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An extremal function for contractions of graphs

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Abstract. The function $c(p)$ is defined for positive integers $p \geqslant 4$ by

$$c(p) = \inf\{c;\ e(G) \geqslant c|G| \quad \text{implies} \quad G > K_p\},$$

where $>$ denotes contraction. Random graph examples show

$$c(p) > 0{\cdot}265p\sqrt{(\log_2 p)}\,(1+o(1)).$$

In 1968 Mader showed that $c(p) \leqslant 8(p-2)\lfloor \log_2 (p-2)\rfloor$ and more recently Kostochka showed that $p\sqrt{(\log p)}$ is the correct order for $c(p)$. We present a simple argument showing $c(p) \leqslant 2{\cdot}68p\sqrt{(\log_2 p)}\,(1+o(1))$.

If G is a graph and uv is an edge of G, the graph obtained by *contracting* the edge uv, denoted by G/uv, is the graph obtained from $G-\{u,v\}$ by adding a vertex z and edges zw for each $w \in \Gamma(u) \cup \Gamma(v) - \{u,v\}$. (We use the notation of [1].) A graph H obtained from G by a sequence of edge deletions and contractions is called a *subcontraction* or *minor* of G, written $H < G$. Interest in contractions of graphs arises in particular from Hadwiger's conjecture, that $\chi(G) = p$ implies $G > K_p$, and from Kuratowski's characterisation of planar graphs in terms of forbidden subcontractions.

When studying contractions it is clearly of some value to know the extremal function $\mathrm{ex}(n;\ > K_p) = \max\{e(G);\ |G| = n \text{ and } G \not> K_p\}$. (Throughout the paper p will denote an integer $\geqslant 4$.) In fact it is more natural to consider the closely related function

$$c(p) = \inf\{c;\ e(G) \geqslant c|G| \quad \text{implies} \quad G > K_p\}.$$

It is easily shown that $c(p)$ exists and is less than 2^p. Mader [4] showed that $c(p) = p-2$ for $p \leqslant 7$, and that $c(p) \leqslant 8(p-2)\lfloor \log_2 (p-2)\rfloor$ for all p. Since a random graph of order n with edges chosen with probability $1-q$ almost certainly does not contract to a K_p where $p = n\sqrt{(\log(1/q)/\log n)}\,(1+o(1))$ (this is the easier half of Bollobás, Catlin and Erdös [2]), we see, on taking $q = 0{\cdot}284$, that $c(p) \geqslant 0{\cdot}265p\sqrt{\log_2 p}$ for large p. Kostochka [3] has recently shown that $c(p) \leqslant 324p\sqrt{\log_2 p}$; I am indebted to P. Duchet for bringing this result to my attention. Our aim here is to present a straightforward proof that $c(p) \leqslant 2{\cdot}68p\sqrt{\log_2 p}$ for large p.

The proof is in two parts. First we show (as do Mader and Kostochka) that to get an upper bound on $c(p)$ we need only consider graphs with large minimal degree. In the remainder we use simple counting arguments to show that we may pick out in such a graph many connected subgraphs which dominate most of the graph; in

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particular between any two such subgraphs there is an edge, so contracting each subgraph to a single vertex yields a large complete graph.

A couple of remarks on terminology: first, if $X \subset V(G)$ and $Y \subset V(G)$ we say X dominates Y if $Y \subset X \cup \Gamma(X)$; secondly, all logarithms are natural unless otherwise specified.

The first lemma looks somewhat curious but is in fact the natural way to achieve the best constant in the main theorem. Define the real number α by $\alpha = 1 + \log 2\alpha$; then $\alpha = 2{\cdot}678\ldots$.

LEMMA 1. *Let* $\mathscr{C}_r = \{G;\ |G| \leqslant \alpha r$ *and* $2\delta(G) - |G| \geqslant \lfloor r \rfloor - 3\}$, *where* r *is real and* $\alpha r \geqslant 3$ *is an integer. If* $G \in \mathscr{C}_r$ *implies* $G > K_{p-2}$, *then* $c(p) \leqslant \alpha r$.

Proof. Choose a graph H_1 which is minimal with respect to subcontraction in the set $\{G;\ e(G) \geqslant \alpha r|G|\}$. We will show $H_1 > K_p$. Note that $e(H_1) = \alpha r|H_1|$, so $\delta(H_1) \leqslant 2\alpha r$. Choose $z \in H_1$ with $d_{H_1}(z) = \delta(H_1)$ and let $H_2 = H_1[\Gamma(z)]$. If $w \in H_2$ then $d_{H_2}(w) \geqslant \alpha r$, for otherwise $e(H_1/zw) \geqslant e(H_1) - \alpha r = \alpha r|H_1/zw|$, contradicting the minimality of H_1.

It is now sufficient to show $H_2 > K_{p-1}$, where $|H_2| \leqslant 2\alpha r$ and $\delta(H_2) \geqslant \alpha r$. Let the real valued function f be defined on the set of graphs by $f(G) = r|G|\,(\log(|G|/r) + 1)/2$, (a curious choice which will be explained later), and let $\mathscr{D} = \{G;\ |G| \geqslant r$ and $e(G) \geqslant f(G)\}$. Note that $H_2 \in \mathscr{D}$ by the definition of α. Let H be a subcontraction of H_2 which is minimal with respect to subcontraction in $\mathscr{D}$. Since even the complete graph on $\lceil r \rceil$ vertices is not in $\mathscr{D}$, it follows that $G \in \mathscr{D}$ implies $|G| \geqslant r + 1$. So the minimality of H implies that for any vertex $u \in H$ and any edge $uv \in E(H)$, both $e(H - u) < f(H - u)$ and $e(H/uv) < f(H/uv)$ hold. Choosing u so that $d_H(u) = \delta(H)$ and putting $G = H[\Gamma(u)]$ we then have, as before, both $|G| \leqslant \lceil r(\log(|H|/r) + 1) \rceil$, since $e(H) = \lceil f(H) \rceil$, and $\delta(G) \geqslant \lceil f(H) \rceil - \lfloor f(H/uv) \rfloor - 1$. Hence

$$\begin{aligned} 2\delta(G) - |G| &\geqslant r|H|\,(\log(|H|/r) + 1) - r(|H| - 1)\,(\log((|H| - 1)/r) + 1) \\ &\quad - r(\log(|H|/r) + 1) - 3 \\ &= r(|H| - 1)\log(|H|/(|H| - 1)) - 3 \\ &> r - 4, \end{aligned}$$

since $|H| \geqslant r + 1$ and $\lambda \log(1 + 1/\lambda) > 1 - 1/\lambda$ for all real $\lambda > 1$. Therefore $G \in \mathscr{C}_r$, implying $G > K_{p-2}$, $H_2 > H > K_{p-1}$, and $H_1 > K_p$, as desired.

(The choice of the function f was made as follows. If we define the function $F(x)$ by $F(|G|) = f(G)$, we want $F(x)$ to be chosen so that for the graph G in the proof of Lemma 1, $2\delta(G) - |G|$ is as large as possible. (This will be needed in the Theorem later.) Now $\delta(G)$ is essentially $F'(|H|)$ and $|G|$ is $2F(|H|)/|H|$. Thus F satisfies the differential equation

$$2F'(x) - 2F(x)/x = \text{constant} = r,$$

giving $2F(x) = rx \log kx$ for some k. The condition $K^r \notin \mathscr{D}$ then determines k, and the condition $H_2 \in \mathscr{D}$ if $2\delta(H_2) \geqslant |H_2|$ determines α.)

LEMMA 2. *Let* G *be a graph of order* n *with* $\delta(G) > n/2$ *and let* Y *be a subset of* $V(G)$. *Then there is a subset* Z *of* $V(G)$ *with* $|Z| \leqslant |Y| - 1$ *such that* $Y \cup Z$ *is connected.*

Proof. Let $Y = \{y_1, \ldots, y_l\}$. For each i, $1 \leqslant i \leqslant l - 1$, the vertices y_i and $y_i + 1$ have a common neighbour z_i, since $\delta(G) > n/2$. Let $Z = \{z_i;\ 1 \leqslant i \leqslant l - 1\}$.

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LEMMA 3. *Let G be a graph of order n with $\delta(G) \geqslant (n+m)/2$, for some non-negative integer $m \leqslant n-3$. Let X be a subset of $V(G)$. Then there are $m+3$ vertices of G each of which dominates at least $|X|/2$ vertices of X.*

Proof. Let $|X| = k$. Let S be the set of ordered pairs $(x, v) \in X \times V(G)$ such that $xv \notin E(G)$. Then each element of X is the first component of at most $n-\delta-1$ ordered pairs in S. If the conclusion of the lemma is false then $k > 0$ and G has at least $n-m-2$ vertices each of which is the second component of at least $\frac{1}{2}(k+1)$ ordered pairs in S. Therefore $k(n-\delta-1) \geqslant |S| \geqslant \frac{1}{2}(k+1)(n-m-2)$ and consequently

$$k\delta \leqslant \tfrac{1}{2}k(n+m) - \tfrac{1}{2}(n-m-2) < \tfrac{1}{2}k(n+m),$$

contradicting the hypothesis that $\delta \geqslant \frac{1}{2}(n+m)$.

LEMMA 4. *Let G be a graph of order n with $\delta(G) \geqslant (n+\lfloor \log_2 k \rfloor)/2$ for some integer k, and let W be a subset of $V(G)$ of order k. Then there is a subset U of $V(G)$ of order at most $2\lfloor \log_2 k \rfloor + 1$ such that $W \cup U$ is connected.*

Proof. It is sufficient to show the existence of a set Y of order $\lfloor \log_2 k \rfloor + 1$ which dominates W, for by Lemma 2 there is a set Z of order $\lfloor \log_2 k \rfloor$ such that $U = Y \cup Z$ is connected; then $W \cup U$ is connected since Y dominates W.

That Y exists is easy to show by induction on k. If $k = 1$ we may take $Y = W$. For larger k, we see by Lemma 3 that there is a vertex $y \in V$ which dominates at least $k/2$ members of W. Put $G' = G - y$ and let W' be the subset of W not dominated by y. Then $|W'| \leqslant \lfloor k/2 \rfloor$ and $\delta(G') \geqslant (n-1+\lfloor \log_2 |W'| \rfloor)/2$. By the induction hypothesis there is a subset Y' dominating W' with $|Y'| \leqslant \lfloor \log_2 (k/2) \rfloor + 1 = \lfloor \log_2 k \rfloor$. Then $Y = Y' \cup \{y\}$ dominates W.

LEMMA 5. *Let G be a graph of order n, with $\delta(G) \geqslant (n+m)/2$ for some non-negative integer $m \leqslant n-3$. Then there are at least $\binom{m+3}{k}$ k-subsets of G each of which dominates all but some $\lfloor 2^{-k}n \rfloor$ vertices of G.*

Proof. The proof is by induction on k; for $k = 0$ there is nothing to prove. We may suppose that there are at least $\binom{m+3}{k-1}$ $(k-1)$-subsets dominating all but $\lfloor 2^{-k+1}n \rfloor$ vertices of G. Let Y be one such $(k-1)$-subset and let X be the set of vertices not dominated by Y. Then $|X| \leqslant 2^{-k+1}n$, and by Lemma 3 there are at least $m-k+4$ vertices of G which are not in Y and which dominate at least $|X|/2$ vertices in X. Any of these $m-k+4$ vertices together with Y forms a k-subset dominating all but $\lfloor 2^{-k}n \rfloor$ vertices of G. Each k-subset can be formed in exactly k ways by the addition of a vertex to a $(k-1)$-subset, and so the number of k-subsets dominating all but $\lfloor 2^{-k}n \rfloor$ vertices of G is at least

$$\binom{m+3}{k-1}\frac{m-k+4}{k} = \binom{m+3}{k}.$$

We are now ready to prove the main theorem. Recall that $\alpha = 2{\cdot}678\ldots$ is defined by $\alpha = 1 + \log 2\alpha$.

THEOREM. *Let $r = r(p)$ be a function of p such that αr is an integer, and such that there is an integer k satisfying both $r - (p-3)l \geqslant k \geqslant 5$ and*

$$(p-3)\binom{\lfloor 2^{-k}\alpha r \rfloor}{k} < \binom{\lfloor r \rfloor - (p-3)l}{k},$$

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where $l = k + 2\lfloor \log_2 k \rfloor + 1$. *Then* $c(p) \leqslant \alpha r(p)$. *In particular,*

$$c(p) \leqslant \alpha p \sqrt{(\log_2 p)}\,(1 + o(1)) \leqslant 2{\cdot}68 p \sqrt{\log_2 p}$$

for large p.

Proof. Let r be a function with the described properties. To prove the first assertion it is enough by Lemma 1 to show that if $G \in \mathscr{C}_r$ then $G > K_{p-2}$.

So let $G \in \mathscr{C}_r$. We shall find successively subgraphs $X_1, \ldots, X_{p-2}$, each of order at most l, such that each X_j is connected and dominates all but at most $\lfloor 2^{-k}\alpha r \rfloor$ vertices of $G - (X_1 \cup \ldots \cup X_{j-1})$, and such that there is an edge between each pair X_i and X_j if $i \neq j$. Suppose we have already found $X_1, \ldots, X_j$, for some j, $0 \leqslant j \leqslant p-3$; let $G' = G - (X_1 \cup \ldots \cup X_j)$ and let $x = |X_1| + \ldots + |X_j|$. Then

$$\delta(G') \geqslant (|G| + \lfloor r \rfloor - 3)/2 - x = (|G'| + \lfloor r \rfloor - x - 3)/2,$$

so by Lemma 5 there are $\binom{\lfloor r \rfloor - x}{k}$ k-subsets of G' which dominate all but at most $\lfloor 2^{-k}|G'| \rfloor \leqslant \lfloor 2^{-k}\alpha r \rfloor$ vertices of G'.

For each i, $1 \leqslant i \leqslant j$, at most $\binom{\lfloor 2^{-k}\alpha r \rfloor}{k}$ of these k-subsets fail to send any edges to X_i. Thus, since

$$j\binom{\lfloor 2^{-k}\alpha r \rfloor}{k} < \binom{\lfloor r \rfloor - x}{k}$$

(because $j \leqslant p-3$ and $x \leqslant jl$), there is some k-subset $Y \subset G'$ which dominates all but $\lfloor 2^{-k}\alpha r \rfloor$ vertices of G' and such that there is at least one edge between Y and X_i, $1 \leqslant i \leqslant j$. Further, since $\lfloor r \rfloor - x \geqslant k \geqslant \lfloor \log_2 k \rfloor + 3$, Lemma 4 shows that Y is contained in a connected subgraph X_{j+1} in G' of order at most l. Hence the described subgraphs X_i can all be found, and by contracting each X_i to a single vertex we see that $G > K_{p-2}$.

To show $c(p) \leqslant \alpha p \sqrt{(\log_2 p)}\,(1 + o(1))$, put $P = \sqrt{(\log_2 (p-3))}$ and choose $\epsilon = \epsilon(p)$ such that both $\epsilon \to 0$ and $E = \log_2 (\alpha/\epsilon) = o(P)$ as $p \to \infty$. Now let

$$k = \lfloor P + E \rfloor = P(1 + o(1)) \quad \text{and} \quad r = \lceil \alpha(p-3)\, l/(1-2\epsilon) \rceil/\alpha.$$

As $p \to \infty$, we have $2^k\epsilon \to \infty$, since $E = o(P) = o(k)$, and $r^{-1}k2^k \to 0$, since $k2^k < 2^{2k} < P$ and $l \to \infty$. In particular $\epsilon r > 2^{-k}\alpha r > k$. Hence for large enough p,

$$\lfloor r \rfloor - (p-3)\, l \geqslant r - 1 - (1-2\epsilon) r > \epsilon r > 2^{-k}\alpha r > k,$$

so it suffices to verify that

$$(p-3)\binom{2^{-k}\alpha r}{k} < \binom{\epsilon r}{k},$$

that is,

$$p - 3 < 2^{k^2}\left(\frac{\epsilon}{\alpha}\right)^k \prod_{i=1}^{k-1} \frac{1 - i/\epsilon r}{1 - i/2^{-k}\alpha r}.$$

Since $\epsilon > 2^{-k}\alpha$, the right-hand product is greater than one, so it is enough to verify that $p - 3 < (2^k\epsilon/\alpha)^k$. But this inequality indeed holds, since

$$k^2 - k \log_2 (\alpha/\epsilon) - \log_2 (p-3) = k^2 - Ek - P^2 > 0.$$

Thus $c(p) \leqslant \alpha r = \alpha p \sqrt{(\log_2 p)}\,(1 + o(1))$.

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