Baral (2003) and Dovier *et al.* (2005), among others, for a presentation of ASP as a tool for declarative problem-solving). In doing this, we will recall the syntax and the main features of lparse/smodels (which is one of the currently available ASP-solvers, and the one we used for this paper – see Syrjänen (1999) for more details). The problem we want to formalise in ASP is the well-known *n-queens* problem: 'given an $n \times n$ chess board, place $n$ queens in such a way that no two of them attack each other'. The following two clauses state that a candidate

[†] Some ASP-solvers are (for details, see the web sites quoted): DLV (`www.dbai.tuwien.ac.at/proj/dlv`); CModels (`www.cs.utexas.edu/users/tag/cmodels`); SModels (`www.tcs.hut.fi/Software/smodels`).

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solution is any disposition of the queens, provided that each column of the board contains one and only one queen:

```
pos(1..n).          1{queen(Col,Row) : pos(Col)}1 :- pos(Row).
```

where the fact that a queen is placed on the $n^{th}$ column and on the $m^{th}$ row is encoded by the atom queen($n$,$m$) (in the syntax of lparse ':-' denotes implication ←, and the value of the constant n occurring in the first clause is a parameter of the program supplied to lparse at run-time).

The second rule is a particular form of constraint that is available in the lparse language. The general form of such a clause is

$$k\{\langle property_def\rangle : \langle range_def\rangle\}m \text{ :- } \langle search_space\rangle$$

where:

— the conditions $\langle search_space\rangle$ in the body define the set of objects of the domain to be checked;
— the atom $\langle property_def\rangle$ in the head defines the property to be checked;
— the conjunction $\langle range_def\rangle$ defines the possible values that the property may take on the objects defined in the body, namely by providing a conjunction of unary predicates each of them defining a range for one of the variables that occur in $\langle property_def\rangle$ but not in $\langle search_space\rangle$;
— $k$ and $m$ are the minimum and maximum number of distinct values that the specified property may take on the specified objects.

(Notice that this form of constraint, which is available in smodels, is actually just syntactic sugar since it can be translated into 'proper' ASP-clauses through negation (Simons 2000; Syrjänen 1999).)

We now introduce two constraints in order to rule out those placements where two queens control either the same row or the same diagonal of the board (here pos(C;R1;R2) is a shorthand notation for the three facts pos(C), pos(R1), pos(R2)):

```
:- queen(C,R1), queen(C,R2), pos(C;R1;R2), R1<R2.
:- queen(C1,R1), queen(C2,R2), pos(C1;C2;R1;R2), R1<R2, abs(C1-C2)==abs(R1-R2).
```

Here are two of the answer sets produced by smodels (in this case we put n= 8):

```
Answer 1.  queen(4,1) queen(6,2) queen(1,3) queen(5,4) queen(2,5) queen(8,6)
           queen(3,7) queen(7,8) ...
Answer 2.  queen(4,1) queen(2,2) queen(8,3) queen(5,4) queen(7,5) queen(1,6)
           queen(3,7) queen(6,8) ...
```

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