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EARLY BOLZANO ON *GROUND-CONSEQUENCE* PROOFS

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Abstract. In his early *Contributions to a Better-Grounded Presentation of Mathematics* (1810) Bernard Bolzano tries to characterize *rigorous proofs* (*strenge Beweise*). *Rigorous* is, *prima facie*, any proof that indicates the grounds for its conclusion. Bolzano lists a number of methodological constraints all rigorous proofs should comply with, and tests them systematically against a specific collection of elementary inference schemata that, according to him, are evidently of ground-consequence-kind. This paper intends to give a detailed and critical account of the fragmentary logic of the *Contributions*, and to point out as well some difficulties Bolzano's attempt runs into, notably as to his methodological ban on 'kind crossing'.

Introduction. In 1810 Bernard Bolzano, at that time professor of the Science of Religion at the Charles-Ferdinand University in Prague, published a booklet entitled *Contributions to a Better-Grounded Presentation of Mathematics*.[1] This booklet consists of two parts, both dedicated to the "theoretical part (*spekulativen Theile*)" of mathematics, that is to mathematics "as a branch of philosophy and as a way of practising correct thinking" (*BM* xi; 88). The first part, *On the Concept of Mathematics and its Classification*, focuses on the search for a definition of mathematics and of a principle of classification for its different branches;[2] the second part is a discourse *on the Mathematical Method*. Bolzano does not content himself—as he did in his previous *Considerations on some Objects of Elementary Geometry* (1804)[3]—with establishing programmatic requirements on proofs that, in the sense specified below, are to be called "rigorous (*streng*)".[4]

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[1] *Beyträge zu einer begründeteren Darstellung der Mathematik*. Henceforth quoted as *BM* followed by the pages of the German text and of the English translation, separated by a semicolon. Responsibility for translations from German is mine, even when I refer to, benefit from, or simply echo published translations. For an overview of Bolzano's life and work see Morscher 2008 and 2013.

[2] Hereto see Casari 2004; Rusnock & Sebestik 2013, 2.

[3] *Betrachtungen über einige Gegenstände der Elementargeometrie*. Henceforth quoted as *Betrachtungen* followed by the pages of the German text and of the English translation, separated by a semicolon.

[4] I translate by "rigorous" Bolzano's talk of "strenge Beweise", "wissenschaftliche Beweise", "streng wissenschaftliche Beweise". Cp. e.g. Bolzano 1817, 7f., 20; *Athanasia*, 19;

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Indeed, here he also tries to transform such *desiderata* into formal conditions to which deductively correct inferences (proceeding from true premises) must conform, in order to provide rigorous proofs of their conclusion.

Bolzano discusses the issues of *rigorous proof* and their structure within the context and set of problems connected with the questions: what is the point of proving something? What is the aim of giving a proof?[5] Do proofs serve the purpose of producing certainty (*Gewissheit*, *Überzeugung*) about the proposition that is to be proven, or do they rather aim at giving the grounds of the proposition that is to be proven (*Begründung*)? Bolzano's elaboration of this set of problems will later constitute an important part of his masterpiece, the *Wissenschaftslehre* (1837),[6] where the contrast in question becomes the contrast between proof as ascertainment (*Gewißmachung*) and proof as foundation or grounding (*Begründung*).[7] He writes:

> [. . .] It is known that Aristotle already[8] and for many centuries after him the scholastics have distinguished two kinds of proofs, namely those which show only the ὅτι, i.e., which show *that* something is the case, and those which show the διότι, i.e., *why* something is the case (*WL* II, 339–391).

In this paper we will not go into these later theorizations by Bolzano (*WL* II, 339–390).[9] Instead, we will often refer to texts from the years of the *BM*, such as the already mentioned *Considerations On Some Objects Of Elementary Geometry*, the second volume of the *Philosophischen Tagebücher 1803–1810*[10] and the *Allgemeine Mathesis,* the first part of the projected continuation of the *Contributions*, also dated 1810.[11]

One might reasonably ask what is the point of focusing exclusively on Bolzano's *Beyträge* and surroundings. It has been observed, more then once, that Bolzano's *Beyträge* have many flaws,[12] that the logic employed is very rudimentary and mostly remains within the tradition, and that Bolzano's terminology is not very precise. This is largely true, as we will specify through a detailed analysis. Yet, the key point is that the *Beyträge* set out to show in a systematic way how it is possible to present mathematics from the

RW I 4f., 7, 14, 27. The literature sometimes uses "explanatory" instead of "rigorous": see e.g. Betti 2010, Mancosu 1999 and 2011, Roski 2014, Sebestik 2011, §7 (who also uses "grounding proofs"). Rusnock 2000, 36, 70 uses both "groundings" and "objective groundings", Rusnock & Sebestik 2013, 6 use "objective grounding".

[5]Cp. Casari 1987, 331.

[6]Henceforth quoted as *WL*.

[7]*WL* IV, 261-2. Also cp. Bolzano 1817, 254 (Engl. transl.); *RW* I, §3, No. 2. On "proofs that" vs. "proofs why" cp., among others, Rusnock 2000, 36; Sebestik 2011, §7.

[8]*An. Post*. 78a 22f.

[9]The classical essay hereto is Buhl 1961. Also cp. Paoli 1991; Sebestik 1992, 258 ff.; Mancosu 1999, 431f. & 2000, 114–116; Centrone 2010, 123–127 & 2011; Roski 2014, 340 ff. Outstanding contributions to a clarification of Bolzano's *Etiology* are to be found in Casari 201x (forthcoming), §§64–66.

[10]Henceforth quoted as *Tb.*

[11]Henceforth quoted as *Allg. Math.*

[12]Cp. e.g. Rusnock 2000, 50; Rusnock& Sebestik 2013, 2ff.

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ground up, on the base of a logic that Bolzano is elaborating at precisely that moment.[13] The very attempt Bolzano undertakes *to recast formally* programmatic requirements proofs have to obey, if they are to be rigorous, and *to reason on formal proofs* as well is, in our opinion, worth consideration notwithstanding the failures it undergoes. It should also be noted that, a few years after having established in the *Beyträge* what a grounded presentation of mathematics should look like, Bolzano writes a number of works in which well known theorems of analysis are re-proved from the ground up,[14] constantly taking into account, as far as possible, the main programmatic requirements set up in his 1810 work. In the Preface to *The Binomial Theorem and as a Consequence from it the Polynomial Theorem and the Series which serve for the Calculation of Logarithmic and Exponential Quantities proved more strictly than before* (1816) he writes:

> In the present work—as a sample of a *new way of developing analysis*—the theorems mentioned in the title are presented in such a way that, I hope, not only have the mistakes just condemned been avoided, but also several other blunders contrary to good method which are frequently allowed. As far as *method* is concerned I have generally kept to the principles set out in the *Beyträge zu einer begründeteren Darstellung der Mathematik*, *First Issue*, 1810, *Second Part*, because I am still convinced of their correctness. None the less (and this I must say to avoid any misunderstanding) the fragmentary form of the present work does not allow those principles to be followed in *all* its parts (Bolzano 1816, 163).

In the Preface to his *Purely Analytic Proof of the Theorem that between any two Values, which give Results of Opposite Sign, there lies at least one real Root of the Equation* (1817) we read:

> A careful reader could already have gathered from the few writings which have appeared *up until now*, but particularly from that outline of a *new logic* given in the first issue of *Beyträge zu einer begründeteren Darstellung der Mathematik* in its *second* part, headed *Über die mathematische Methode*, that I hold certain views which, if they are not found to be altogether wrong, must lead to a *complete reorganization of all purely a priori sciences* (Bolzano 1817, 261).

Under this perspective Bolzano's attempt at the formal translation of programmatic requirements for rigorous proofs makes the *Beyträge* appear as an important step towards the instillation of rigor in mathematics.[15] This is confirmed by the fact that Bolzano broaches the issue of rigorous proofs not only in relation to the connection between premises and conclusion, but also, as we shall see, in relation to the question whether the premises of a proof have been appropriately proven.

[13] Cp. Rusnock & Sebestik 2013.
[14] Cp. Bolzano 1816, 1817, 1817a.
[15] Cp. Kline 1972, Ch. 40; Detlefsen 2010; Detlefsen & Arana 2011.

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From his beginnings onwards Bolzano works with the Leibnizian-Wolffian idea that there is a certain objective connection among truths, independent of the cognitive activity of the subject: certain truths are the 'grounds (*Gründe*)' of others and the latter are 'consequences (*Folgen*)' of the former.[16] A *rigorous proof* is a proof that indicates the grounds of its conclusion. In the *Contributions* he writes:

> we must [. . .] understand by the [*rigorous*][17] *proof* of a truth the representation of the *objective dependence* (*objective Abhängigkeit*) of it on *other truths*, i.e., the derivation of it from those truths which must be considered as *the ground* (*Grund*) *for it*—not fortuitously, but *actually and necessarily*—while the truth itself, in contrast, must be considered as their *consequence* (*Folge*) (*BM* 64; 110).

As Sebestik & Rusnock 2013 rightly stress: "The doctrine of the objective connection of truths is the seed from which all of Bolzano's logical theories will grow. It means that the real purpose of a scientific exposition is the organisation of truths according to their relations of objective dependence, and not simply certainty and strength of conviction."[18]

What should a proof look like in order to be rigorous? Already in his first writing, the *Considerations On Some Objects Of Elementary Geometry*, Bolzano had maintained that rigorous proofs (i) should begin with propositions which—to put it with Leibniz—"neither admit of a proof nor need it"[19], and (ii) should not use any *fortuitous alien* concept (*fremd(artig)e Mittelbegriffe*). They should ban any *metábasis eis állo génos*, any crossing to another kind.[20] They should not be roundabout. Such a requirement evidently echoes what Aristotle writes in the *Posterior Analytics* ". . . you cannot prove anything by crossing from another kind, e.g. something geometrical by arithmetic".[21] He writes:[22]

> Firstly, I propose for myself the rule that the obviousness of a proposition does not free me from the obligation to continue searching for a proof of it, at least until I clearly realize that absolutely no proof could ever be required, and why [. . .]
>
> Secondly, I must point out that I believed I could never be satisfied with a completely strict proof if it were not derived from the same

[16] On the ground-consequence relation in the *BM* see a.o. Centrone 2012 & 2012a; Roski 2014, 25–89; Rusnock 2000, 31–55; Rusnock & Sebestik 2013, 4f. In the *WL* the ground-consequence relation (*Abfolge*) in introduced in §162, and developed, albeit incompletely, in the same volume in §§198–222. Hereto cp., among others, Buhl 1961; Berg 1962; 151–164; Casari 1992 & 201x, Ch. 7; Centrone 2010, 123 ff., & 2011; Mancosu 1991, Sebestik 1992, 258ff.

[17] Bolzano writes here "scientific". He uses "rigorous", "scientific" and "rigorous scientific" as synonyms. Cp. fn. 4 above.

[18] Rusnock & Sebestik 2013, 4–5.

[19] Leibniz 1704, IV 9, §2.

[20] *Betrachtungen* ii-v; 31–32.

[21] *An. Post.* 75a 38–39.

[22] *Betrachtungen* ii-v; 31–32.

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> concepts which the thesis to be proved contained, but rather made use of some fortuitous, alien, intermediate concept [*Mittelbegriff*], which is always an erroneous metábasis eis állo génos.

In the later *Contributions* Bolzano states, as to the first aim, that the distinguishing feature of an axiom does not lie in its intuitiveness (*Anschaulichkeit*), because such a feature is insufficiently suited for providing a firm basis for the classification of all truths in axioms and theorems (*BM* 59–60; 109). "If [. . .] the word 'axiom' is to be taken in an *objective* sense we must understand by it a truth we *not only do not know how to prove but which is in itself unprovable*" (*BM* 59–63; 110). As to the second aim, he tries to recast formally the methodological ban on "kind crossing", as well as other interesting constraints on proofs. He identifies and investigates four simple inference schemata yielding an affirmative conclusion and four simple inference schemata yielding a negative conclusion, all of which are such—in his opinion—that the connection between the premises and the conclusion is of ground-consequence-kind.

In Section 1, I present the inference schemata—those having an affirmative conclusion (A-Rules) and those having a negative conclusion (N-Rules)—taken into consideration by Bolzano. I then propose an interpretation of the components of the premises- and conclusion-forms of such inference schemata that appears to fit well with Bolzano's text. In Section 2, I discuss some requirements Bolzano imposes on *rigorous* proofs and critically analyse some "metatheorems" demonstrated by Bolzano, saying that any proof made up of inferences that are instances of some of his inference schemata satisfies those requirements. In Section 3, I discuss some serious difficulties that arise from Bolzano's contention that each of the ground-consequence inference schemata he is considering satisfies all the requirements he imposes on rigorous proofs.

Preliminarily, it should be noticed that early Bolzano, following the tradition, calls "judgments (*Urtheile*)" the premises and conclusions of inferences. Judgments, however, are propositional mental acts that occur at a certain time and need a bearer. In the *Wissenschaftslehre* Bolzano will clearly distinguish between acts of judgment and their contents (*Stoffe*) and will characterize the latter as abstract logical objects that have no spatial or temporal determinations, do not stand in causal relations and are, *individually* and *generically*, independent of their bearers, that is, independent of this or that bearer as well as from the very fact that there are bearers. In the *Wissenschaftslehre* Bolzano will call the contents of acts of judgment "propositions in themselves (*Sätze an sich*)" and their nonsentential parts "ideas in themselves (*Vorstellungen an sich*)" or "objective ideas (*objective Vorstellungen*)". In the modern literature it is customary to call them "propositions" and "concepts", respectively. Propositions are, according to the *Theory of Science*, (ultimately) built up of concepts and they are either true or false. In that place, Bolzano will call true propositions "truths in themselves (*Wahrheiten an sich*)". In the following I will maintain early Bolzano's terminology and also talk of "judgments" when he talks of

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"judgments". However, I take him to be saying that judgments that occur as premises and conclusions of some inference are not propositional mental acts but their abstract logical contents, i.e., propositions. Accordingly, I take him to understand under "theorem (*Satz*)" the proposition conveyed by a judgment appearing as the last item of a chain of correct inferences that do the job of proving the truth of that proposition.

§1. The Fragmentary Logic of the *Contributions*.

1.1. The Quartet of Simple Inference Rules with affirmative conclusion: A-Rules. In §12 of the *Contributions* Bolzano poses the following question: "*how many simple* (*einfache*) *and essentially different kinds of inferences* (*wesentlich verschiedene Schlußarten*)" of ground-consequence-kind "*are there?*" By "simple and essentially different" kinds of inferences he means mutually independent schematic inference rules that cannot be reduced to simpler ones. For what concerns *inferences having an affirmative conclusion*, he *tentatively* recognizes *four* simple inference rules—let us call them, for the sake of brevity, "A-Rules"—that, in his opinion, are "ways in which one truth can depend on another truth" (*BM* 64; 110).[23] We said "tentatively", because Bolzano writes: "it is not without hesitation that I proceed to put forward my opinion" (loc. cit.) and explicitly adds that these four inference rules are those "which have occurred to me so far" (*BM* 65; 110). The A-Rules are as follows:[24]

(I) *Barbara* (Figure 1)

$$\frac{\text{S is M} \qquad \text{M is P}}{\text{S is P}}$$

(II) 'cum'-*introduction in the predicate*

$$\frac{\text{A is (or contains) B} \qquad \text{A is (or contains) C}}{\text{A is (or contains) (B cum C)}}$$

(III) 'et'- *introduction in the subject*

$$\frac{\text{A is (or contains) M} \qquad \text{B is (or contains) M}}{\text{(A et B) is (or contains) M}}$$

(IV) 'cum'- *introduction in the subject*

$$\frac{\text{A is (or contains) M} \qquad \text{(A cum B) is possible}}{\text{(A cum B) is (or contains) M}}$$

[23] Rusnock 2000 just mentions the four simple inference patterns, without discussing them any further: "Bolzano believes that the concept 'inference' is simple, and hence unanalysable; as a consequence, he does not seek a classification of basic inference types, but instead makes a provisional list. These are extremely limited in power, and I will simply record them here" (Rusnock 2000, 37).

[24] This quartet of inference rules is slightly different from the one Bolzano actually presents in the *Contributions*. The original Bolzanian formulation of the A-Rules presents nontrivial interpretative problems on which I focus in Centrone 2012, 14ff. In the original formulation, rule (II) is an 'et'-*introduction in the subject* and not a 'cum'-*introduction in the subject* as it is here. However, in the former formulation one of the premises of the rule turns out to be superfluous, given the reading of the term-connective 'et' apparently intended by Bolzano – and this is in contrast with Bolzano's explicit request (see Sect. 2 below) that in each inference of ground-consequence-kind the premises should be irredundant. The substitution of 'et' by 'cum' in the original rule (II) is supported, among others, by Bolzano's reflections on the term-connective 'et' in *Allg. Math.* 34–35, as well as in the coeval *Tb.* 222–223; 238 under '3.a'.

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The letters 'A', 'B', 'S', 'M' stand for (singular or general) terms. Bolzano uses the schematic form 'A is B' as placeholder for sentences such as 'Caius is a man' as well as 'every man is mortal' (*BM* 65; 110). Thus, singular propositions are assimilated to general propositions.[25]

According to Bolzano, the expressions 'A is B', 'A contains B' and 'A is a kind of B' amount to the same: "the connecting concept in these judgments is the concept of the *belonging* of a certain *property*, or ... of the *inclusion* of a certain thing, as *individual* or *kind*, under a certain *genus*" (*BM* 74; 113).[26] On the basis of the *extensional reading* suggested by Bolzano's explanation, in the following I will replace all such expressions by 'A $\subseteq$ B'.[27]

Two term-connectives occur in the four rules: 'cum' and 'et'. Bolzano's understanding of 'cum' results from his reflections on what a *definition* of a *complex* concept is—a concept that cannot be defined is a *simple* concept (*BM* 43; 104). By a definition Bolzano understands "the *statement of the most immediate components* (two or more) *out of which a given concept is composed*" (*BM* 42–43; 104). For instance, "a drake is a male duck" is an appropriate definition of the concept expressed by the word "drake". In general, a definition has the form:

(1) *a, which is* α*, is A;* or: *(a cum* α*)* $= A$.

In (1) "neither a, nor α, on their own, may be $=$ A" (*BM* 43; 104), and the concepts a and α must be different. Hereby the interpretation of the term-connective 'cum' is determined. We can say: the *concept* expressed by a term of the form 'a cum α' is a concept that is composed by a genus-concept, which is expressed by the term in the 'a'-position, and by another concept, which is in turn expressed by the term in the 'α'-position, in such a way that it (i.e., the composed concept 'a cum α') coincides with the species-concept, which is expressed by 'A'. The *extension* of a term of the form 'a cum α' is thus the *intersection* between the class of things, to which the term in the 'a'-position applies and the class of things, to which the term in the 'α'-position applies. Henceforth, I will replace 'cum' with the symbol '$\cap$' of set-theoretic intersection.

Bolzano's understanding of 'et' results both from his use of it and from some of his reflections contained in *Allg. Math.* Here Bolzano poses questions about the possible combinations of (*terms* expressing) concepts and says:

> There is, at first, a combination of two concepts in a third. Such combination is denoted by the word '*and*'. For example, *inkwell* and *pouce pot* give rise together to the concept of *writing things*. [...] *This combination is possible between any two concepts*. [...] *Writing things*

[25] *WL* I, 248–50; II, 264, 400. Cp. Künne 2008, 236.

[26] Also cp. *BM* 114–116; 124–126. In *Tb.* 182 f., on 09.10.1809, he writes: "Would it not be better to render the form of judgments I have indicated until now, with much inconvenience, by 'A contains B', by 'A is contained in B' ... [?]".

[27] We identify A with {A}, in case A is a singular term.

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> [. . .] is neither an object, which is an *inkwell*, nor an object, which is a *pouce pot*, but rather a sum of *inkwell* and *pouce pot*.[28]

Notice that Bolzano will maintain this conception of the term-connective 'et' also in his latest works. In his posthumously published *Paradoxes of Infinite* (1851) he will reaffirm that the concept expressed by 'and' is simple and that 'and' ('und', as he writes there) designates that conceptual operation that is responsible for set-formation (*PdU* 3-4). Henceforth, I will replace 'et' with the symbol '∪' of set-theoretic union.

On the basis of the preceding observations we can rephrase Bolzano's A-Rules as follows:[29]

(I) *Modus Barbara*

$$\frac{A \subseteq M \quad M \subseteq B}{A \subseteq B}$$

(II) ∩-*introduction in the predicate*

$$\frac{A \subseteq B \quad A \subseteq C}{A \subseteq (B \cap C)}$$

(III) ∪-*introduction in the subject*

$$\frac{A \subseteq M \quad B \subseteq M}{(A \cup B) \subseteq M}$$

(IV) ∩- *introduction in the subject*

$$\frac{A \subseteq M \quad (A \cap B) \text{ is possible}}{(A \cap B) \subseteq M}$$

Rule (I) is the *Modus Barbara* of the tradition (except that the premises are not in the usual order). Bolzano says that "concerning the *syllogism* . . . there is only a *single*, simple form of this, namely *Barbara* or Γραμματα in the *first* figure. Every other figure and form of the syllogism seems to me to be either not essentially different from *Barbara*, or not completely simple" (*BM* 65; 110). However, he neither explains why he believes that all other valid syllogistic modi (presumably, those having an affirmative conclusion) are 'not completely simple', nor does he provide hints as to how the latter should be reduced to *Barbara*.

As to the rules (II) and (III), Bolzano observes that they are *invertible*, that is

(II$^{\mathbf{I}}$.1) ∩- *elimination from the predicate*

$$\frac{A \subseteq (B \cap C)}{A \subseteq B}$$

(II$^{\mathbf{I}}$.2) ∩- *elimination from the predicate*:

$$\frac{A \subseteq (B \cap C)}{A \subseteq C}$$

(III$^{\mathbf{I}}$.1) ∪-*elimination from the subject*

$$\frac{(A \cup B) \subseteq M}{A \subseteq M}$$

(III$^{\mathbf{I}}$.2) ∪-*elimination from the subject*:

$$\frac{(A \cup B) \subseteq M}{B \subseteq M}$$

are indeed *correct inference rules*. He however insists that such rules are *not* of ground-consequence-kind: "the first two propositions [in (II) and (III)]

[28] "Es gibt fürs erste eine Zusammensetzung 2er Begriffe zu einem dritten, die das Wort '*und*' andeutet, z.B. [*Dintzeug*] und [*Streusandfaß*] zusammen geben den Begriff eines *Schreibzeuges*. [. . .] *Diese Zusammensetzung ist zwischen jeglichen 2 Begriffen möglich*. [. . .]*Schreibzeug* ist nicht ein Gegenstand, der *Dintfaß* ist, weder ein Gegenstand, der *Streusandfaß* ist, sondern eine Summe von *Dintfaß* und *Streusandfaß*" (*Allg. Math.* 34-35).

[29] *BM* 65–68; 110–111. In *BM* 114–115 and in *Th.* 205 he puts Modus Barbara as last rule of the table.

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can be considered as *ground* of the third and not, indeed, conversely" (*BM* 66; 111).

Rule (IV) represents a sort of "limitation of the subject". In this context it is important to point out that Bolzano classifies judgments of the form 'A is B is possible', or 'A *can be* B', as *possibility* judgments. In his view, judgments of this kind express a truth iff the assumption of an A, which is B, is *consistent*. He probably thinks that the truth of the judgment that $A \subseteq M$ presupposes that $(A \cap M)$ is possible. Thus, the second premise of Rule (IV) does not say that something falls under the concept of an A which is B, but only that something *can* fall under this concept.[30] Thus, Rule (IV) fails to permit the inference from, say, 'every man is mortal' to 'every man, which is omniscient, is mortal'—for no man *can be* omniscient. However, the inference from the same premise to 'Every man, which is born on the moon, is mortal' is permitted according to Rule (IV)—even if no man *is actually* born on the moon. Otherwise said, 'A is B is possible' or 'A *can be* B' does not express an existential presupposition in the sense of a presupposition of *real* existence, but rather expresses a '*consistency*-presupposition'.

1.2. The Quartet of Simple Inferences with negative conclusion: N-Rules. As to *inference schemata having a negative conclusion*, Bolzano mentions explicitly *four* inference rules—let us call them "N-Rules"—that in his opinion guarantee, similarly to the A-Rules, that the connection between premises and conclusion is of ground-consequence-kind. The N-Rules are as follows:[31]

(I^{Neg}) *Celarent* (Figure 1)

Whatever is a kind of M is not a kind of B
A is a kind of M

A is not a kind of B

(II^{Neg}) 'et'-*introduction in the predicate*

A is not a kind of M
A is not a kind of N

A is not a kind of (M et N) = B

(III^{Neg}) 'et'- *introduction in the subject*

M is not a kind of B
N is not a kind of B

(M et N) is not a kind of B

(VI^{Neg}) 'cum'- *introduction in the subject*

Whatever is a kind of M is not a kind of B
M can be a kind of N

(M cum N) = A is not a kind of B

Concerning negative judgments of the form 'A is not a kind of B', Bolzano writes: "in a *negative* judgment subject and predicate are mutually exclusive" (*BM* 115; 125). Taking into account that the expressions 'is a kind of'/'is not

[30] At this point I do not agree with Jan Berg, the Editor of the *BGA*, who maintains that the second premise of Rule (IV) is saying that $(A \cap B)$ is different from the empty class. Cp. fn. 199 to *Tb.* 238. Indeed, Bolzano repeatedly uses the verb "can (*können*)" and not "is (*ist*)": "[Man] könnte [**Rule IV**] folgende Gestalt erteilen: **A** ist eine Art von **M** [. . .]; **A** ist eine Art von Dingen, welche **B** *sein können*; [also:] Einige Dinge welche **B** *sein können* sind **M**; oder **B** *kann sein* **M**" (loc. cit.; my emphasis).

[31] *BM* 114-115; 125. Bolzano presents the N-Rules in a different order, starting from Rule (II^{Neg}) and ending with the syllogism. For the sake of convenience, I present this second rule-quartet in the same order as the first one. Notice that the N-Rules do not mirror exactly the A-Rules. Indeed, Rule (II) above is a 'cum'-introduction in the predicate while Rule (II^{Neg}) is an 'et'-introduction in the predicate. As I already explained (fn. 22) the emendation of the A-Rules was necessary for reasons of irredundancy of the premises, which do not arise in the case of the N-Rules.

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a kind of' designate a relation "between the extensions of [. . .] two concepts" (*BM* 113; 124), it is now natural to interpret the negative proposition-forms occurring in the N-Rules as, to put it in traditional terminology, universally negative propositions: 'A is not a kind of B' is *not* the contradictory opposite of 'A is B', i.e., 'Some A are not B', but rather its contrary opposite, 'No A is B'. We can thus rephrase Bolzano's N-Rules as follows:[32]

(I$^{\text{Neg}}$) *Modus Celarent*

$$\frac{M \cap B = \emptyset \quad A \subset M}{A \cap B = \emptyset}$$

(II$^{\text{Neg}}$) $\cup$*-introduction in the predicate*

$$\frac{A \cap M = \emptyset \quad A \cap N = \emptyset}{A \cap (M \cup N) = \emptyset}$$

(III$^{\text{Neg}}$) $\cup$*-introduction in the subject*

$$\frac{M \cap B = \emptyset \quad N \cap B = \emptyset}{(M \cup N) \cap B = \emptyset}$$

(IV$^{\text{Neg}}$) $\cap$*- introduction in the subject*

$$\frac{M \cap B = \emptyset \quad M \cap N \text{ is possible}}{(M \cap N) \cap B = \emptyset}$$

§2. Main Requirements on Rigorous Proofs. Rigorous proofs are characterized, *prima facie*, as proofs that provide the ground(s) of their conclusion. According to Bolzano, there are a number of further formal constraints that proofs should satisfy in order to be called rigorous. Namely, rigorous proofs should:

(a) proceed from the general to the particular;
(b) proceed from the simpler to the more complex;
(c) be such that all premises are relevant to obtain the conclusion;
(d) exclude alien intermediate concepts.

Let us, first of all, observe, that the requirement (a) of generality-decrease and the requirement (b) of increasing-complexity are strongly connected, under the assumption of the *Canon of reciprocity* (discussed below): indeed, proceeding from the general to the particular (extensionally intended) corresponds, from an intensional point of view, to adding conceptual marks to the subject-concept or to the predicate-concept of a categorical proposition, i.e., to proceeding from the simple to the more complex. Requirement (a) is notoriously endorsed by Aristotle: "[I]f demonstrations are probative deductions which give the explanation and the reason why, and if universal are more explanatory [. . .]—then universal demonstrations are better. For it is rather that they give the explanation and the reason why."[33] Again, if we think of proofs as processes of backtracking from a truth to its reasons, "we seek the reason why up to a certain point",[34] that is, such a procedure halts when reaching the most simple and most general truths. The whole tradition follows Aristotle on this point. Arnauld and Nicole are particularly explicit

[32] *BM* 114-115; 125. Notice that Rule (II$^{\text{Neg}}$) and Rule (III$^{\text{Neg}}$) are essentially the same rule, up to the commutativity of '$\cap$'.

[33] *An. Post.* 85b 23-29.

[34] *An. Post.* 85b 30.

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on this point: "always to start with the simplest and most general things, to proceed next to the more composite and particular".[35]

Bolzano says he finds himself forced to list the above formal requirements "about the *organization and succession of propositions* in a scientific system, and about the nature of their proofs" because "they are still not being *followed* and it seems they are not sufficiently acknowledged as being *indispensable*" (*BM* 101-102; 121). Taken together, these requirements are *de facto* intended to provide proofs that are to be called rigorous with a kind of *directness property*—in contrast to proofs that are not rigorous—to the effect that, in each such proof, every concept that occurs in the proof is also an "ingredient" of the conclusion. It is worth observing that such a demand is strongly reminiscent of the *subformula property* enjoyed by normal-form derivations and cut-free derivations in Gentzen's natural deduction calculi, respectively sequent calculi: all formulas occurring in a derivation in normal form (in a cut-free derivation) are subformulas of some formulas occurring in the assumptions or in the conclusion of the derivation (in the end-sequent of the derivation).

Having established requirements (a)–(d) as essential to rigorous proofs, Bolzano next tries (unsuccessfully, as we shall see) to give evidence for the claim that his two rule-quartets respect those requirements by demonstrating that all proofs that are constructed by means of the A-Rules, as well as all proofs that are constructed by means of the N-Rules (let us call them "A-Proofs" and "N-Proofs", respectively) satisfy them. More precisely, he discusses in detail requirements (a)–(c) *only* with respect to A-Proofs, and requirement (d) also with respect to N-Proofs.

I will present below Bolzano's claims with respect to requirements (a)–(c), in the form of three corresponding "metatheorems" (as I would like to call them), together with their (alleged) demonstration. The discussion of the analogous claim with respect to requirement (d) will be deferred to the next section.

A preliminary word of caution is in order here. As the reader will easily realize, Bolzano's *demonstrations* of his "metatheorems"—regardless of the circumstance that the first two metatheorems are, *as they stand*, false—rest now and again upon assumptions that he neither explains nor justifies adequately. Nevertheless, I think it is worth having a closer look at them: they represent (surprising) early examples of "reasoning about (formal) proofs," of a (rudimentary) proof-theoretic approach, as we might say.

To start with, we have to explain the meanings of some "technical" notions that enter in Bolzano's own formulation of the metatheorems, in particular the conceptual pairs expressed by "wider/narrower" and "simpler/more composite", and the notion of "a proposition *following* another proposition" (in a proof).

Indeed, Bolzano recasts the ban on kind crossing, as to the A-Rules, as the constraint that the intermediate concepts should *not* be *narrower* than the subject or *wider* than the predicate, while he recasts it, as to the N-Rules,

[35] Arnauld & Nicole 1662, 256. Cp. hereto Roski 2014, 14.

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as the constraint that the intermediate concepts should be *wider* than the subject *or wider* than the predicate:

> [S]everal [. . .] *intermediate concepts* can also appear in the proof. However, if the proof is to contain nothing superfluous, then *for an affirmative proposition there should only appear intermediate concepts which are not narrower than the subject and not wider than the predicate. But for a negative proposition they should only be wider than the subject or wider than the predicate* (*BM* 110; 123).

Bolzano explicitly endorses an extensional reading of "wider" and "narrower". He writes: "[It] will be seen without any trouble" that in all A-Rules the intermediate concepts are *between* the subject and the predicate "if one always makes use (*in Anwendung bringt*) of the relationship which has been asserted between the extensions (*Umfangsverhältnisse*) of the two concepts [the subject and the predicate] of an affirmative judgment" (*BM* 113; 124). We can therefore render "A is wider (narrower) than B" by 'A $\supset$ B' (or 'A $\supseteq$ B', depending on the context), respectively 'A $\subset$ B' ('A $\subseteq$ B', depending on the context).

Concerning the second conceptual pair, namely 'more composite/simpler', Bolzano does not explicitly state how he wants 'more composite' and 'simpler' to be understood. On the basis of the text it is however quite natural to assume that he takes 'P is more composite than P′' to mean 'the content of P has more conceptual marks than the content of P′' which, according to Bolzano, is tantamount to: P has the form 'P′ cum π' for some concept π (cp. e.g. *BM* 42–43; 104).

The third technical notion, that of "a proposition *following* another proposition" in a given proof, is employed by Bolzano in the statement of the first two "metatheorems", see below. As is clear from the analysis of his "demonstrations" of these claims, this notion has to be taken literally: a proposition P_1 *follows* a proposition P_2 in a proof D just in case P_1 *appears below* P_2 in D. Or, if we visualize proofs as (proof-) trees, just in case there is a path in D going (in the top-bottom direction) from P_2 to P_1; which is tantamount to saying that there is a chain of inferences leading from P_2, possibly together with other auxialiary premises, to the conclusion P_1.

Let us now come to the "metatheorems" which Bolzano "demonstrates" about A-Proofs. The first claim he puts forward is that the *requirement of generality-decrease* is satisfied by all those proofs. This is Bolzano's

> **Metatheorem (1)**. If two propositions appearing in an A-Proof have the same predicate, *then the proposition with the narrower subject must follow that with the wider subject, and not conversely* (*BM* 102; 121).

Demonstration (Bolzano, *loc. cit.*): For if $M \subseteq B$ and $A \subseteq B$ are two (true) propositions with the same predicate, and the subject of the second proposition is narrower than that of the first, then either [i] $A = (M \cap C)$ is valid or, *in the case that the concept A is simple*, [ii] $A \subset M$ is valid. In case [i] the conclusion $A \subseteq B$ follows from the two premises $M \subseteq B$ and $(M \cap C)$ *is possible* according to Rule (IV), that is by $\cap$-introduction in the subject.

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In case [ii], $A \subseteq B$ follows from $A \subseteq M$ and $M \subseteq B$ according to Rule (I), i.e., according to an inference in Modus Barbara; for $A \subset M$ implies $A \subseteq M$ in virtue of the definition of proper inclusion. In both cases the proposition with the narrower subject is derived from the proposition with the wider subject—and another premise.

Notice that in the above demonstration Bolzano is wrong (at least) in so far as he does not distinguish, further to the cases [i] and [ii], the *critical* case in which $A = (A_1 \cup A_2)$. And in fact, as anticipated, the claim is (literally) false: A-Proofs containing applications of Rule (III), ∪-*introduction in the subject*, trivially do not respect the generality-decrease requirement: premises and conclusion of this rule have the same predicate, but the subject of both premises (A, B) is narrower than the subject of the conclusion (A ∪ B).[36]

The second claim Bolzano puts forward is that the *requirement of increasing complexity* is satisfied by all A-Proofs. This is Bolzano's

> **Metatheorem (2)**. If two propositions appearing in a proof have the same subject, "then the proposition with the *more composite* predicate must follow that with the simpler predicate, and not conversely" (*BM* 104–105; 122; my emphasis).

Demonstration (Bolzano, *loc. cit.*). Let $A \subseteq C$ and $A \subseteq (B \cap C)$ be two (true) propositions with the same subject. Then the proposition with the *simpler* predicate must precede in a proof that with the *more composite* predicate. $A \subseteq (B \cap C)$ can in fact be derived from $A \subseteq B$ and $A \subseteq C$ according to Rule (II), '∩'-introduction in the predicate.

As in the case of Metatheorem (1), notice that this second claim too cannot be true in general: Rule (I) satisfies Metatheorem (2) only if the term M in the premises is not more composite than the term B in the conclusion.[37] Indeed, what is wrong with Bolzano's demonstration is the tacit, unjustified assumption from which he starts, namely the equivalence of "C' is more composite than C" with "$C' = (B \cap C)$ for some concept B".

On Metatheorem (2) Bolzano expresses himself as follows: "This truth has forced itself particularly clearly upon those who have reflected upon the nature of scientific exposition: '*One must*' they said, '*always teach more, not less, in the later propositions than in the preceding ones*' (*BM* 105;122). Thereafter Bolzano observes that "[i]t is obvious here that we cannot extend our assertion *further* and replace the expression, '*the proposition with the more composite predicate*', with the more general one [!], '*the proposition*

[36] A *Note* added by Bolzano (*BM* 103-104; 121–122), whose interpretation is unfortunately not so simple, might perhaps explain why he would not have considered this as a real counterexample to his claim. There are propositions—he says—which *seem* to have a wider subject than other propositions (with the same predicate) only because of the way they are expressed; such propositions are in fact (equivalent to the conjunction of) two propositions. Thus, following this line of thought, he might have replied that the counterexample is illusory, in so far as the proposition (A ∪ B) ⊆ M is in fact (equivalent to) a pair of propositions, namely A ⊆ M and B ⊆ M (trivializing however, in this way, his rule (III)!)

[37] A related difficulty will indeed be pointed out by Bolzano in his later *Wissenschaftslehre*, see the end of Section 3 below.

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with the narrower predicate"' (loc. cit., my emphasis). Thus, a further claim Bolzano puts forward is

> **Claim (2a)**. In Metatheorem **(2)** 'more composite' ('simpler') cannot be substituted by 'narrower' ('wider') *salva veritate* (*BM* 105–106; 122).

Or, in other words, the following statement does not hold:

> **(2*)**. If two propositions appearing in a proof have the same subject, then the proposition with the *narrower* predicate must follow that with the *wider* predicate, and not conversely.

Bolzano disproves **(2*)** by means of a counterexample. Let us assume that $A \subseteq B$ and $A \subseteq M$ are two propositions with the same subject, and that the predicate of the second is narrower than the predicate of the first, i.e., $M \subset B$. In this case it may well happen that $A \subseteq B$, i.e., the proposition with the *wider* predicate, *follows* from $A \subseteq M$, i.e., from the proposition with the *narrower* subject, together with truth $M \subset B$ by means of an inference in Modus Barbara. In this case the proposition with the wider predicate would follow that with the narrower.

Let us make here a digression and explain why Bolzano, in the passage quoted above, says that (2*) would *extend* Metatheorem (2), that is, why he claims that

$$\textbf{(2*)} \Rightarrow \textbf{(2)}.$$

In the *Contributions*—and in his later *Theory of Science* as well—Bolzano questions the validity of the traditional principle that the intension and the extension of a concept stand in an inverse relation (*Canon of Reciprocity*). In one of its possible formulations, the *Canon* says that the extension of a concept A is properly included in the extension of a concept B iff the content of B is part of the content of A, in symbols:

$$\textbf{(K)}\ \text{Extension (A)} \subset \text{Extension (B)} \leftrightarrow \text{Content (B)} \sqsubset \text{Content (A)},$$

where '$\sqsubset$' stands for the mereological relation of 'being part of'.
Let us write down *in extenso* the two directions of (K):

$$\textbf{(K}_\textbf{1}\textbf{)}\ \text{Content (B)} \sqsubset \text{Content (A)} \rightarrow \text{Extension (A)} \subset \text{Extension (B)}$$

$$\textbf{(K}_\textbf{2}\textbf{)}\ \text{Extension (A)} \subset \text{Extension (B)} \rightarrow \text{Content (B)} \sqsubset \text{Content (A)},$$

It is clear that 'narrower' and 'more composite' amount to the same under the assumption of the validity of *both* directions of the *Canon*. In the *Contributions* Bolzano "officially" accepts the *Canon*.[38] However, his acceptance of the direction (K_2) does not harmonize with his simultaneous assumption that there are *simple* species-concepts. Indeed, if a concept A is simple, then it can very well happen that its extension is included in the extension of a concept B, yet B cannot be part of the content of A because A has no

[38] In the *Theory of Science* Bolzano will instead deny the validity of *each* of the directions of the *Canon*. *WL* I, 568–571.

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parts.[39] Anyway, one easily sees that the reason Bolzano says that (2*) *would extend* (2) is that he is tacitly taking for granted the direction (K_1) of the *Canon*.

Let us now come back to the third of Bolzano's "metatheorems" on A-Proofs, saying that the *requirement of relevance* is satisfied by all A-Proofs:

> **Metatheorem (3)**. *If the subject* (or the hypothesis) *of a proposition is as wide as it can be so that the predicate* (or the thesis) *can be applied to it, then in any correct proof of this proposition all characteristics of the subject must be used, i.e., they must be applied in the derivation of the predicate, and if this does not happen the proof is incorrect* (*BM* 106-107; 122-123).

The intuition behind this third Metatheorem can be made more clear, we think, if we momentarily set aside the Bolzanian *subject-predicate format* of propositions and consider instead (as the expressions "or the hypothesis" and "or the thesis" within brackets in the above quotation perhaps suggest) true *conditional propositions* p saying that a certain conclusion (thesis) q holds under the assumptions (hypotheses) $p_1, p_2, \ldots p_n$. Then, the Metatheorem says, in a rigorous proof of a proposition of this kind *all* the assumptions $p_1, p_2, \ldots p_n$ are used in an essential way to derive the conclusion q. For, if at least one of the assumptions (say p_n) is not used along the derivation of q in the proof, then (i) either it is true that q follows already from $p_1, p_2, \ldots p_{n-1}$, but then the proposition proved in the proof is not the "strongest" that can be proved, or (ii) it is false that q follows from $p_1, p_2, \ldots p_{n-1}$, and so the proof itself contains a mistake and is therefore incorrect. Thus, it is tacitly understood that the conclusion of a rigorous proof should be not liable to a suitable strengthening.

A *prima facie* reformulation of Metatheorem (3) using Bolzano's format of propositions we might tentatively consider is:

> **(3a).** In a rigorous proof of a proposition $A \subseteq B$ all the characteristics of the concept A must be relevantly used, if it is the case that $\neg\exists\alpha(A \subset \alpha \,\&\, \alpha \subseteq B)$.

The proviso "$\neg\exists\alpha(A \subset \alpha \,\&\, \alpha \subseteq B)$" is intended to rephrase the Bolzanian condition "*If the subject . . . of a proposition is as wide as it can be so that the predicate . . . can be applied to it*". Unfortunately this proviso is, trivially, never satisfied (just take B for α!) in the extensional reading implied, as we stressed above, in Bolzano's talk of "wide/narrow". Admittedly, such a trivialization might be removed just by considering "$\neg\exists\alpha(A \subset \alpha \,\&\, \alpha \subset B)$" as the right way to rephrase the condition in question; but still, doing so would not be of much help to really understand the argument Bolzano proposes in order to support his claim. Indeed, by looking at the "demonstration" of Metatheorem (3)—or rather "intuitive justification" of it, none of the A-rules being mentioned in particular—provided by Bolzano, one can realize that he freely mixes extensional and intensional talk; in particular, he thinks

[39] *BM* 47-48; 105-106. Hereto cp. Centrone 2010a, 212-213.

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of the subject A of the proposition $A \subseteq B$ whose proof is under consideration as *presented through its conceptual marks*, that is, extensionally, as the class of those objects falling under all those conceptual marks. On this basis, we are led to propose the following improvement of the reformulation **(3a)**:

> **(3b).** In a rigorous proof of a proposition $A \subseteq B$, where $A = \alpha_1 \cap \cdots \cap \alpha_n$, all the characteristics (the α_i'*s*) of the concept A must be relevantly used, if it is not the case that, for some i ($1 \leq i \leq n$), $\alpha_1 \cap \cdots \cap \alpha_{i-1} \cap \alpha_{i+1} \cap \cdots \cap \alpha_n \subseteq B$ holds true.

Think of $\alpha_1, \ldots, \alpha_n$ as the (extensions of) the conceptual marks of the subject A of the proved proposition $A \subseteq B$. Then, Bolzano says, if some conceptual mark α_i of the subject A is not relevantly used in the proof ("if one were not to *use* some characteristic of the subject in the proof at all, i.e., not to derive any consequences from it", *BM* 107; 123), then either a stronger proposition can indeed be proved, namely $A' \subseteq B$ with $A \subset A' = \alpha_1 \cap \cdots \cap \alpha_{i-1} \cap \alpha_{i+1} \cap \cdots \cap \alpha_n$ ("either the theorem itself is expressed too narrowly and contains superfluous restrictions", *loc. cit.*), or "if this is not the case, the proof itself contains some hidden false conclusion" (*loc. cit.*).

§3. The problem of *intermediate concepts* (*Mittelbegriffe*). According to the Logic of the *Contributions* a proof is rigorous only if some constraints on the intermediate concepts *(termini medii)* occurring in a two-premise inference are satisfied. These constraints, according to Bolzano, are as follows:

(i) if the proposition that is to be proven is affirmative, then the intermediate concepts must be *wider than the subject and narrower than the predicate*,[40] and
(ii) if the proposition that is to be proven is negative, then the intermediate concepts must be "*wider than the subject, or wider than the predicate*" (*BM* 110; 123).

Let us consider requirement (i) first. It can be formulated also as follows:

***Medius*-Principle.** $\forall m$ ((m is an intermediate concept in an A-Proof of a proposition of the form 'A $\subseteq$ B') $\rightarrow$ (A $\subseteq m \wedge m \subseteq$ B)).

Incidentally, we observe that Bolzano's constraints on intermediate concepts are reminiscent of those traditionally imposed on the search for the *medium* within the framework of the problem of the *inventio medii*. Indeed, the question of producing a suitable syllogism for any given problem[41] is recast, by Aristotle himself, as the search for the right premises for a certain

[40] When stating this requirement for the first time, Bolzano actually writes something different, namely "*not narrower than the subject, and not wider than the predicate*" (*BM* 110; 123). But immediately after, and in all subsequent places, he writes "wider" (symbolized by '$>$') instead of "not narrower" as well as "narrower" (symbolized by '$<$') instead of "not wider".

[41] *An. Pr.* A 27–31.

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proposition that is to be proven.[42] One of the ways to find the right premises is, as Aristotle himself will maintain in the *Posterior Analytics*, to find the middle term (*meson*).[43] The *inventio medii* is applied to find the premises of a universal affirmative proposition. One enumerates the conceptual marks that follow from the subject and imply the predicate of the proposition that is to be proven and takes as middle term the one which is determined by the set of marks that are common to the subject and the predicate of the proposition that is to be proven.[44] Bolzano writes:

> [I]f in a proof there appear *intermediate concepts* which are, for example, *narrower* than the subject, then the proof is obviously defective; it is what is otherwise usually called a *metábasis eis állo génos.* Sometimes, of course, it is clear right at the first glance that a certain concept, being completely alien, does not belong to the proof of a proposition. . . —In other cases it can only be decided whether or not some superfluous concept is involved, through a detailed analysis of the whole proof into its simple components (*BM* 117–118; 126).

Do inferences according to A-Rules, and consequently A-Proofs, satisfy the *Medius*-principle? Trivially, inferences according to Rule (I), i.e., to Modus Barbara, do satisfy it, since in this case $A \subseteq M \subseteq B$ (in Bolzano's notation: $M > A$ *and* $M < B$). However, Bolzano claims: "[B]y whichever of the four *simple forms of inference* mentioned above [. . .] it [an affirmative judgment 'A is B'] can be proved, it is apparent that the intermediate concepts used are *always* [!] wider than A and narrower than B" (*BM* 112–113; 124).

However, the other three simple forms of inference apparently violate the *Medius*-Principle. Let us consider them one by one.

Rule **(II)** $\cap$-introduction in the predicate

$$\frac{A \subseteq B \quad A \subseteq C}{A \subseteq (B \cap C).}$$

Inferences according to this rule can violate the *Medius*-Principle. In Rule (II) 'A' represents the extension of the subject-concept and '$(B \cap C)$' the extension of the predicate-concept of the proposition that is to be proven, whereas 'B' and 'C' stand for the extension of the intermediate concepts of the argument. Rule (II) satisfies the *Medius*-Principle only in case that $B \subseteq (B \cap C)$ and $C \subseteq (B \cap C)$. But this, clearly, does not always hold. Thus, Bolzano seems not to have noticed the conflict between his *Medius*-Principle and Rule (II).

Rule **(III)** $\cup$-introduction in the subject

$$\frac{A \subseteq M \quad B \subseteq M}{(A \cup B) \subseteq M.}$$

[42] *An. Pr.* A 27.
[43] *An. Post.* B 3, 90 a 35.
[44] *An. Pr.* A 28, 44b 8–10.

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Inferences according to this rule can as well violate the *Medius*-principle: indeed, Rule (III) satisfies the *Medius*-principle only in case that $(A \cup B) \subseteq A$ and $(A \cup B) \subseteq B$. But this, clearly, does not always hold. Again: Bolzano seems not to be aware of the conflict between his *Medius*-principle and his Rule (III).

Rule **(IV)** $\cap$-introduction in the subject

$$\frac{A \subseteq M \qquad (A \cap B) \text{ is possible}}{(A \cap B) \subseteq M.}$$

Inferences according to this rule violate the *Medius*-Principle, too. In Rule (IV) the letters 'A' and 'B' stand for the extensions of the intermediate concepts. This rule satisfies the *Medius*-principle only if in each inference according to it holds:

(1) $(A \cap B) \subseteq A \subseteq M$,
(2) $(A \cap B) \subseteq B \subseteq M$.

Condition (1) is indeed always satisfied ('$(A \cap B) \subseteq A$' is valid, and '$A \subseteq M$' holds because it is the first premise). Condition (2) is, however, not always satisfied: '$(A \cap B) \subseteq B$' is obviously valid, too; but '$B \subseteq M$' represents by no means always a truth. Again: Bolzano seems not to be aware of the conflict between his *Medius*-principle and Rule (IV). He writes indeed, without further explanations, that

> *A* > *(A cum B) and A* < *M*, and
> *B* > *(A cum B) and B* < *M*

always hold by inferences according to Rule (IV).[45]

One encounters similar difficulties relatively to the N-Proofs. Bolzano writes:

> In a *negative* judgment subject and predicate are mutually exclusive, so one cannot say of either of the two that it is the *wider* or the *narrower*. But it is clear, as previously, that the *intermediate concepts* which are required in the proof of such negative judgments *A is not a kind of B*, must always be *wider* than the *subject A or wider than the predicate B* (*BM* 115; 125).

This was requirement (ii) above, which can also be formulated as follows:

***Medius*-Principle^{Neg}** $\forall m$ ($(m$ is an intermediate concept in a N-Proof of a proposition of the form '$A \cap B = \emptyset$') $\rightarrow$ ($A \subseteq m \vee B \subseteq m$)).

Do inferences according to the N-Rules (and N-Proofs in turn) always satisfy the *Medius*-Principle^{Neg}? Undoubtedly, inferences according to Rule (I^{Neg}), i.e., to Modus Celarent, do satisfy it (the intermediate concept is wider than the subject, in Bolzano's notation: *M* > *A*).

[45] Cp. *BM* 114; 125. *A*, *B* and *M* are the terms involved in the premises and the conclusion of Rule (IV) (corresponding to the *third* rule in Bolzano's list; see fn. 27 above).

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Also inferences according to Rule ($\mathrm{IV}^{\mathrm{Neg}}$) do satisfy the *Medius*-Principle$^{\mathrm{Neg}}$. Indeed, in

Rule $(\mathbf{IV}^{\mathbf{Neg}})$ $\quad \cap$- *introduction in the subject*

$$\frac{\begin{array}{c}\mathrm{M} \cap \mathrm{B} = \emptyset \\ \mathrm{M} \cap \mathrm{N} \neq \emptyset\end{array}}{(\mathrm{M} \cap \mathrm{N}) \cap \mathrm{B} = \emptyset.}$$

'$(\mathrm{M} \cap \mathrm{N})$' stands for the extension of the subject-concept, 'B' for the extension of the predicate-concept and 'M' and 'N' stand for the extensions of the intermediate concepts. Since it is true that the intermediate concepts are wider than the subject (both $(\mathrm{M} \cap \mathrm{N}) \subseteq \mathrm{M}$ and $(\mathrm{M} \cap \mathrm{N}) \subseteq \mathrm{N}$ are valid) Rule ($\mathrm{IV}^{\mathrm{Neg}}$) satisfies the *Medius*-Principle$^{\mathrm{Neg}}$.

Nevertheless, the other two N-Rules violate the *Medius*-Principle$^{\mathrm{Neg}}$. Let us consider the

Rule $(\mathbf{II}^{\mathbf{Neg}})$ $\quad \cup$-*introduction in the predicate*

$$\frac{\begin{array}{c}\mathrm{A} \cap \mathrm{M} = \emptyset \\ \mathrm{A} \cap \mathrm{N} = \emptyset\end{array}}{\mathrm{A} \cap (\mathrm{M} \cup \mathrm{N}) = \emptyset}$$

'A' stands for the extension of the subject-concept, '$(\mathrm{M} \cup \mathrm{N})$' for the extension of the predicate-concept and 'M' and 'N' stand for the extensions of the intermediate concepts. Rule ($\mathrm{II}^{\mathrm{Neg}}$) satisfies the *Medius*-Principle$^{\mathrm{Neg}}$ if and only if *both* ($\mathrm{A} \subseteq \mathrm{M}$ *or* $(\mathrm{M} \cup \mathrm{N}) \subseteq \mathrm{M}$) *and* ($\mathrm{A} \subseteq \mathrm{N}$ *or* $(\mathrm{M} \cup \mathrm{N}) \subseteq \mathrm{N}$) hold (given the premises) – which obviously is in general not the case!

The same applies, *mutatis mutandis*, to Rule ($\mathrm{III}^{\mathrm{Neg}}$) which, as we said, is just Rule ($\mathrm{II}^{\mathrm{Neg}}$) up to the commutativity of $\cap$.

Bolzano is thus again wrong in writing that the *Medius*-Principle$^{\mathrm{Neg}}$ is satisfied by Rules ($\mathrm{II}^{\mathrm{Neg}}$) and ($\mathrm{III}^{\mathrm{Neg}}$). His claim rests on the false assumption that (in his notation) $M > M\ et\ N$ and $N > M\ et\ N$ do hold. By the way, when Bolzano discusses the affirmative Rule (II) ($\mathrm{A} \subseteq \mathrm{M}$, $\mathrm{A} \subseteq \mathrm{N}$ / $\mathrm{A} \subseteq \mathrm{M} \cup \mathrm{N}$), he writes, in order to prove that the *Medius*-Principle does hold, "$M < M\ et\ N$" and "$N < M\ et\ N$" (*BM* 115; 124), which is exactly the opposite of what he is claiming here (*BM* 115; 124)!

Thus, on a closer inspection the *Medius*-Principle turns out, *contra* Bolzano's claims, to be invalid both in the case of A-Proofs and in the case of N-Proofs. All in all, we can say, this early Bolzanian attempt to tackle at a "formal" level the fundamental issue of the ban on the *metábasis eis állo génos* in the context of his investigations on *rigorous proofs* is unsuccessful. Presumably, the difficulties Bolzano's approach runs into are due to his adoption of a too simple, *extensional* characterization of the notion of an *intermediate* concept occurring in a proof: this crucial notion is indeed entirely characterized in terms of the relational notions of "being wider than" and "being narrower than" which in turn, as we stressed[46], he interprets in a purely extensional way. The topic of intermediate concepts will indeed be taken up again by Bolzano in the *Theory of Science*, within a more

[46] See p. 12.

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refined setting provided by his mereological account of the notion of the *content* of a concept. Without going into details, we would like to conclude by giving a hint at one of Bolzano's later, less *naive* claims concerning the issue of intermediate concepts.

In §221 of the *Theory of Science* Bolzano reformulates the requirement of *increasing complexity* (see Sect. 2) on ground-consequence proofs as follows: "The propositions that make up the objective ground of a truth [. . .] cannot contain, taken singularly, more simple parts than their conclusion" (*WL* II, 384). In the light of this requirement, he now maintains that he would *not* recognize the syllogism in Barbara, whose standard form in the *WL* is

$$\frac{\text{What has } a\text{, has } b \qquad \text{What has } b\text{, has } c}{\text{What has } a\text{, has } c}$$

as a ground-consequence inference schema, "if either the concept b is more complex (*zusammengesetzter*) than a,[47] or b is more complex than c. For in the first case the second premise, and in the second case the first premise, would be more complex than the conclusion" (*WL* II, 384). In other words, in contrast with his claims in the *BM*, Bolzano is now saying that *not all* concrete inferences instantiating the syllogistic *schema* Barbara are of ground-consequence-kind. An instance of Barbara is of ground-consequence-kind *only if* the middle term b (expressing the intermediate concept) satisfies a sort of *Medius*-Principle which is now stated, at variance with the extensional characterization of the *BM*, in nonextensional terms of mereological complexity: b should not be more complex than the subject a and the predicate c.

§4. Final Considerations. In his *Contributions to a Better-Grounded Presentation of Mathematics*, as part of an effort to characterize what makes a proof *rigorous*, Bernard Bolzano puts forward—as we have seen—some general methodological principles as well as a fragmentary logical calculus consisting of a number of simple inference patterns. Bolzano also claims that proofs constructed by means of his simple inference patterns satisfy those methodological principles. The paper, while stressing the undeniable interest of Bolzano's early "proof-theoretical" approach, shows that such a claim is largely wrong. "Still", as P. Rusnock and J. Sebestik rightly stress, "there can be no doubt about the heuristic force of the motivating ideas."[48]

Can we learn anything general about Bolzano's understanding of proof, or of rigor, or of grounding, by looking at the constraints he suggests? We think the answer is positive: those constraints help to place Bolzano within a tradition of searching for purity of methods that begins (as we already mentioned) with Aristotle, and continues through ancient and medieval mathematics

[47]This is what Bolzano writes. The Editor of the BGA Jan Berg corrects Bolzano and replaces (BGA I 12.2, 199, fn. *g*) "*b* zusammengesetzter als *a*" by "*a* zusammengesetzter als *b*". However, Bolzano is right here, as otherwise his argument wouldn't make sense!

[48]Rusnock & Sebestik 2013, 9.

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(usually as the request to exclude arithmetical considerations from geometry), Descartes[49] and Leibniz[50] till Bolzano's program for reform, centred on what M. Detlefsen once called the "probative *ideal* of rigor":

> This was a standard for mathematical reasoning which urged the retrogressive development of proofs, a persistent and systematic tracing back of a theorem to such premises as left no further room for probative development. Judged by such a standard, a proof was taken to be rigorous only if it was *probatively complete*—that is, only if every proposition on which it relied was either itself proved or was such as to not admit of further proof.[51]

That this is at issue is confirmed, as we stressed in the Introduction, by the fact that the question of rigorous proofs concerns, according to Bolzano, not only the connection between premises and conclusion, but also the status of the premises of a proof—that is, whether they have been appropriately proven.

One might well say that this tradition finds an adequate formal expression in contemporary proof theory, in particular in the investigation of formal properties connected with the notion of derivation in normal form.

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[49] Cp. e.g. Descartes 1637, Section II, 7–13.

[50] Cp. e.g. Leibniz 1704, Ch. 17, §§1-3.

[51] Detlefsen 2010, §2.

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